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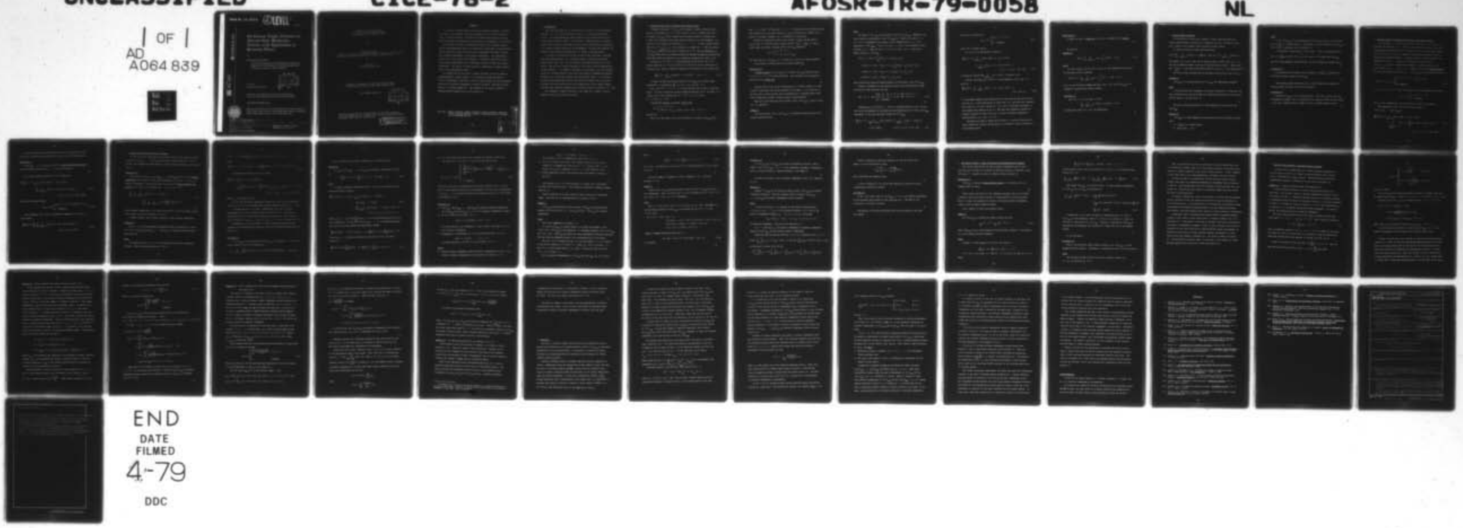

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On Poisson Traffic Processes in Discrete State Markovian Systems with Applications to Queueing Theory

by

BENJAMIN MELAMED

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July 1977
Revised February 1978

Computer, Information & Control Engineering Program
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ON POISSON TRAFFIC PROCESSES IN DISCRETE STATE MARKOVIAN SYSTEMS WITH APPLICATIONS TO QUEUEING THEORY

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Abstract

We consider a regular Markov process with continuous parameter, countable state space, and stationary transition probabilities, over which we define a class of traffic processes. The feasibility that multiple traffic processes constitute mutually independent Poisson processes is investigated in some detail.

We show that a variety of independence conditions on a traffic process and the underlying Markov process are equivalent or sufficient to ensure Poisson related properties; these conditions include independent increments, renewal, weak pointwise independence, and pointwise independence. Two computational criteria for Poisson traffic are developed: a necessary condition in terms of weak pointwise independence, and a sufficient condition in terms of pointwise independence. The utility of these criteria is demonstrated by sample applications to queueing-theoretic models.

It follows that, for the class of traffic processes as per this paper in queueing-theoretic context, Muntz's $M \Rightarrow M$ property, Gelenbe and Muntz's notion of completeness, and Kelly's notion of quasi-reversibility are essentially equivalent to pointwise independence of traffic and state. The latter concept, however, is the most general one. The relevance of the theory developed to queueing network decomposition is also pointed out.

Key words: Markov Processes, Traffic Processes, Poisson Processes, Queueing Theory, Queueing Networks, Traffic in Queueing Networks, Decomposition of Queueing Networks

1. Introduction

This paper has grown out of previous work on traffic in certain queueing networks ([4], [19], [20]) whose state process is a discrete state Markov process. The paper generalizes several aspects of the discussion and results in the papers alluded to above. In particular, a general notion of a traffic process over a discrete state Markov process will be defined and the feasibility of it being a Poisson process will be investigated. We shall also exemplify the utility of the results by applying them to a number of queueing models.

In the way of motivation, we point out that traffic processes in networks with flow characteristics (e.g., queueing networks, communication networks, machine repair shops, etc.) are an important operating characteristic of such models. They are also of major importance to the study of valid decompositions of such networks. It is common to postulate, in such models, that the incoming traffic is a Poisson process, a fact that often renders a mathematical analysis tractable; it is also based on many real-life empirical data. If, in addition, one may validly assume that traffic flows within the network are also Poisson processes, then this could give rise to decompositions of the original network such that each component subnetwork may be validly studied in isolation ([4], [20]).

The treatment of traffic processes in this paper will, however, be more general--at the level of Markov processes.

2. Traffic Processes over a Discrete State Markov Process

Throughout the paper, $\{C(t)\}_{t\geq a}$ will designate a right-continuous Markov process with parameter set $[a, \infty)$ for some real a, and a countable state set Γ. We assume $\{C(t)\}_{t\geq a}$ to have standard and stationary transition probabilities, so that the associated infinitesimal generator matrix Q is time homogenous; its transition rate elements are denoted $q(\gamma, \delta)$, $\gamma, \delta \in \Gamma$. We shall further assume that the $q(\gamma) \triangleq \sum_{\delta\in\Gamma-\{\gamma\}} q(\gamma, \delta)$ are bounded as γ ranges over Γ. Thus the process $\{C(t)\}_{t\geq a}$ is regular in the sense of Çinlar [8] p. 251 and our assumptions on $\{C(t)\}_{t\geq a}$ imply that the associated Forward and Backward Kolmogorov Equations have unique and identical solutions for the transition probabilities ([12] p. 475).

Denoting $c_t(\gamma) \triangleq P[C(t) = \gamma]$ and premultiplying the matrix form of the Forward Equations (cf. [11] pp. 240-241) by a row vector initial condition with components $c_a(\gamma)$ yields a system of equations in the absolute state probabilities

$$\frac{\partial}{\partial t} c_t(\gamma) = \sum_{\xi\in\Gamma-\{\gamma\}} c_t(\xi)q(\xi, \gamma) - c_t(\gamma)q(\gamma), \quad t \geq a, \gamma \in \Gamma. \tag{2.1}$$

We shall say that equilibrium prevails if $\{C(t)\}_{t\geq a}$ is in steady state; equivalently, in equilibrium, $\frac{\partial}{\partial t} c_t(\gamma) \equiv 0$, $t \geq a$, for all $\gamma \in \Gamma$.

Next, let $\Theta \subset \Gamma^2 - \{(\gamma, \gamma): \gamma \in \Gamma\}$ be an arbitrary set of pairs of distinct states. To avoid trivialities we shall always assume that $\Theta \neq \Phi$. For each $\gamma \in \Gamma$, Θ gives rise to the following sets $\Theta(\gamma, \cdot) \triangleq \{\delta: (\gamma, \delta) \in \Theta\}$ and $\Theta(\cdot, \gamma) \triangleq \{\beta: (\beta, \gamma) \in \Theta\}$.

Consider the sequence of epoches $\{T_n\}_{n=0}^{\infty}$ where

$$T_n = \begin{cases} a, & \text{if } n = 0 \\ \inf\{t: t > T_{n-1}, (C(t-), C(t)) \in \Theta\}, & \text{if } n > 0 \end{cases}$$

induced by Θ.

Thus, T_n is the epoch of the n-th occurrence of a jump in $\{C(t)\}_{t\geq a}$ from some $\gamma \in \Gamma$ to some $\delta \in \Gamma$ such that $(\gamma, \delta) \in \Theta$. We adopt here the view that certain state transitions in the underlying $\{C(t)\}_{t\geq a}$ are interpreted as traffic due to entities (customers, messages, etc.) moving about in the system.

Instead of studying the traffic point process $\{T_n\}_{n=0}^{\infty}$, one may equivalently elect to study the traffic interval process $\{T_{n+1} - T_n\}_{n=0}^{\infty}$, or equivalently again the traffic counting process $\{K(t)\}_{t\geq a}$ defined by

$$K(t) = \begin{cases} 0, & \text{if } t = a \\ n, & \text{if } T_n \leq t < T_{n+1}. \end{cases}$$

The state space of $\{K(t)\}_{t\geq a}$ is $N \cup \{0\}$ where N is the set of natural numbers.

In this paper, we shall adopt the following terminology.

<u>Definition 2.1</u>

A <u>traffic process</u> over $\{C(t)\}_{t\geq a}$ is a process $\{K(t)\}_{t\geq a}$ induced by some $\Theta \subset \Gamma^2 - \{(\gamma, \gamma): \gamma \in \Gamma\}$ as described above. The inducing Θ will henceforth be referred to as a <u>traffic set</u>.

□

The particular choice of the representation of a traffic process is a mere technical convenience serving the purposes of this paper. It is simply due to the fact that a Poisson process can be represented as a counting process whose state probabilities satisfy a simple system of birth equations.

