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by

Naftali A. Langberg[1], Ramón V. León[2,3], and Frank Proschan[3]

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Characterization of Partially Ordered
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ABSTRACT

In this paper we obtain characterizations of life distributions F such that (a) $G^{-1}(F)$ is convex (concave) and alternatively (b) $G^{-1}(F)$ is starshaped (antistarshaped), where G is an absolutely continuous life distribution with positive, bounded, right continuous density. These characterizations generalize earlier results for the IFR(DFR) and IFRA(DFRA) classes, and should prove useful in unifying the study of the class of distributions with decreasing density, comparing Weibull (Gamma) distributions with different shape parameters, etc.

1. Introduction.

In a previous paper (Langberg, León, and Proschan, 1978), we obtain characterizations of large classes of standard nonparametric life distributions, such as the IFR(DFR), IFRA(DFRA), etc. (See Section 2 for definitions and notation.) These characterizations are obtained under the weakest possible assumptions that we can make concerning the life distributions being characterized.

In the present paper, we continue our characterization work and also obtain additional results in one direction of implication, but now we focus on more general classes of distributions, many of them of interest and applicable in reliability. We consider classes of distributions F such that $G^{-1}(F)$ is convex (concave) or starshaped (antistarshaped), where G is a known distribution. The assumption that G is known is reasonable in many practical situations, as seen from the following pairs (F, G) such that $G^{-1}(F)$ is convex: (1) G is exponential, F is IFR; (2) G is uniform, F has decreasing density; (3) G is Weibull (Gamma) with shape parameter α, F is Weibull (Gamma) with shape parameter $\beta(>\alpha)$; etc. Similar pairs can be displayed for $G^{-1}(F)$ is starshaped. Since we assume G known and F unknown, we make convenient smoothness assumptions for G, but as few assumptions about F as possible.

As pointed out in Barlow and Proschan (1975), Barlow and Doksum (1972), Barlow and Van Zwet (1970), the advantage of considering these more general classes is that many results, tests, methods of inference, methods of proof, etc., of use in the IFR(DFR), IFRA(DFRA) classes carry over with minor modifications to the corresponding more general classes.

In Section 2, we present definitions, notation, and elementary properties. In Section 3, we obtain characterization results for the Barlow-Doksum transform, a generalization of the well known total time on test transform. These results are not simply of theoretical interest, but can be used to develop tests as to whether a set of underlying data come from a Weibull with greater shape parameter than say $\alpha(>0)$. For example, Barlow (personal communication) has found that increasing stress often leads to increasing shape parameter of the Weibull governing lifelength. In Section 4, characterization results for convex and concave ordering are obtained in terms of order statistics or their spacings. Section 5 is devoted to characterization of starshaped and antistarshaped orderings in terms of order statistics; spacings are not useful in these characterizations.

2. Preliminaries.

Let F be a life distribution, that is, $F(0-) = 0$. We use the following notation and conventions: $F^{-1}(t) \equiv \inf\{x: F(x) > t\}$, $t \in [0, 1)$; $F^{-1}(1) \equiv \sup\{x: F(x) < 1\}$; $\overline{F} \equiv 1 - F$; $R \equiv -\ell n\overline{F}$. We use "increasing" in place of "nondecreasing" and "decreasing" in place of "nonincreasing". Throughout the paper we assume that G is a fixed absolutely continuous life distribution with positive, bounded, and right continuous density g on the interval $(G^{-1}(0), G^{-1}(1))$. Let $X_1, X_2, \ldots, X_n$ $(Y_1, Y_2, \ldots, Y_n)$ be a random sample of size n from F(G) and let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ $(Y_{1:n} < Y_{2:n} < \ldots < Y_{n:n})$ be the corresponding order statistics.

Definition 2.1. The life distribution F is convex with respect to G, written $F \underset{c}{<} G$, if either (i) F is degenerate or (ii) $G^{-1}F$ is convex on $(-\infty, F^{-1}(1))$.

Definition 2.2. The life distribution F is concave with respect to G, written $F \underset{cv}{<} G$ if $G^{-1}F$ is concave on $(F^{-1}(0), \infty)$.

