

AD-A065 101

RENSSELAER POLYTECHNIC INST TROY N Y DEPT OF MECHANI--ETC F/G 9/1  
TEMPERATURE DEPENDENCE OF THE RESONANT FREQUENCY OF ELECTRODED --ETC(U)  
JAN 79 B K SINHA, H F TIERSTEN

N00014-76-C-0368

UNCLASSIFIED

TR-27

NL

OF  
ADA  
065101



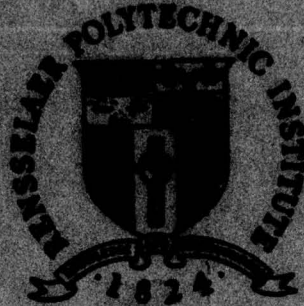
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|--|--|--|--|--|--|--|--|--|--|--|--|--|--|--|
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |

END  
DATE  
FILMED  
4 -79  
DDC

A065101

DDC FILE COPY:

(12) LEVEL 1



Rensselaer Polytechnic Institute  
Troy, New York 12181

TEMPERATURE DEPENDENCE OF THE RESONANT FREQUENCY OF ELECTRODED  
DOUBLY-ROTATED QUARTZ THICKNESS-MODE RESONATORS

by

B.K. Sinha and H.F. Tiersten

Office of Naval Research  
Contract N00014-76-C-0368  
Project NR 318-009  
Technical Report No. 27

DDC  
RECEIVED  
MR 1 1979  
B

January 1979

Distribution of this document is unlimited. Reproduction  
in whole or in part is permitted for any purpose of the  
United States Government.



**Rensselaer Polytechnic Institute  
Troy, New York 12181**

⑥ **TEMPERATURE DEPENDENCE OF THE RESONANT FREQUENCY OF ELECTRODED  
DOUBLY-ROTATED QUARTZ THICKNESS-MODE RESONATORS**

by

⑩ **B.K. Sinha and H.F. Tiersten**

⑮  
Office of Naval Research  
Contract **N00014-76-C-0368**  
Project NR 318-009  
⑨ **Technical Report No. 27**

⑪ **January 1979**

⑫ **56p.**

Distribution of this document is unlimited. Reproduction in whole or in part is permitted for any purpose of the United States Government.

409 359



Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE  |                       | READ INSTRUCTIONS<br>BEFORE COMPLETING FORM                                  |
|--|-----------------------|--|
| 1. REPORT NUMBER<br>No. 27 ✓   | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER  |
| 4. TITLE (and Subtitle)<br>TEMPERATURE DEPENDENCE OF THE RESONANT<br>FREQUENCY OF ELECTRODED DOUBLY-ROTATED<br>QUARTZ THICKNESS-MODE RESONATORS  |                       | 5. TYPE OF REPORT & PERIOD COVERED<br>Technical Report                       |
|  |                       | 6. PERFORMING ORG. REPORT NUMBER   |
| 7. AUTHOR(s)<br>H.F. Tiersten and B.K. Sinha   |                       | 8. CONTRACT OR GRANT NUMBER(s)<br>N00014-76-C-0368 ✓                         |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Department of Mechanical Engineering, ✓<br>Aeronautical Engineering & Mechanics<br>Rensselaer Polytechnic Institute<br>Troy, New York 12181   |                       | 10. PROGRAM ELEMENT, PROJECT, TASK<br>AREA & WORK UNIT NUMBERS<br>NR 318-009 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Office of Naval Research<br>Physics Branch  |                       | 12. REPORT DATE<br>January 1979  |
|  |                       | 13. NUMBER OF PAGES<br>52  |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)  |                       | 15. SECURITY CLASS. (of this report)<br>Unclassified                         |
|  |                       | 15a. DECLASSIFICATION/DOWNGRADING<br>SCHEDULE                                |
| 16. DISTRIBUTION STATEMENT (of this Report)<br>Distribution of this document is unlimited.   |                       |  |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)   |                       |  |
| 18. SUPPLEMENTARY NOTES  |                       |  |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)<br>Frequency Changes    Thermoelectricity    Perturbations<br>Quartz Resonators    Plate Equations    Nonlinear Elasticity<br>Thickness Vibrations    Electroded Plates    Temperature Dependence<br>Thermal Bias    Extension    Crystal Plates<br>Thermal Stresses    Electrode Stresses    Piezoelectricity  |                       |  |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br>A system of approximate equations for the determination of thermal stresses in piezoelectric plates with large thin films of a different material plated on the surfaces is derived. The plate equations are obtained by making a suitable expansion of the pertinent variables in the thickness coordinate, inserting the expansion in the appropriate variational principle and integrating with respect to the thickness in the manner of Mindlin. Conditions resulting in both extensional and flexural |                       |  |

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE  
S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

79 02 23 033



**Unclassified**

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

stresses are considered and the full anisotropy of the quartz is included in the treatment. The particular case of purely extensional thermal stresses resulting from large electrodes of equal thickness plated on the major surfaces of doubly-rotated quartz thickness-mode resonators is treated in detail. The changes in resonant frequency resulting from the thermally induced biasing stresses and strains are determined from an existing perturbation equation. Calculations, using the newly defined first temperature derivatives of the fundamental elastic constants of quartz, are performed for large gold electrodes on doubly-rotated quartz plates.

**Unclassified**

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

~~0~~

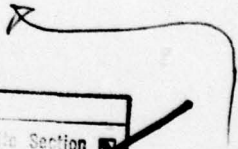
TEMPERATURE DEPENDENCE OF THE RESONANT FREQUENCY OF ELECTRODED  
DOUBLY-ROTATED QUARTZ THICKNESS-MODE RESONATORS

H.F. Tiersten and B.K. Sinha  
Department of Mechanical Engineering,  
Aeronautical Engineering & Mechanics  
Rensselaer Polytechnic Institute  
Troy, New York 12181



ABSTRACT

A system of approximate equations for the determination of thermal stresses in piezoelectric plates with large thin films of a different material plated on the surfaces is derived. The plate equations are obtained by making a suitable expansion of the pertinent variables in the thickness coordinate, inserting the expansion in the appropriate variational principle and integrating with respect to the thickness in the manner of Mindlin. Conditions resulting in both extensional and flexural stresses are considered and the full anisotropy of the quartz is included in the treatment. The particular case of purely extensional thermal stresses resulting from large electrodes of equal thickness plated on the major surfaces of doubly-rotated quartz thickness-mode resonators is treated in detail. The changes in resonant frequency resulting from the thermally induced biasing stresses and strains are determined from an existing perturbation equation. Calculations, using the newly defined first temperature derivatives of the fundamental elastic constants of quartz, are performed for large gold electrodes on doubly-rotated quartz plates.



|                                 |   |
|---------------------------------|---|
| ACCESSION for                   |   |
| NTIS                            | Write Section <input checked="" type="checkbox"/> |
| DDC                             | Diff Section <input type="checkbox"/>             |
| UNANNOUNCED                     | <input type="checkbox"/>                          |
| JUSTIFICATION                   |   |
| BY                              |   |
| DISTRIBUTION/AVAILABILITY CODES |   |
| Dist. Avail. and/or SPECIAL     |   |
| A                               |   |



1. Introduction

A perturbation analysis of the linear electroelastic equations for small fields superposed on a bias has been performed<sup>1</sup>. The change in resonant frequency due to any bias such as, e.g., a residual stress may readily be obtained from the resulting equation for the first perturbation of the eigenvalue if the bias is known. The use of this perturbation equation has already been shown to be extremely accurate in the determination of changes in the surface wave velocity of crystals due to flexural biasing stresses<sup>2</sup>.

In this paper a system of approximate plate equations for the determination of thermal stresses in thin piezoelectric plates coated with much thinner films is derived in the manner of Mindlin<sup>3-5</sup>. The resulting approximate equations simplify the treatment of many thermal stress problems considerably, and the three-dimensional detail not included in the approximate description is not deemed to be important for our purposes. In order to keep the derivation of the thermoelastic plate equations clear and not introduce extraneous complications that can lead to confusion at the outset, we first ignore the elastic constants that cause coupling between shear and extension in the constitutive equations. Both extensional and flexural plate equations are obtained. After we obtain the thermoelastic plate equations under the aforementioned simplifying assumptions, we extend them to the general anisotropic case. It should be noted that coupling between shear and extension exists and is important even in the case of rotated Y-cut quartz. The general anisotropic version of the thermoelastic plate equations is applied to the case of purely extensional thermal stresses arising from large rectangular identical electrodes on doubly-rotated quartz plates and a simple exact solution of the extensional plate equations is obtained. This solution of the static



approximate thermoelastic plate equations for the arbitrarily anisotropic plated crystal plate with rectangular electrodes is readily shown to hold for large electrodes of essentially arbitrary shape. The approximate three-dimensional displacement field resulting from the solution of the plate equations is readily determined from the description. This three-dimensional displacement field is required in order to obtain the change in resonant frequency due to the thermally induced biasing state from the equation for the first perturbation of the eigenvalue.

Since the total change in frequency of quartz plates is due not only to the thermal stresses and strains but also to the change in the fundamental elastic constants with temperature, in order to complete the above-mentioned calculations of the resulting actual change in frequency per degree change in temperature, the newly defined<sup>6</sup> temperature derivatives of the fundamental elastic constants of quartz must be employed. These newly defined<sup>6</sup> temperature derivatives of the fundamental elastic constants of quartz are used, along with the aforementioned thermal stress analysis of the quartz plate and the perturbation equation, in performing calculations of the change in frequency per degree change in temperature for rectangular gold electrodes on doubly rotated quartz plates. The results indicate that there are whole ranges of doubly-rotated orientations for which the change in frequency with temperature due to the electrodes is more than an order of magnitude less than that of the AT-cut.

## 2. Perturbation Equations

The equation for the first perturbation of the eigenvalue obtained from the perturbation analysis mentioned in the Introduction may be written in the form

$$\Delta_\mu = H_\mu / 2\omega_\mu, \quad \omega = \omega_\mu - \Delta_\mu, \quad (2.1)$$

where  $\omega_\mu$  and  $\omega$  are the unperturbed and perturbed eigenfrequencies, respectively, and

$$H_\mu = \int_S N_L [\tilde{K}_{LY}^l \sigma_Y^\mu - k_{LY}^\mu u_Y + \tilde{D}_L^l \dot{\tilde{\varphi}}^\mu - d_L^\mu \dot{\tilde{\varphi}}] dS + \int_V [\tilde{K}_{LY,L}^n \sigma_Y^\mu + \tilde{D}_{L,L}^n \dot{\tilde{\varphi}}^\mu] dV, \quad (2.2)$$

in which  $S$  is a surface enclosing a volume  $V$ . In (2.2)  $N_L$  denotes the unit normal to the undeformed surface at the reference temperature,  $u_Y$  denotes the mechanical displacement vector and  $\tilde{\varphi}$  denotes the electric potential. The linear stress  $\tilde{K}_{LY}^l$  and electric displacement vector  $\tilde{D}_L^l$  are given by the usual linear piezoelectric constitutive relations

$$\begin{aligned} \tilde{K}_{LY}^l &= c_{2LYM\alpha} u_{\alpha,M} + e_{MLY} \tilde{\varphi}_{,M}, \\ \tilde{D}_L^l &= e_{LMY} u_{Y,M} - \epsilon_{LM} \tilde{\varphi}_{,M}, \end{aligned} \quad (2.3)$$

and as usual satisfy

$$\tilde{K}_{LY,L}^l = \rho \ddot{u}_Y, \quad \tilde{D}_{L,L}^l = 0. \quad (2.4)$$

The quantities  $c_{2LYM\alpha}$ ,  $e_{MLY}$  and  $\epsilon_{LM}$  denote the second order elastic, piezoelectric and dielectric constants, respectively, and  $\rho$  denotes the mass density. The nonlinear contributions to the Piola-Kirchhoff stress tensor  $\tilde{K}_{LY}^n$  and reference electric displacement vector  $\tilde{D}_L^n$  take the respective forms<sup>1</sup>

$$\begin{aligned}\tilde{K}_{LY}^n &= (\hat{c}_{LYM\alpha} + \Delta c_{2LYM\alpha}) u_{\alpha, M} + \hat{e}_{MLY} \tilde{\varphi}_{, M}, \\ \tilde{D}_L^n &= \hat{e}_{LMY} u_{Y, M} - \hat{\epsilon}_{LM} \tilde{\varphi}_{, M},\end{aligned}\quad (2.5)$$

where  $\hat{c}_{LYM\alpha}$ ,  $\hat{e}_{MLY}$  and  $\hat{\epsilon}_{LM}$  are effective constants that depend on the biasing state<sup>1</sup> and  $\Delta c_{2LYM\alpha}$  denotes a small change in the fundamental elastic constants due in this instance to a change in temperature. Clearly, the total dynamic Piola-Kirchhoff stress tensor and reference electric displacement vector are given by

$$\tilde{K}_{LY} = \tilde{K}_{LY}^l + \tilde{K}_{LY}^n, \quad \tilde{D}_L = \tilde{D}_L^l + \tilde{D}_L^n. \quad (2.6)$$

The vector  $g_Y^\mu$  denotes the normalized mechanical displacement for the  $\mu$ th unperturbed mode and  $\hat{F}^\mu$  denotes the normalized electric potential for the  $\mu$ th mode, i.e.,

$$g_Y^\mu = \frac{u_Y^\mu}{N_\mu}, \quad \hat{F}^\mu = \frac{\varphi^\mu}{N_\mu}, \quad (2.7)$$

where

$$N_\mu^2 = \int_V \rho u_Y^\mu u_Y^\mu dV. \quad (2.8)$$

The normalized stress tensor  $k_{LY}^\mu$  and electric displacement vector  $d_L^\mu$ , both for the normalized  $\mu$ th mode, are given by

$$\begin{aligned}k_{LY}^\mu &= c_{LYM\alpha} g_{\alpha, M}^\mu + e_{MLY} \hat{F}_{, M}^\mu, \\ d_L^\mu &= e_{LMY} g_{Y, M}^\mu - \epsilon_{LM} \hat{F}_{, M}^\mu.\end{aligned}\quad (2.9)$$

The upper case notation for many dynamic variables and the capital Latin and lower case Greek index notation is being employed for consistency with Ref. 1, as is the remainder of the notation in this section. The fact that the capital Latin and lower case Greek indices refer to the reference and intermediate positions of material points respectively, is not important here, and in this



work they may be used interchangeably. We employ Cartesian tensor notation, the summation convention for repeated tensor indices, the convention that a comma followed by an index denotes partial differentiation with respect to a reference coordinate and the dot notation for differentiation with respect to time.

