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SUMMARY

In this paper we consider the time-dependent behaviour of a single-server queue with Poisson input and constant service time, where the input rate equals the reciprocal of the service time. Formulae for the probabilities associated with different queue-lengths have been derived by an empirical method which included inspection of some numerical results. These formulae are now proved correct by induction.

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I INTRODUCTION

The problems studied in this paper were raised by Attwooll when preparing a lecture[1] on time-dependent queues. It is well known that a single-server queue with Poisson arrivals and constant service time (an M/D/1 queue) has a steady-state queue-length distribution if and only if the arrival rate is less than the service rate. (The service rate is the reciprocal of the service time.) If the arrival rate equals (or is greater than) the service rate the queue is 'unstable' and the average queue length increases indefinitely with time. The steady-state solution for the M/D/1 queue was published[2] in 1909. The time-dependent solution for the M/M/1 queue (exponential distribution of service times) was first published in 1953 [3].

In this paper we consider the case where the arrival rate is exactly equal to the service rate. Without loss of generality we define unit time to equal the service time (therefore equal also to the mean time between arrivals).

Let $q(n)$ be the probability that there are i customers in the queue at time n. We shall only consider integer values of n. The "number in the system" includes the customer being served, if there is one.

The number of arrivals during a unit interval has a Poisson distribution with mean 1, so the probabilities of 0, 1, 2, j arrivals are respectively $\exp(-1)$, $\exp(-1)$, $\exp(-1)/2!$, $\exp(-1)/j!$. The number of departures during a unit interval is 1 (if there is a customer in the system at the start of the interval) or 0 (if the system is empty at the start of the interval). It is easy to write down equations giving the queue-length probabilities at time $(n + 1)$ in terms of the probabilities at time n:

$$q_0(n + 1) = \{q_0(n) + q_1(n)\} \exp(-1) \tag{1}$$

$$q_1(n + 1) = \{q_0(n) + q_1(n) + q_2(n)\} \exp(-1) \tag{2}$$

$$q_2(n + 1) = \{q_0(n)/2! + q_1(n)/2! + q_2(n) + q_3(n)\} \exp(-1) \tag{3}$$

$$\vdots$$

$$q_i(n + 1) = \left\{q_0(n)/_i! + q_1(n)/_i! + q_2(n)/(i - 1)! + \ldots q_i(n) + q_{i+1}(n)\right\} \exp(-1). \qquad \ldots\ldots(4)$$

We consider the case where the system starts empty, so $q_0(0) = 1$ and all the other initial probabilities equal 0. Values of the different probabilities for $n = 1, 2, 3$, etc can then be obtained numerically by repeated solution of the above set of equations, using a computer. Attwooll did this and noticed that the probability of no customers waiting was given by the formula:

$$q_0(n) + q_1(n) = \frac{(n+1)^{n+1}}{(n+1)!} \exp(-n) \quad . \tag{5}$$

He was unable to prove this formula. We shall proceed by assuming *its correctness* in order to deduce results for other queue-lengths, then verify our results by induction.

2 DERIVATION OF FORMULAE

Using (5) to substitute in equation (1):

$$q_0(n+1) = \frac{(n+1)^{n+1}}{(n+1)!} \exp(-n-1)$$

so *obviously*

$$q_0(n) = \frac{n^n}{n!} \exp(-n) \quad . \tag{6}$$

This expression is undefined for $n = 0$. We assume $q_0(0) = 1$, *ie* the system starts empty.

To simplify the equations we introduce the notations

$$Q_i(n) = q_i(n) \exp(n) \tag{7}$$

and

$$[n] = n^n/n!$$

$$[0] = 1$$

so equation (6) can be put in the form

$$Q_0(n) = [n] \quad . \tag{8}$$

From equations (1) to (4) we can derive the system of equations:

$$Q_0(n+1) = Q_0(n) + Q_1(n) \tag{9}$$

$$Q_1(n+1) = Q_0(n) + Q_1(n) + Q_2(n) \tag{10}$$

$$Q_2(n+1) = Q_0(n)/2! + Q_1(n)/2! + Q_2(n) + Q_3(n) \tag{11}$$

$$\vdots$$

$$Q_j(n+1) = Q_0(n)/j! + Q_1(n)/j! + Q_2(n)/(j-1)! + \ldots + Q_j(n) + Q_{(j+1)}(n). \quad \ldots\ldots(12)$$