Let F be nondegenerate, strictly increasing on $(F^{-1}(0), F^{-1}(1))$ and $G^{-1}(1) = \infty$. Then $F \underset{c}{<} G$ if and only if $G \underset{cv}{<} F$. This relationship is the reason only convex ordering is usually defined in the literature (see for example Barlow and Proschan, 1975, p. 106). However without assumptions on F, the two orderings are not so easily related.

We define the increasing failure rate (IFR) and shifted decreasing failure rate (SDFR) classes of life distributions.

Definition 2.3. The life distribution F is IFR if either (i) F is degenerate or (ii) R(x) is convex on $(-\infty, F^{-1}(1))$.

Definition 2.4. The life distribution F is SDFR if R(x) is concave on $(F^{-1}(0), \infty)$.

If G is the exponential distribution with mean 1, then $G^{-1}F = R$. Hence F is IFR(SDFR) if and only if F is convex (concave) with respect to any exponential distribution.

If G is the uniform distribution on $[0, a]$, $a > 0$, then $F \le_{cv} G$ is equivalent to F having a decreasing density. If $F(G)$ denotes the gamma distribution with shape parameter $\alpha(\beta)$, $\alpha \le \beta$ then $F \le_{c} G$. The Weibull family is similarly ordered with respect to its shape parameter (Van Zwet, 1964, and Barlow and Proschan, 1966).

Definition 2.5. The life distribution F is starshaped (antistarshaped) with respect to G, written $F \le_{*} G$ $(F \le_{a*} G)$, if $\frac{1}{t}\, G^{-1}F(t)$ is increasing (decreasing) in t $(0 < t < F^{-1}(1))$.

We define the increasing failure rate average (IFRA) and the decreasing failure rate average (DFRA) classes of life distributions.

Definition 2.6. The life distribution F is IFRA(DFRA) if $\frac{1}{t}R(t)$ is increasing (decreasing) in t $(0 < t < F^{-1}(1))$.

Let G be any exponential distribution. Then $F \le_{*} G$ $(F \le_{a*} G)$ if and only if F is IFRA(DFRA). Let G be the uniform distribution on $[0, a)$, and F have a density. Then $F \le_{a*} G$ is equivalent to

$$f(t)t \le F(t) \text{ for } t > 0. \tag{2.1}$$

Note that the class of distributions satisfying (2.1) contains the class of distributions with decreasing densities.

Finally we remark that $F \le_{c} G$ implies that $F \le_{*} G$. Similarly, $F \le_{cv} G$ implies that $F \le_{a*} G$.

3. Properties of the Barlow-Doksum Transform.

For a fixed G let $H_F^{-1}(t) \equiv \int_0^{F^{-1}(t)} gG^{-1}F(u)du$. This transform of F was first introduced in connection with isotonic tests of convex ordering by Barlow and Doksum (1972). Hence we call H_F^{-1} the Barlow-Doksum (B-D) transform. When G is the exponential distribution H_F^{-1} is the usual total time on test transform studied by Barlow (1977), Barlow and Campo (1975), and Langberg, León, and Proschan (1978), among others. We should remark that Chandra and Singpurwalla (1978) have pointed out the close relationship between the total time on test transform and the Lorenz curve used by econometrists. In this section we develop some properties of H_F^{-1} which we use in the proofs of Section 4.

Before stating the first theorem we need two definitions.

Definition 3.1. A point x is a point of increase of F if $F(x - h) < F(x) < F(x + h)$ for every $h > 0$.

Definition 3.2. A sequence $\{(k_r, n_r)\}_{r=1}^{\infty}$ of ordered pairs of natural numbers is a t-sequence $(0 \le t \le 1)$ if (i) $1 \le k_r \le n_r < n_{r+1}$ for all r, and (ii) $k_r/n_r \to t$ as $r \to \infty$.