What can be said about the joint process $\{(C(t), K(t))\}_{t\geq a}$? First, we show (cf. [4], Theorem 1)

<u>Lemma 2.1</u>

The joint process $\{(C(t), K(t))\}_{t\geq a}$ is a conservative Markov process with bounded transition rates.

Proof

The jumps of $\{K(t)\}_{t\geq a}$ are contained in those of $\{C(t)\}_{t\geq a}$. Therefore, the joint process is conservative, since $\{C(t)\}_{t\geq a}$ is. Clearly, for every $s < u$, $K(u) - K(s)$ is measurable with respect to the σ-algebra $\sigma\{C(t): s < t \leq u\}$ generated by $\{C(t)\}_{s<t\leq u}$. Let $a < t_1 < t_2 < \ldots < t_r < u$ be a partition of the interval $[a, u]$. Then, for any $\gamma_j \in \Gamma$, $n_j \in N \cup \{0\}$, $1 \leq j \leq r$, and by the Markov property of $\{C(t)\}_{t\geq a}$

$$P[C(u) = \gamma,\ K(u) = n \,|\, \bigcap_{j=1}^{r} \{C(t_j) = \gamma_j,\ K(t_j) = n_j\}]$$

$$= P[C(u) = \gamma,\ K(u) - K(t_r) = n - n_r \,|\, \bigcap_{j=1}^{r} \{C(t_j) = \gamma_j,\ K(t_j) = n_j\}]$$

$$= P[C(u) = \gamma,\ K(u) - K(t_r) = n - n_r \,|\, C(t_r) = \gamma_r,\ K_{t_r} = n_r]$$

$$= P[C(u) = \gamma,\ K(u) = n \,|\, C(t_r) = \gamma_r,\ K_{t_r} = n_r]$$

which verifies the requisite Markov property of the process $\{(C(t), K(t))\}_{t\geq a}$.

Finally, boundedness of the transition rates of the joint process follows from the fact that they have the form

$$\tilde{q}((\gamma, i), (\delta, j)) = \begin{cases} q(\gamma, \delta), & \text{if } (\gamma, \delta) \in \Theta \text{ and } 0 \leq i = j - 1 \\ q(\gamma, \delta), & \text{if } (\gamma, \delta) \notin \Theta \text{ and } 0 \leq i = j \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

$\square$

Denoting $P_t(\gamma, n) \triangleq P[C(t) = \gamma, K(t) = n]$ and with the aid of (2.2), we can now derive the equations in the absolute state probabilities for $\{(C(t), K(t))\}_{t\geq a}$, analogously to the ones previously derived for $\{C(t)\}_{t\geq a}$.

$$\frac{\partial}{\partial t} P_t(\gamma, n) = \sum_{\xi \notin \Theta(\cdot,\gamma)\cup\{\gamma\}} P_t(\xi, n) q(\xi, \gamma) + \sum_{\xi \in \Theta(\cdot,\gamma)} P_t(\xi, n-1) q(\xi, \gamma)$$

$$- P_t(\gamma, n) q(\gamma), \qquad t \geq a,\ (\gamma, n) \in \Gamma \times (N \cup \{0\}). \tag{2.3}$$

The initial conditions are

$$P_a(\gamma, k) = \begin{cases} c_a(\gamma), & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

since $K(a) = 0$ almost surely.

Eq. (2.3) can be equivalently written as

$$\frac{\partial}{\partial t} P_t(\gamma, n) = \sum_{\xi \in \Gamma - \{\gamma\}} P_t(\xi, n) q(\xi, \gamma) - P_t(\gamma, n) q(\gamma)$$

$$+ \sum_{\eta \in \Theta(\cdot, \gamma)} (P_t(\eta, n-1) - P_t(\eta, n)) q(\eta, \gamma),$$

$$t \geq a, \ (\gamma, n) \in \Gamma \times (N \cup \{0\}), \tag{2.5}$$

by adding and subtracting $\sum_{\eta \in \Theta(\cdot, \gamma)} P_t(\eta, n) q(\eta, \gamma)$ from Eq. (2.3).

Finally, denoting $k_t(n) = P[K(t) = n]$ and summing Eq. (2.5) over $\gamma \in \Gamma$ gives us

$$\frac{\partial}{\partial t} k_t(n) = \sum_{\gamma \in \Gamma} \sum_{\eta \in \Theta(\cdot, \gamma)} (P_t(\eta, n-1) - P_t(\eta, n)) q(\eta, \gamma),$$

$$t \geq a, \ n \in N \cup \{0\}. \tag{2.6}$$

To interchange summation and differentiation in the above we have used the fact that the $P_t(\gamma, n)$ have derivatives of every order in t, and that every countable sum of the $P_t(\gamma, n)$ over a subset of $\Gamma \times (N \cup \{0\})$ is uniformly convergent on each compact time interval of $[a, \infty)$. This fact will henceforth justify all termwise operations on sums of the $P_t(\gamma, n)$ such as termwise integration, differentiation, etc. ([24], 1.1, 1.7).

Throughout the paper we denote $M(t) \triangleq E[K(t)]$. To avoid trivialities we shall, henceforth, restrict the discussion to substantive traffic processes in the following sense:

Definition 2.2

A traffic process is nontrivial if $M(t) \not\equiv 0$; otherwise it is trivial.

$\square$

We now show

Theorem 2.1

$$M(t) = \sum_{\gamma \in \Gamma} \sum_{\eta \in \Theta(\cdot,\gamma)} q(\eta, \gamma) \int_a^t c_\tau(\eta) d\tau, \quad t \geq a. \tag{2.7}$$

Proof

For every fixed $j \in N$ sum (2.6) over $n \geq j$; then integrate both sides of the resultant sum thus obtaining

$$P\{K(t) \geq j\} = \sum_{\gamma \in \Gamma} \sum_{\eta \in \Theta(\cdot,\gamma)} q(\eta, \gamma) \int_a^t P_\tau(\eta, j - 1) d\tau, \quad t \geq a.$$

Eq. (2.7) now follows by summing the above over $j \in N$, since $\{K(t)\}_{t \geq a}$ is a nonnegative integer-valued random variable.

$\square$

Corollary 2.1

$M(t) \equiv \lambda t$, $t \geq 0$, for some $\lambda \geq 0$ iff

$$\sum_{\gamma \in \Gamma} \sum_{\eta \in \Theta(\cdot,\gamma)} c_t(\eta) q(\eta, \gamma) \equiv \text{const.}, \quad t \geq a.$$

In particular, $M(t) \equiv \lambda t$, $t \geq a$, in equilibrium.

$\square$

3. Poisson Traffic Processes

In this section we shall give a number of simple characterizations of Poisson related traffic processes over a Markovian process. We shall see that only a subset of the ordinary Poisson axioms will here suffice.

To simplify notation we shall henceforth denote

$$m(t) \triangleq \sum_{\gamma\in\Gamma} \sum_{\eta\in\Theta(\cdot,\gamma)} c_t(\eta)q(\eta, \gamma) = \frac{\partial}{\partial t} M(t) \quad \text{and} \quad m(t, \gamma) \triangleq \sum_{\eta\in\Theta(\cdot,\gamma)} c_t(\eta)q(\eta, \gamma).$$

Intuitively, $m(t)$ is the total rate of expected traffic count, while $m(t, \gamma)$ is the rate of expected traffic count due to transitions into state γ. Observe that $m(t) = \sum_{\gamma\in\Gamma} m(t, \gamma)$ and that in equilibrium both $m(t)$ and $m(t, \gamma)$ are independent of t.

The first theorem characterizes an arbitrary Poisson process over $\{C(t)\}_{t\geq a}$.

Theorem 3.1

$\{K(t)\}_{t\geq a}$ is a Poisson process iff $\{K(t)\}_{t\geq a}$ has independent increments.

Proof

A Poisson process has independent increments by definition. Conversely, the only counting process with unit jumps and continuous mean function is the Poisson process (see e.g., Çinlar [8] Ch. 4).

$\square$

The second theorem characterizes a time homogenous Poisson process over $\{C(t)\}_{t\geq a}$.

Theorem 3.2

$\{K(t)\}_{t\geq a}$ is a time homogenous Poisson process iff the following conditions hold:

i) $\{T_n\}_{n=0}^{\infty}$ is a renewal process

ii) $m(t) \equiv m(a), \quad t \geq a.$

Proof

($\Rightarrow$) If $\{K(t)\}_{t\geq a}$ is a time homogenous Poisson process then it is well-known that $\{T_n\}_{n=0}^{\infty}$ is a renewal process. Furthermore, the rate function of $\{K(t)\}_{t\geq a}$ is $m(t) \equiv m(a)$ as required, due to Corollary 2.1.

($\Leftarrow$) Conversely, suppose that i) and ii) hold. Since the renewal function $R(t)$ of $\{T_n\}_{n=0}^{\infty}$ is $R(t) = M(t) = \int_a^t m(\tau)d\tau = m(a)t$, it follows that $\{K(t)\}_{t\geq a}$ must be a time homogenous Poisson process, as $R(t)$ determines a renewal process.

$\square$

Corollary 3.1

In equilibrium, the renewal process property of $\{T_n\}_{n=0}^{\infty}$ is equivalent to the Poisson process property of $\{K(t)\}_{t\geq a}$.