When the electrical and electroelastic nonlinearities are ignored we have

$$\hat{\epsilon}_{LM} = 0, \quad \hat{\epsilon}_{ML\gamma} = 0, \quad (2.10)$$

and

$$\hat{c}_{L\gamma M\alpha} = T_{LM}^1 \delta_{\gamma\alpha} + c_{3L\gamma M\alpha AB} E_{AB}^1 + c_{2L\gamma KM} w_{\alpha,K} + c_{2LKM\alpha} w_{\gamma,K}, \quad (2.11)$$

where

$$T_{LM}^1 = c_{2LMKN} E_{KN}^1, \quad E_{KN}^1 = \frac{1}{2} (w_{K,N} + w_{N,K}), \quad (2.12)$$

$c_{3L\gamma M\alpha AB}$  denote the third order elastic constants and  $w_K$  denotes the static biasing displacement field. Thus, in this description the present position  $\underline{y}$  is related to the reference position  $\underline{x}$  by

$$\underline{y}(X_L, t) = \underline{x} + \underline{w}(X_L) + \underline{u}(X_L, t). \quad (2.13)$$

For the electroded crystal plate with shorted electrodes, which is of interest here, the boundary conditions take the form<sup>7</sup>

$$N_L \tilde{K}_{L\gamma} = \delta_{\gamma B} 2h' \gamma'_{ABCD} u_{C,DA} - 2h' \rho' \ddot{u}_{\gamma}, \quad \tilde{\varphi} = 0, \quad \text{on } S, \quad (2.14)$$

where A, B, C, D take values confined to the surface of the crystal plate and skip the value associated with the normal,  $2h'$  is the thickness of the electrode and  $\rho'$  the electrode mass density. Since the electrodes are isotropic, the plating stiffnesses are given by

$$\gamma'_{ABCD} = \lambda'_0 \delta_{AB} \delta_{CD} + \mu' (\delta_{AC} \delta_{BD} + \delta_{AD} \delta_{BC}), \quad (2.15)$$

where the plate Lamé constant  $\lambda'_0$  is given by

$$\lambda'_0 = 2\mu' \lambda' / (\lambda' + 2\mu'), \quad (2.16)$$

and  $\lambda'$  and  $\mu'$  are the Lamé constants for the plating material. The boundary conditions satisfied by the normalized  $\mu$ th eigensolution of the unperturbed linear system are given by

$$N_{L\gamma}^{\mu} = \delta_{\gamma B} 2h' \gamma'_{ABCD} g_{C,D}^{\mu} - 2h' \rho' g_{\gamma}^{\mu}, \quad \hat{F}^{\mu} = 0, \quad \text{on } S. \quad (2.17)$$

Taking  $u_{\gamma}$  and  $\tilde{\varphi}$  to be the unperturbed normalized  $\mu$ th eigensolution  $g_{\gamma}^{\mu}$  and  $\hat{F}^{\mu}$ , respectively, substituting from (2.6)<sub>1</sub>, (2.14), (2.17), (2.5)<sub>2</sub> and (2.10) into (2.2) and employing the divergence theorem, we obtain

$$H_{\mu} = - \int_V \tilde{K}_{L\gamma}^n g_{\gamma,L}^{\mu} dv. \quad (2.18)$$

Now, the substitution of (2.5)<sub>1</sub> and (2.10)<sub>2</sub> along with the aforementioned selection of  $u_{\gamma}$  and  $\tilde{\varphi}$  into (2.18) yields

$$H_{\mu} = - \int_V (\hat{c}_{L\gamma M\alpha} + \Delta c_{2L\gamma M\alpha}) g_{\alpha,M}^{\mu} g_{\gamma,L}^{\mu} dv, \quad (2.19)$$

which, with (2.1), (2.11) and the given small change in the fundamental elastic constants  $\Delta c_{2L\gamma M\alpha}$  due in this instance to a change in temperature<sup>6</sup>, gives the change in resonant frequency due to a biasing deformation and the change in elastic constants resulting from a change in temperature.

### 3. Static Plate Equations

A schematic diagram of the plated crystal plate is shown in Fig.1 along with the associated coordinate system. Referred to this coordinate system the static form of Mindlin's plate equations may be written<sup>3</sup>

$$K_{AJ,A}^{(n)} - nK_{2J}^{(n-1)} + F_J^{(n)} = 0, \quad n=0,1,2, \quad (3.1)$$

where A, B, C, D take the values 1 and 3 and skip 2 and

$$K_{IJ}^{(n)} = \int_{-h}^h x_2^n K_{IJ} dx_2, \quad F_J^{(n)} = [x_2^n K_{2J}]_{-h}^h. \quad (3.2)$$

The linear thermoelastic constitutive equations for the mth order stress resultants for homogeneous temperature excursions  $(T - T_0)$  take the form

$$K_{IJ}^{(m)} = c_{IJKL} \sum_{n=0}^2 H_{mn} E_{KL}^{(n)} - d^{(m)} \nu_{IJ} (T - T_0), \quad (3.3)$$

where we have taken the liberty of dropping the lower script 2 on the  $c_{IJKL}$ , the  $\nu_{IJ}$  denote the thermoelastic constants and

$$H_{mn} = \begin{cases} 2h^{(m+n+1)} / (m+n+1) & m+n \text{ even} \\ 0 & m+n \text{ odd,} \end{cases} \quad (3.4)$$

$$d^{(m)} = \begin{cases} 2h^{(m+1)} / (m+1) & m \text{ even} \\ 0 & m \text{ odd.} \end{cases} \quad (3.5)$$

The nth order plate strains take the usual form

$$E_{KL}^{(n)} = \frac{1}{2} [w_{K,L}^{(n)} + w_{L,K}^{(n)} + (n+1)(\delta_{2L} w_K^{(n+1)} + \delta_{2K} w_L^{(n+1)})], \quad (3.6)$$

where

$$w_A = \sum_{n=0}^2 x_2^n w_A^{(n)}, \quad w_2 = \sum_{n=0}^3 x_2^n w_2^{(n)}, \quad (3.7)$$

$$E_{AB} = \sum_{n=0}^2 x_2^n E_{AB}^{(n)}, \quad E_{22} = \sum_{n=0}^3 x_2^n E_{22}^{(n)}. \quad (3.8)$$

The thermoelastic constants  $\nu_{IJ}$  are related to the coefficients of linear expansion  $\alpha_{KL}$  by the usual relation<sup>8</sup>

$$\nu_{IJ} = c_{IJKL} \alpha_{KL}. \quad (3.9)$$



The static form of Mindlin's simplified extensional equations for the very thin electrode platings may be written<sup>9</sup>

$$K_{AB,A}^{(0)'} + F_B^{(0)'} = 0, \quad F_B^{(0)'} = [K_{2B}'] h', \quad (3.10)$$

where

$$K_{AB}^{(0)'} = 2h' \gamma'_{ABCD} E_{CD}^{(0)'} - 2h' v_{AB}^{*'} (T - T_0), \quad (3.11)$$

and since the platings are isotropic in the plane

$$\begin{aligned} \gamma'_{ABCD} &= \lambda'_0 \delta_{AB} \delta_{CD} + \mu' (\delta_{AC} \delta_{BD} + \delta_{AD} \delta_{BC}), \\ \lambda'_0 &= 2\mu' \lambda' / (\lambda' + 2\mu'), \quad v_{AB}^{*'} = [2\mu' (3\lambda' + 2\mu') / (\lambda' + 2\mu')] \alpha' \delta_{AB}, \end{aligned} \quad (3.12)$$

which are for the upper plating and similar equations hold for the bottom plating, but with double primes replacing the primes. In order to obtain the static equations for the plated crystal plate we need the boundary conditions at all interfaces. The traction conditions take the form

$$\begin{aligned} K_{2A}'(h') &= 0, \quad K_{2A}''(-h'') = 0, \\ K_{2A}'(-h') &= K_{2A}(h), \quad K_{2A}''(h'') = K_{2A}(-h), \end{aligned} \quad (3.13)$$

and the conditions of continuity of mechanical displacement take the form

$$\begin{aligned} w_A^{(0)'} &= w_A^{(0)} + h w_A^{(1)} + h^2 w_A^{(2)}, \\ w_A^{(0)''} &= w_A^{(0)} - h w_A^{(1)} + h^2 w_A^{(2)}, \end{aligned} \quad (3.14)$$

since the thickness displacements of the thin platings  $w_2^{(0)'}$  and  $w_2^{(0)''}$  take place freely.

In this section in order to keep the derivation of the extensional and flexural equations for the plated crystal plate clear we simplify the treatment somewhat and ignore the elastic and thermoelastic constants that cause coupling between shear and extension in the constitutive equations.

This simplification and attendant restriction of the equations is removed in the next section. When flexure and extension are uncoupled, from (3.1) we may write the extensional plate equations in the form

$$\begin{aligned} K_{AB,A}^{(0)} + F_B^{(0)} &= 0, \quad K_{A2,A}^{(1)} - K_{22}^{(0)} + F_2^{(1)} = 0, \\ K_{AB,A}^{(2)} - 2K_{2B}^{(1)} + F_B^{(2)} &= 0. \end{aligned} \quad (3.15)$$

In order to allow for the free<sup>10</sup> thickness strains  $E_{22}^{(0)}$  and  $E_{22}^{(2)}$ , we take

$$K_{22}^{(0)} = 0, \quad K_{22}^{(2)} = 0. \quad (3.16)$$

In addition, in order to eliminate the first order extensional equation in (3.15) completely<sup>11</sup>, we take

$$K_{2A}^{(1)} = 0. \quad (3.17)$$

From (3.16) and the reduced form of (3.3), with (3.4) and (3.5), we obtain

$$\begin{aligned} E_{22}^{(0)} &= -\frac{c_{21}}{c_{22}} E_{11}^{(0)} - \frac{c_{23}}{c_{22}} E_{33}^{(0)} - \frac{\nu_{22}}{c_{22}} (T - T_0), \\ E_{22}^{(2)} &= -\frac{c_{21}}{c_{22}} E_{11}^{(2)} - \frac{c_{23}}{c_{22}} E_{33}^{(2)}, \end{aligned} \quad (3.18)$$

the substitution of which in the reduced form of (3.3) yields

$$\begin{aligned} K_{11}^{(0)} &= 2h[c_{11}^* E_{11}^{(0)} + c_{13}^* E_{33}^{(0)}] + \frac{2}{3} h^3 [c_{11}^* E_{11}^{(2)} + c_{13}^* E_{33}^{(2)}] - 2h\nu_{11}^* (T - T_0), \\ K_{33}^{(0)} &= 2h[c_{13}^* E_{11}^{(0)} + c_{33}^* E_{33}^{(0)}] + \frac{2}{3} h^3 [c_{13}^* E_{11}^{(2)} + c_{33}^* E_{33}^{(2)}] - 2h\nu_{33}^* (T - T_0), \\ K_{11}^{(2)} &= \frac{2}{3} h^3 [c_{11}^* E_{11}^{(0)} + c_{13}^* E_{33}^{(0)}] + \frac{2}{5} h^5 [c_{11}^* E_{11}^{(2)} + c_{33}^* E_{33}^{(2)}] - \frac{2}{3} h^3 \nu_{11}^* (T - T_0), \\ K_{33}^{(2)} &= \frac{2}{3} h^3 [c_{13}^* E_{11}^{(0)} + c_{33}^* E_{33}^{(0)}] + \frac{2}{5} h^5 [c_{13}^* E_{11}^{(2)} + c_{33}^* E_{33}^{(2)}] - \frac{2}{3} h^3 \nu_{33}^* (T - T_0), \\ K_{13}^{(0)} &= 2hc_{55} \left[ E_{13}^{(0)} + \frac{h^2}{3} E_{13}^{(2)} \right], \\ K_{13}^{(2)} &= 2h^3 c_{55} \left[ \frac{1}{3} E_{13}^{(0)} + \frac{h^2}{5} E_{13}^{(2)} \right], \end{aligned} \quad (3.19)$$

where

$$c_{11}^* = c_{11} - \frac{c_{12}^2}{c_{22}}, \quad c_{13}^* = c_{13} - \frac{c_{12}c_{23}}{c_{22}}, \quad c_{33}^* = c_{33} - \frac{c_{32}^2}{c_{22}},$$

$$v_{11}^* = v_{11} - v_{22}c_{12}/c_{22}, \quad v_{33}^* = v_{33} - v_{22}c_{32}/c_{22}. \quad (3.20)$$

From (3.15), with (3.16) and (3.17), we obtain the resulting second order extensional equations

$$K_{AB,A}^{(0)} + F_B^{(0)} = 0, \quad K_{AB,A}^{(2)} + F_B^{(2)} = 0. \quad (3.21)$$

In this approximation from (3.6) the pertinent plate strains  $E_{AB}^{(n)}$  ( $n=0,2$ ) take the form

$$E_{11}^{(n)} = w_{1,1}^{(n)}, \quad E_{33}^{(n)} = w_{3,3}^{(n)}, \quad E_{13}^{(n)} = \frac{1}{2} (w_{1,3}^{(n)} + w_{3,1}^{(n)}). \quad (3.22)$$

When written out, the constitutive equations, (3.11), for the upper electrode plating take the form

$$K_{11}^{(0)'} = 2h' [ (\lambda_0' + 2\mu') w_{1,1}^{(0)'} + \lambda_0' w_{3,3}^{(0)'} ] - 2h' v^{*'} (T - T_0),$$

$$K_{33}^{(0)'} = 2h' [ (\lambda_0' + 2\mu') w_{3,3}^{(0)'} + \lambda_0' w_{1,1}^{(0)'} ] - 2h' v^{*'} (T - T_0),$$

$$K_{13}^{(0)'} = 2h' \mu' (w_{1,3}^{(0)'} + w_{3,1}^{(0)'}), \quad (3.23)$$

and similar equations exist for the lower electrode plating, but with primes replaced by double primes. In the case of these extensional equations the displacements  $w_A^{(0)'}$  and  $w_A^{(0)''}$ , in the upper and lower electrode platings, respectively, are given by

$$w_A^{(0)'} = w_A^{(0)} + h^2 w_A^{(2)}, \quad w_A^{(0)''} = w_A^{(0)} + h^2 w_A^{(2)}. \quad (3.24)$$

The equations for the plated crystal plate are obtained by employing (3.13) in (3.10) for both the upper and lower platings and then inserting (3.10) for both platings in (3.2)<sub>2</sub>, with the result



$$F_B^{(0)} = K_{AB,A}^{(0)'} + K_{AB,A}^{(0)''}, \quad F_B^{(2)} = h^2 (K_{AB,A}^{(0)'} + K_{AB,A}^{(0)''}). \quad (3.25)$$

The substitution of (3.25) into (3.21) yields

$$[K_{AB}^{(0)} + K_{AB}^{(0)'} + K_{AB}^{(0)''}]_{,A} = 0, \quad [K_{AB}^{(2)} + h^2 (K_{AB}^{(0)'} + K_{AB}^{(0)''})]_{,A} = 0, \quad (3.26)$$

which are the second order extensional equations of equilibrium of the plated crystal plate. Clearly, Eqs. (3.26) are consistent with the integral forms

$$\oint_C N_A \chi_{AB}^{(0)} ds = 0, \quad \oint_C N_A \chi_{AB}^{(2)} ds = 0, \quad (3.27)$$

where

$$\chi_{AB}^{(0)} = K_{AB}^{(0)} + K_{AB}^{(0)'} + K_{AB}^{(0)''}, \quad \chi_{AB}^{(2)} = K_{AB}^{(2)} + h^2 (K_{AB}^{(0)'} + K_{AB}^{(0)''}), \quad (3.28)$$

and  $N_A$  denotes the outwardly directed unit normal to the undeformed position of the closed curve  $c$  in the plane of the plate. Hence, on an edge of the plated crystal plate, the traction boundary conditions that accompany (3.26) are

$$\begin{aligned} N_A \chi_{AB}^{(0)} &= \bar{K}_B^{(0)}, \\ N_A \chi_{AB}^{(2)} &= \bar{K}_B^{(2)}, \end{aligned} \quad (3.29)$$

where the  $\bar{K}_B^{(n)}$  represent applied traction resultants on the edge. The alternative displacement conditions are on  $w_B^{(0)}$  and  $w_B^{(2)}$ . The specification of appropriate combinations of these prescribed conditions may readily be shown to be unique to within homogeneous rigid plate rotations of zero and second order by means of the usual Neumann type procedure<sup>12,13,4</sup>. At an edge of discontinuity separating one region from another, we require the continuity of

$$N_A \chi_{AB}^{(0)}, \quad N_A \chi_{AB}^{(2)}, \quad w_B^{(0)}, \quad w_B^{(2)}. \quad (3.30)$$