Let $T_G(X_{k:n}) \equiv \sum_{i=1}^{k} g(G^{-1}(\frac{i-1}{n}))(X_{i:n} - X_{i-1:n})$. If G is the exponential distribution with mean 1, then $T_G(X_{k:n}) = n^{-1}T(X_{k:n})$, where $T(X_{k:n}) \equiv \sum_{i=1}^{k} (n - i + 1)(X_{i:n} - X_{i-1:n})$, is the total time on test statistics commonly used in reliability theory (see for example Barlow and Proschan, 1975, p. 61). If n items are placed on test at time 0 and successive failures are observed at times $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$, then $T(X_{k:n})$ represents the total test time observed between 0 and $X_{k:n}$.

We may now state and prove the following theorem.

Theorem 3.3. Let $F^{-1}(t)$ be a point of increase of F, and let (k, n) range over a t-sequence. Then

$$T_G(X_{k:n}) \to H_F^{-1}(t) \text{ a.s. as } n \to \infty.$$

Proof. Let F_n denote the empirical distribution function of F. Then $T_G(X_{k:n}) = H_{F_n}^{-1}(\frac{k-1}{n}) = \int_0^{X_{k:n}} g\, G^{-1}F_n(u)du$. Also for (k, n) ranging over a t-sequence, $X_{k:n} \to F^{-1}(t)$ a.s. as $n \to \infty$ since $F^{-1}(t)$ is a point of increase of F (see Rao, 1973, p. 423). The desired result follows by the Glivenko-Cantelli Theorem (Chung, 1974, p. 133) and the continuity of $g\, G^{-1}$. ||

Next we note that if EX_1 is finite, then $EX_{k:n}$ and $ET_G(X_{k:n})$ are also finite. This follows since $0 \le X_{k:n} \le \sum_{i=1}^{n} X_i \equiv n\overline{X}_n$ and
$T_G(X_{k:n}) \le (\max_{1\le i\le n} g\, G^{-1}(\frac{i-1}{n})) \sum_{i=1}^{k} (X_{i:n} - X_{i-1:n}) \le (\sup_{0\le x<\infty} g(x))X_{k:n}$.

The above inequalities can be used to show also that whenever EX_1 is finite, $\{T_G(X_{k_r:n_r})\}_{r=1}^{\infty}$ is uniformly integrable for every t-sequence $\{(k_r, n_r)\}_{r=1}^{\infty}$. Since a uniformly integrable sequence which converges almost surely converges in mean (see Breiman, 1973, p. 91), we have thus shown:

Theorem 3.4. Let t, k, and n be as in Theorem 3.3 and let EX_1 be finite. Then $E|T_G(X_{k:n}) - H_F^{-1}(t)| \to 0$ a.s. as $n \to \infty$. In particular, $ET_G(X_{k:n}) \to H_F^{-1}(t)$ as $n \to \infty$.

As shown in Langberg, León, and Proschan (1978), neither Theorem 3.3 nor Theorem 3.4 is true if $F^{-1}(t)$ is not a point of increase of F. Theorems 3.3 and 3.4 contain as special cases similar theorems of Langberg, León, and Proschan (1978) concerning the total time on test statistics. Also these theorems are related to a theorem of Barlow and van Zwet (Theorem 2.2 of Barlow and Doksum, 1972).

Let ${}^{+}f(x_0)$ denote the right-hand derivative of f at the point x_0. We will need the following lemma in the proof of Theorem 3.6 below.

Lemma 3.5. Let x be a point of increase and of continuity of F. Then ${}^{+}H_F^{-1}(F(x))$ exists and is nonzero if and only if ${}^{+}F(x)$ exists and is nonzero. In this case, ${}^{+}(G^{-1}F(x))$ exists and is nonzero, and ${}^{+}H_F^{-1}(F(x))\ {}^{+}(G^{-1}F(x)) = 1$.

Proof. Note that in a neighborhood of x, F^{-1} behaves like the usual inverse function of F. The result follows using standard differentiation results. ||

The following theorem relates convex (concave) ordering to the B-D transform.

Theorem 3.6. Let F be a life distribution. Then $F \underset{c}{\leq} G$ ($F \underset{cv}{\leq} G$) if and only if H_F^{-1} is concave (convex) on [0, 1).

We will need the following simple properties of H_F^{-1} in the proof of Theorem 3.6.

(3.1) $H_F^{-1}(0) = F^{-1}(0)$.