$\square$

The preceding characterizations give us some information as regards non-Poisson traffic processes, by way of elimination.

Corollary 3.2

Suppose $\{K(t)\}_{t\geq a}$ is not a Poisson process. Then $\{K(t)\}_{t\geq a}$ does not have independent increments, and, in equilibrium, the respective point process $\{T_n\}_{n=0}^{\infty}$ is not even a renewal process (though it may be a delayed renewal process).

$\square$

7. Multiple Poisson Traffic: Queueing-Theoretic Examples

4. Multiple Traffic Processes over a Discrete State Markov Process

Let $\{K_1(t)\}_{t\geq a}, \ldots, \{K_\ell(t)\}_{t\geq a}$ be traffic processes over $\{C(t)\}_{t\geq a}$, for some fixed but arbitrary $\ell \in N$. For the i-th traffic process above, the associated entities are denoted Θ_i for its traffic set, $M_i(t)$ for its mean function, $k_t^{(i)}(n) \triangleq P[K_i(t) = n]$, etc.; in general, we append the appropriate index to such previously defined symbols. To simplify notation we shall denote in the sequel $K(t) \triangleq (K_1(t), \ldots, K_\ell(t))$ to be the vector traffic process, $n \triangleq (n_1, \ldots, n_\ell)$ to be a vector with nonnegative integer components, and $k_t(n) \triangleq P[K_1(t) = n_1, \ldots, K_\ell(t) = n_\ell]$. Lemma 2.1 still holds mutatis mutandis for the joint process $\{(C(t); K_1(t), \ldots, K_\ell(t))\}_{t\geq a}$; the new transition rates are

$$\tilde{q}((\gamma, i), (\delta, j)) = \begin{cases} q(\gamma, \delta), \text{ if } & i = j - \sum_{i=1}^{\ell} \chi_i(\gamma, \delta)e_i \\ 0, \text{ otherwise} \end{cases} \tag{4.1}$$

$(\gamma, i), (\delta, j) \in \Gamma \times (N \cup \{0\})^\ell$; in the above χ_i is the characteristic function

$$\chi_i(\gamma, \delta) = \begin{cases} 1, & \text{if } (\gamma, \delta) \in \Theta_i \\ 0, & \text{otherwise} \end{cases}$$

and e_i is the n-dimensional unit vector with 1 in the i-th coordinate.

The counterpart of Eq. (2.5) for the joint process $\{C(t), K_1(t), \ldots, K_\ell(t))\}_{t\geq a}$ is

$$\begin{aligned} \frac{\partial}{\partial t} P_t(\gamma, n) = & \sum_{\xi\in\Gamma-\{\gamma\}} P_t(\xi, n)q(\xi, \gamma) - P_t(\gamma, n)q(\gamma) \\ & + \sum_{\eta\in \bigcup_{i=1}^{\ell} \Theta_i(\cdot,\gamma)} P_t(\eta, n - \sum_{i=1}^{\ell} \chi_i(\eta, \gamma)e_i) - P_t(\eta, n))q(\eta, \gamma), \end{aligned} \tag{4.2}$$

$$t \geq a, (\gamma, n) \in \Gamma \times (N \cup \{0\})^\ell.$$

For reasons that will become apparent later on, we shall restrict the discussion to traffic processes which are disjoint in the following sense:

Definition 4.1

$\{K_1(t)\}_{t\geq a}, \ldots, \{K_\ell(t)\}_{t\geq a}$ are said to be disjoint traffic processes if their associated traffic sets $\Theta_1, \ldots, \Theta_\ell$ are disjoint sets.

$\square$

For disjoint traffic processes, Eq. (4.2) reduces to

$$\frac{\partial}{\partial t} P_t(\gamma, n) = \sum_{\xi\in\Gamma-\{\gamma\}} P_t(\xi, n)q(\xi, \gamma) - P_t(\gamma, n)q(\gamma)$$

$$+ \sum_{i=1}^{n} \sum_{\eta\in\Theta_i(\cdot,\gamma)} (P_t(\eta, n - e_i) - P_t(\eta, n))q(\eta, \gamma), \tag{4.3}$$

$$t \geq a, \ (\gamma, n) \in \Gamma \times (N \cup \{0\})^\ell.$$

The initial condition becomes

$$P_a(\gamma, n) = \begin{cases} c_a(\gamma), & \text{if } n_i = 0 \text{ for all } 1 \leq i \leq \ell \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

The counterpart of Eq. (2.6) is obtained by summing Eq. (4.3) over $\gamma \in \Gamma$ thus yielding

$$\frac{\partial}{\partial t} k_t(n) = \sum_{i=1}^{\ell} \sum_{\gamma\in\Gamma} \sum_{\eta\in\Theta_i(\cdot,\gamma)} (P_t(\eta, n - e_i) - P_t(\eta, n))q(\eta, \gamma), \tag{4.5}$$

$$t \geq a, \ n \in (N \cup \{0\})^\ell.$$

5. Multiple Disjoint Poisson Traffic Processes

In this section we investigate the possibility that disjoint multiple traffic processes $\{K_1(t)\}_{t\geq a}, \ldots, \{K_\ell(t)\}_{t\geq a}$ have Poisson related properties. In particular, the upcoming discussion applies to single traffic processes as the special case $\ell = 1$.

Definition 5.1

The processes $\{C(t)\}_{t\geq a}, \{K_1(t)\}_{t\geq a}, \ldots, \{K_\ell(t)\}_{t\geq a}$ are said to be pointwise independent if for every $t \geq a$ the random variables $C(t), K_1(t), \ldots, K_\ell(t)$ are mutually independent. The processes above are said to be weakly pointwise independent if for every $t \geq a$ and every $(n_1, \ldots, n_\ell) \in (N \cup \{0\})^\ell$,

$$\sum_{i=1}^{\ell} \sum_{\gamma\in\Gamma} \sum_{\eta\in\Theta_i(\cdot,\gamma)} P_t(\eta, n_1, \ldots, n_\ell)q(\eta, \gamma)$$
$$= \sum_{i=1}^{\ell} \sum_{\gamma\in\Gamma} \sum_{\eta\in\Theta_i(\cdot,\gamma)} c_t(\eta)\Big(\prod_{j=1}^{\ell} k_t^{(j)}(n_j)\Big)q(\eta, \gamma). \tag{5.1}$$

□

Since pointwise independence is of central interest here, the disjointness assumption is made so as not to preclude it a priori.

We begin, however, with a characterization of weak pointwise independence.

Theorem 5.1

$K_1(t), \ldots, K_n(t)$ have mutually independent Poisson distributions for every $t \geq a$ iff $\{C(t)\}_{t\geq a}, \{K_1(t)\}_{t\geq a}, \ldots, \{K_\ell(t)\}_{t\geq a}$ are weakly pointwise independent processes.

Proof

($\Rightarrow$) Suppose the $K_i(t)$, $1 \leq i \leq \ell$, are distributed as mutually independent Poissons. Then the generating function of $K(t)$ is

$$\varphi_t(y_1, \ldots, y_\ell) = \exp\left(\sum_{i=1}^{\ell} M_i(t)(y_i - 1)\right), \; t \geq a, \; |y_i| \leq 1, \; 1 \leq i \leq \ell, \tag{5.2}$$

whence

$$\frac{\partial}{\partial t} \varphi_t(y_1, \ldots, y_\ell) = \varphi_t(y_1, \ldots, y_\ell) \sum_{i=1}^{\ell} m_i(t)(y_i - 1), \; t \geq a, \; |y_i| \leq 1, \; 1 \leq i \leq \ell. \tag{5.3}$$

On equating coefficients in (5.3) we obtain

$$\frac{\partial}{\partial t} k_t(n) = \sum_{i=1}^{\ell} (k_t(n - e_i) - k_t(n))m_i(t)$$

$$= \sum_{i=1}^{\ell} \left(\prod_{j=1}^{\ell} k_t^{(j)}(n_j - \delta_{j,i}) - \prod_{j=1}^{\ell} k_t^{(j)}(n_j)\right) \sum_{\gamma \in \Gamma} \sum_{\eta \in \Theta_i(\cdot, \gamma)} c_t(\eta) q(\eta, \gamma), \tag{5.4}$$

$$t \geq a, \; n = (n_1, \ldots, n_\ell) \in (N \cup \{0\})^\ell$$

where $\delta_{j,i}$ is Kronecker's delta.

Eq. (5.1) now follows by equating the right side of Eq. (5.4) to the right side of Eq. (4.5), via a straightforward multiple induction on $n = (n_1, \ldots, n_\ell)$.

($\Rightarrow$) Assume that Eq. (5.1) holds. Substituting (5.1) into (4.5) and rearranging terms in the resultant equation yields Eq. (5.4). The latter is equivalent to Eq. (5.3) whose unique solution is given by Eq. (5.2), since the initial condition is $\varphi_a(y_1, \ldots, y_\ell) \equiv 1$, $|y_i| \leq 1$, $1 \leq i \leq \ell$, by virtue of (4.4).

Consequently, $k_t(n)$ corresponds to ℓ Poisson-distributed processes with respective rate functions $m_i(t)$; moreover, the $K_i(t)$ are mutually independent for every $t \geq a$.

