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COMBINATORIAL INEQUALITIES, MATRIX NORMS, AND GENERALIZED NUMER--ETC(U)

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COMBINATORIAL INEQUALITIES, MATRIX NORMS, AND GENERALIZED NUMERICAL RADII

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ABSTRACT. Two new combinatorial inequalities are presented. The main result states that if γ_j, $1 \le j \le n$, are fixed complex scalars with $\sigma \equiv |\sum \gamma_j| > 0$ and $\delta \equiv \max_{i,j} |\gamma_i - \gamma_j| > 0$, and if $\underset{\sim}{V}$ is a normed vector space over the complex field, then

$$\max_\pi |\Sigma_j \gamma_j a_{\pi(j)}| \ge [\sigma\delta/(2\sigma + \delta)] \max_j |a_j| ,$$

$$\forall a_1,\ldots,a_n \in \underset{\sim}{V} ,$$

π varying over permutations of n letters. Next, we consider an arbitrary generalized matrix norm N and discuss methods to obtain multiplicativity factors for N, i.e., constants $\nu > 0$ such that νN is submultiplicative. Using our combinatorial inequalities, we obtain multiplicativity factors for certain C-numerical radii which are generalizations of the classical numerical radius of an operator.

1. SOME NEW COMBINATORIAL INEQUALITIES

In a recent paper [5] we studied a somewhat less general version of the following problem: Given fixed complex scalars $\gamma_1,\ldots,\gamma_n$, and a normed vector space $\underset{\sim}{V}$ over the complex field $\mathbf{C}$, can we find a constant $K > 0$ such that the inequality

$$(1.1) \qquad \max_{\pi \in S_n} \left| \sum_{j=1}^{n} \gamma_j a_{\pi(j)} \right| \ge K \cdot \max |a_j|, \qquad \forall a_1,\ldots,a_n \in \underset{\sim}{V} ,$$

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is satisfied? Here S_n is the symmetric group of n letters, and $|a_j|$ is the norm of the vector a_j.

We start with the following lemma.

LEMMA 1.1. <u>For any</u> $\gamma_1,\dots,\gamma_n \in \mathbf{C}$ <u>and</u> $a_1,\dots,a_n \in \underset{\sim}{V}$,

$$\max_{\pi} \left|\sum_j \gamma_j\, a_{\pi(j)}\right| \geq \frac{1}{2} \max_{i,j} |\gamma_i - \gamma_j| \cdot \max_{i,j} |a_i - a_j| .$$

<u>Proof</u>. We may rearrange the γ_j and the a_j so that

$$|\gamma_1 - \gamma_n| = \max_{i,j}|\gamma_i - \gamma_j| , \qquad |a_1 - a_n| = \max_{i,j}|a_i - a_j| .$$

Now, consider the vectors

$$b_1 = \gamma_1 a_1 + \gamma_2 a_2 + \cdots + \gamma_{n-1} a_{n-1} + \gamma_n a_n ,$$
$$b_2 = \gamma_1 a_n + \gamma_2 a_2 + \cdots + \gamma_{n-1} a_{n-1} + \gamma_n a_1 .$$

We have

$$\begin{aligned} \max_{\pi} \left|\sum_j \gamma_j\, a_{\pi(j)}\right| &\geq \max\{|b_1|,|b_2|\} \geq \frac{1}{2} |b_1 - b_2| \\ &= \frac{1}{2} |\gamma_1 a_1 + \gamma_n a_n - \gamma_1 a_n - \gamma_n a_1| \\ &= \frac{1}{2} |\gamma_1 - \gamma_n| \cdot |a_1 - a_n| , \end{aligned}$$

and the proof is complete. $\square$

Denoting

$$\sigma = \left|\sum_j \gamma_j\right| , \qquad \delta = \max_{i,j} |\gamma_i - \gamma_j| , \tag{1.2}$$

we prove the following result.

THEOREM 1.2. <u>There exists a constant</u> $K > 0$ <u>that satisfies</u> (1.1) <u>if and only if</u> $\sigma\delta > 0$. <u>If</u> $\sigma\delta > 0$ <u>then</u> (1.1) <u>holds with</u> $K = \sigma\delta/(2\sigma + \delta)$.