Since the upper and lower electrodes can have different thicknesses  $2h'$  and  $2h''$ , flexure can occur even though the temperature change  $(T - T_0)$  is homogeneous. In order to obtain the flexural equations we first write the plate equations in the form

$$K_{A2,A}^{(0)} + F_2^{(0)} = 0, \quad K_{AB,A}^{(1)} - K_{2B}^{(0)} + F_B^{(1)} = 0, \quad K_{A2,A}^{(2)} - 2K_{22}^{(1)} + F_2^{(2)} = 0. \quad (3.31)$$

In order to allow for the free thickness-strains<sup>10</sup>, we take

$$K_{22}^{(1)} = 0. \quad (3.32)$$

In order to eliminate the second order flexural equation in (3.31) completely<sup>14</sup>, we take

$$K_{2A}^{(2)} = 0. \quad (3.33)$$

In addition, as usual in the elementary theory of flexure we take the thickness-shear plate strains  $E_{2A}^{(0)}$  to vanish<sup>15</sup>, which with (3.6) yields

$$w_A^{(1)} = -w_{2,A}^{(0)}. \quad (3.34)$$

From (3.32) and the reduced form of (3.3), with (3.4) and (3.5), we obtain

$$E_{22}^{(1)} = -\frac{c_{21}}{c_{22}} E_{11}^{(1)} - \frac{c_{23}}{c_{22}} E_{33}^{(1)}, \quad (3.35)$$

the substitution of which in the reduced form of (3.3) yields the flexural constitutive equations in the form

$$\begin{aligned} K_{11}^{(1)} &= \frac{2}{3} h^3 [c_{11}^* E_{11}^{(1)} + c_{13}^* E_{33}^{(1)}], \quad K_{13}^{(1)} = \frac{2}{3} h^3 c_{55} E_{13}^{(1)} \\ K_{33}^{(1)} &= \frac{2}{3} h^3 [c_{13}^* E_{11}^{(1)} + c_{33}^* E_{33}^{(1)}], \end{aligned} \quad (3.36)$$

where  $c_{11}^*$ ,  $c_{33}^*$  and  $c_{13}^*$  are given in (3.20). From (3.6) and (3.33) we find that the pertinent plate strains take the form

$$E_{AB}^{(1)} = \frac{1}{2} (w_{A,B}^{(1)} + w_{B,A}^{(1)}) = -w_{2,AB}^{(0)} \quad (3.37)$$

From (3.31) - (3.33), we obtain the usual flexural equations

$$K_{B2,B}^{(0)} + F_2^{(0)} = 0, \quad K_{AB,A}^{(1)} - K_{2B}^{(0)} + F_B^{(1)} = 0, \quad (3.38)$$

where we have taken proper account of the fact that the thickness-shear stress resultants  $K_{2A}^{(0)}$  exist even though the associated thickness-shear plate strains  $E_{2A}^{(0)}$  vanish. The substitution of (3.38)<sub>2</sub> into (3.38)<sub>1</sub> yields

$$K_{AB,AB}^{(1)} + F_{B,B}^{(1)} + F_2^{(0)} = 0, \quad (3.39)$$

which is the equation of the flexure of thin plates. The equation of flexure for the plated crystal plate is obtained by employing (3.13) in (3.10) for both the upper and lower platings and then inserting (3.10) for both platings in (3.2)<sub>2</sub>, with the result

$$F_2^{(0)} = 0, \quad F_B^{(1)} = h[K_{AB,A}^{(0)'} - K_{AB,A}^{(0)''}] \quad (3.40)$$

In this case of flexure the displacements  $w_A^{(0)'}$  and  $w_A^{(0)''}$ , in the upper and lower electrode platings, respectively, are given by

$$w_B^{(0)'} = hw_B^{(1)} = -hw_{2,B}^{(0)}, \quad w_B^{(0)''} = -hw_B^{(1)} = hw_{2,B}^{(0)} \quad (3.41)$$

The substitution of (3.40) into (3.39) yields

$$[K_{AB}^{(1)} + h(K_{AB}^{(0)'} - K_{AB}^{(0)''})]_{,AB} = 0, \quad (3.42)$$

which is the equation of static flexure for the plated crystal plate.

The substitution of (3.40) into (3.38) yields

$$K_{B2,B}^{(0)} = 0, \quad [K_{AB}^{(1)} + h(K_{AB}^{(0)'} - K_{AB}^{(0)''})]_{,A} - K_{2B}^{(0)} = 0, \quad (3.43)$$



which are consistent with the integral forms

$$\oint_C N_B K_{B2}^{(0)} ds = 0, \quad \oint_C N_A \mathcal{M}_{AB} ds = \int_A K_{2B}^{(0)} ds, \quad (3.44)$$

where

$$\mathcal{M}_{AB} = K_{AB}^{(1)} + h(K_{AB}^{(0)'} - K_{AB}^{(0)''}), \quad (3.45)$$

and  $A$  is the area enclosed by  $C$ . On account of (3.34), the independent edge conditions may not be obtained directly from (3.44). However, since Eqs. (3.34), (3.36), (3.37) and (3.42) - (3.44) are identical in form with the equations of the elementary theory of the flexure of thin plates in the absence of applied loading, the edge conditions for the plated plate are the same as in the elementary theory of flexure of thin plates. The Neumann type uniqueness theorem<sup>15</sup> for the elementary theory of the flexure of thin plates shows that the traction boundary conditions take the form

$$N_A \mathcal{M}_{AB} N_B = \bar{m}, \quad K_{N2}^{(0)} + \frac{\partial}{\partial s} \mathcal{M}_{Ns} = \bar{v} + \frac{\partial \bar{t}}{\partial s}, \quad (3.46)$$

where

$$K_{N2}^{(0)} = N_B K_{2B}^{(0)}, \quad \mathcal{M}_{Ns} = N_A \mathcal{M}_{AB} \mathcal{J}_B, \quad (3.47)$$

and  $\mathcal{J}_B$  is a unit vector tangent to  $c$  in the counterclockwise direction and  $\bar{m}$ ,  $\bar{t}$  and  $\bar{v}$  are the prescribed bending moment, twisting moment and vertical shearing force, respectively, applied on an edge. The alternate kinematic conditions are on  $w_2^{(0)}$  and  $\partial w_2^{(0)} / \partial N$ . At an edge of discontinuity separating one region from another, we require the continuity of

$$N_A \mathcal{M}_{AB} N_B, \quad K_{N2}^{(0)} + \frac{\partial}{\partial s} \mathcal{M}_{Ns}, \quad w_2^{(0)}, \quad \frac{\partial}{\partial N} w_2^{(0)}. \quad (3.48).$$

#### 4. Static Plate Equations for General Anisotropy

In this section we extend both the extensional and flexural plate equations obtained in Sec.3 under simplifying restrictive assumptions to the general anisotropic case. However, we still assume that the basic assumptions of extension and elementary flexure used in the last section hold. Under these circumstances the extensional and flexural equations uncouple and may be obtained separately. Hence, we again obtain the extensional equations (3.15) and flexural equations (3.31), respectively, from (3.1).

In order to eliminate flexure from the extensional equations, from (3.31) it is clear that we must have

$$K_{A2}^{(0)} = 0, \quad K_{AB}^{(1)} = 0, \quad K_{22}^{(1)} = 0, \quad K_{A2}^{(2)} = 0, \quad (4.1)$$

along with

$$F_B^{(1)} = 0, \quad (4.2)$$

since

$$F_2^{(0)} \equiv 0, \quad F_2^{(2)} \equiv 0. \quad (4.3)$$

In addition, in order to allow for the free thickness strains  $E_{22}^{(0)}$  and  $E_{22}^{(2)}$ , we again have (3.16). Furthermore, since we have allowed for the free thickness strain  $E_{22}^{(0)} = w_2^{(1)}$ , we must eliminate the first order extensional equation in (3.15) completely again with the conditions in (3.17). From (4.1), (3.16) and (3.17), we have

$$K_{2K}^{(0)} = 0, \quad K_{2K}^{(2)} = 0, \quad K_{KL}^{(1)} = 0. \quad (4.4)$$

Since the temperature change  $(T - T_0)$  is homogeneous, from (4.4), and (3.3) - (3.5) all first order plate strains  $E_{KL}^{(1)}$  are implicitly taken to vanish in the constitutive equations for the first order stress

resultants even though the first order plate strains  $E_{2A}^{(1)}$ , which from (3.6) are given by

$$E_{2A}^{(1)} = \frac{1}{2} (w_{2,A}^{(1)} + 2w_A^{(2)}), \quad (4.5)$$

actually exist in the case of extension by virtue of the fact that  $w_2^{(1)}$  and  $w_A^{(2)}$  are nonzero for extension. If the first order plate strains  $E_{2A}^{(1)}$  are to be included in the constitutive equations for the  $K_{KL}^{(1)}$ , the conditions (3.16)<sub>1</sub> and (3.17) must be eliminated and  $w_2^{(1)}$  must be retained in the description on the same footing as the  $w_A^{(0)}$  and  $w_A^{(2)}$ . These comments apply even in the case of the isotropic plate and pertain to the description developed in Sec.3. In the resulting extensional description obtained with the full inclusion of the variable  $w_2^{(1)}$ , at the very least flexural deformations will be induced in the general anisotropic case. These flexural deformations will not occur in the isotropic case or in the case of the restricted anisotropy considered in Sec.3. The inclusion of the  $E_{2A}^{(1)}$  in the constitutive equations for the  $K_{KL}^{(1)}$  and the attendant relaxation of conditions imposed and extension of the description is quite cumbersome and known to have a negligible influence on the results in the isotropic case and deemed to have a sufficiently small influence on the results in the general anisotropic case that it may be omitted without appreciable error.

Employing (3.16)<sub>1</sub> and (3.17), which still have been assumed to hold in this general anisotropic case, in (3.15) we again obtain (3.21) as the second order extensional plate equations. From (4.4)<sub>1-2</sub> and (3.3), we obtain

$$E_W^{(0)} + \frac{h^2}{3} E_W^{(2)} = -c_{WV}^{-1} c_{VS} \left( E_S^{(0)} + \frac{h^2}{3} E_S^{(2)} \right) + c_{WV}^{-1} v_V (T - T_0),$$



$$E_W^{(0)} + \frac{3}{5} h^2 E_W^{(2)} = -c_{WV}^{-1} c_{VS} \left( E_S^{(0)} + \frac{3}{5} h^2 E_S^{(2)} \right) + c_{WV}^{-1} \nu_V (T - T_0), \quad (4.6)$$

where we have introduced the usual compressed matrix notation<sup>16</sup> for tensor indices according to the scheme

$$R, S = 1, 3, 5; \quad W, V = 2, 4, 6. \quad (4.7)$$

Substituting from (4.6) into the constitutive equations for the nonzero extensional stress resultants in (3.3), we obtain

$$\begin{aligned} K_R^{(0)} &= 2h \gamma_{RS} \left[ E_S^{(0)} + \frac{h^2}{3} E_S^{(2)} \right] - 2h \beta_R (T - T_0), \\ K_R^{(2)} &= \frac{2}{3} h^3 \gamma_{RS} \left[ E_S^{(0)} + \frac{3}{5} h^2 E_S^{(2)} \right] - \frac{2}{3} h^3 \beta_R (T - T_0), \end{aligned} \quad (4.8)$$

where

$$\gamma_{RS} = c_{RS} - c_{RW} c_{WV}^{-1} c_{VS}, \quad \beta_R = \nu_R - c_{RW} c_{WV}^{-1} \nu_V. \quad (4.9)$$

The  $\gamma_{RS}$  are Voigts' anisotropic plate elastic constants and the  $\beta_R$  are the associated anisotropic plate thermoelastic constants. At this point we note that the  $E_{2A}^{(0)}$ , which exist in the general anisotropic case, correspond to flexural type plate deformations, i.e., vertical plate shearing strains, which, however, vanish identically in ordinary flexure. The same sort of statement holds, of course, for the  $E_{2A}^{(2)}$ . For the case of anisotropic extension considered here the three-dimensional strains  $E_{KL}$ , which we need, are related to the plate strains by

$$E_{KL} = \frac{1}{2} (w_{K,L} + w_{L,K}) = E_{KL}^{(0)} + \kappa_2^2 E_{KL}^{(2)}, \quad (4.10)$$

in which we have taken the first order plate strains  $E_{2A}^{(1)}$  to vanish as in the constitutive equations for the first order stress resultants.

The associated three-dimensional displacement fields are still given by (3.7). The plate strains  $E_{AB}^{(n)}$  ( $n=0, 2$ ) are given by

$$E_{AB}^{(n)} = \frac{1}{2} (w_{A,B}^{(n)} + w_{B,A}^{(n)}), \quad (4.11)$$

and the remaining plate strains may be obtained by solving (4.6) with the result

$$E_W^{(0)} = -c_{WV}^{-1} c_{VS} E_S^{(0)} + c_{WV}^{-1} v_V (T - T_0), \quad E_W^{(2)} = -c_{WV}^{-1} c_{VS} E_S^{(2)}, \quad (4.12)$$

and we note that the relaxation of plate stress resultants results in plate strains and zero plate rotations. The  $n$ th order plate rotations take the form<sup>17</sup>

$$\Omega_{KL}^{(n)} = \frac{1}{2} [w_{L,K}^{(n)} - w_{K,L}^{(n)} + (n+1) (\delta_{2K} w_L^{(n+1)} - \delta_{2L} w_K^{(n+1)})], \quad (4.13)$$

where

$$\Omega_{KL} = \frac{1}{2} (w_{L,K} - w_{K,L}) = \sum_{n=0}^{\infty} x_2^n \Omega_{KL}^{(n)}. \quad (4.14)$$

As a consequence, we note for later use that

$$w_A^{(1)} = E_{2A}^{(0)}, \quad (4.15)$$

since

$$E_{2A}^{(0)} = \frac{1}{2} (w_{2,A}^{(0)} + w_A^{(1)}), \quad \Omega_{2A}^{(0)} = \frac{1}{2} (-w_{2,A}^{(0)} + w_A^{(1)}) = 0. \quad (4.16)$$

The extensional equations for the plated crystal plate are obtained by employing (3.13) in (3.10) for both the upper and lower platings and then inserting (3.10) for both platings in (3.2)<sub>2</sub> as in Sec.3, which results in (3.25), the substitution of which into (3.21) again yields (3.26) as the second order extensional equations of equilibrium of the, in this instance, general anisotropic plate. However, in this general anisotropic case  $K_{AB}^{(0)}$  and  $K_{AB}^{(2)}$  are given by (4.8), rather than (3.19). From (3.28) it is clear that (3.26) can be written in the form

$$\chi_{AB,A}^{(0)} = 0, \quad \chi_{AB,A}^{(2)} = 0, \quad (4.17)$$

where, from (3.28), (4.8), (3.14), (4.15) and (3.11) for the upper plating and the equivalent of (3.11), but with primes replaced by double primes, for the lower plating, for the general anisotropic case we have

$$\begin{aligned}
 \chi_{AB}^{(0)} &= 2h \left( \gamma_{ABCD} + \frac{h'}{h} \gamma'_{ABCD} + \frac{h''}{h} \gamma''_{ABCD} \right) E_{CD}^{(0)} + 2h^3 \left( \frac{1}{3} \gamma_{ABCD} + \right. \\
 &\left. \frac{h'}{h} \gamma'_{ABCD} + \frac{h''}{h} \gamma''_{ABCD} \right) E_{CD}^{(2)} - 2h \left( \beta_{AB} + \frac{h'}{h} \nu_{AB}^{*'} + \frac{h''}{h} \nu_{AB}^{*''} \right) (T - T_0) - \\
 &h^2 \left( \frac{h'}{h} \gamma'_{ABCD} - \frac{h''}{h} \gamma''_{ABCD} \right) (k_{2CBA} E_{BA,D}^{(0)} + k_{2DBA} E_{BA,C}^{(0)} - \zeta_{2C}^T{}_{,D} - \zeta_{2D}^T{}_{,C}), \\
 \chi_{AB}^{(2)} &= \frac{2}{3} h^3 \left( \gamma_{ABCD} + \frac{3h'}{h} \gamma'_{ABCD} + \frac{3h''}{h} \gamma''_{ABCD} \right) E_{CD}^{(0)} + \frac{2}{5} h^5 \left( \gamma_{ABCD} + \right. \\
 &\left. \frac{5h'}{h} \gamma'_{ABCD} + \frac{5h''}{h} \gamma''_{ABCD} \right) E_{CD}^{(2)} - \frac{2}{3} h^3 \left( \beta_{AB} + \frac{3h'}{h} \nu_{AB}^{*'} + \frac{3h''}{h} \nu_{AB}^{*''} \right) (T - T_0) - \\
 &h^4 \left( \frac{h'}{h} \gamma'_{ABCD} - \frac{h''}{h} \gamma''_{ABCD} \right) (k_{2CBA} E_{BA,D}^{(0)} + k_{2DBA} E_{BA,C}^{(0)} - \zeta_{2C}^T{}_{,D} - \zeta_{2D}^T{}_{,C}), \quad (4.18)
 \end{aligned}$$

where

$$k_{WS} = c_{WV}^{-1} c_{VS}, \quad \zeta_W = c_{WV}^{-1} \nu_V, \quad (4.19)$$

in the compressed notation and we note that  $T_{,D}$  vanishes for the homogeneous temperature states considered here. The edge (or boundary) conditions at a junction are still given by (3.29), (3.30) and the discussion in between.