$\square$

<u>Corollary 5.1</u>

If $\{C(t)\}_{t \geq a}$ is in equilibrium and $\{K(t)\}_{t \geq a}$ is a singleton ($\ell = 1$) Poisson traffic process over it, then necessarily

$$\sum_{\gamma \in \Gamma} \sum_{\eta \in \Theta(\cdot, \gamma)} \frac{\partial^r}{\partial t^r} P_t(\eta, 0) q(\eta, \gamma) \equiv (-1)^r (m(a))^{r+1} \exp(-m(a)t), \; r \in N \cup \{0\}.$$

$\square$

Next we characterize pointwise independence of traffic and state.

Theorem 5.2

$\{C(t)\}_{t\geq a}$, $\{K_1(t)\}_{t\geq a}$, ..., $\{K_\ell(t)\}_{t\geq a}$ are pointwise independent processes iff

$$\sum_{i=1}^{\ell} m_i(t, \gamma) = c_t(\gamma) \sum_{i=1}^{\ell} m_i(t), \quad t \geq a, \quad \text{for every} \quad \gamma \in \Gamma. \tag{5.5}$$

Proof

($\Rightarrow$) Suppose pointwise independence holds. Eq. (4.3) is equivalent to the generating function equation

$$\begin{aligned}\frac{\partial}{\partial t}[c_t(\gamma)\varphi_t(y_1, \ldots, y_\ell)] = &\sum_{\xi\in\Gamma-\{\gamma\}} c_t(\xi)\varphi_t(y_1, \ldots, y_\ell)q(\xi, \gamma) \\ &- c_t(\gamma)\varphi_t(y_1, \ldots, y_\ell)q(\gamma) \\ &+ \sum_{i=1}^{\ell} \sum_{\eta\in\Theta_i(\cdot,\gamma)} c_t(\eta)\varphi_t(y_1, \ldots, y_\ell)(y_i - 1)q(\eta, \gamma), \\ &t \geq a, \ |y_i| \leq 1, \ 1 \leq i \leq \ell, \ \gamma \in \Gamma,\end{aligned} \tag{5.6}$$

where $\varphi_t(y_1, \ldots, y_\ell) = \exp(\sum_{i=1}^{\ell} M_i(t)(y_i - 1))$ is the generating function of $K(t)$ due to Theorem 5.1. We use this form of $\varphi_t(y_1, \ldots, y_\ell)$ in differentiating the left side of (5.6) which after some manipulation becomes

$$\frac{\partial}{\partial t}[c_t(\gamma)\varphi_t(y_1, \ldots, y_\ell)] = \varphi_t(y_1, \ldots, y_\ell)\left(\frac{\partial}{\partial t} c_t(\gamma) + c_t(\gamma) \sum_{i=1}^{\ell} m_i(t)(y_i - 1)\right).$$

Since $\varphi_t(y_1, \ldots, y_\ell)$ may be cancelled on both sides of (5.6), the latter reduces to

$$\frac{\partial}{\partial t} c_t(\gamma) + c_t(\gamma) \sum_{i=1}^{\ell} m_i(t)(y_i - 1) = \frac{\partial}{\partial t} c_t(\gamma) + \sum_{i=1}^{\ell} m_i(t, \gamma)(y_i - 1). \tag{5.7}$$

Eq. (5.5) now follows from the above by equating the relevant coefficients.

($\Leftarrow$) Suppose Eq. (5.5) holds. It can be checked directly that

$$P_t(\gamma, n_1, \ldots, n_\ell) \triangleq \begin{cases} \dfrac{\sum_{i=1}^{\ell} m_i(t,\gamma)}{\sum_{i=1}^{\ell} m_i(t)} \prod_{j=1}^{\ell} \exp(-M(t)) \dfrac{(M_i(t))^{n_j}}{n_j!}, & \text{if } \sum_{i=1}^{\ell} m_i(t) > 0 \\ 0, \text{ otherwise} \end{cases} \tag{5.8}$$

solves Eq. (4.3) and is consistent with the initial condition (4.4). An easy proof of this assertion involves the transformation of (5.8) into the appropriate $c_t(\gamma)\varphi_t(y_1, \ldots, y_\ell)$ and then working the way backwards from (5.7) to (5.6) which is equivalent to (4.3).

$\square$

Corollary 5.2

a) $\{C(t)\}_{t\geq a}$, $\{K_1(t)\}_{t\geq a}$, ..., $\{K_\ell(t)\}_{t\geq a}$ are mutually pointwise independent iff $\{C(t)\}_{t\geq a}$ and $\{K_i(t)\}_{t\geq a}$, $1 \leq i \leq \ell$, are pointwise independent in pairs.

b) Eq. (5.5) holds iff for every $1 \leq i \leq \ell$,

$$m_i(t, \gamma) = c_t(\gamma)L_i(t), \quad t \geq a, \quad \gamma \in \Gamma,$$

for some functions $L_i(t)$ depending on t only; in fact, for every $1 \leq i \leq \ell$, $L_i(t) \equiv m_i(t)$, necessarily.

c) Consequently, in equilibrium, Eq. (5.5) holds iff for every $1 \leq i \leq \ell$,

$$m_i(t, \gamma) = c_t(\gamma)L_i, \qquad t \geq a, \quad \gamma \in \Gamma$$

for some constants L_i; in fact, $L_i = m_i$ for every $1 \leq i \leq \ell$.

$\square$

Proof

a) Mutual pointwise independence implies pointwise independence in pairs. Conversely, pointwise independence in pairs implies for every $1 \leq i \leq \ell$,

$$m_i(t, \gamma) = c_t(\gamma) m_i(t), \quad t \geq a, \quad \gamma \in \Gamma.$$

This becomes Eq. (5.5) on summing both sides over $1 \leq i \leq \ell$.

b) If Eq. (5.5) holds, then from a) the condition holds for $L_i(t) \equiv m_i(t)$. Conversely, by summing both sides of $m_i(t, \gamma) = c_t(\gamma) L_i(t)$ over $\gamma \in \Gamma$ we deduce $L_i(t) \equiv m_i(t)$; summing it over $1 \leq i \leq \ell$ then yields Eq. (5.5).

c) Follows immediately from b) and from the time stationarity of the $m_i(t, \gamma)$ and $m_i(t)$.

$\square$

The relation of Eq. (5.5) and Corollary 5.2 to Muntz [22], and Gelenbe and Muntz [13] should be noted. A more detailed discussion is deferred, however, until Sec. 8.

Before proceeding to the main theorem we shall now prove two supporting lemmas. The first one is a generalization of Corollary 1 in [4].

Lemma 5.1

$\{C(t)\}_{t \geq a}$ and the multiple traffic process $\{K(t)\}_{t \geq a}$ are pointwise independent iff for any fixed $s \geq a$, $\{C(t)\}_{t \geq s}$ and $\{K(t) - K(s)\}_{t \geq s}$ are pointwise independent.

Proof

($\Leftarrow$) Follows immediately by taking $s = a$.

($\Rightarrow$) Since $\{C(t)\}_{t \geq s}$ is a Markov process, it follows from Lemma 2.1 that $\{(C(t), K(t) - K(s))\}_{t \geq s}$ is also Markovian. To distinguish between $\{(C(t), K(t))\}_{t \geq a}$ and $\{(C(t), K(t) - K(s))\}_{t \geq s}$ we denote the various mathematical entities associated with the latter by appending tildas to the corresponding ones in the former.

Thus, Eq. (4.3) is satisfied by $\tilde{P}_t(\gamma, n)$ over the domain $t \in [s, \infty)$, subject to the initial condition (4.4) with $a = s$. Since $c_t(\gamma) \equiv \tilde{c}_t(y)$ for every $\gamma \in \Gamma$ and $t \geq s$, it also follows that $m_i(\gamma, t) = \tilde{m}_i(\gamma, t)$ and $m_i(t) = \tilde{m}_i(t)$ for any $t \geq s$, $1 \leq i \leq \ell$ and $\gamma \in \Gamma$.

Now, by pointwise independence of $\{C(t)\}_{t \geq a}$ and $\{K(t)\}_{t \geq a}$, Eq. (5.5) holds, whence

$$\sum_{i=1}^{\ell} \tilde{m}_i(t, \gamma) = \tilde{c}_t(\gamma) \sum_{i=1}^{\ell} \tilde{m}_i(t), \quad t \geq s, \; \gamma \in \Gamma, \tag{5.9}$$

also holds. The Lemma now follows from (5.9) by applying Theorem 5.2 in the other direction.

$\square$

The second lemma is tantamount to Burke's argument in [5]. (See also, Theorem 3 in [4]).

Lemma 5.2

Suppose that $\{C(t)\}_{t\geq a}$ and the multiple traffic process $\{K(t)\}_{t\geq a}$ are pointwise independent. Then, for every fixed $t \geq a$, the σ-algebras $\sigma\{K(t) - K(s): s \leq t\}$ and $\sigma\{C(u), K(u) - K(t): u \geq t\}$ are independent.