<u>Proof</u>. Suppose $\sigma\delta = 0$. If $\sigma = 0$, take $a_j = a$, $1 \leq j \leq n$, for some $a \neq 0$; if $\delta = 0$, then the γ_j are equal, so choose a_j not all zero with $\sum a_j = 0$. In both cases,

$$\max_{\pi} \left| \sum_j \gamma_j \, a_{\pi(j)} \right| = 0 \quad \text{but} \quad \max_j |a_j| > 0 \; ;$$

hence no $K > 0$ satisfies (1.1).

Conversely, suppose $\sigma\delta > 0$ and let us show that $K = \sigma\delta/(2\sigma + \delta)$ satisfies (1.1). The following proof, which is shorter than the original one in [5], is due to Redheffer and Smith [8].

Order the a_j so that

$$a_1 = \max_j |a_j| \; , \quad |a_1 - a_n| = \max_j |a_1 - a_j| \equiv \theta |a_1| \qquad (0 \leq \theta \leq 2) \; .$$

Thus, by Lemma 1.1,

$$\max_{\pi} \left| \sum_j \gamma_j \, a_{\pi(j)} \right| \geq \frac{\theta\delta}{2} \max_j |a_j| \; . \tag{1.3}$$

Next, consider the vectors

$$c_j = \gamma_j a_{1+j} + \gamma_2 a_{2+j} + \cdots + \gamma_n a_{n+j} \; , \qquad j = 1,\ldots,n \; ,$$

where $k + j = (k + j) \bmod n$. We have

$$\begin{aligned} \max_{\pi} \left| \sum_j \gamma_j \, a_{\pi(j)} \right| &\geq \max_j |c_j| \geq \frac{1}{n} |c_1 + \cdots + c_n| \\ &= \frac{\sigma}{n} |a_1 + \cdots + a_n| \\ &= \frac{\sigma}{n} |na_1 - (a_1 - a_2) - (a_1 - a_3) - \cdots - (a_1 - a_{n-1})| \\ &\geq \frac{\sigma}{n} \{n|a_1| - (n-1)|a_1 - a_n|\} \\ &= \sigma(1 - \frac{n-1}{n}\theta) \max_j |a_j| \; . \end{aligned} \tag{1.4}$$

By (1.3) and (1.4), therefore,

$$\max_{\pi} \left| \sum_j \gamma_j \, a_{\pi(j)} \right| \geq \max \left\{ \frac{\theta\delta}{2} \, , \, \sigma(1 - \frac{n-1}{n}\theta) \right\} \cdot \max_j |a_j| \; . \tag{1.5}$$

The expressions in the braces are functions of θ describing straight lines with opposite slopes and intersecting value $\sigma\delta/(2\sigma + \delta - 2\sigma/n.)$ Thus, for any θ,

$$\max \left\{ \frac{\theta\delta}{2} \, , \, \sigma(1 - \frac{n-1}{n}\theta) \right\} \geq \frac{\sigma\delta}{2\sigma + \delta - 2\sigma/n} > \frac{\sigma\delta}{2\sigma + \delta} \; . \tag{1.6}$$

By (1.5) and (1.6), the theorem follows. □

What is the best (greatest) possible K which satisfies (1.1)? In answer to that question, Redheffer and Smith proved the following [8].

THEOREM 1.3. *If* $\sigma\delta > 0$, *then the best* K *for* (1.1) *satisfies*

$$\frac{\sigma\delta}{2\sigma + \delta - 2\sigma/n} \le K \le \min\left\{\sigma\,,\ \frac{\sigma\delta}{2\sigma + \delta - 2\sigma/n - 2\delta/n}\right\}, \tag{1.7}$$

and the inequality on the right becomes an equality when the γ_j *and* a_j *are real numbers*.

We note that the left-hand inequality in (1.7) was established already in the proof of Theorem 1.2. For the complete proof of Theorem 1.3, see [2].

From Theorem 1.3, Redheffer and Smith immediately conclude that *while the Goldberg-Straus constant in Theorem* 1.2 *is not optimal for any* n, *it is the best that can be chosen independently of* n, *even if the* γ_j *and* a_j *are real*.

Under certain restrictions on the γ_j, we can improve the constant obtained in Theorem 1.2.