In order to eliminate extension, which has already been considered, from the flexural equations, it is clear from (3.15) that we must have

$$K_{AB}^{(0)} = 0, \quad K_{A2}^{(1)} = 0, \quad K_{22}^{(0)} = 0, \quad K_{AB}^{(2)} = 0, \quad (4.20)$$

along with

$$F_B^{(0)} = 0, \quad F_B^{(2)} = 0, \quad (4.21)$$

since

$$F_2^{(1)} \equiv 0. \quad (4.22)$$



In addition, in order to allow for the free thickness strain  $E_{22}^{(1)}$ , we again have (3.32). Furthermore, since we have allowed for the free thickness strain  $E_{22}^{(1)}$ , we must eliminate the second order flexural equation in (3.31) completely again with the condition (3.33) since

$$F_2^{(2)} \equiv 0. \quad (4.23)$$

Moreover, as usual in the elementary theory of flexure, the constitutive equations for the zero order shear stress resultants  $K_{2A}^{(0)}$  are ignored and the zero order thickness-shear plate strains  $E_{2A}^{(0)}$  are taken to vanish and we again have (3.34). From (3.32) and (4.20)<sub>2</sub> with (3.3) - (3.5) for  $m=1$ , we obtain

$$E_W^{(1)} = -c_{WV}^{-1} c_{VS} E_S^{(1)}, \quad (4.24)$$

where we have introduced the conventions shown in (4.7). Substituting from (4.24) into the nonzero equations for the first order stress resultants in (3.3), we obtain

$$K_R^{(1)} = \frac{2}{3} h^3 \gamma_{RS} E_S^{(1)}, \quad (4.25)$$

where the  $\gamma_{RS}$  are defined in (4.9)<sub>1</sub>. The remainder of the equations and discussion for the elementary theory of flexure presented in Sec.3 hold without change with the exception of Eqs.(3.36) which are replaced by (4.25) for the general anisotropic case considered in this section.

##### 5. Large Electrodes on Quartz Plates

A plan view of the electroded plate is shown in Fig.2. The  $X_2$ -coordinate axis, which is normal to the major surfaces of the plate at  $T=T_0$ , is arbitrarily oriented with respect to the crystal axes. Since the outside edges of the plate are traction free, from (3.29) and (4.17) we have

$$\chi_{AB}^{(0)} = 0, \chi_{AB}^{(2)} = 0, \quad (5.1)$$

for all the unelectroded portions of the plate. Consequently, from (3.30) the plate edge conditions that determine the biasing displacement field in the electroded region take the form

$$\begin{aligned} \chi_{11}^{(0)} = 0, \chi_{11}^{(2)} = 0, \chi_{13}^{(0)} = 0, \chi_{13}^{(2)} = 0 \text{ at } x_1 = \pm l_1, \quad |x_3| < l_3, \\ \chi_{13}^{(0)} = 0, \chi_{13}^{(2)} = 0, \chi_{33}^{(0)} = 0, \chi_{33}^{(2)} = 0 \text{ at } x_3 = \pm l_3, \quad |x_1| < l_1. \end{aligned} \quad (5.2)$$

The solution satisfying (5.2) and (4.17) takes the form

$$\chi_{AB}^{(0)} = 0, \chi_{AB}^{(2)} = 0 \text{ everywhere.} \quad (5.3)$$

This solution is unique to within static homogeneous plate rotations of zero and second order. Substituting from (4.18) into (5.3) for identical electrodes on both surfaces, we obtain

$$\begin{aligned} \left( \gamma_{RS} + \frac{2h'}{h} \gamma'_{RS} \right) E_S^{(0)} + h^2 \left( \frac{\gamma_{RS}}{3} + \frac{2h'}{h} \gamma'_{RS} \right) E_S^{(2)} = \left( \beta_R + \frac{2h'}{h} \nu_R^{*'} \right) (T - T_0), \\ \left( \gamma_{RS} + \frac{6h'}{h} \gamma'_{RS} \right) E_S^{(0)} + h^2 \left( \frac{3}{5} \gamma_{RS} + \frac{6h'}{h} \gamma'_{RS} \right) E_S^{(2)} = \left( \beta_R + \frac{6h'}{h} \nu_R^{*'} \right) (T - T_0). \end{aligned} \quad (5.4)$$

Equations (5.4) constitute six homogeneous linear equations which may readily be solved for the six plate strains  $E_S^{(0)}$  and  $E_S^{(2)}$ . Clearly, if we define the six dimensional vectors and matrices  $A_\alpha$ ,  $B_\alpha$  and  $a_{\alpha\beta}$  by

$$\begin{aligned} [A_\alpha] = [E_S^{(0)}, E_S^{(2)}], \quad \alpha = 1, 2 \dots 6, \\ [B_\beta] = \left[ \left( \beta_R + \frac{2h'}{h} \nu_R^{*'} \right), \left( \beta_R + \frac{6h'}{h} \nu_R^{*'} \right) \right], \quad \beta = 1, 2 \dots 6, \end{aligned}$$

$$[a_{\alpha\beta}] = \begin{bmatrix} \left( \gamma_{RS} + \frac{2h'}{h} \gamma'_{RS} \right) h^2 \left( \frac{\gamma_{RS}}{3} + \frac{2h'}{h} \gamma'_{RS} \right) \\ \left( \gamma_{RS} + \frac{6h'}{h} \gamma'_{RS} \right) h^2 \left( \frac{3}{5} \gamma_{RS} + \frac{6h'}{h} \gamma'_{RS} \right) \end{bmatrix}, \quad (5.5)$$

then the solution to (5.4) can be written

$$A_{\alpha} = a_{\alpha\beta}^{-1} B_{\beta} (T - T_0), \quad (5.6)$$

which yields  $E_S^{(0)}$  and  $E_S^{(2)}$  as known expressions linear in  $(T - T_0)$ . Note that the solution is independent of  $l_1$  and  $l_3$  and further, that since  $\chi_{AB}^{(0)}$  and  $\chi_{AB}^{(2)}$  constitute two planar tensors that vanish everywhere, the plate edge conditions

$$N_A \chi_{AB}^{(0)} = 0, \quad N_A \chi_{AB}^{(2)} = 0, \quad (5.7)$$

at the edge of an electrode are satisfied for the edge having a variable  $N_A$  and, hence, being a curve. Consequently, the solution is valid for electrodes of arbitrary shape provided the electrodes are large compared to the thickness of the quartz plate<sup>18</sup>. When  $E_S^{(0)}$  and  $E_S^{(2)}$  have been determined from (5.6),  $E_W^{(0)}$  and  $E_W^{(2)}$  are readily determined from (4.12). Then the three-dimensional strain can be evaluated from (4.10), which holds for this purely extensional case. There is no flexure because the temperature change  $(T - T_0)$  is homogeneous and the electrodes are identical.

It is well known that in static linear elasticity the solution to a boundary value problem is unique only to within a static homogeneous (global) infinitesimal rigid rotation<sup>19</sup>. In addition, the change in frequency due to a homogeneous infinitesimal rigid rotation is shown to vanish in the Appendix. Consequently, without any loss in generality, we may select the homogeneous rigid rotation to take any value that is convenient and in particular to vanish. Accordingly, we take



$$\Omega_{KL}^1 = \frac{1}{2} (w_{L,K} - w_{K,L}) = 0, \quad (5.8)$$

which with (4.10) yields

$$w_{L,K} = E_{KL} = E_{KL}^{(0)} + X_2^2 E_{KL}^{(2)}, \quad (5.9)$$

which with (5.6) provides the biasing displacement gradients  $w_{K,N}$  as a known linear function of  $(T - T_0)$ . Thus, we may now substitute from (5.6) into (4.12) and then from (5.6) and (4.12) into (5.9), which may then be substituted into (2.11) with (2.12) to obtain  $\hat{c}_{LYM\alpha}$  as a known linear function of  $(T - T_0)$ .

#### 6. Thickness Modes in Piezoelectric Plates

Thickness modes in piezoelectric plates are solutions depending on the thickness coordinate only, which satisfy the linear piezoelectric equations and the boundary conditions on the major surfaces of the plate, but do not satisfy any conditions on the minor surfaces. The thickness solutions are of practical importance because plates with lateral dimensions very large compared to the thickness are employed in resonators and filters, and, consequently, the actual mode in the bounded plate is not that different from the thickness mode. In any event in this first treatment of temperature induced frequency changes in electroded quartz plates only the thickness modes will be considered.

Since  $X_2$  is the thickness coordinate, we substitute from (2.3) into (2.4) and retain  $X_2$ -dependence only to obtain

$$\begin{aligned} c_{22\gamma\alpha 2} u_{\alpha,22} + e_{22\gamma} \tilde{\phi}_{,22} &= \rho \ddot{u}_{\gamma}, \\ e_{22\gamma} u_{\gamma,22} - \epsilon_{22} \tilde{\phi}_{,22} &= 0, \end{aligned} \quad (6.1)$$

which are the differential equations that must be satisfied by the thickness solution. For the thickness eigensolutions of interest here, from (2.14) for the linear case only, with (2.3)<sub>1</sub> and retaining  $X_2$ -dependence only, we obtain the boundary conditions

$$\bar{c}_{2\gamma\alpha 2} u_{\alpha,2} + e_{22\gamma} \tilde{\varphi}_{,2} = \mp 2h' \rho' \ddot{u}_{\gamma}, \quad \tilde{\varphi} = 0, \quad \text{at } x_2 = \pm h, \quad (6.2)$$

since the electrodes are shorted. The solution for thickness-modes in arbitrarily anisotropic piezoelectric plates with shorted electrodes on the major surfaces may be written in the form<sup>20</sup>

$$u_{\gamma} = \sum_{n=1}^3 B^{(n)} \beta_{\gamma}^{(n)} \sin \eta_n X_2 e^{i\omega t},$$

$$\tilde{\varphi} = \left[ \frac{e_{22\gamma}}{\epsilon_{22}} \sum_{n=1}^3 B^{(n)} \beta_{\gamma}^{(n)} \sin \eta_n X_2 + L X_2 \right] e^{i\omega t}, \quad (6.3)$$

which satisfies (6.1) provided

$$(\bar{c}_{2\gamma\alpha 2} - \bar{c}^{(n)} \delta_{\gamma\alpha}) \beta_{\alpha}^{(n)} = 0, \quad \bar{c}^{(n)} = \rho \omega^2 / \eta_n^2, \quad (6.4)$$

and the  $\beta_{\gamma}^{(n)}$  are the normalized eigenvectors of the linear homogeneous algebraic system in (6.4) for the eigenvalues  $\bar{c}^{(n)}$  of the piezoelectrically stiffened elastic constants, which are given by

$$\bar{c}_{2\gamma\alpha 2} = c_{2\gamma\alpha 2} + e_{22\gamma} e_{22\alpha} / \epsilon_{22}. \quad (6.5)$$

In order that (6.2)<sub>2</sub> be satisfied we must have

$$L = -\frac{1}{h} \sum_{n=1}^3 B^{(n)} \frac{e_{22\gamma}}{\epsilon_{22}} \beta_{\gamma}^{(n)} \sin \eta_n h, \quad (6.6)$$

and (6.2)<sub>1</sub> are satisfied if the  $B^{(n)}$  are given by

$$B^{(n)} = -\beta_{\alpha}^{(n)} e_{22\alpha} L / \bar{c}^{(n)} \eta_n [\cos \eta_n h - R \eta_n h \sin \eta_n h], \quad (6.7)$$

where we have employed (6.4) and

$$\beta_Y^{(n)} \beta_Y^{(m)} = \delta_{mn}, \quad R = 2\rho'h'/\rho h, \quad (6.8)$$

in obtaining (6.7) from (6.2)<sub>1</sub>. Substituting from (6.7) into (6.6), we obtain

$$L \left[ 1 - \sum_{n=1}^3 (k^{(n)})^2 / \eta_n h (\cot \eta_n h - R \eta_n h) \right] = 0, \quad (6.9)$$

where

$$(k^{(n)})^2 = \frac{\beta_Y^{(n)} e_{22\gamma} \beta_\alpha^{(n)} e_{22\alpha}}{\bar{c}^{(n)} \epsilon_{22}}. \quad (6.10)$$

The condition for a nontrivial solution of the scalar equation (6.9) is

$$\sum_{n=1}^3 (k^{(n)})^2 / \eta_n h (\cot \eta_n h - R \eta_n h) = 1, \quad (6.11)$$

the roots of which determine the resonant frequencies of thickness vibration of piezoelectric plates driven by the application of a voltage across the surface electrodes. From this unperturbed thickness eigensolution we can determine the normalized eigensolution we need for the perturbation formulation in Sec.2 simply by writing

$$g_Y = u_Y/N, \quad \hat{f} = \tilde{\varphi}/N, \quad (6.12)$$

where, from (2.8), including the electrode platings, (6.3)<sub>1</sub> and (6.8)<sub>2</sub>

we find

$$N^2 = \rho \sum_{m=1}^3 \sum_{n=1}^3 B^{(m)} \beta_Y^{(m)} B^{(n)} \beta_Y^{(n)} \left[ \frac{\sin(\eta_m - \eta_n)h}{\eta_m - \eta_n} - \frac{\sin(\eta_m + \eta_n)h}{\eta_m + \eta_n} + 2Rh \sin \eta_n h \sin \eta_m h \right]. \quad (6.13)$$



### 7. Temperature Dependence of Resonant Frequency

The change in the frequency of thickness vibrations with temperature of any electroded quartz plate may now be determined from (2.1), which we rewrite here for any one mode in the form

$$\Delta_M = H_M / 2\omega_M, \quad \omega = \omega_M - \Delta, \quad (7.1)$$

where for the case of thickness vibrations<sup>21</sup> we have

$$H_M = - \int_{-h}^h \tilde{K}_{2\gamma}^n g_{\gamma,2}^M dx_2. \quad (7.2)$$

From (2.5) and (2.10) for the thickness-mode being perturbed here, we have

$$\tilde{K}_{2\gamma}^n = (\hat{c}_{2\gamma 2\alpha} + \Delta c_{22\gamma 2\alpha}) g_{\alpha,2}^M, \quad (7.3)$$

where  $\hat{c}_{2\gamma 2\alpha}$  defined in (2.11), with (2.12), is known as a linear expression in  $(T - T_0)$  from (5.9), with (5.5), (5.6) and (4.12) and

$$\Delta c_{22\gamma 2\alpha} = (dc_{22\gamma 2\alpha} / dT) (T - T_0). \quad (7.4)$$