Proof

Let $\Lambda \in \sigma\{C(u), K(u) - K(t): u \geq t\} = \sigma\{C(u): u \geq t\}$. Now, from Lemma 5.1, and the Markov property of $\{C(t)\}_{t\geq a}$, we can write for any $s \leq t$, $\gamma \in \Gamma$ and $n \in (N \cup \{0\})^{\ell}$,

$$\begin{aligned}
&P[\Lambda, C(t) = \gamma, K(t) - K(s) = n] \\
&\quad = P[\Lambda | C(t) = \gamma, K(t) - K(s) = n] \cdot P[C(t) = \gamma, K(t) - K(s) = n] \\
&\quad = P[\Lambda | C(t) = \gamma] \cdot P[C(t) = \gamma] \cdot P[K(t) - K(s) = n] \\
&\quad = P[\Lambda, C(t) = \gamma] \cdot P[K(t) - K(s) = n]
\end{aligned}$$

whence on summing both sides above over $\gamma \in \Gamma$,

$$P[\Lambda, K(t) - K(s) = n] = P[\Lambda] \cdot P[K(t) - K(s) = n] \tag{5.10}$$

as required.

$\square$

Corollary 5.3

If $\{C(t)\}_{t\geq a}$ and $\{K(t)\}_{t\geq a}$ are pointwise independent processes, then by Lemma 5.2 each $\{K_i(t)\}_{t\geq a}$, $1 \leq i \leq \ell$, has independent increments; consequently, each is a Poisson process by combining Theorem 5.1 and Lemma 5.2.

$\square$

We shall now proceed to show a stronger independence result, (cf. Theorem 4 in [4]).

Theorem 5.3

Suppose $\{C(t)\}_{t\geq a}$ and the multiple traffic process $\{K(t)\}_{t\geq a}$ are pointwise independent processes. Then the component traffic processes $\{K_1(t)\}_{t\geq a}$, ..., $\{K_\ell(t)\}_{t\geq a}$ are mutually independent Poisson processes.

Proof

In view of Corollary 5.3 it suffices to show that for each partition $a = t_0 < t_1 < t_2 < \ldots < t_r = t$ of an arbitrary interval $[a, t]$, and for any choice of nonnegative integers n_{ij}, $1 \leq i \leq \ell$, $1 \leq j \leq r$, the events

$$E_{i,j} \triangleq [K_i(t_j) - K_i(t_j - 1) = n_{ij}],\ 1 \leq i \leq \ell,\ 1 \leq j \leq r,$$

are mutually independent. The proof is by induction on r.

If $r = 1$, then the $E_{i,r}$ are mutually independent by pointwise independence among the $\{K_i(t)\}_{t\geq a}$, and the induction base is established.

Assume now that the Theorem holds for $r = p$, $p \geq 1$, and show it for $r = p + 1$.

Since $[\bigcap_{i=1}^{\ell} \bigcap_{j=1}^{p} E_{i,j}] \in \sigma\{K(t_p) - K(s)\colon s \leq t_p\}$ and $[\bigcap_{i=1}^{\ell} E_{i,p+1}] \in \sigma\{K(u) - K(t_p)\colon u \geq t_p\}$,

we can write by virtue of Eq. (5.10),

$$P[\bigcap_{i=1}^{\ell} \bigcap_{j=1}^{p+1} E_{i,j}] = P[(\bigcap_{i=1}^{\ell} \bigcap_{j=1}^{p} E_{i,j}) \cap (\bigcap_{i=1}^{\ell} E_{i,p+1})] = P[\bigcap_{i=1}^{\ell} \bigcap_{j-1}^{p} E_{i,j}]\cdot P[\bigcap_{i=1}^{\ell} E_{i,p+1}].$$

Finally, applying the induction hypothesis to the first factor, and Lemma 5.1 to the second factor yields

$$P[\bigcap_{i=1}^{\ell} \bigcap_{j=1}^{p+1} E_{i,j}] = \prod_{i=1}^{\ell} \prod_{j=1}^{p+1} P[E_{i,j}]$$

which establishes the induction step.

$\square$

In view of Theorem 5.3, we now see that Theorem 5.2 provides us with a computational criterion as follows:

<u>Corollary 5.4</u>

If Eq. (5.5) holds, then the $\{K_i(t)\}_{t\geq a}$, $1 \leq i \leq \ell$, are mutually independent Poisson processes with respective rate functions $m_i(t)$. The same is true if any part in Corollary 5.2 holds.

$\square$

Applications of the theory developed thus far are furnished in the next two sections.

6. Non-Poisson Traffic: Atomic Processes and Queueing-Theoretic Examples

This section demonstrates how weak pointwise independence may be used to show non-Poisson traffic by violating the necessary condition in Theorem 5.1 and Corollary 5.1. Consider the class of traffic processes defined by

Definition 6.1

$\{K(t)\}_{t\geq a}$ is called an atomic traffic process if its traffic set Θ is a singleton pair of states.

$\square$

Atomic traffic processes are the elementary building blocks of all traffic processes, since every traffic process is a superposition of disjoint traffic atoms. We shall now exemplify the utility of the weak pointwise independence concept vis-à-vis atomic traffic processes.

First, however, we show a more general result.

Lemma 6.1

Let $\{K(t)\}_{t\geq a}$ be a nontrivial traffic process such that

$$\left(\bigcup_{\xi\in\Gamma} \Theta(\cdot, \xi)\right) \cap \left(\bigcup_{\xi\in\Gamma} \Theta(\xi, \cdot)\right) = \Phi. \tag{6.1}$$

Then $\{K(t)\}_{t\geq a}$ is not a time homogenous Poisson process; moreover, in equilibrium it is not a Poisson process altogether.

Proof

Setting $n = 0$ and letting $t \to a+$ in Eq. (2.5) gives us

$$\frac{\partial}{\partial t} P_a(\gamma, 0) = \frac{\partial}{\partial t} c_a(\gamma) - m(a, \gamma), \quad \gamma \in \Gamma.$$

If $\eta \in \Theta(\cdot, \gamma)$ for some $\gamma \in \Gamma$, then $\Theta(\cdot, \eta) = \Phi$ by (6.1) so that $m(a, \eta) = 0$.

Hence

$$\frac{\partial}{\partial t} P_a(\eta, 0) = \frac{\partial}{\partial t} c_a(\eta) \quad \text{for any} \quad \eta \in \Theta(\cdot, \gamma), \quad \gamma \in \Gamma.$$

Substituting the above into the left side of (5.1) for $\ell = 1$ and differentiating yields for $t \to a+$

$$\sum_{\gamma\in\Gamma} \sum_{\eta\in\Theta(\cdot,\gamma)} \frac{\partial}{\partial t} P_a(\eta, 0) q(\eta, \gamma) = \sum_{\gamma\in\Gamma} \sum_{\eta\in\Theta(\cdot,\gamma)} \frac{\partial}{\partial t} c_a(\eta) q(\eta, \gamma) = \frac{\partial}{\partial t} m(a). \qquad (6.2)$$

Now assume $\{K(t)\}_{t\geq a}$ is a Poisson process. By weak pointwise independence of $\{C(t)\}_{t\geq a}$ and $\{K(t)\}_{t\geq a}$ (see Theorem 5.1)

$$\sum_{\gamma\in\Gamma} \sum_{\eta\in\Theta(\cdot,\gamma)} \frac{\partial}{\partial t} P_a(\eta, 0) q(\eta, \gamma) = \lim_{t\to a+} \frac{\partial}{\partial t} [m(t)\cdot\exp(-M(t))]$$

$$= \lim_{t\to a+} [m(t)\cdot\exp(-M(t))\cdot(-m(t)) + \exp(-M(t))\cdot\frac{\partial}{\partial t} m(t)]$$

$$= \frac{\partial}{\partial t} m(a) - (m(a))^2. \qquad (6.3)$$

A comparison of (6.2) and (6.3) gives us necessarily $m(a) = 0$. But if $\{K(t)\}_{t\geq a}$ is time homogenous, then $m(t) \equiv 0$ from ii) in Theorem 3.2, which contradicts the nontriviality of $\{K(t)\}_{t\geq a}$. Finally, in equilibrium, $\{K(t)\}_{t\geq a}$ is necessarily time homogenous from Corollary 2.1, whence the rest of the Theorem follows.

$\square$

We can now assert,

Corollary 6.1

None of the nontrivial atomic traffic processes over $\{C(t)\}_{t\geq a}$ is a time homogenous Poisson process. Furthermore, in equilibrium, none is a Poisson process.

Proof

The Corollary follows trivially since every singleton traffic set $\Theta = \{(\alpha, \beta)\}$ satisfies Eq. (6.1).

$\square$

Thus, in equilibrium, we have the intuitively curious situation where none of the nontrivial traffic atoms is a Poisson process; however, an arbitrary superposition of traffic atoms may or may not be a Poisson process. In fact, examples of both cases abound in the queueing-theoretic literature (see next section).

We point out that if a superposition of point processes forms a Poisson process, then either all superposed components are independent Poisson processes or none is. Most superposition results are variants of the first type (see, e.g. Çinlar [9]). What we have just shown is a nonvacuous example that falls within the scope of the second type.

To further illustrate the utility of Corollary 6.1 we note that the departure process (exclusive of the loss stream) from an M/M/1/0 queue in equilibrium is not a Poisson process. In the same spirit we can deduce that any departure stream of customers from a Markovian queueing network, such that departing customers leave behind a prescribed network state, cannot be a Poisson process in equilibrium.