Using equation (8) to substitute for both $Q_0(n)$ and $Q_0(n+1)$ in equation (9), and re-arranging terms, we find:

$$Q_1(n) = [n+1] - [n] \tag{13}$$

Using equations (8) and (13) to substitute for $Q_1(n+1)$, $Q_1(n)$ and $Q_0(n)$ in equation (10):

$$Q_2(n) = [n+2] - 2[n+1] \tag{14}$$

Proceeding, in the same way, to substitute into equation (11), we find:

$$Q_3(n) = [n+3] - 3[n+2] + \frac{3}{2}[n+1] \tag{15}$$

and it can also be shown that:

$$Q_4(n) = [n+4] - 4[n+3] + \frac{4.2}{2!}[n+2] - \frac{4}{3!}[n+1] \tag{16}$$

$$Q_5(n) = [n+5] - 5[n+4] + \frac{5.3}{2!}[n+3] - \frac{5.2^2}{3!}[n+2] + \frac{5}{4!}[n+1] \tag{17}$$

$$Q_6(n) = [n+6] - 6[n+5] + \frac{6.4[n+4]}{2!} - \frac{6.3^2}{3!}[n+3] + \frac{6.2^3}{4!}[n+2] - \frac{6[n+1]}{5!} \quad \ldots\ldots(18)$$

$$\vdots$$

$$Q_i(n) = -i \sum_{j=0}^{i-1} \frac{(j-i)^{j-1}}{j!}[n+i-j] \qquad (i > 1)\,. \tag{19}$$

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The distribution of queue lengths can now be calculated, using equation (7) to transform equations (8), (13) and (19) back into the original variables q_i :

$$q_0(n) = \frac{n^n}{n!} \exp(-n) = [n] \exp(-n) \tag{20}$$

$$q_1(n) = \left\{ \frac{(n+1)^{n+1}}{(n+1)!} - \frac{n^n}{n!} \right\} \exp(-n) = \Big([n+1] - [n] \Big) \exp(-n) \tag{21}$$

$$q_i(n) = -\, i \exp(-n) \sum_{j=0}^{i-1} \frac{(j-i)^{j-1} [n+i-j]}{j!} \qquad (i > 1) \; . \tag{22}$$

By assuming the correctness of Attwooll's empirical formula (equation (5)) we have generalised his result and obtained formulae (equations (20), (21), (22)) for all queue lengths.

3 PROOF OF CORRECTNESS

We shall prove these formulae by induction, *ie* we shall prove first that the formulae are correct for $n = 0$, and secondly, provided the formulae are true for all times up to and including some value n they are also correct for time $(n + 1)$.

For $n = 0$, straightforward substitution in equations (20) and (21) gives:

$$q_0(0) = 1 \; , \qquad q_1(0) = 0 \; . \tag{23}$$

Putting $n = 0$ in equation (22) gives:

$$q_i(0) = -\, i \sum_{j=0}^{i-1} \frac{(j-i)^{j-1}}{J!} \, \frac{(i-j)^{i-j}}{(i-j)!} \qquad (i > 1)$$

hence

$$q_i(0) = \frac{1}{(i-1)!} \sum_{j=0}^{i-1} (-1)^j (i-j)^{i-1} \binom{i}{j}$$

writing k for $(i - j)$ and summing in the reverse order:

$$q_i(0) \;=\; \frac{1}{(i-1)!}\sum_{k=1}^{i}(-1)^{i-k}k^{i-1}\binom{i}{k}$$

$$q_i(0) \;=\; \frac{(-1)^i}{(i-1)!}\sum_{k=1}^{i}(-1)^{k}k^{i-1}\binom{i}{k} \qquad (i > 1) \;.$$

In the Appendix it is shown that

$$\sum_{k=1}^{i}(-1)^{k}k^{i-1}\binom{i}{k} \;=\; 0$$

so that

$$q_i(0) \;=\; 0 \qquad\qquad (i > 1) \;. \tag{24}$$

Together with equations (23) this completes the first part of the proof.

Now assuming equations (20) to (22) are correct for some value n, the probabilities at time $(n + 1)$ are found by substituting for the $q_i(n)$ on the right-hand side of equations (1) to (4). Thus we find that

$$q_0(n+1) \;=\; \frac{(n+1)^{n+1}}{(n+1)!}\exp(-n-1)$$

and this is obviously of the correct form, so equation (20) holds at time $(n + 1)$.

$$\begin{aligned} q_1(n+1) \;&=\; \Big([n] + [n+1] - [n] + [n+2] - 2[n+1]\Big)\exp(-n-1) \\ &=\; \Big([n+2] - [n+1]\Big)\exp(-n-1) \end{aligned}$$

and this is also of the correct form, so equation (21) holds at time $(n + 1)$.