The  $dc_{22\gamma 2\alpha} / dT$  are obtained from the first temperature derivatives of the fundamental elastic constants of quartz<sup>6</sup>  $\tilde{c}_{2DEFG} / dT$  referred to the principal axes by the tensor transformation relation

$$\frac{d}{dT} c_{2\gamma 2\alpha} = a_{2D} a_{\gamma E} a_{2F} a_{\alpha G} \frac{d}{dT} \tilde{c}_{2DEFG}, \quad (7.5)$$

where the  $a_{\gamma E}$  are the matrix of direction cosines for the transformation from the principal axes to the coordinate system containing the axis normal to the plane of the plate. When the conventional IEEE notation<sup>22</sup> for doubly-rotated plates is written in the form  $(Y, X, w, l)\varphi, \theta$ , where  $\psi = 0$ , the rotation angles  $\varphi$  and  $\theta$  are the first two Euler angles, and the  $a_{\beta G}$  are given by

$$a_{\beta G} = \begin{bmatrix} \cos \varphi & \sin \varphi & 0 \\ -\cos \theta \sin \varphi & \cos \theta \cos \varphi & \sin \theta \\ \sin \theta \sin \varphi & -\sin \theta \cos \varphi & \cos \theta \end{bmatrix}, \quad (7.6)$$

since  $\psi = 0$ . Clearly, the transformation relations for the second and third order elastic, piezoelectric and dielectric constants, and coefficients of linear expansion may be written in the respective forms

$$\begin{aligned} c_{2KLMN} &= a_{KD} a_{LE} a_{MF} a_{NG} \tilde{c}_{2DEFG}, \\ c_{3KLMNAB} &= a_{KD} a_{LE} a_{MF} a_{NG} a_{AH} a_{BI} \tilde{c}_{3DEFGHI}, \\ e_{KLM} &= a_{KD} a_{LE} a_{MF} \tilde{e}_{DEF}, \\ \epsilon_{KL} &= a_{KM} a_{LN} \tilde{\epsilon}_{MN}, \quad \alpha_{KL} = a_{KM} a_{LN} \tilde{\alpha}_{MN}, \end{aligned} \quad (7.7)$$

where the tensor quantities with the upper cycle are referred to the principal axes of the crystal.

Calculations of the resonant frequency, called  $\omega_M$  here, for the particular unperturbed thickness-mode of interest proceed by numerically finding the value of  $\omega_M$  satisfying (6.4) and (6.11). After the  $\omega_M$  of interest has been determined, the full Mth eigensolution is obtained by substituting the  $\beta_{\alpha}^{(n)}$  and  $\eta_n$  from (6.4) and the  $B^{(n)}$  from (6.7) into (6.3). The normalization integral in (6.13) is then evaluated and the normalized eigensolution obtained from (6.12). Then the perturbation integral  $H_M$  is evaluated by employing (7.3) and (6.12)<sub>1</sub> in (7.2) and performing the integrations. On account of (5.9), with (5.5), (5.6), (4.12) and (6.3)<sub>1</sub> two distinct types of integrals arise in (7.2), the evaluation of which yields

$$\int_{-h}^h g_{\alpha,2}^M g_{\gamma,2}^M dx_2 = \frac{1}{N_M} \sum_{p=1}^3 \sum_{q=1}^3 B^{(p)} \beta_{\alpha}^{(p)} B^{(q)} \beta_{\gamma}^{(q)} \eta_p \eta_q \left[ \frac{\sin(\eta_p + \eta_q)h}{(\eta_p + \eta_q)} + \frac{\sin(\eta_p - \eta_q)h}{(\eta_p - \eta_q)} \right],$$

$$\int_{-h}^h x_2^2 g_{\alpha,2}^M g_{\gamma,2}^M dx_2 = \frac{1}{N_M} \sum_{p=1}^3 \sum_{q=1}^3 B^{(p)} \beta_{\alpha}^{(p)} B^{(q)} \beta_{\gamma}^{(q)} \eta_p \eta_q \left[ \left( \frac{h^2}{\eta_p + \eta_q} - \frac{2}{(\eta_p + \eta_q)^3} \right) \sin(\eta_p + \eta_q)h + \frac{2h}{(\eta_p + \eta_q)^2} \cos(\eta_p + \eta_q)h + \left( \frac{h^2}{\eta_p - \eta_q} - \frac{2}{(\eta_p - \eta_q)^3} \right) \sin(\eta_p - \eta_q)h + \frac{2h}{(\eta_p - \eta_q)^2} \cos(\eta_p - \eta_q)h \right]. \quad (7.8)$$

Calculations have been performed using the known values of the second order elastic, piezoelectric and dielectric constants of quartz<sup>23</sup>, the third order elastic<sup>24</sup> and thermoelastic<sup>25</sup> constants of quartz and the recently obtained<sup>6,26</sup> temperature derivatives of the fundamental elastic constants of quartz. Specific calculations have been made for doubly-rotated quartz plates 1.7 mm thick, with 4000 Å thick gold electrodes on the major surfaces, vibrating in the fundamental thickness-mode. Typical results of the calculations are shown in Figs.3-8. Figure 3 shows the actual change in frequency for an electroded quartz plate for a fixed value of  $\theta = -49.2167^\circ$ , which corresponds to the BT-cut for  $\varphi = 0^\circ$ , as a function of  $\varphi$ . The lower and upper curves are for the B- and C-modes, respectively, which are defined<sup>26</sup> so that for each cut and mode number  $n$  the sequence  $f_A > f_B > f_C$  is followed. The curves in the figures repeat because of the symmetry of quartz. Figure 4 shows the change in frequency due to the presence of the electrodes for the same orientations and modes of the electroded plate shown in Fig.3. Note that the change in frequency due to the electrodes is about two orders of magnitude smaller than the



actual change shown in Fig.3. Since a portion of the aging rate is a result of the relaxation of residual stress in the electrodes<sup>27</sup>, the change in frequency shown in Fig.4 is the portion of the actual change in frequency that contributes to the aging rate. Figure 5 shows the actual change in frequency for an electroded quartz plate for a fixed value of  $\theta = 35.25^\circ$ , which corresponds to the AT-cut for  $\varphi = 0^\circ$ , as a function of  $\varphi$ . Note that the C-mode, which is the piezoelectrically active mode for the AT-cut, i.e., for  $\varphi = 0^\circ$ , is relatively flat and has no zero crossings, while the B-mode has wide excursions and zero crossings. The zero crossings for the B-mode correspond closely to points on the locus of zero temperature coefficients shown in Fig.2 of Ref.26. Figure 6 shows the change in frequency due to the presence of the electrodes for the cases for which the actual changes in frequency are shown in Fig.5. The dotted line in Fig.7 represents the actual change in frequency near a zero crossing to an enlarged scale for a fixed value of  $\varphi = 5^\circ$  for the B-mode, and the solid line represents the change in frequency due to the presence of the electrodes. Note that the two lines have different scales, one being two orders of magnitude smaller than the other. The difference in  $\theta$  for the zero crossings is related to what is called the apparent shift in angle of the zero temperature cut with electrode thickness. The vertical distance from the intersection of the dotted line with the horizontal axis, which is the actual zero temperature cut of the 1.7 mm thick quartz plate with 4000 Å thick electrodes, to the solid line measured on the inner scale represents the change in frequency that can contribute to aging. Thus, it is desirable to find actual zero temperature cuts which have that ordinate as small as possible for a given electrode thickness. The dotted

line in Fig.8 represents the actual change in frequency near a zero crossing to an enlarged scale for a fixed value of  $\varphi = 30^\circ$  for the C-mode, and the solid line represents the change in frequency due to the presence of the electrodes. The zero crossings in both Figs.7 and 8 correspond closely to points on the locus of zero temperature coefficients shown in Fig.2 of Ref.26, as they should.

#### Acknowledgements

We wish to thank D. Stevens of Rensselaer Polytechnic Institute for help with the calculations.

This work was supported in part by the Army Research Office under Grant No. DAAG29-76-G-0173, the Office of Naval Research under Contract No. N00014-76-C-0368 and the National Science Foundation under Grant No. ENG 72-04223.

## REFERENCES

1. H.F. Tiersten, "Perturbation Theory for Linear Electroelastic Equations for Small Fields Superposed on a Bias," J. Acoust. Soc. Am., 64, 832 (1978).
2. B.K. Sinha and H.F. Tiersten, "On the Influence of a Flexural Biasing State on the Velocity of Piezoelectric Surface Waves," Wave Motion, 1, 37 (1979).
3. R.D. Mindlin, "An Introduction to the Mathematical Theory of the Vibration of Elastic Plates," U.S. Army Signal Corps Eng. Lab., Fort Monmouth, New Jersey (1955). Signal Corps Contract DA-36-039SC-56772.
4. R.D. Mindlin, "High Frequency Vibrations of Crystal Plates," Quart. Appl. Math., 19, 51 (1961).
5. H.F. Tiersten, Linear Piezoelectric Plate Vibrations (Plenum, New York, 1969), Chap.13.
6. B.K. Sinha and H.F. Tiersten, "First Temperature Derivatives of the Fundamental Elastic Constants of Quartz," to be published in the J. Appl. Phys.
7. Ref.5, Chap.14, Sec.4.
8. J. Tasi and G. Herrmann, "Thermoelastic Dissipation in High Frequency Vibrations of Crystal Plates," J. Acoust. Soc. Am., 36, 100 (1964).
9. R.D. Mindlin, "High Frequency Vibrations of Plated Crystal Plates," in Progress in Applied Mechanics (Macmillan, New York, 1963), pp.73-84.
10. This procedure, which follows the work of Mindlin, results in a valuable reduction and simplification in the equations.
11. Since  $w_2^{(1)}$  does not vanish, the conditions in Eq. (3.17) are implied by  $(3.15)_2$  and  $(3.16)_1$ . Certain consequences of (3.17) are discussed in detail in Sec.4.
12. A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, 4th ed. (Cambridge University Press, Cambridge, 1927) also (Dover, New York, 1944) Sec.118.
13. Ref.5, Chap.13, Sec.3.
14. Since  $w_2^{(2)}$  does not vanish, the conditions in (3.33) are implied by  $(3.31)_3$  and (3.32). The second order plate equations are not important in the elementary theory of the flexure of thin plates.



15. Ref.3, Sec.6.04.
16. Ref.5, Chap.7, Sec.1.
17. The  $n$ th order plate rotations are obtained by substituting from the thickness expansion for the mechanical displacement field into the expression for the three-dimensional local small rotation and proceeding exactly as in the determination of the  $n$ th order plate strains in (3.6).
18. Although the approximate plate equations yield accurate results for points not too close to the edges of the electrodes, the results are not at all meaningful in the near vicinity of the electrode edges. This is a result of the fact that the three-dimensional elastic strain field has a singularity along the edge of the electrode. However, even in the least ductile of real materials, the inelastic behavior prevents the singularity from occurring in the actual stress and strain distributions, which remain finite. The stress and strain distributions in the vicinity of the electrode edges obtained from the approximate plate equations are quite different from the actual ones that occur even when the inelastic behavior is taken into account. Nevertheless, this difference is not important for our purposes because in the case of large area electrodes, in which we are interested, the strain distribution obtained from the approximate plate equations is accurate over almost the entire electroded region, and the small area in the vicinity of the electrode edges does not have a significant influence.
19. Ref.12, Secs.18 and 118.
20. H.F. Tiersten, "Thickness Vibrations of Piezoelectric Plates," J. Acoust. Soc. Am., 35, 53 (1963).
21. For a wide plate with wide electrodes the influence of energy trapping can be ignored.
22. "Standards on Piezoelectric Crystals, 1949," Proc. IRE, 37, 1378 (1949).
23. R. Bechmann, "Elastic and Piezoelectric Constants of Alpha-Quartz," Phys. Rev., 110, 1060 (1958).
24. R.N. Thurston, H.J. McSkimin and P. Andreatch, Jr., "Third Order Elastic Constants of Quartz," J. Appl. Phys., 37, 267 (1966).
25. F. Kohlrausch, Lehrbuch der prakt. Physik, 16. aufl. 5. 158 (1930).
26. R. Bechmann, A.D. Ballato and T.J. Lukaszek, "Higher Order Temperature Coefficients of the Elastic Stiffnesses and Compliances of Alpha Quartz," Proc. IRE, 50, 1812 (1962).

27. E.P. Eer Nisse, "Simultaneous Thin-Film Stress and Mass-Change Measurements Using Quartz Resonators," *J. Appl. Phys.*, 43, 1330 (1972).
28. Ref.5, Chap.8, Sec.7. In this instance the operations are performed for more than one region.
29. The sign of the second term in Eq. (2.12)<sub>2</sub> of Ref.1 should be changed from a minus to a plus.

#### FIGURE CAPTIONS

- Figure 1 Schematic diagram of the plated crystal plate
- Figure 2 Plan view of rectangular electrode on quartz plate
- Figure 3 Actual relative change in the fundamental resonant frequency per  $^{\circ}\text{K}$  for a 1.7 mm thick doubly-rotated quartz plate with 4000 Å thick gold electrodes as a function of  $\varphi$  with  $\theta = -49^{\circ}13'$ .
- Figure 4 Relative change in the fundamental resonant frequency per  $^{\circ}\text{K}$  due to the electrodes for the cases treated in Fig.3.
- Figure 5 Actual relative change in the fundamental resonant frequency per  $^{\circ}\text{K}$  for a 1.7 mm thick doubly-rotated quartz plate with 4000 Å thick gold electrodes as a function of  $\varphi$  with  $\theta = 35^{\circ}15'$ .
- Figure 6 Relative change in the fundamental resonant frequency per  $^{\circ}\text{K}$  due to the electrodes for the cases treated in Fig.5
- Figure 7 Relative change in the fundamental resonant frequency per  $^{\circ}\text{K}$  for a 1.7 mm thick doubly-rotated quartz plate with 4000 Å thick gold electrodes near a zero crossing as a function of  $\theta$  with  $\varphi = 5^{\circ}$ . The dotted and solid lines indicate the actual and electrode induced changes in frequency, respectively.
- Figure 8 Relative change in the fundamental resonant frequency per  $^{\circ}\text{K}$  for a 1.7 mm thick doubly-rotated quartz plate with 4000 Å thick gold electrodes near a zero crossing as a function of  $\theta$  with  $\varphi = 30^{\circ}$ . The dotted and solid lines indicate the actual and electrode induced changes in frequency, respectively.