An important class of conjectured non-Poisson traffic in queueing networks consists in most traffic on arcs having direct or indirect feedback ([6]; [19], Conjecture 5.1). Intuitively, a recycling of customers takes place which deprives the traffic of independent increments. Burke [7] proves directly the non-Poisson conjecture for the total input into an equilibrium M/M/1 queue with feedback; an extension of this result to Jackson queueing networks (see Example 7.1) with single server nodes appears in Melamed [21]. The latter is based on Corollary 5.1. The non-Poisson conjecture bears, in particular, on all traffic in closed queueing networks such as the ones in Gordon and Newell [14].

7. Multiple Poisson Traffic: Queueing-Theoretic Examples

In this section we demonstrate how to apply pointwise independence to certain traffic processes in a number of queueing networks whose discrete state is represented by a Markov process. These applications utilize the computational criterion of Theorem 5.2 as set forth in Corollary 5.2.

Example 7.1: Jackson queueing networks (see Jackson [15]).

A Jackson network consists of J service stations with infinite line capacities. Each station j houses s_j parallel independent exponential servers with respective rates σ_j. Exogenous customers arrive at the stations according to independent Poisson processes with respective rates α_j. On service completion at station j a customer is routed to station k, $0 \le k \le J$, with probability p_{jk} (a routing to $k = 0$ designates leaving the network altogether). All arrival, service, and routing processes are mutually independent.

The vector valued process of the J line sizes is a Markov process with state space $\Gamma = \{\gamma = (n_1, \ldots, n_J): n_j \in N \cup \{0\}\}$. Next, suppose the equations

$$\delta_j = \alpha_j + \sum_{i=1}^{J} \delta_i p_{ij}, \quad 1 \le j \le J, \tag{7.1}$$

have a nonnegative solution in the δ_j, $1 \le j \le J$. This is always the case when the network is open in the sense that it is possible to leave the network from every node through some finite sequence of routings (see [20], Ch. 4).

Suppose the network is open such that $\rho_j \triangleq \frac{\delta_j}{\sigma_j s_j} < 1$, $1 \le j \le J$. Then the state equilibrium distribution is $c_t(n_1, \ldots, n_J) \equiv \prod_{j=1}^{J} c_t(n_j)$ where

$$c_t(n_j) = \begin{cases} (1-\rho_j)\dfrac{\rho_j^{n_j}}{n_j!}, & \text{if } n_j \le s_j \\[2ex] (1-\rho_j)\dfrac{\rho_j^{n_j}}{s_j!\,s_j^{n_j-s_j}}, & \text{if } n_j > s_j \end{cases}$$

(see [15] p. 520).

Let $\{K_j(t)\}_{t\geq a}$ be the equilibrium traffic process of customers that leave the network from station j. Thus $\Theta_j = \{(\gamma + e_j, \gamma): \gamma \in \Gamma\}$ and $\Theta_j(\cdot, \gamma) = \{\gamma + e_j\}$.

Denoting $\sigma_j(i) \triangleq \min\{i, s_j\}\sigma_j$ we compute for any $\gamma = (n_1, \ldots, n_J) \in \Gamma$,

$$m_j(t, \gamma) \equiv c_t(\gamma + e_j)\sigma_j(n_j + 1)p_{j0}$$

$$= c_t(\gamma)\frac{\rho_j}{\min\{n_j+1, s_j\}}\sigma_j(n_j + 1)p_{j0}$$

$$= c_t(\gamma)\rho_j\sigma_j p_{j0} = c_t(\gamma)\delta_j p_{j0}, \quad 1 \le j \le J.$$

Hence, part c) of Corollary 5.2 holds for $L_j = \delta_j p_{j0}$, $1 \le j \le J$.

It now follows from Corollary 5.4 that the $\{K_j(t)\}_{t\geq a}$, $1 \le j \le J$, are mutually independent Poisson processes with respective rates $\delta_j p_{j0}$.

$\square$

We point out that this result includes as a special case the well-known result by P. J. Burke [5] that the equilibrium departure process from a M/M/s queue is a Poisson process with the same rate as the arrival process; this result was arrived at by examining the interdeparture intervals. The same result was later attained by E. Reich [23] through the use of reversibility. A related derivation was demonstrated by F. P. Kelly [17]; his results apply to a large class of Márkovian queueing networks to be described in the sequel.

Example 7.2: Kelly's networks with random routings (see Kelly [17]).

In this queueing model we have J service stations with infinite waiting line capacities and I types of customers. Exogenous customers type i, $1 \le i \le I$, arrive at station j, $1 \le j \le J$, according to independent Poisson processes with respective rates $\alpha_j(i)$. Each station j houses an exponential server with rate $\sigma_j \varphi_j(n_j)$, where n_j is the total number of customers at station j. The routing probabilities $p_{jk}(i)$ depend on the type of customer routed. In addition, the ℓ-th customer in line j is allocated a proportion $f_j(\ell, n_j)$ of the service effort in station j. A customer arriving at station j is inserted in the ℓ-th position there with probability $g_j(\ell, n_j + 1)$. All arrival, service and routing processes are mutually independent. The vector-valued process of line configurations is a Markov process with state space $\Gamma = \{(\gamma_1, \ldots, \gamma_J): c_j \in I^*\}$ where I^* is the set of all finite strings $\gamma_j(1)\gamma_j(2) \ldots \gamma_j(n_j)$ where $\gamma_j(\ell)$ is the type of the ℓ-th customer in station j (I^* includes the empty string). The transition rates of the state process are defined by

$$q(\gamma, T_{j\cdot\ell\cdot}(\gamma)) = \sigma_j \varphi_j(n_j) p_{j0}(\gamma_j(\ell)) f_j(\ell, n_j)$$

$$q(\gamma, T^i_{\cdot j\cdot\ell}(\gamma)) = \alpha_j(i) g_j(\ell, n_j + 1)$$

$$q(\gamma, T_{jk\ell m}(\gamma)) = \sigma_j \varphi_j(n_j) p_{jk}(\gamma_j(\ell)) f_j(\ell, n_j) g_k(m, n_k + 1)$$

where $T_{j\cdot\ell\cdot}$ is the operator that removes the ℓ-th customer at station j from the network; $T^i_{\cdot j\cdot\ell}$ is the operator that inserts a customer of type i in the ℓ-th position at station j; $T_{jk\ell m}$ is the operator that moves the ℓ-th customer in station j to the m-th position in station k.

When the network is open with respect to every customer type i, $1 \le i \le I$, Eq. (7.1) has unique solutions $\delta_j(i)$ for given $\alpha_j = \alpha_j(i)$ and $p_{jk} = p_{jk}(i)$, $1 \le j,k \le J$, and we denote $\rho_j(i) \triangleq \dfrac{\delta_j(i)}{\sigma_j}$. Under certain conditions (see [17],

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20. Abstract continued.

Theorem 2) the equilibrium distribution has the form

$$c_t(\gamma_1, \ldots, \gamma_J) \equiv b\prod_{j=1}^{J} A_j(\gamma_j) \tag{7.2}$$

where b is a positive constant and

$$A_j(\gamma_j) \triangleq \begin{cases} \prod_{\ell=1}^{n_j} \dfrac{\rho_j(\gamma_j(\ell))}{\varphi_j(\ell)}, & \text{if } n_j \geq 1 \\ 1, & \text{otherwise} \end{cases}.$$

Let $\{K_{ij}(t)\}_{t\geq a}$ be the equilibrium traffic process of customers type i which depart the network from station j. Thus,

$\Theta_{ij} = \{(T^i_{.j.\ell}(\gamma), \gamma)\colon \gamma \in \Gamma, 1 \leq \ell \leq n_j + 1\}$ and $\Theta_{ij}(\cdot, \gamma) = \{T^i_{.j.\ell}(\gamma)\colon 1 \leq \ell \leq n_j + 1\}$.

For any $\gamma = (\gamma_1, \ldots, \gamma_J) \in \Gamma$ we now compute using the identity

$$c_t(T^i_{.j.\ell}(\gamma)) = c_t(\gamma)\,\frac{\rho_j(i)}{\varphi_j(n_j+1)},$$

$$\begin{aligned} m_{ij}(t, \gamma) &\equiv \sum_{\ell=1}^{n_j+1} c_t(T^i_{.j.\ell}(\gamma))q(T^i_{.j.\ell}(\gamma), \gamma) \\ &= \sum_{\ell=1}^{n_j+1} c_t(\gamma)\,\frac{\rho_j(i)}{\varphi_j(n_j+1)}\, q(T^i_{.j.\ell}(\gamma), T_{j.\ell.}(T^i \quad (\gamma))) \\ &= c_t(\gamma) \sum_{\ell=1}^{n_j+1} \frac{\delta_j(i)}{\sigma_j\varphi_j(n_j+1)}\, \sigma_j\varphi_j(n_j + 1)p_{j0}(i)f_j(\ell, n_j + 1) \\ &= c_t(\gamma)\delta_j(i)p_{j0}(i), \quad 1 \leq i \leq I, \quad 1 \leq j \leq J. \end{aligned}$$

Again, part c) of Corollary 5.2 holds for $L_{ij} = \delta_j(i)p_{j0}(i)$, $1 \leq i \leq I$, $1 \leq j \leq J$. It now follows from Corollary 5.4 that the $\{K_{ij}(t)\}_{t\geq a}$ are mutually independent Poisson processes with respective rates $\delta_j(i)p_{j0}(i)$, in agreement with [17] p. 553.