Equation (3) is just a special case of equation (4) which can be written as:

$$q_i(n+1) \;=\; \left\{\frac{q_0(n)}{i!} + \frac{q_1(n)}{i!} + \sum_{j=2}^{i+1}\frac{q_j(n)}{(i+1-j)!}\right\}\exp(-1) \;. \tag{25}$$

Using equations (20) to (22) to substitute for the $q(n)$ on the right-hand side of equation (25)

$$q_i(n+1) = \exp(-n-1)\left\{\frac{[n]}{(i)!} + \frac{[n+1]-[n]}{(i)!} - \sum_{j=2}^{i+1}\sum_{k=0}^{j-1} j\,\frac{(k-j)^{k-1}}{k!}[n+j-k]\right\}$$

$$q_i(n+1) = \exp(-n-1)\left\{\sum_{j=1}^{i+1}\sum_{k=0}^{j-1} j\,\frac{(k-j)^{k-1}}{k!}\,[n+j-k]\right\} . \qquad (26)$$

Now let $m = i + 1 - j + k$ so that $j = i + k + 1 - m$. To transform the summation it is helpful to draw up a table, showing which combinations of j and k are to be summed over, and the value of m for each summable combination see Fig 1. Substituting for j , and summing over m and k instead of over j and k :

$$q_i(n+1) = \exp(-n-1)\left\{\sum_{m=0}^{i}\sum_{k=0}^{m} \frac{(i+k+1-m)(m-i-1)^{k-1}}{k!}\,[n+1+i-m]\right\}$$

$$q_i(n+1) = \exp(-n-1)\left\{\sum_{m=0}^{i}\left\{[n+1+i-m]\sum_{k=0}^{m}\frac{(i+k+1-m)(m-i-1)^{k-1}}{k!}\right\}\right\} \qquad \ldots\ldots(27)$$

but

$$\sum_{k=0}^{m}\frac{(i+k+1-m)(m-i-1)^{k-1}}{k!} = \sum_{k=0}^{m}(-1)^{k-1}\,\frac{(i+1-m+k)(i+1-m)^{k-1}}{k!}$$

$$= \sum_{k=0}^{m}(-1)^{k-1}\left\{\frac{(i+1-m)^{k}}{k!} + \frac{(i+1-m)^{k-1}(k\neq 0)}{(k-1)!}\right\}$$

$$= (-1)^{m-1}\,\frac{(i+1-m)^{m}}{m!} \qquad (28)$$

substituting this last expression for the inner sum in equation (27) we obtain:

$$q_i(n+1) = -\exp(-n-1) \sum_{m=0}^{i} (-1)^m \frac{(m-i-1)^{m-1}[n+1+i-m]}{m!} . \quad (29)$$

By comparing equations (29) and (22), it can be seen that the formula is correct for time $(n + 1)$. This completes the second part of the proof.

We have now shown that, provided equations (20), (21) and (22) are correct up to time n, they are also true for time $(n + 1)$. But we showed earlier that the equations are correct for time 0, therefore they are also correct for time 1, therefore they are also correct at time 2, and so on. Thus equations (2) to (22) are true for all integer values of the time n.

We have thus proved the correctness of equations (20) to (22). Attwooll's empirical result, equation (5), can easily be proved by adding equation (20) to equation (21).

4 MEAN QUEUE LENGTH

The mean queue length may be defined as

$$\bar{q}(n) = \sum_{i=2}^{\infty} (i-1)q_i(n) \quad (30)$$

(the customer being served is not included). By multiplying each of equations (1) to (4) by an appropriate factor, and summing over i, Attwooll obtained the result

$$\bar{q}(n+1) = \bar{q}(n) + \{q_0(n) + q_1(n)\}e^{-1} . \quad (31)$$

Using equation (5), which we have proved in section 3

$$\bar{q}(n+1) - \bar{q}(n) = \frac{(n+1)^{n+1}}{(n+1)!} \exp(-n-1) \quad (32)$$

hence

$$\bar{q}(n) = \sum_{m=1}^{n} \frac{m^m}{m!} \exp(-m) . \quad (33)$$

For large values of m, the value of $m!$ is given approximately by Stirling's formula

$$m! = (2\pi m)^{0.5} m^m \exp(-m) \quad .$$

Thus for large values of n

$$\bar{q}(n) - \bar{q}(n-1) \doteqdot (2\pi n)^{-0.5} \quad . \tag{34}$$

Hence it may be shown that

$$\bar{q}(n) \doteqdot \sqrt{\frac{2n+1}{\pi}} \quad . \tag{35}$$

Comparing approximate numerical values obtained from equation (35) with the corresponding exact values obtained from equation (33), it was found that a very good approximation is given by the equation

$$\bar{q}(n) \doteqdot \sqrt{\frac{2n+1.2}{\pi}} - 0.66 \; . \tag{36}$$

5 SUMMARY OF RESULTS

The following results (except for the last equation (A-2)) relate to the queue system M/D/1 with arrival rate equal to service rate.