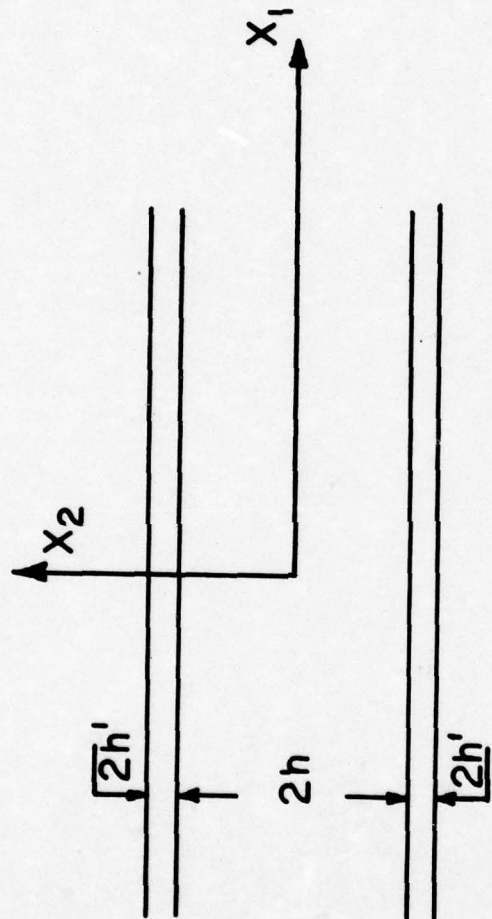


Fig.1

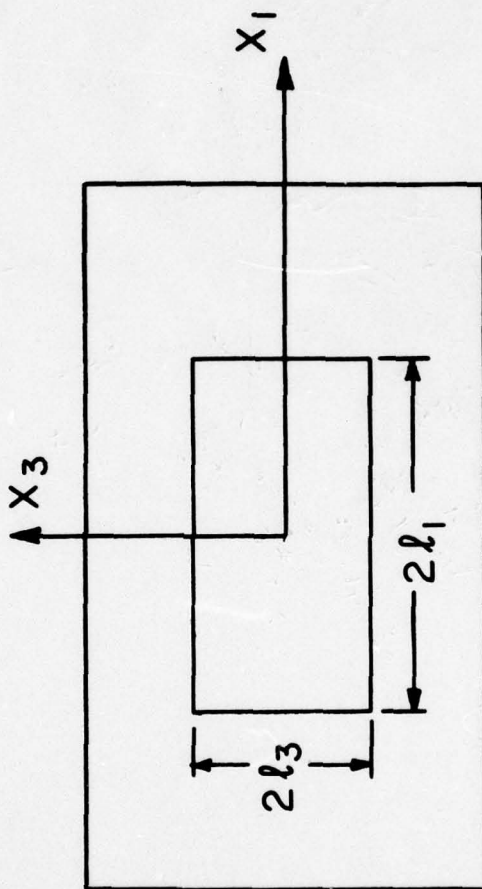


Fig.2

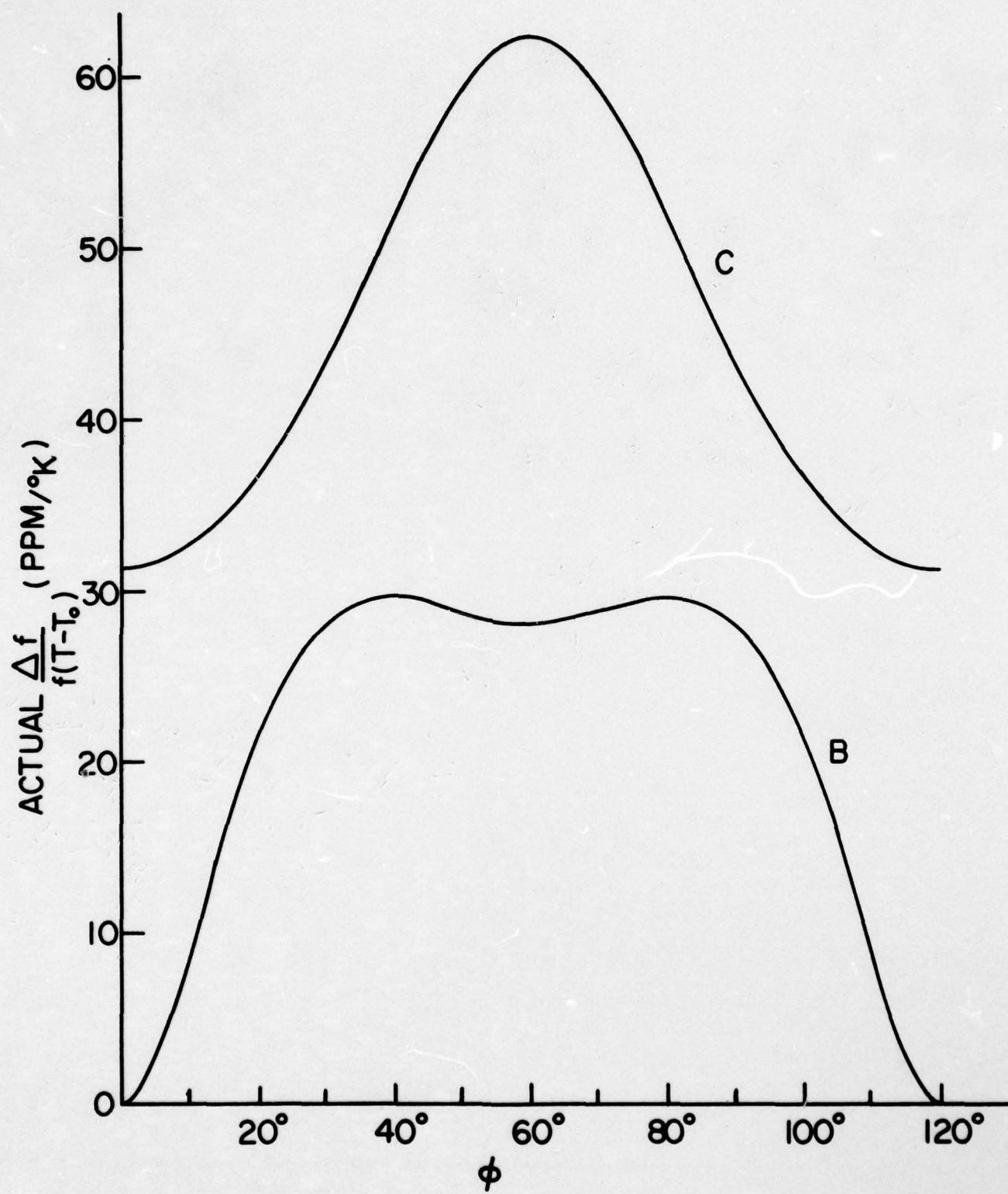


Fig. 3



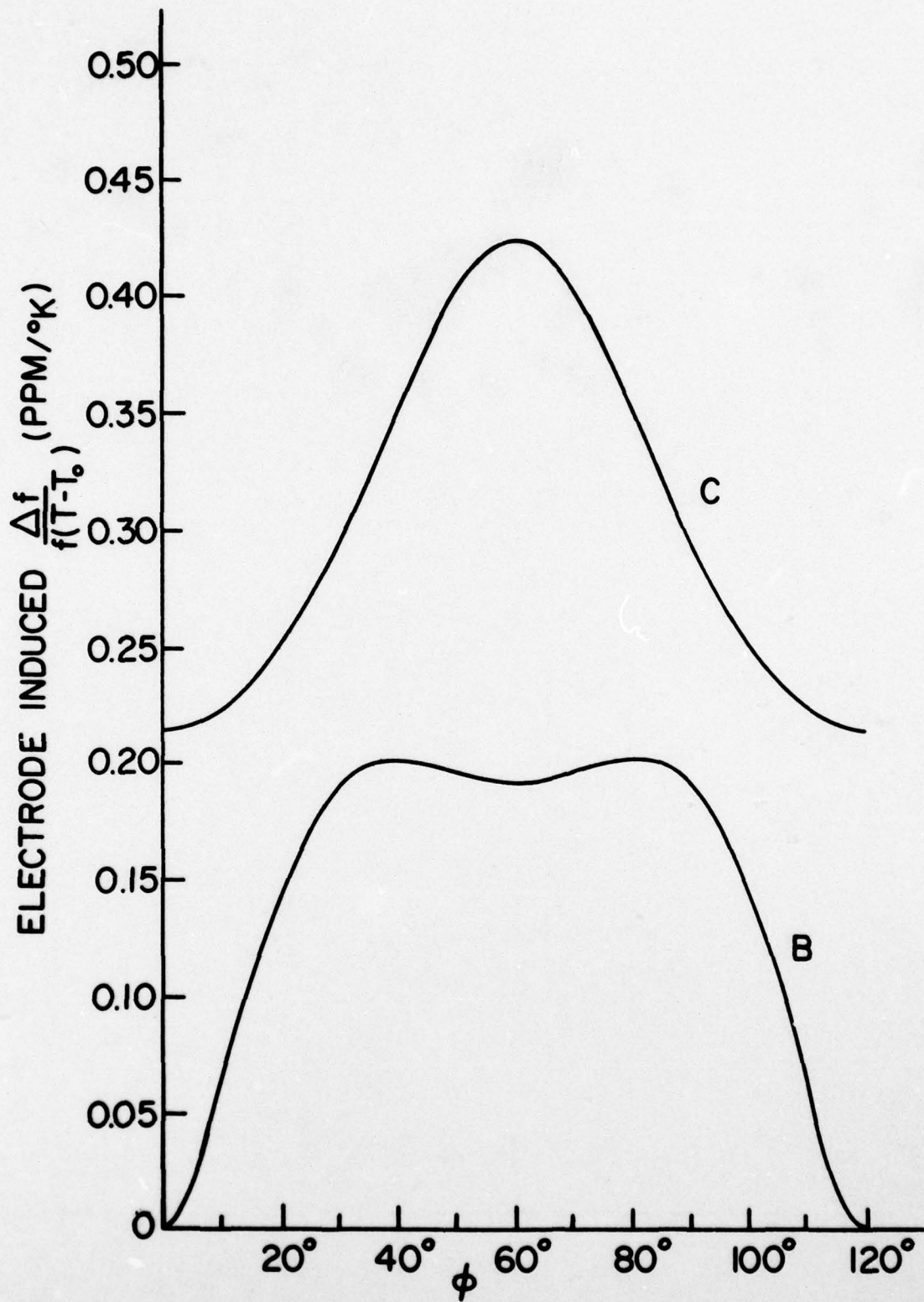


Fig.4

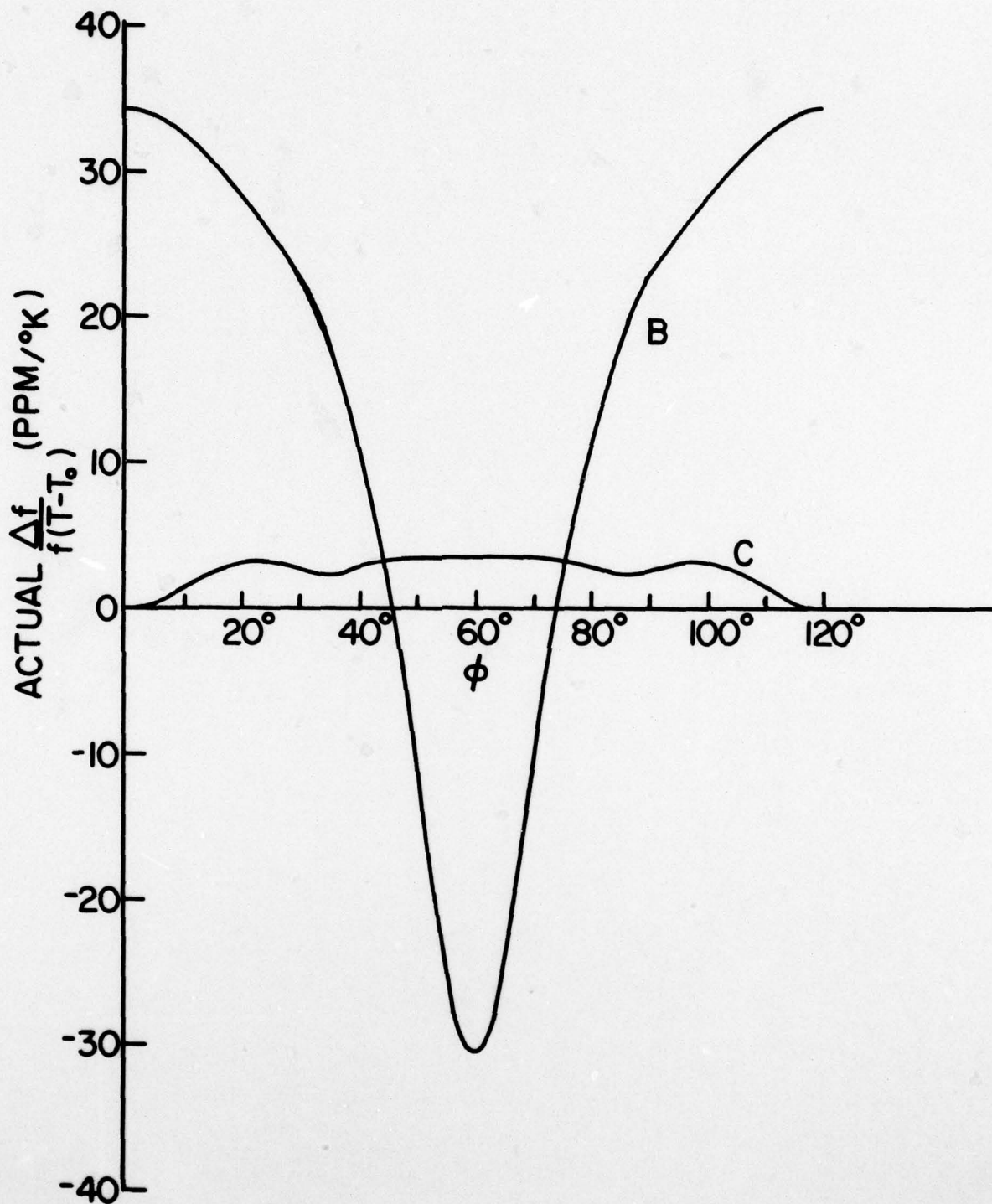


Fig.5

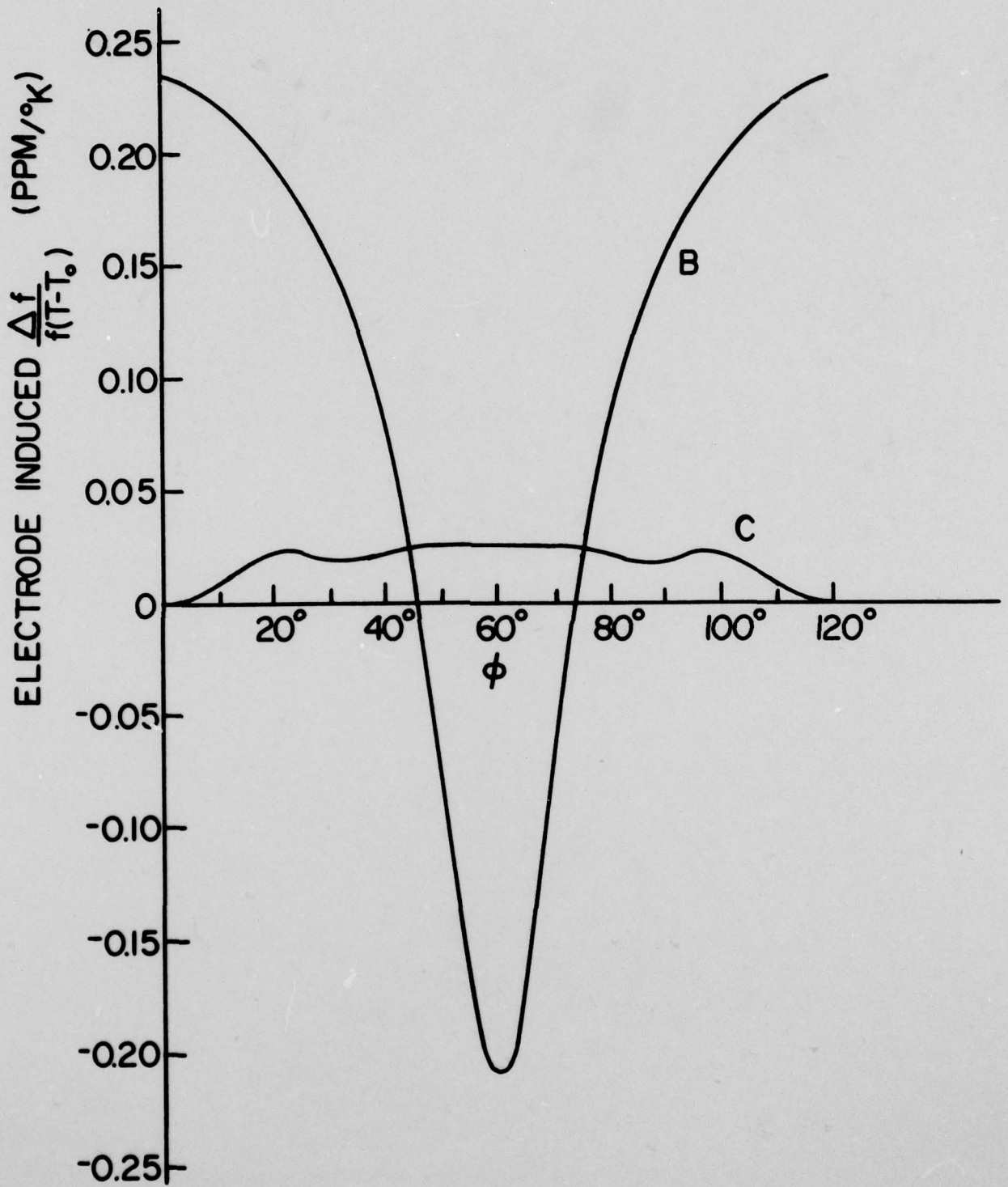


Fig.6



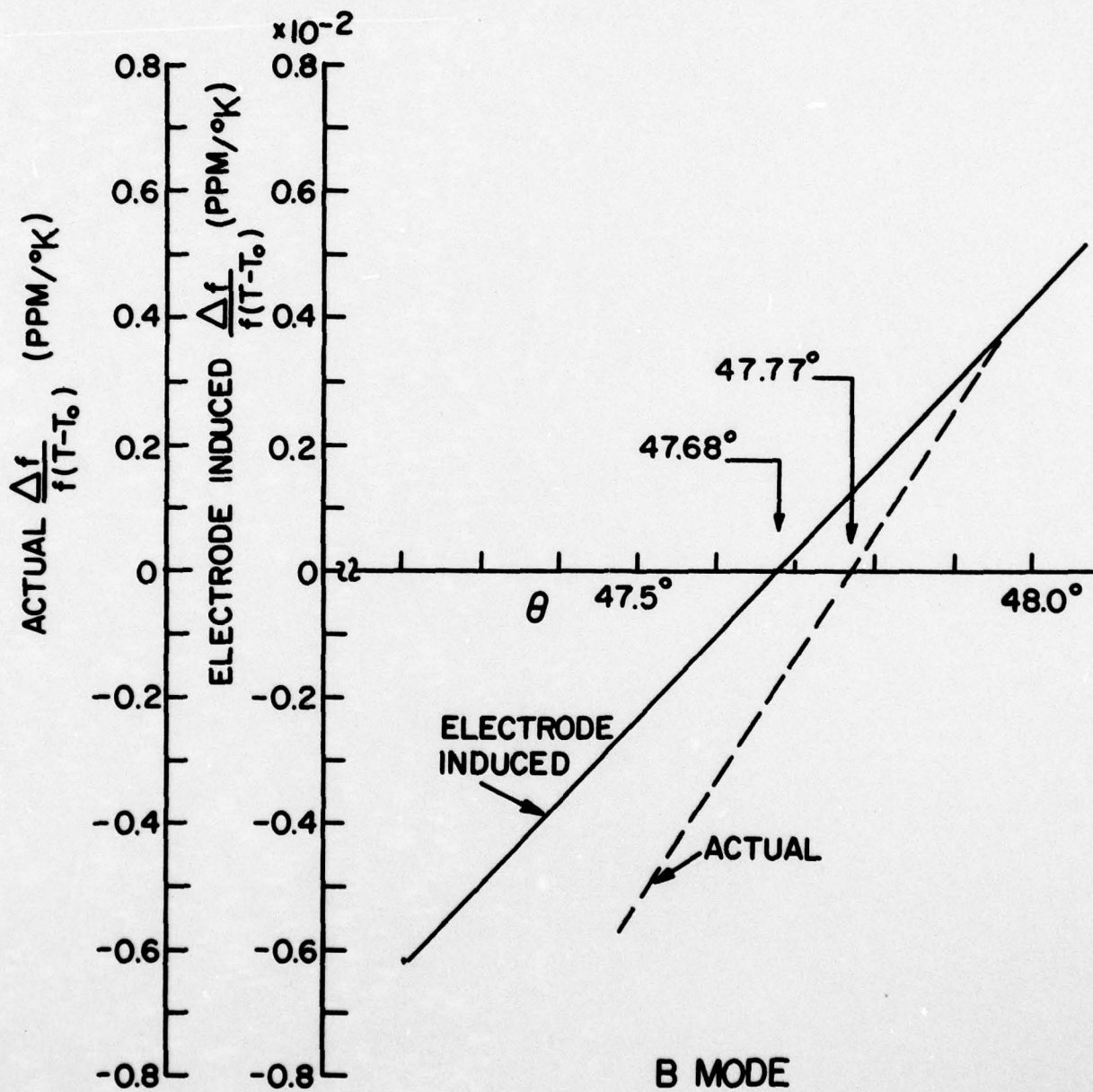


Fig.7

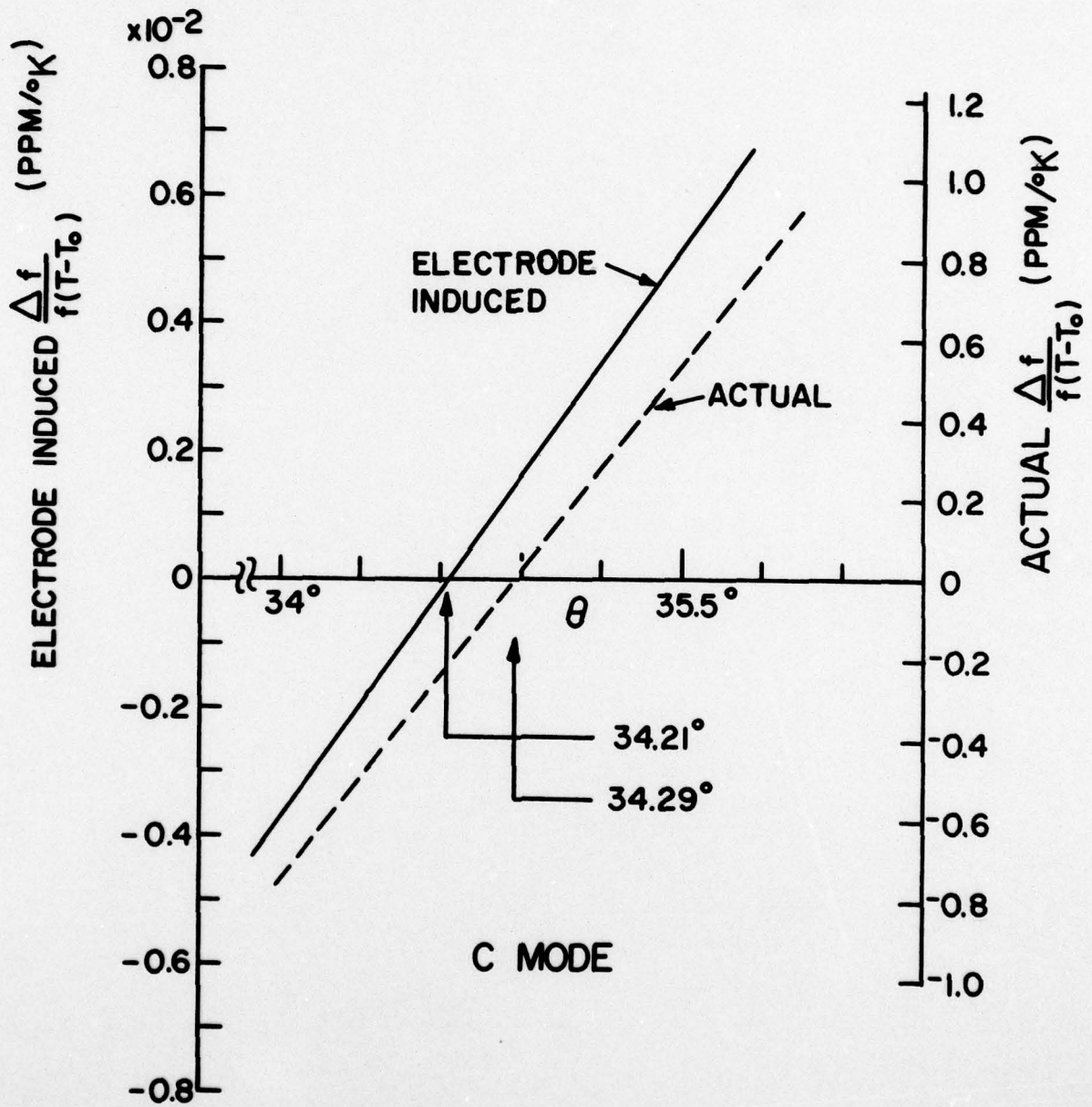


Fig. 8

## APPENDIX

We now show from the perturbation integral for  $\Delta_\mu$  in (2.1) and (2.2) that  $\Delta_\mu$  vanishes for an arbitrary pure homogeneous infinitesimal rigid rotation  $\Omega_{KL}^1$ , which is given by

$$\Omega_{KL}^1 = \frac{1}{2} (w_{L,K} - w_{K,L}). \quad (A1)$$

Since under these circumstances the strain  $E_{KL}^1$  vanishes as does the stress  $T_{KL}^1$ , from (2.12)<sub>2</sub> and (A1) we have

$$w_{L,K} = \Omega_{KL}^1, \quad \Omega_{KL}^1 = -\Omega_{LK}^1, \quad (A2)$$

and from (2.11) we obtain

$$\hat{c}_{LYMV} = c_{2LYKM} \Omega_{KV}^1 + c_{2LKMV} \Omega_{KY}^1. \quad (A3)$$

Since the biasing displacement field resulting from the nonlinear behavior of the electrode plating has not been included in the description, i.e., in Eq. (2.14)<sub>1</sub>, the form of the perturbation integral in (2.2) does not actually result in a zero  $\Delta_\mu$  for nonzero  $\Omega_{KL}^1$ . However, if the bias due to the nonlinear behavior of the electrode plating is properly included in the description, the amended form of the perturbation integral does result in a zero  $\Delta_\mu$  for nonzero  $\Omega_{KL}^1$ . The unbiased plating equations, which enable the entire effect of the plating to be treated as a homogeneous boundary condition at the surface of the piezoelectric plate and thereby result in a major simplification in the analysis, are based on a number of simplifying thin plate assumptions<sup>7</sup> which result in the occurrence of Voigts' anisotropic linear plate constants. Since the use of the approximate thin plate assumptions in the case of the biased plating will result in the occurrence of effective anisotropic plate constants other than the Voigt constants, it does not appear to be particularly



purposeful to obtain the biased approximate thin plating equations. Consequently, in this appendix we take the alternative course of extending the perturbation integral for  $\Delta_\mu$  to the case where the electrode platings are included as additional three-dimensional regions attached to the piezoelectric plate. We then show that the extended perturbation integral results in a zero  $\Delta_\mu$  for nonzero  $\Omega_{KL}^1$ . However, in order to keep the demonstration simple and clear and not introduce additional complications we first show that, for no electrode plating on a purely elastic plate, the perturbation integral presently in (2.18) vanishes for an arbitrary  $\Omega_{KL}^1$ , and we further note that since the actual electrode plating is very thin, the perturbation integral in (2.18) very nearly vanishes for thin electrode platings on the piezoelectric plate.