$\square$

<u>Example 7.3</u>: Kelly's networks with fixed routes and gamma-distributed service (see Kelly [18]).

This model is a variation on the basic setup of J stations and I types of customers, where we conveniently take $\sigma_j = 1$, $1 \le j \le J$. For $1 \le i \le I$, customers type i arrive according to mutual independent Poisson processes with respective rates $\alpha(i)$. A customer traces a fixed route $r(i, 1), r(i, 2), \ldots, r(i, S(i))$ of $S(i)$ stages through the network and then exits. At node $r(i, s)$ en route, a customer requires a gamma distributed (Erlang) service composed of $z(i, s)$ phases of mutually independent exponential services each with mean $d(i, s)$. We require, however, that $f_j \equiv g_j$ for all $1 \le j \le J$. All arrival and service processes are mutually independent.

The state process is Markovian over the state space Γ consisting of all J-tuples $\gamma = (\gamma_1, \ldots, \gamma_J)$ where each γ_j is a finite (possibly empty) string over the set $\{(i, s, p): 1 \le i \le I, 1 \le s \le S(i), 1 \le p \le z(i, s)\}$. Define $\delta_j(i, s) \stackrel{\Delta}{=} \alpha(i)d(i, s)\delta_{j,r(i,s)}$, $1 \le j \le J$, $1 \le i \le I$, $1 \le s \le S(i)$, where $\delta_{j,r(i,s)}$ is Kronecker's delta.

Under certain conditions, the equilibrium state distribution is again given by Eq. (7.2) provided we redefine

$$A_j(c_j) = \begin{cases} \prod_{\ell=1}^{n_j} \dfrac{\delta_j(t_j(\ell), s_j(\ell))}{\varphi_j(\ell)}, & \text{if } n_j > 0 \\ 1, & \text{otherwise} \end{cases} \tag{7.3}$$

where $t_j(\ell)$ and $s_j(\ell)$ are the type and stage respectively of the ℓ-th customer in line configuration γ_j, and n_j is the length of γ_j.

Let the $\{K_{ij}(t)\}_{t \ge a}$ be as in the previous example. Thus,

$\Theta_{ij} = \{(T^e_{\cdot j \cdot \ell}(\gamma), \gamma): e = (i, S(i), z(i, S(i))), \gamma \in \Gamma, 1 \le \ell \le n_j + 1\}$ and

$\Theta_{ij}(\cdot, \gamma) = \{T^e_{\cdot j \cdot \ell}(\gamma): e = (i, S(i), z(i, S(i))), 1 \le \ell \le n_j + 1\}$.

Here $T^e_{.j.\ell}$ is the operator that inserts a customer with attribute set e as above (i.e., a customer type i in his last stage of the route and last phase in service) into the ℓ-th position in station j. Observing that $c_t(T^e_{.j.\ell}(\gamma)) = c_t(\gamma)\frac{\delta_j(i,S(i))}{\varphi_j(n_j+1)}$ we compute,

$$m_{ij}(t, \gamma) = \sum_{\ell=1}^{n_j+1} c_t(T^e_{.j.\ell}(\gamma))q(T^e_{.j.\ell}(\gamma), \gamma)$$

$$= c_t(\gamma) \sum_{\ell=1}^{n_j+1} \frac{\delta_j(i,S(i))}{\varphi_j(n_j+1)} \varphi_j(n_j + 1)f_j(\ell, n_j + 1)$$

$$= c_t(\gamma)\delta_j(i, S(i)), \quad 1 \le i \le I, \quad 1 \le j \le J.$$

We conclude that the $\{K_{ij}(t)\}_{t\geq 0}$ are mutually independent Poisson processes with respective rates $\delta_j(i, S(i))$, in agreement with [18] p. 423.

$\square$

Analogous results can be similarly obtained for the class of Kelly's networks in Sec. 3 of [18] where the f_j are allowed to differ from the g_j, but the service requirements are constrained to be exponential.

Suppose the rate of type i arrivals is $\alpha(i, \gamma)$; i.e., it is also a function of the instantaneous state of the system. Kelly ([18], Sec. 5) considers the case $\alpha(i, \gamma) = \alpha(i)\cdot \prod_{W\in 2^I : i\in W} \psi(N(\gamma, W))$, where ψ: $N \cup \{0\} \to [0, \infty)$ is a given function, and $N(\gamma, W) \triangleq \sum_{i=1}^{I} N(\gamma, i)$ where $N(\gamma, i)$ is the number of type i customers in network configuration γ. He shows that under certain conditions the equilibrium state distribution has the form

$$c_t(\gamma) = b\cdot B(\gamma)\cdot \prod_{j=1}^{J} A_j(\gamma_j)$$

where

$$B(\gamma) = \prod_{W\in 2^I} \prod_{n=0}^{N(\gamma,W)-1} \psi(n)$$

and the $A_j(\gamma_j)$ are still defined by (7.3). Thus, in the notation of Example 7.3, $B(T^e_{\cdot\ell\cdot j}(\gamma)) = \prod_{W:i\in W} \psi(N(\gamma, W))$ for any $e = (i, S(i), z(i, S(i)))$, whence

$$c_t(T^e_{\cdot\ell\cdot j}(\gamma)) = c_t(\gamma)\cdot\Big(\prod_{W:i\in W}\psi(N(\gamma, W))\Big)\cdot \frac{\delta_j(i,S(i))}{\varphi_j(n_j+1)} .$$

It follows in an analogous calculation that

$$m_{ij}(t, \gamma) = c_t(\gamma)\delta_j(i, S(i))\cdot \prod_{W:i\in W} \psi(N(\gamma, W)).$$

Hence, $\{C(t)\}_{t\geq a}$ and $\{K_{ij}(t)\}_{t\geq a}$ are pointwise independent iff $\prod_{W:i\in W}\psi(N(\gamma, W)) = L'$ independent of $\gamma \in \Gamma$, which is generally not the case. (Notice, however, that when the product above does depend on $\gamma \in \Gamma$, this does not, in general, exclude $\{K_{ij}(t)\}_{t\geq a}$ from being a Poisson process, albeit pointwise dependent on the state.[†]) A similar phenomenon takes place in Jackson [16] and in the following.

Example 7.4: The BCMP queueing networks (see Baskett et al. [2]).

These networks consist of four types of stations, all related to Kelly's networks in [17]. There are, however, three differences: customers arrive according to state dependent Poisson processes; they require type dependent services which are mixtures of sums of exponentials; and, on service completion, customers are allowed to change types in a Markovian manner.

Based on the equilibrium state distributions derived in [2], it can be rigorously shown that the $m_{ij}(t)$ factor into $c_t(\gamma)$ and another product. The latter contains the instantaneous arrival rate as a state dependent factor. Consequently, the $\{K_{ij}(t)\}_{t\geq a}$ and $\{C(t)\}_{t\geq a}$ are not, in general, pointwise

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[†]A trivial case in point is an arrival process to a Jackson network which is Poisson by definition. However, it can be easily verified that it is pointwise dependent on the state, say in equilibrium.

independent when the network is in equilibrium. However, it can be rigorously checked that the above are pointwise independent provided the arrival rates are fixed. The latter fact agrees with Theorem 13 in [13].

$\square$

The author is unaware of any result in the queueing-theoretic literature enunciating Poisson traffic (over a discrete state Markov process)that cannot be explained by means of pointwise independence of traffic count and state.

8. Discussion

The class of intuitive traffic processes that can be modeled via distinguished state transitions in an underlying Markov process $\{C(t)\}_{t\geq a}$ is reasonably comprehensive vis-à-vis applications. In particular it includes all traffic processes in the queueing-theoretic literature with the exception of certain feedback traffic processes.

Consider a feedback stream of customers that after service completion in station j immediately rejoin the waiting line of that station in such a way that the state of the system remains unchanged (notice that this situation never arises for traffic processes between distinct nodes or for traffic streams that leave the network altogether). In this case, defining the relevant $\{T_n\}_{n=0}^{\infty}$ becomes impossible since a consideration of any traffic set Θ is insufficient to determine the epoches in question. Moreover, a direct appeal to Lemma 2.1 is now invalid, even though the result of the lemma may be correct.

To remedy this situation one may attempt to proceed in two ways. First, it may be possible to modify $\{C(t)\}_{t\geq a}$ into a new Markov process $\{\tilde{C}(t)\}_{t\geq a}$ with state space $\tilde{\Gamma}$ for which all feedback epochs correspond to discernible state transitions. This technique was used by Kelly ([19], Sec. 2.1), and earlier by Daley ([10], p. 399) to treat balking arrivals to an M/M/s queue. The second approach is to define directly the requisite joint process $\{(C(t), K(t))\}_{t\geq a}$ and to show it to be Markovian by another technique (e.g., *via a stochastic integral* representation as in [3] and [20]). Either way, chances are that the rest of the theory in this paper would still be applicable, as was the case in [20] and [21].