(3.2) $H_F^{-1}(t+) = H_F^{-1}(t)$ for $t \in [0, 1]$.

(3.3) H_F^{-1} is increasing on [0, 1].

(3.4) For $y \in [0, \infty)$, the set $\{s: H_F^{-1}(s) = y\} = [a, b)$, where $0 \leq a < b \leq 1$, if and only if $P(X_1 = F^{-1}(a)) = b - a$.

(3.5) For $0 \leq a < 1$, $H_F^{-1}(a-) = H_F^{-1}(a)$ if and only if $F(F^{-1}(a-)) = F(F^{-1}(a))$. In particular, H_F^{-1} is continuous on [a, b) if and only if every point in $(F^{-1}(a), F^{-1}(b-))$ is a point of increase of F.

Proof of Theorem 3.6. Let H_F^{-1} be concave on [0, 1). Since H_F^{-1} is increasing on [0, 1), there exists a real number A in [0, 1) such that H_F^{-1} is strictly increasing on [0, A] and constant on [A, 1). If A = 0, H_F^{-1} is constant on

$[0, 1)$ and consequently $F \underset{c}{<} G$ since in this case F is degenerate at $F^{-1}(0)$. Next suppose that $A > 0$. It follows that ${}^{+}H_F^{-1}(t) > 0$ for all $t \in (0, A)$. Equivalently, ${}^{+}H_F^{-1}(t) > 0$ for x in $(F^{-1}(0), F^{-1}(1))$ since $F^{-1}(A-) = F^{-1}(1)$. By (3.4) and (3.5) every point of $(F^{-1}(0), F^{-1}(1))$ is a point of increase and of continuity of F. Hence by Lemma 3.5, the concavity of H_F^{-1} implies that ${}^{+}(G^{-1}F(x))$ exists and is increasing on $(F^{-1}(0), F^{-1}(1))$, that is, $G^{-1}F$ is convex on $(F^{-1}(0), F^{-1}(1))$. Since by (3.5), $F(F^{-1}(0)) = 0$, it follows that $F \underset{c}{<} G$.

Next let $F \underset{c}{<} G$. Then either F is degenerate, in which case H_F^{-1} is constant and thus concave, or $G^{-1}F$ is convex on $S \equiv (F^{-1}(0), F^{-1}(1))$ and $F(F^{-1}(0)) = 0$. We assume the latter case. If $x \in S$ and $h > 0$, then

$$0 < \frac{G^{-1}F(x) - G^{-1}F(F^{-1}(0))}{x - F^{-1}(0)} < \frac{G^{-1}F(x + h) - G^{-1}F(x)}{h}.$$

Consequently ${}^{+}(G^{-1}F)$ exists and is positive on S. Now since ${}^{+}F = {}^{+}(G^{-1}F) \cdot g(G^{-1}F)$ is positive on S, S contains only points of increase and of continuity of F. Thus by Lemma 3.5, ${}^{+}H_F^{-1}F$ is decreasing on S, that is, ${}^{+}H_F^{-1}$ is concave on $(0, 1)$. By (3.2), H_F^{-1} is concave on $[0, 1)$.

The counterpart result for concave ordering can be proved similarly. ||

Barlow and Doksum (1972) obtained the conclusion of Theorem 3.6, but under stronger regularity conditions on F.

<u>Corollary 3.7</u>. (Barlow and Campo, 1975). Let G be any exponential distribution. Then the life distribution F is IFR(SDFR) if and only if H_F^{-1} is concave (convex) on $[0, 1]$.

Our proof of Corollary 3.7 avoids some technical difficulties which arise in the limiting argument used in the Barlow and Campo proof of the "if" part of Corollary 3.7.

4. Convex (Concave) Ordering and Order Statistics.

In this section we present a series of results concerning convex (concave) ordering and order statistics. Our first theorem gives a sufficient condition for $F \underset{c}{<} (\underset{cv}{<})\ G$.

Theorem 4.1. Let F and G be life distributions with finite means. Suppose F is continuous and let $E(X_{k:n} - X_{k-1:n})/E(Y_{k:n} - Y_{k-1:n})$ be decreasing (increasing) in k (k = 2, ..., n) for infinitely many values of $n \geq 2$. Then $F \underset{c}{<} (\underset{cv}{<})\ G$.