THEOREM 1.4. *If* $\gamma_1,\ldots,\gamma_n$ *are of the same argument, then* (1.1) *holds with* $K = \delta/2$.

Proof. We may assume that

$$\gamma_1 \ge \cdots \ge \gamma_n\ .$$

Arrange the a_j so that

$$|a_1| = \max_j |a_j|\ ,$$

and let P be a projection of $\underset{\sim}{V}$ in the direction of a_1. We write

$$Pa_j = \lambda_j a_j\ , \qquad j = 1,\ldots,n\ ,$$

and set

$$\rho_j = \operatorname{Re}\lambda_j\ , \qquad j = 1,\ldots,n\ .$$

Since

$$\lambda_1 = 1 \geq |\lambda_j| , \qquad j = 2,\ldots,n ,$$

it follows that

$$\rho_1 = 1 \geq |\rho_j| , \qquad j = 2,\ldots,n .$$

So we may order $a_2,\ldots,a_n$ to satisfy

$$1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n .$$

We have

$$(1.8) \qquad \max_{\pi} \left| \sum_j \gamma_j a_{\pi(j)} \right| \geq \max_{\pi} \left| P\left(\sum_j \gamma_j a_{\pi(j)}\right) \right|$$
$$= \max_{\pi} \left| \sum_j \gamma_j \lambda_j \right| \cdot |a_1| \geq \max_{\pi} \left| \mathrm{Re}\left(\sum_j \gamma_j \lambda_{\pi(j)}\right) \right| \cdot |a_1|$$
$$= \max_{\pi} \left| \sum_j \gamma_j \rho_{\pi(j)} \right| \cdot \max_j |a_j| .$$

Now, if $\rho_n \geq 0$, then

$$\max_{\pi} \left| \sum_j \gamma_j \rho_{\pi(j)} \right| = \sum \gamma_j \rho_j \geq \gamma_1 \rho_1 \geq \frac{1}{2}(\gamma_1 - \gamma_n) = \frac{\delta}{2} ;$$

and if $\rho_n < 0$, then, by Lemma 1.1,

$$\max_{\pi} \left| \sum_j \gamma_j \rho_{\pi(j)} \right| \geq \frac{\delta}{2} \max_{i,j} |\rho_i - \rho_j| = \frac{\delta}{2}(\rho_1 - \rho_n) \geq \frac{\delta}{2} .$$

This together with (1.8) completes the proof. □

Note that when the γ_j are of the same argument, then $\delta > 0$ implies $\sigma > 0$, in which case

$$\frac{\delta}{2} > \frac{\sigma\delta}{2\sigma + \delta} .$$

That is, the constant of Theorem 1.4 is indeed an improvement over the K of Theorem 1.2.

2. MATRIX NORMS AND GENERALIZED NUMERICAL RADII

In this section we review (mainly without proof) some of the results in [5] which lead to applications of our combinatorial inequalities.

We start with the following definitions [7]: let $C_{n\times n}$ denote the algebra of $n \times n$ complex matrices. A mapping

$$N : \mathbb{C}_{n\times n} \to \mathbb{R}$$

is a <u>seminorm</u> if for all $A,B \in \mathbb{C}_{n\times n}$ and $\alpha \in \mathbb{C}$,

$$N(A) \geq 0 ,$$

$$N(\alpha A) = |\alpha| \, N(A) ,$$

$$N(A + B) \leq N(A) + N(B) \, .$$

If in addition

$$N(A) > 0 , \quad \forall A \neq 0 ,$$

then N is a <u>generalized matrix norm</u>. Finally, if N is also (sub) multiplicative, i.e.,

$$N(AB) \leq N(A)N(B) ,$$

we say that N is a <u>matrix norm</u>.

EXAMPLES. (i) If $|\cdot|$ is any norm on $\mathbb{C}^n$, then

$$\|A\| = \max\{|Ax| : |x| = 1\}$$

is a matrix norm on $\mathbb{C}_{n\times n}$. In particular, we recall the <u>spectral norm</u>

$$\|A\|_2 = \max\{(x^*A^*Ax)^{1/2} : x^*x = 1\} .$$

(ii) The <u>numerical radius</u>,

$$r(A) = \max\{|x^*Ax| : x^*x = 1\} ,$$

is a nonmultiplicative generalized matrix norm (e.g., [6, §173,176], [3]).