A broader class of traffic processes over Markovian processes $\{C(t)\}_{t\geq a}$ may be defined by allowing the traffic epochs $\{T_n\}_{n=0}^{\infty}$ to be affected by past history of $\{C(t)\}_{t\geq a}$. One may then attempt to redefine a Markovian "state" process $\{\tilde{C}(t)\}_{t\geq a}$ with a new $\tilde{\Gamma}$ and $\tilde{\Theta}$ such that $\{\tilde{C}(t)\}_{t\geq a}$ "remembers by state" the relevant information in the past history of the old $\{C(t)\}_{t\geq a}$.

The approach and definitions of this paper shed a new light on the differential equations (2.1). The traditional heuristic interpretation is that the "probability rate of being in state γ" is the difference between the "flow rate into γ" and "the flow rate out of γ." On the other hand, let us define

$\Theta_{\gamma_{in}} = \{(\xi, \gamma): \xi \in \Gamma - \{\gamma\}\}$ and $\Theta_{\gamma_{out}} = \{(\gamma, \xi): \xi \in \Gamma - \{\gamma\}\}$.

Then clearly for any $\gamma \in \Gamma$, $\frac{\partial}{\partial t} c_t(\gamma) = m_{\gamma_{in}}(t) - m_{\gamma_{out}}(t)$, or equivalently upon integration $c_t(\gamma) = c_a(\gamma) + E[K_{\gamma_{in}}(t) - K_{\gamma_{out}}(t)]$, $t \geq a$.

From this equation it can be easily shown that for any $s \leq t$

$$c_t(\gamma) - c_s(\gamma) = E[K_{\gamma_{in}}(s, t) - K_{\gamma_{out}}(s, t)]$$

where $K(s, t) \triangleq K(t) - K(s)$. Thus, from a traffic oriented vantage point, the probability difference of being in state γ at the extreme points of any time interval [s, t] equals the expected difference of the number of times the system entered and left state γ in the aforesaid interval.

It is interesting to note how the Markov property of the underlying $\{C(t)\}_{t\geq a}$ affects the feasibility of $\{K(t)\}_{t\geq a}$ being a Poisson related process. It turns out that various notions of independence play a significant role in this respect: independent increments in $\{K(t)\}_{t\geq a}$ already ensure it to be a Poisson process (theorem 3.1); a renewal $\{T_n\}_{n=0}^{\infty}$ and a time invariant m(t) already ensure the same thing (Theorem 3.2); weak pointwise independence already ensures that disjoint $K_1(t), \ldots, K_\ell(t)$, $t \geq a$, are distributed as mutually independent Poissons (Theorem 5.1); and finally, pointwise independence already ensures that disjoint $\{K_1(t)\}_{t\geq a}, \ldots, \{K_\ell(t)\}_{t\geq a}$ are mutually independent Poisson processes (Theorem 5.3).

A number of concepts essentially equivalent to pointwise independence have been discussed in the literature. Muntz [22] discusses departure processes from an equilibrium queueing system with different types of customers whose arrival rates are λ_i, $1 \leq i \leq I$. Suppose each customer type arrives according to independent Poisson processes such that

$$\forall \gamma \in \Gamma, \quad \sum_{\eta\in\Theta_i} \frac{c_t(\eta)q(\eta,\nu)}{c_t(\gamma)} = \lambda_i$$

where Θ_i is the traffic set of the respective departure process. Muntz calls this condition the M $\Rightarrow$ M (Markov implies Markov) property to indicate that each such departure process is Poisson when the arrival process is. The above condition is a special case of Eq. (5.5); it is easily seen to be equivalent to pointwise independence in equilibrium.

In Sec. 5 of [13], Gelenbe and Muntz discuss Markovian queues with Poisson arrivals at a fixed rate λ; they define such systems to be complete (ibid. p. 52) if the departure process $\{K(t)\}_{t\geq a}$ satisfies

$$\lim_{t\to\infty} P[K(t) - K(t - \Delta t) = n | C(t) = \gamma] = \begin{cases} \lambda\Delta t + o(\Delta t), & \text{if } n = 1 \\ o(\Delta t), & \text{if } n \geq 1 \\ 1 - \lambda\Delta t + o(\Delta t), & \text{if } n = 0 \end{cases}$$

for any $\gamma \in \Gamma$.

Then, they proceed to give a heuristic derivation of equilibrium analogues of Corollary 5.2. By virtue of Lemma 5.1, we can recognize completeness as pointwise independence of $\{C(t)\}_{t\geq a}$ and $\{K(t)\}_{t\geq a}$ when the former is in equilibrium.

In Sec. 6 of [18], Kelly describes a queueing network with Poisson arrivals; the network is represented by a Markov state process $\{C(t)\}_{t\geq a}$ in equilibrium, and each departing customer is classified into one of I groups depending (perhaps stochastically) on the network's past history. Such a queue is quasi-reversible if (see p. 428 <u>ibid</u>.):

a) departures of group i customers, for i = 1, 2, ..., I, form independent Poisson processes; and

b) the state of the network at time t is independent of departures from the network up until time t.

Suppose the I departure streams can be modeled by traffic processes $\{K_1(t)\}_{t\geq a}, \ldots, \{K_I(t)\}_{t\geq a}$ via traffic sets Θ_i, $1 \leq i \leq I$. Then quasi-reversibility clearly implies pointwise independence of $\{C(t)\}_{t\geq a}$ and the $\{K_i(t)\}_{t\geq a}$, $1 \leq i \leq I$, (Condition b) above). However, Theorem 5.3 shows that pointwise independence of $\{C(t)\}_{t\geq a}$ and the $\{K_i(t)\}_{t\geq a}$, $1 \leq i \leq I$, already implies Condition a) above (i.e., b) implies a)). It follows that for the class of departure processes defined as traffic processes in the sense of this paper, quasi-reversibility is logically equivalent to pointwise independence (i.e., to Condition b) alone).

As a matter of fact, for the class of traffic processes in this paper over an underlying $\{C(t)\}_{t\geq a}$ in equilibrium, Kelly's quasi-reversibility, Muntz's $M \Rightarrow M$ property, Gelenbe and Muntz's completeness and our concept of pointwise independence, all boil down to essentially the same thing. Although all four concepts are largely equivalent, the pointwise independence formulation enjoys the generality and convenience of being stated in purely probabilistic terms without any allusion to queueing-theoretic context or an underlying equilibrium assumption.

The utility of the pointwise independence concept is greatly enhanced by Corollary 5.2 and 5.4. The former provides a convenient computational test for pointwise independence which, in view of the latter, serves as a sufficient condition for mutually independent Poisson processes; its ease of application has been demonstrated in the examples of Sec. 7.

The utility of the weak pointwise independence concept derives from Theorem 5.1 and, in equilibrium situations, from Corollary 5.1. These may serve as necessity conditions for Poisson traffic processes by checking the actual behavior of $\lim_{t\to a+} \frac{\partial}{\partial t} P_t(\gamma, n)$ against the hypothesized one. This approach was demonstrated in Sec. 6; a more substantive application of this strategy can be found in [21] concerning traffic processes on the so-called nonexit arcs of a Jackson network.

The concept of pointwise independence (of traffic and state) has considerable relevance to the study of queueing network decomposition. A typical Markovian queueing network is postulated to have Poisson arrivals, independent servers and independent routing switches--the above being mutually independent processes. The problem of valid decompositions arises when one wishes to study one or more subnetworks in isolation via the theory available for the original network. In other words, under what conditions does a subnetwork satisfy all the postulates of the original network? In the aforementioned typical queueing network it is required that all incoming streams into subnetwork nodes be mutually independent Poisson processes which, in addition, are also independent of the service and routing mechanisms operating within that subnetwork.

Now, certain subnetworks may have a state process (an appropriately selected subvector of the original vector valued state process), which still retains the Markov property. Consider the departure streams from such a subnetwork. As we have seen in the examples of Sec. 7, these departure streams and the compressed state are quite likely to be pointwise independent, in equilibrium. Consequently, if there is a subnetwork whose incoming customer streams are either exogenous or from the subnetwork's complement, that subnetwork will indeed satisfy all postulates of the original network, thus constituting an equilibrium original network in miniature. The reader is referred to [4] for an example of this situation from the domain of Jackson queueing networks.

Finally, we point out the plausibility of extending the results of this paper to traffic processes over Markov processes with time dependent transition rates or with continuous parameter and uncountably infinite state space. The latter could enable one to treat queues and queueing networks with more general arrivals and services, such as the limiting cases considered by Kelly [18] and Barbour [1].

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We consider a regular Morkov process with continuous parameter, countable state space, and stationary transition probabilities, over which we define a class of traffic processes. The feasability that multiple traffic processes constitute mutually independent Poisson processes is investigated in some detail.

We show that a variety of independence conditions on a traffic process and the underlying Markov process are equivalent or sufficient to ensure Poisson related properties; these conditions include independent increments,

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renewal, weak pointwise independence, and pointwise independence. Two computational criteria for Poisson traffic are developed: a necessary condition in terms of weak pointwise independence, and a sufficient condition in terms of pointwise independence. The utility of these criteria is demonstrated by sample applications fo queueing-theoretic models.

It follows that, for the class of traffic processes as per this paper in queueing-theoretic contexts, Muntz's M $\Rightarrow$ M property, Gelenbe and Muntz's notion of completeness, and Kelly's notion of quasi-reversibility are essentially equivalent to pointwise independence of traffic and state. The relevance of the theory to queueing network decomposition is also noted.

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