In order to prove Theorem 4.1 we need the following lemma.

Lemma 4.2. Let the conditions of Lemma 4.1 be satisfied. Then the support of F is the interval $[F^{-1}(0),\ F^{-1}(1)]$.

Proof. The support of a continuous distribution is a closed set without isolated points (see Chung, 1974, p. 10). It follows that if S, the support of F, is not an interval, then we can find a, b, and ε such that $(a - \varepsilon,\ a] \subset s$, $(a,\ b) \subset \{x:\ x \notin s\}$, and $[b,\ b + \varepsilon)\ \varepsilon\ S$. Let $t = F(a) = F(b)$, $t_1 = \frac{t + F(a - \varepsilon)}{2}$. Also let $h > 0$ be small enough so that $[t_1 - h,\ t_2 + h] \subset (t,\ F(b + \varepsilon))$ and $[t_2 - h,\ t_2 + h] \subset (t,\ F(b + \varepsilon))$.

By hypothesis, $\dfrac{Eg\ G^{-1}(\frac{i-1}{n})(X_{i:n} - X_{i-1:n})}{Eg\ G^{-1}(\frac{i-1}{n})(Y_{i:n} - Y_{i-1:n})}$ is decreasing (increasing) in i (i = 2, 3, ..., n) for infinitely many values of n. Now observe that if $\{a_i\}_{i=2}^n$ and $\{b_i\}_{i=2}^n$ are sequences of positive real numbers such that a_i/b_i is decreasing (increasing) in i (i = 2, ..., n), then $\sum_{i=k}^{k+j} a_i / \sum_{i=k}^{k+j} b_i$ is decreasing (increasing) in k (k = 2, ..., n - j) for each j (j = 1, ..., n - 1). Thus we obtain for each one of the infinitely many n that

$$\frac{ET_G(X_{([n(t_1-h)]+[n(2h)]):n}) - ET_G(X_{[n(t_1-h)]:n})}{ET_G(Y_{([n(t_1-h)]+[n(2h)]:n} - ET_G(Y_{[n(t_1-h)]:n})}$$

$$\text{(4.1)} \qquad \geq(\leq) \frac{ET_G(X_{([n(t-h)]+[n(2h)]):n}) - ET_G(X_{[n(t-h)]:n})}{ET_G(Y_{([n(t-h)]+[n(2h)]):n} - ET_G(Y_{[n(t-h)]:n})}$$

$$\geq \leq \frac{ET_G(X_{([n(t_2-h)]+[n(2h)]):n} - ET_G(X_{[n(t_2-h)]:n})}{ET_G(Y_{([n(t_2-h)]+[n(2h)]):n} - ET_G(Y_{[n(t_2-h)]:n})}$$

The points at which F equals $t_1 - h$, $t_1 + h$, $t - h$, $t + h$, $t_2 - h$, and $t_2 + h$ are in the interior of S and are consequently points of increase of F. Applying Theorem 3.4 to the chain of inequalities (4.1), we conclude that

$$\text{(4.2)} \qquad \begin{aligned} \frac{H_F^{-1}(t_1 + h) - H_F^{-1}(t_1 - h)}{2h} &\geq(\leq) \frac{H_F^{-1}(t + h) - H_F^{-1}(t - h)}{2h} \\ &\geq(\leq) \frac{H_F^{-1}(t_2 + h) - H_F^{-1}(t_2 - h)}{2h} . \end{aligned}$$

Since $H_F^{-1}(\cdot) \equiv \int_0^{F^{-1}(\cdot)} g\ G^{-1}F(u)du$ is continuous at t_1 and t_2, by letting $h \to 0$ in (4.2), we conclude that $\lim_{h\to 0}[H_F^{-1}(t + h) - H_F^{-1}(t - h)] = 0$. But since H_F^{-1} is increasing, this implies that H_F^{-1} is continuous at t, or equivalently, that F^{-1} is continuous at t. This contradicts the fact that Fis constant on (a, b). It follows that S must be an interval, as was to be shown. ||