In [5] we introduced the following generalization of the numerical radius: Given matrices $A,C \in \mathbb{C}_{n\times n}$, the C-numerical radius of A is the nonnegative quantity

$$r_C(A) = \max\{|\mathrm{tr}(CU^*AU)| : U \ \ n \times n \ \ \text{unitary}\} .$$

It is not hard to see that

$$r(A) = r_C(A) \quad \text{with} \quad C = \mathrm{diag}(1,0,\dots,0) \ ;$$

thus $R(A)$ is a special case of $r_C(A)$.

It follows from the definition that for each C, r_C is a seminorm on $\mathbf{C}_{n\times n}$. We may then ask whether r_C is a generalized matrix norm. Since the situation is trivial for $n = 1$, we hereafter assume that $n \geq 2$.

THEOREM 2.1 ([5]). r_C <u>is a generalized matrix norm on</u> $\mathbf{C}_{n\times n}$ <u>if and only if</u> C <u>is a nonscalar matrix and</u> $\mathrm{tr}\, C \neq 0$.

Next, we consider multiplicativity, which seems to be a complicated question.

For a given seminorm N and a constant $\nu > 0$, evidently

$$N_\nu \equiv \nu N$$

is a seminorm, too. Similarly, if N is a generalized matrix norm, then so is N_ν. In each case the new norm may or may not be multiplicative. If it is, we call ν a <u>multiplicativity factor</u> for N.

It is an interesting fact that seminorms do not have multiplicativity factors, while generalized matrix norms always do. More precisely, we have the following result.

THEOREM 2.2 ([5]). (i) <u>A nontrivial seminorm has multiplicativity factors if and only if it is a generalized matrix norm</u>.

(ii) <u>If</u> N <u>is a generalized matrix norm, then</u> ν <u>is a multiplicativity factor if and only if</u>

$$\nu \geq \nu_N \equiv \max_{A,B\neq 0} \frac{N(AB)}{N(A)N(B)} .$$

Theorems 2.1 and 2.2 guarantee that r_C <u>has multiplicativity factors if and only if</u> C <u>is nonscalar and</u> $\mathrm{tr}\, C \neq 0$. In practice, however, Theorem 2.2 was of no help to us since we were unable to apply it to C-numerical radii.

An alternative way of obtaining multiplicativity factors is suggested by the following theorem of Gastinel [2] (originally in [1]).

THEOREM 2.3. *Let* N *be a generalized matrix norm,* M *a matrix norm, and* $\eta \geq \xi > 0$ *constants such that*

$$\xi M(A) \leq N(A) \leq \eta M(A)\,, \qquad \forall A \in \mathbf{C}_{n\times n}\,.$$

Then any $\nu \geq \eta/\xi^2$ *is a multiplicativity factor for* N.

Proof. For $\nu \geq \eta/\xi^2$, we have

$$\begin{aligned} N_\nu(AB) \equiv \nu N(AB) &\leq \nu\eta M(AB) \leq \nu\eta M(A)M(B) \leq \frac{\nu\eta}{\xi^2} N(A)N(B) \\ &\leq \nu^2 N(A)N(B) = N_\nu(A)N_\nu(B)\,, \end{aligned}$$

and the proof is complete. □

Since any two generalized matrix norms on $\mathbf{C}_{n\times n}$ are equivalent, constants $\xi \geq \eta > 0$ as required in Theorem 2.3 always exist.

Having Gastinel's theorem and the inequalities of Section 1, we are now ready to obtain multiplicativity factors for C-numerical radii with Hermitian C.

Combining Lemmas 9 and 10 of [5], we state:

LEMMA 2.3. *If* C *is Hermitian with eigenvalues* γ_j, *and if* K *satisfies* (1.1), *then*

$$\left[\frac{K}{2}\right] \|A\|_2 \leq r_C(A) \leq \left[\sum_j |\gamma_j|\right] \|A_2\|\,, \qquad \forall A \in \mathbf{C}_{n\times n}\,.$$

Using the notation of (1.2), we prove:

THEOREM 2.4. *Let* C *be Hermitian, nonscalar, with* $\operatorname{tr} C \neq 0$ *and eigenvalues* γ_j. *Then any* ν *with*

$$\nu \geq 4 \sum |\gamma_j| \left(\frac{2\sigma + \delta}{\sigma\delta}\right)^2$$

is a multiplicativity factor for r_C; i.e., $\nu r_C \equiv r_{\nu C}$ *is a matrix norm.*

Proof. Since C is nonscalar, the γ_j are not all equal; and since $\operatorname{tr} C \neq 0$, $\sum \gamma_j \neq 0$. Thus $\sigma\delta > 0$, so inequality (1.1) is satisfied by the positive constant K of Theorem 1.2. By Lemma 2.3, therefore,

$$\frac{1}{2} \cdot \frac{\sigma\delta}{2\sigma + \delta} \|A\|_2 \leq r_C(A) \leq \sum |\gamma_j| \; \|A\|_2 , \qquad \forall A \in \mathbf{C}_{n\times n} ,$$

and Gastinel's theorem completes the proof. □

For Hermitian definite C, we improve Theorem 2.4 as follows.

THEOREM 2.5. Let C be Hermitian nonnegative (nonpositive) definite. If C is nonscalar with eigenvalues γ_j, then any ν with $\nu \geq 16\sigma/\delta^2$ is a multiplicativity factor for r_C.

Proof. Since C is Hermitian definite, the γ_j are of the same sign. So (1.1) holds with K of Theorem 1.4, and Lemma 2.3 implies that

$$\frac{\delta}{4} \|A\|_2 \leq r_C(A) \leq \sum |\gamma_j| \; \|A\|_2 = \sigma \|A\|_2 , \qquad \forall A .$$

Since C is nonscalar, the γ_j are not all equal; so $\delta > 0$, and Theorem 2.3 completes the proof. □

The optimal (least) multiplicativity factor for r, ν_r, is the subject of our last result.

THEOREM 2.6. νr is a matrix norm if and only if $\nu \geq 4$. That is, $\nu_r = 4$.

Proof. It is well known (e.g., [6, §173]) that

$$\frac{1}{2} \|A\|_2 \leq r(A) \leq \|A\|_2 , \qquad \forall A \in \mathbf{C}_{n\times n} .$$

Thus, by Gastinel's theorem, $\nu \geq 4$ is a multiplicativity factor for r, and by Theorem 2.2, $\nu_r \leq 4$.

To show that $\nu_r \geq 4$, consider the $n \times n$ matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0_{n-2} , \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus 0_{n-2} .$$

A simple calculation shows that $r(A) = r(B) = 1/2$ and $r(AB) = 1$. Hence $r_\nu \equiv \nu r$ satisfies

$$r_\nu(AB) \leq r_\nu(A) r_\nu(B)$$

if and only if $\nu \geq 4$, and the theorem follows. □

Note that the results of Theorems 2.4 - 2.6 depend neither on the dimension n nor on the space $\underset{\sim}{V}$.

REFERENCES

1. N. Gastinel, Matrices du Second Degré et Normes Générales en Analyse Numérique Linéaire. Thesis, Université de Grenoble, 1960.

2. N. Gastinel, Linear Numerical Analysis, Academic Press, New York, 1970.

3. M. Goldberg, On certain finite dimensional numerical ranges and numerical radii, Linear and Multilinear Algebra (1979), to appear.

4. M. Goldberg and E.G. Straus, Elementary inclusion relations for generalized numerical ranges, Linear Algebra Appl. 18 (1977), 1-24.

5. M. Goldberg and E.G. Straus, Norm properties of C-numerical radii, Linear Algebra Appl. (1979), to appear.

6. P.R. Halmos, A Hilbert Space Problem Book, Van Nostrand, New York, 1967.

7. A. Ostrowski, Über Normen von Matrizen, Math. Z., 63 (1955), 2-18.

8. R. Redheffer and C. Smith, On a surprising inequality of Goldberg and Straus, to appear.

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20. Abstract continued.

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π varying over permutations of n letters. Next, we consider an arbitrary generalized matrix norm N and discuss methods to obtain multiplicativity factors for N, i.e., constants $\nu > 0$ such that νN is submultiplicative. Using our combinatorial inequalities, we obtain multiplicativity factors for certain C-numerical radii which are generalizations of the classical numerical radius of an operator.