Substituting from (A3) into (2.19) for zero temperature change, we obtain

$$H_\mu = - \int_V [c_{2LYKM} \Omega_{K\alpha}^1 + c_{2LKMA} \Omega_{K\gamma}^1] g_{\alpha, M}^\mu g_{\gamma, L}^\mu dV, \quad (A4)$$

which by virtue of the symmetry of the  $c_{2LJKM}$  and the fact that  $\Omega_{K\alpha}^1$  is homogeneous, may be written

$$H_\mu = - 2\Omega_{K\alpha}^1 \int_V c_{2KMLY} g_{\gamma, L}^\mu g_{\alpha, M}^\mu dV. \quad (A5)$$

Substituting from (2.9)<sub>1</sub> for the purely elastic case ( $e_{MLY} = 0$ ), we obtain

$$H_\mu = - 2\Omega_{K\alpha}^1 \int_V k_{KM}^\mu g_{\alpha, M}^\mu dV, \quad (A6)$$

which with the aid of the divergence theorem and the normalized eigen-solution form of (2.4)<sub>1</sub> yields

$$H_{\mu} = - 2\Omega_{K\alpha}^{-1} \left[ \int_S N_M K_{MK}^{\mu} g_{\alpha}^{\mu} dS + \omega_{\mu}^2 \int_V \rho g_{K\alpha}^{\mu} g_{\alpha}^{\mu} dV \right]. \quad (A7)$$

Since for traction-free boundary conditions in the linear eigenvibration problem we have

$$N_M K_{MK}^{\mu} = 0 \quad \text{on } S, \quad (A8)$$

Eqs. (2.1), (A2)<sub>2</sub> and (A7) yield

$$\Delta_{\mu} = 0. \quad (A9)$$

In order to obtain the appropriate form of the perturbation integral for the electroded piezoelectric plate, we return to Eqs. (3.17) and (3.18) of Ref.1, which have the form

$$\tilde{K}_{LY,L}^{\mu} + \rho^0 \omega_{\mu}^2 u_{\gamma}^{\mu} = 0, \quad \tilde{D}_{L,L}^{\mu} = 0, \quad (A10)$$

$$\tilde{K}_{LY,L}^l + \tilde{K}_{LY,L}^n + \rho^0 \omega^2 u_{\gamma} = 0, \quad \tilde{D}_{L,L}^l + \tilde{D}_{L,L}^n = 0, \quad (A11)$$

where (A10) are the equations satisfied by the  $\mu$ th eigensolution and (A11) are the equations satisfied by the nearby perturbed solution at frequency  $\omega$ . To (A10) and (A11) we must adjoin the equations for the nonpiezoelectric electrodes, which may be written in the form

$$\tilde{K}_{LY,L}^{m\mu} + \rho^{m0} \omega^2 u_{\gamma}^{m\mu} = 0, \quad \tilde{\varphi}^{m\mu} = 0, \quad (A12)$$

$$\tilde{K}_{LY,L}^{ml} + \tilde{K}_{LY,L}^{mn} + \rho^{m0} \omega^2 u_{\gamma}^m = 0, \quad \tilde{\varphi}^{ml} = 0, \quad (A13)$$

where  $m$  represents the  $m$ th electrode, say  $m=1$  for the top electrode and  $m=2$  for the bottom electrode, and since we are concerned with perturbations in which the electrodes remain shorted, we have (A12)<sub>2</sub> and (A13)<sub>2</sub>.