Proof of Theorem 4.1. Let t_1, t_2, and h be such that $0 \leq t_1 < t_2 < t_2 + h \leq 1$. Using the argument in the proof of Lemma 4.2 yielding (4.2), we obtain:

$$\text{(4.3)} \qquad H_F^{-1}(t_1 + h) - H_F^{-1}(t_1) \geq(\leq) H_F^{-1}(t_2 + h) - H_F^{-1}(t_2).$$

Since (4.3) is true for all t_1, t_2, and h satisfying the above constraints, H_F^{-1} must be concave (convex) on $[0, 1)$. By Theorem 3.6, this implies that $F \underset{c}{<} (\underset{cv}{<})\ G$. ||

Theorem 4.3 is a partial converse to Theorem 4.1.

<u>Theorem 4.3</u>. Let F and G be life distributions with finite means and suppose $F \underset{c}{<} (\underset{cv}{<})\ G$. Then

$$\text{(4.4)} \qquad \lim_{n\to\infty} \frac{E(X_{[n(t+h)]:n} - X_{[nt]:n})}{E(Y_{[n(t+h)]:n} - Y_{[nt]:n})}$$

is decreasing (increasing) in t $(0 < t < t + h < 1)$ for all h $(0 < h < 1)$.

<u>Proof</u>. Let $F \underset{c}{<} G$. Note that every element of $(F^{-1}(0), F^{-1}(1))$ is a point of increase of F since $F \underset{c}{<} G$ and every element of $(G^{-1}(0), G^{-1}(1))$ is a point of increase of G. Thus $X_{[nt]:n} \to F^{-1}(t)$ a.s. and $Y_{[nt]:n} \to G^{-1}(t)$ a.s. as $n \to \infty$ (see Rao, 1973, p. 423). We show $X_{[nt]:n}$ is uniformly integrable. We have $P(X_{[nt]:n} > x) = P(B(n, \overline{F}(x)) > n - [nt] + 1)$, where $B(n, \overline{F}(x))$ denotes a binomial random variable with parameters n and $\overline{F}(x)$. Thus

$$\text{(4.5)} \qquad P(X_{[nt]:n} > x) \le \frac{n}{n - [nt] + 1} \overline{F}(x)$$

since $P(Z > A) \le EZ/A$ for any nonnegative random variable Z and any $A > 0$. Hence

$$EX_{[nt]:n} I[X_{[nt]:n} \ge A] = \int_A^\infty P[X_{[nt]:n} > x]dx + AP[X_{[nt]:n} \ge A]$$

[by integration by parts]

$$\le \frac{n}{n - [nt] + 1}\left(\int_A^\infty \overline{F}(x)dx + A\overline{F}(A)\right)$$

[by (4.5)]

$$\le \frac{1}{1 - \frac{[nt]}{n} + \frac{1}{n}} (EX_1 I[X_1 \ge A]).$$

It follows that $X_{[nt]:n}$ (and similarly $Y_{[nt]:n}$) is a uniformly integrable sequence in n. Consequently, $EX_{[nt]:n} \to F^{-1}(t)$ and $EY_{[nt]:n} \to G^{-1}(t)$ as $n \to \infty$. Hence the limit in (4.4) exists and equals $\dfrac{F^{-1}(t + h) - F^{-1}(t)}{G^{-1}(t + h) - G^{-1}(t)}$. But since $G^{-1}F$ is convex on $(-\infty, F^{-1}(1))$ on $(-\infty, F^{-1}(1))$ for $0 < t_1 < t_2 < t_2 + h < 1$; $0 < h < 1$, then

$$\frac{G^{-1}F(F^{-1}(t_1 + h)) - G^{-1}F(F^{-1}(t_1)}{F^{-1}(t_1 + h) - F^{-1}(t_1)} \leq \frac{G^{-1}F(F^{-1}(t_2 + h) - G^{-1}F(F^{-1}(t_2))}{F^{-1}(t_2 + h) - F^{-1}(t_2)} .$$

Equivalently, the limit in (4.4) is decreasing in t $(0 < t < t + h < 1)$ for all h $(0 < h < 1)$.