The probability of i customers in the system (including the customer being served) at time n is given by the formulae

$$q_0(n) = [n] \exp(-n) \tag{20}$$

$$q_1(n) = ([n+1] - [n]) \exp(-n) \tag{21}$$

$$q_i(n) = -i \exp(-n) \sum_{j=0}^{i-1} \frac{(j-i)^{j-1}[n+i-j]}{j!} \tag{22}$$

where $[r] = r^r/r!$
and $[0] = 1$.

The expected queue-length (excluding the customer being served) at time n is given by the formula

$$\bar{q}(n) = \sum_{m=1}^{n} [m] \exp(-m) \tag{33}$$

and by the approximate formula

$$\bar{q}(n) \doteqdot \sqrt{\frac{2n + 1.2}{\pi}} - 0.66 \quad . \tag{36}$$

Also we have proved the interesting combinatorial identity

$$\sum_{k=1}^{n} (-1)^k k^r \binom{n}{k} = 0 \qquad 0 \leqslant r < n \; . \tag{A-2}$$

Appendix

SUMMING A FINITE SERIES

In this Appendix we find the sum of the series:

$$S(n,r) = \sum_{k=1}^{n} (-1)^k k^r \binom{n}{k} . \qquad \text{(A-1)}$$

From (A-1) it follows that

$$\sum_{r=0}^{\infty} \frac{S(n,r)z^r}{r!} = \sum_{r=0}^{\infty} \left[\sum_{k=1}^{n} (-1)^k \frac{k^r z^r}{r!} \binom{n}{k} \right]$$

$$= \sum_{k=1}^{n} \left[(-1)^k \binom{n}{k} \sum_{r=0}^{\infty} \frac{k^r z^r}{r!} \right]$$

$$= \sum_{k=1}^{n} (-1)^k \binom{n}{k} \exp(kz)$$

$$= \left[1 - \exp(z) \right]^n$$

$$= \left(- z - \frac{z^2}{2!} - \frac{z^3}{3!} - \ldots \right)^n$$

$$\sum_{r=0}^{\infty} \frac{S(n,r)z^r}{r!} = (-z)^n \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots \right)^n .$$

By equating coefficients of z^r it can be seen that

$$S(n,r) = 0 \qquad 0 \leqslant r < n$$

ie

$$\sum_{k=1}^{n} (-1)^k k^r \binom{n}{k} = 0 . \qquad 0 \leqslant r < n . \qquad \text{(A-2)}$$

The special case $S(n,1) = 0$ is included in Ref 4, on page 4 as formula 0.154.2.

The special case $S(n, n - 1) = 0$

ie

$$\sum_{k=1}^{n} (-1)^k k^{n-1} \binom{n}{k} = 0 \qquad \text{(A-3)}$$

is used in section 3 of this Memorandum.

It can also be easily shown that

$$\sum_{k=1}^{n} (-1)^k k^{n} \binom{n}{k} = (-1)^n n! \qquad \text{(A-4)}$$

and that

$$\sum_{k=1}^{n} (-1)^k k^{n+1} \binom{n}{k} = (-1)^n (n+1)!n/2 \quad . \qquad \text{(A-5)}$$

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	j = 1	j = 2	j = 3	......	j = i + 1
k = i					m = i
k = i - 1					m = i - 1
k = i - 2					m = i - 2
.					.
.					.
.					.
k = 2			m = i	......	m = 2
k = 1		m = i	m = i - 1	......	m = 1
k = 0	m = i	m = i - 1	m = i - 2	......	m = 0

Fig 1 Values of m for combinations of j and k that are summed over (see equation 26)

REPORT DOCUMENTATION PAGE

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17. Abstract

In this paper we consider the time-dependent behaviour of a single-server queue with Poisson input and constant service time, where the input rate equals the reciprocal of the service time. Formulae for the probabilities associated with different queue-lengths have been derived by an empirical method which included inspection of some numerical results. These formulae are now proved correct by induction.

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