From the first of the conditions in (A10) - (A13), we form

$$\int_{V^i} [(\tilde{K}_{LY,L}^{\mu} + \rho^0 \omega_{\mu}^2 u_{\gamma}^{\mu}) u_{\gamma}^{\mu} - (\tilde{K}_{LY,L}^l + \tilde{K}_{LY,L}^n + \rho^0 \omega^2 u_{\gamma}) u_{\gamma}^{\mu}] dV +$$

$$\sum_m \int_{V^m} [(\tilde{K}_{LY,L}^{ml} + \rho^{mo} \omega^2 u_Y^m) u_Y^m - (\tilde{K}_{LY,L}^{ml} + \tilde{K}_{LY,L}^{mn} + \rho^{mo} \omega^2 u_Y^m) u_Y^{m\mu}] dv = 0, \quad (A14)$$

and at this point for our purposes, from (2.3)<sub>1</sub> and (2.5)<sub>1</sub>, we note that

$$\tilde{K}_{LY}^{ml} = \hat{C}_{LYM\alpha}^m u_{\alpha,M}^m, \quad \tilde{K}_{LY}^{mn} = \hat{C}_{LYM\alpha}^m u_{\alpha,M}^m, \quad (A15)$$

where from (2.11) for zero biasing stress  $T_{LY}^{ml}$  we have

$$\hat{C}_{LYM\alpha}^m = \frac{C_{LYM\alpha AB}^m E_{AB}^{ml}}{3} + \frac{C_{LYKM}^m \alpha_{,K}}{2} + \frac{C_{LKMA}^m \alpha_{,K}}{2}. \quad (A16)$$

Performing the usual operations<sup>28</sup>, employing (2.3), (A10)<sub>2</sub>, (A11)<sub>2</sub> and the divergence theorem in the usual manner, we obtain

$$\begin{aligned} & (\omega_\mu^2 - \omega^2) \left[ \int_{V^i} \rho^o u_Y^\mu u_Y^\mu dv + \sum_m \int_{V^m} \rho^{mo} u_Y^{m\mu} u_Y^m dv \right] = \int_{S^i} N_L [\tilde{K}_{LY}^{l\mu} u_Y^\mu \\ & - \tilde{K}_{LY}^{l\mu} u_Y^\mu + \tilde{\beta}_L^l \tilde{\varphi}^\mu - \tilde{\beta}_L^{l\mu} \tilde{\varphi}] ds + \int_{V^i} [\tilde{K}_{LY,L}^{ln} u_Y^\mu + \tilde{\beta}_{L,L}^n \tilde{\varphi}^\mu] dv \\ & + \sum_m \left[ \int_{S^m} N_L (\tilde{K}_{LY}^{ml} u_Y^{m\mu} - \tilde{K}_{LY}^{l\mu} u_Y^m) ds + \int_{V^m} \tilde{K}_{LY,L}^{mn} u_Y^{m\mu} dv \right]. \end{aligned} \quad (A17)$$

Since the perturbed solution is nearby the unperturbed solution, we have

$$\Delta = \omega_\mu - \omega, \quad |\Delta| \ll \omega_\mu, \quad u_Y^\mu - u_Y = \eta_Y^\mu, \quad |\eta_Y^\mu| \ll |u_Y^\mu|. \quad (A18)$$

Substituting from (A18) into (A17), neglecting products of small quantities, introducing the normalization integral and employing (2.3) for the  $\mu$ th mode, (2.7) and (2.9), we obtain

$$\Delta = H_\mu / 2\omega_\mu, \quad (A19)$$

where



$$\begin{aligned}
H_{\mu} = & \int_{S^i} N_L [k_{LY}^l g_Y^{\mu} - k_{LY}^{\mu} g_Y^l + d_L^{l\mu} \hat{f}^{\mu} - d_L^{\mu l} \hat{f}^l] ds + \int_{V^i} [k_{LY, L}^n g_Y^{\mu} + d_{L, L}^n \hat{f}^{\mu}] dv \\
& + \sum_m \left[ \int_{S^m} N_L (k_{LY}^{ml} g_Y^{m\mu} - k_{LY}^{m\mu} g_Y^m) ds + \int_{V^m} k_{LY, L}^{mn} g_Y^{m\mu} dv \right], \quad (A20)
\end{aligned}$$

and

$$\begin{aligned}
k_{LY}^l &= \frac{\tilde{K}_{LY}^l}{N(\mu)}, \quad g_Y^l = \frac{u_Y}{N(\mu)}, \quad d_L^{l\mu} = \frac{\tilde{D}_L^{l\mu}}{N(\mu)}, \quad \hat{f}^{\mu} = \frac{\alpha}{N(\mu)}, \quad k_{LY}^n = \frac{\tilde{K}_{LY}^n}{N(\mu)}, \\
d_L^{n\mu} &= \frac{\tilde{D}_L^{n\mu}}{N(\mu)}, \quad k_{LY}^{ml} = \frac{\tilde{K}_{LY}^{ml}}{N(\mu)}, \quad g_Y^{m\mu} = \frac{u_Y^{m\mu}}{N(\mu)}, \quad k_{LY}^{m\mu} = \frac{\tilde{K}_{LY}^{m\mu}}{N(\mu)}, \quad g_Y^m = \frac{u_Y^m}{N(\mu)}, \quad (A21)
\end{aligned}$$

with

$$N_{(\mu)}^2 = \int_{V^i} \rho^o u_Y^{\mu} u_Y^{\mu} dv + \sum_m \int_{V^m} \rho^{mo} u_Y^{m\mu} u_Y^{m\mu} dv. \quad (A22)$$

In (A20) the integrals over  $S^m$  are of two types, one in which the surface abuts free-space and the other in which the surface abuts the piezoelectric body. Similarly, since the piezoelectric body is only partially electroded, the integral over  $S^i$  is of two types also, one in which the surface abuts free-space and the other in which the surface abuts an electrode. At the interfaces between the electrodes and the insulator, the surface element in (A20) occurs twice, once from each side with outwardly directed unit normals  $N_L$ , which then are oppositely directed. If we agree to count each such surface only once and adopt the convention that the normal  $N_L$  is positive when directed out of the insulator into the electrode at a contiguous surface, (A20) can be written

$$\begin{aligned}
H_\mu = & \int_{\mathcal{S}^{if}} N_L [k_{LY}^l g_Y^\mu - k_{LY}^\mu g_Y^\mu - d_L^{l\lambda\mu} \hat{f}^\mu - d_L^{\mu\lambda} \hat{f}^\mu] dS + \sum_m \int_{\mathcal{S}^{ie}} N_L [(k_{LY}^l - k_{LY}^{ml}) g_Y^\mu \\
& - (k_{LY}^\mu - k_{LY}^{m\mu}) g_Y^\mu + d_L^{l\lambda\mu} \hat{f}^\mu - d_L^{\mu\lambda} \hat{f}^\mu] dS + \sum_m \int_{\mathcal{S}^{mf}} N_L (k_{LY}^{ml} g_Y^{m\mu} - k_{LY}^{m\mu} g_Y^m) dS \\
& + \int_{V^i} [k_{LY,L}^n g_Y^\mu + d_{L,L}^n \hat{f}^\mu] dV + \sum_m \int_{V^m} k_{LY,L}^{mn} g_Y^{m\mu} dV, \tag{A23}
\end{aligned}$$

in which  $\mathcal{S}^{if}$  denotes the portion of the surface of the insulator abutting free space,  $\mathcal{S}^{ie}$  denotes the portion of the surface of the insulator abutting an electrode and  $\mathcal{S}^{mf}$  denotes the portion of the surface of the electrodes abutting free-space, and where we have employed the conditions

$$g_Y^{m\mu} = g_Y^\mu, \quad g_Y^m = g_Y, \tag{A24}$$

along the surfaces on which the electrodes are attached to the insulator. Since the  $\mu$ th piezoelectric eigensolution satisfies the boundary conditions

$$\begin{aligned}
N_L k_{LY}^\mu &= 0, \quad N_L d_L^\mu = 0, \quad \text{on } \mathcal{S}^{if}, \\
N_L (k_{LY}^\mu - k_{LY}^{m\mu}) &= 0, \quad \hat{f}^\mu = 0 \quad \text{on } \mathcal{S}^{ie}, \\
N_L k_{LY}^{m\mu} &= 0, \quad \text{on } \mathcal{S}^{mf}, \tag{A25}
\end{aligned}$$

from (A23)  $H_\mu$  can be written in the form

$$\begin{aligned}
H_\mu = & \int_{\mathcal{S}^{if}} N_L (k_{LY}^l g_Y^\mu + d_L^{l\lambda\mu} \hat{f}^\mu) dS + \sum_m \int_{\mathcal{S}^{ie}} N_L [(k_{LY}^l - k_{LY}^{ml}) g_Y^\mu - d_L^{\mu\lambda} \hat{f}^\mu] dS \\
& + \sum_m \int_{\mathcal{S}^{mf}} N_L k_{LY}^{ml} g_Y^{m\mu} dS + \int_{V^i} (k_{LY,L}^n g_Y^\mu + d_{L,L}^n \hat{f}^\mu) dV + \sum_m \int_{V^m} k_{LY,L}^{mn} g_Y^{m\mu} dV. \tag{A26}
\end{aligned}$$

In the perturbation integral in (A26) the quantities without the  $\mu$  are perturbation terms which are to be determined from the  $\mu$ th eigensolution due to the presence of the bias. The boundary conditions that exist in the presence of the bias are

$$\begin{aligned} N_L (k_{LY}^l + k_{LY}^n) &= 0, \quad N_L (d_L^l + d_L^n) = 0, \quad \text{on } S^{if}, \\ N_L (k_{LY}^l + k_{LY}^n - k_{LY}^{ml} - k_{LY}^{mn}) &= 0, \quad \hat{f} = 0, \quad \text{on } S^{ie}, \\ N_L (k_{LY}^{ml} + k_{LY}^{mn}) &= 0 \quad \text{on } S^{mf}. \end{aligned} \quad (\text{A27})$$

Substituting from (A27) into (A26), rewriting the surface integrals so that normals to all surfaces are positive as in Eq. (A20) and employing the divergence theorem, we obtain

$$H_\mu = - \int_{V^i} (k_{LY}^n g_{Y,L}^\mu + d_{L,L}^{n\mu}) dv - \sum_m \int_{V^m} k_{LY}^{mn} g_{Y,L}^{m\mu} dv, \quad (\text{A28})$$

which is the form of the perturbation integral of particular interest to us. In (A28) the variables with the superscript  $n$  take the values given by the  $\mu$ th orthonormal eigensolution  $g_Y^\mu$ ,  $\hat{f}^\mu$ ,  $g_Y^{m\mu}$  in the presence of the bias, and consequently, from (2.5) for zero temperature change, (A15)<sub>2</sub> and (A21) we have

$$\begin{aligned} k_{LY}^n &= \hat{c}_{LYMQ} g_{\alpha,M}^\mu + \hat{e}_{MLY} \hat{f}_{,M}^\mu \\ d_L^n &= \hat{e}_{LMQ} g_{\alpha,M}^\mu - \hat{e}_{LM} \hat{f}_{,M}^\mu, \\ k_{LY}^{mn} &= \hat{c}_{LYMQ}^m g_{\alpha,M}^{m\mu}, \end{aligned} \quad (\text{A29})$$

where from (2.12) of Ref.1, including the terms depending on the biasing deformation only, which are the only ones needed for our purposes here, we have<sup>29</sup>

$$\hat{e}_{LMQ} = -k_{LMQBC}^1 E_{BC}^1 + e_{LMK}^w \alpha_{,K}, \quad \hat{e}_{LM} = -2\epsilon_0 J_{ML}^1 E_{ML}^1, \quad (\text{A30})$$



where  $k_{LMQBC}^1$ ,  $\epsilon_0$  and  $J^1$  denote the first order electroelastic constants, the electric permittivity of free-space and the Jacobian of the biasing deformation, which is approximately unity. The substitution of (A29) into (A28) yields

$$H_\mu = - \int_{V^i} (\hat{c}_{LYM\alpha}^m g_{\alpha, M}^{\mu} g_{\gamma, L}^{\mu} + 2\hat{e}_{ML\gamma}^m \hat{f}_{, M}^{\mu} g_{\gamma, L}^{\mu} - \hat{e}_{LM, L}^m \hat{f}_{, M}^{\mu}) dV - \sum_m \int_{V^m} \hat{c}_{LYM\alpha}^m g_{\alpha, M}^{\mu} g_{\gamma, L}^{\mu} dV. \quad (A31)$$

Equation (A31) is the form of the perturbation integral we need to show that  $\Delta\omega_\mu$  vanishes for a nonzero pure homogeneous infinitesimal rigid rotation  $\Omega_{KL}^1$  of the electroded piezoelectric body.

For a homogeneous infinitesimal rigid rotation of the electroded piezoelectric body, from (A2), (A16) and (A30), we have

$$\begin{aligned} \hat{c}_{LYM\alpha}^m &= c_{2LYKM}^m \Omega_{K\alpha}^1 + c_{2LKMQ}^m \Omega_{K\gamma}^1, \\ \hat{e}_{LM\alpha} &= e_{LMK} \Omega_{K\alpha}^1, \quad \hat{e}_{LM} = 0, \end{aligned} \quad (A32)$$

along with (A3), since  $E_{KL}^1$  vanishes and the electrodes are attached to the piezoelectric body. Substituting from (A3) and (A32) into (A31), we obtain

$$\begin{aligned} H_\mu &= - \int_{V^i} [(c_{2LYKM}^m \Omega_{K\alpha}^1 + c_{2LKMQ}^m \Omega_{K\gamma}^1) g_{\alpha, M}^{\mu} g_{\gamma, L}^{\mu} + 2e_{MLK} \Omega_{K\alpha}^1 \hat{f}_{, M}^{\mu} g_{\alpha, L}^{\mu}] dV \\ &\quad - \sum_m \int_{V^m} [c_{2LYKM}^m \Omega_{K\alpha}^1 + c_{2LKMQ}^m \Omega_{K\gamma}^1] g_{\alpha, M}^{\mu} g_{\gamma, L}^{\mu} dV, \end{aligned} \quad (A33)$$

which by virtue of the symmetry of the  $c_{2LJKM}^m$ ,  $e_{MLK}$  and  $c_{2LJKM}^m$  and the fact that  $\Omega_{K\alpha}^1$  is homogeneous, may be written

$$H_{\mu} = - 2\Omega^1 \int_{\mathcal{K}\alpha} \int_{V_i} (c_{2KMLY} g_{Y,L}^{\mu} + e_{LMK} \hat{F}_{,L}^{\mu}) g_{\alpha,M}^{\mu} dV + \sum_m \int_{V^m} \frac{c_m}{2KMLY} g_{Y,L}^{m\mu} g_{\alpha,M}^{m\mu} dV. \quad (A34)$$

Substituting from (2.9)<sub>1</sub> and (A15), with (A21) and omitting the superscript  $\ell$ , we obtain

$$H_{\mu} = - 2\Omega^1 \int_{\mathcal{K}\alpha} \int_{V_i} k_{MK}^{\mu} g_{\alpha,M}^{\mu} dV + \sum_m \int_{V^m} k_{MK}^{m\mu} g_{\alpha,M}^{m\mu} dV, \quad (A35)$$

which with the aid of the divergence theorem, the normalized forms of (A10)<sub>1</sub> and (A12)<sub>1</sub> and the conventions introduced in (A23) yields

$$H_{\mu} = - 2\Omega^1 \left[ \int_{\mathcal{K}\alpha} \int_{if} N_M k_{MK}^{\mu} g_{\alpha}^{\mu} dS + \sum_m \int_{ie} N_M (k_{MK}^{\mu} - k_{MK}^{m\mu}) g_{\alpha}^{\mu} dS + \sum_m \int_{mf} N_M k_{MK}^{m\mu} g_{\alpha}^{m\mu} dS + \omega_{\mu}^2 \left( \int_{V_i} \rho^o g_K^{\mu} g_{\alpha}^{\mu} dV + \sum_m \int_{V^m} \rho^{mo} g_K^{m\mu} g_{\alpha}^{m\mu} dV \right) \right], \quad (A36)$$

where we have employed (A24) in obtaining (A36). Now, from (A19) and (A36), with the conditions in (A2)<sub>2</sub> and (A25), we have

$$\Delta_{\mu} = 0. \quad (A37)$$