A similar argument yields the result when $F \underset{cv}{<} G$. ||

Our next result concerning convex (concave) ordering and order statistics is an immediate consequence of the following lemma of Barlow and Proschan (1966).

Lemma 4.4. (Lemma 3.5 of Barlow and Proschan, 1966). Let $F_{i:n}$ denote the distribution of the ith order statistic in a sample of size n from a continuous distribution F defined on $(-\infty, \infty)$. Suppose h(x) changes signs k times and

$$h(i, n) = \int_{\infty}^{\infty} h(x)dF_{i:n}(x)$$

converges absolutely. Then (i)h(i, n) changes signs at most k times as a function of i = 1, 2, ..., n for fixed n, and changes sign at most k times as a function of n = 1, 2, ..., for fixed i. Furthermore, if h(i, n) changes sign exactly k times as a function of i(n), then h(i, n) must have the same (opposite) arrangement of signs in i(n) as does h(x), where x, i, and n traverse their respective domains from left to right.

(ii) $h(n - i, n)$ changes sign at most k times as a function of $n = 1, 2, \ldots$; if $h(n - i, n)$ actually does change sign in n exactly k times, then the changes occur in the same order as do those of $h(x)$.

Before stating our result, we observe that $F \underset{c}{<} G$ and $EY < \infty$ imply that $EX < \infty$ and consequently that $EX_{i:n} < \infty$ for all i and n ($i = 1, 2, \ldots, n$; $n \geq 1$).

<u>Theorem 4.5</u>. Let $F \underset{c}{<} (\underset{cv}{<})\ G$, F be continuous at $F^{-1}(1)$, and $EY < \infty$ (continuous at $F^{-1}(0)$, $EY < \infty$, and $EX < \infty$). Then (i) for all $a \geq 0$ and $b \geq 0$, $a\ EX_{i:n} - EY_{i:n} - b$ changes signs at most twice in $i = 1, 2, \ldots n$ ($n = 1, 2, \ldots$), and if twice, from negative to positive to negative (positive to negative to positive; (ii) for all $a \geq 0$, $b \geq 0$, $a\ EX_{n-i:n} - EY_{n-i:n} - b$ changes signs at most twice in $n = 1, 2, \ldots$, and if twice, from negative to positive to negative (positive to negative to positive).

<u>Proof</u>. Let $F \underset{c}{<} G$ and let $\phi(x) = G^{-1}F(x)$. Then ϕ is convex. Thus for $a \geq 0$, $b \geq 0$, $(ax - b) - \phi(x)$ changes signs at most twice, and if twice, from negative to positive to negative. Hence by Lemma 4.4(i),

$$h(i, n) \equiv \int_0^\infty (ax - b - \phi(x))dF_{i:n}$$

$$= aEX_{i:n} - b - EY_{i:n}$$

changes sign at most twice in $i = 1, 2, \ldots, n$ ($n = 1, 2, \ldots$), and if twice, from negative to positive to negative. Thus (i) follows.

A similar argument using part (ii) of Lemma 4.4 yields (ii). For the case $F \underset{cv}{<} G$, the proof is similar. ||

We now present a converse to Theorem 4.5 (ii).

Theorem 4.6. Let the support of F be an interval and let F be continuous at $F^{-1}(0)$. Suppose for all $a \geq 0$ and $b \geq 0$, and infinitely many $n \geq 1$, $aEX_{i:n} - EY_{i:n} - b$ changes signs at most twice in $i = 1, 2, \ldots, n$, and if twice, from negative to positive to negative. Then $F \underset{c}{<} G$.

Proof. Let $a \geq 0$, $b \geq 0$. Since $F^{-1}(t)$ and $G^{-1}(t)$ are points of increase for all t $(0 < t < 1)$, then $aF^{-1}(t) - G^{-1}(t) - b = \lim_{n\to\infty}(aEX_{[nt]:n} - EY_{[nt]:n} - b)$ changes signs at most twice in t $(0 < t < 1)$, and if twice, from negative to positive to negative. Letting $t = F(x)$, we get that $ax - b - G^{-1}F(x)$ changes signs at most twice in x $(F^{-1}(0) < x < F^{-1}(1))$. Since F is continuous at $F^{-1}(0)$ and $G^{-1}F$ is strictly increasing in $x(-\infty < x < \infty)$, $G^{-1}F$ is convex on $(-\infty, F^{-1}(1)$, as desired. ||

A result similar to Theorem 4.6 is available for concave ordering but we omit it.

The next theorem concerns the ratios of order statistics.

Theorem 4.8. Let $F \underset{c}{<} (\underset{cv}{<})$ G and let F be continuous at $F^{-1}(1)$ (at $F^{-1}(0)$). Then

$$\frac{Y_{i+1:n} - Y_{i:n}}{Y_{i:n} - Y_{i-1:n}} \overset{st}{\geq} (\overset{st}{\leq}) \frac{X_{i+1:n} - X_{i:n}}{X_{i:n} - X_{i-1:n}}$$

for all i and n $(i = 2, 3, \ldots, n - 1;\ n \geq 2)$.

Proof. Let $F \underset{c}{<} G$ and let $Y'_{i:n} = G^{-1}F(X_{i:n})$ for $i = 2, 3, \ldots, n - 1;\ n \geq 2$. Then $(Y'_{1:n}, \ldots, Y'_{n:n}) \overset{st}{=} (Y_{1:n}, \ldots, Y_{n:n})$ since $F \underset{c}{<} G$ and F continuous at $F^{-1}(1)$ imply that F is continuous. Since $G^{-1}F$ is convex, then

$$\frac{Y'_{i+1:n} - Y'_{i:n}}{Y'_{i:n} - Y'_{i-1:n}} \geq \frac{X_{i+1:n} - X_{i:n}}{X_{i:n} - X_{i-1:n}}$$

for $i = 2, 3, \ldots, n-1$; $n \geq 2$. Since

$$\frac{Y_{i+1:n} - Y_{i:n}}{Y_{i:n} - Y_{i-1:n}} \stackrel{st}{=} \frac{Y'_{i+1:n} - Y'_{i:n}}{Y'_{i:n} - Y'_{i-1:n}},$$

the conclusion follows in the case $F \underset{c}{<} G$.

A similar argument yields the conclusion when $F \underset{cv}{<} G$. ||

If $F \underset{c}{<} (\underset{cv}{<}) G$ it is reasonable to expect that information about the order statistics $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$ yields information about the order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. Theorem 4.9 shows one way this expectation is fulfilled. Other examples will follow.

Theorem 4.9. Let $F \underset{c}{<} (\underset{cv}{<}) G$ and the support of F be an interval. Let $1 \leq i \leq j < \ell \leq n$, $i < k \leq \ell$, and $a > 0$. Then

$$P[Y_{\ell:n} - Y_{j:n} \geq a(Y_{k:n} - Y_{i:n})] \geq (\leq) P[X_{\ell:n} - X_{j:n} \geq a(X_{k:n} - X_{i:n})].$$

Proof. Let $F \underset{c}{<} G$ and $Y'_{1:n}, Y'_{2:n}, \ldots, Y'_{n:n}$ be as in the proof of Theorem 4.5. Let $\phi(y)$ be the concave function $F^{-1}G$. Then for $1 \leq i \leq j < \ell \leq n$ and $i < k \leq \ell$,

$$\frac{\phi(Y'_{k:n}) - \phi(Y'_{i:n})}{Y'_{k:n} - Y'_{i:n}} \geq \frac{\phi(Y'_{\ell:n}) - \phi(Y'_{j:n})}{Y'_{\ell:n} - Y'_{j:n}}$$

(see Royden, 1968, p. 108). Hence

$$\frac{Y'_{\ell:n} - Y'_{j:n}}{Y'_{k:n} - Y'_{i:n}} \geq \frac{X_{\ell:n} - X_{j:n}}{X_{k:n} - X_{i:n}} .$$

Thus for $a > 0$,

$$P\left[\frac{X_{\ell:n} - X_{j:n}}{X_{k:n} - X_{i:n}} \geq a\right] \leq P\left[\frac{Y_{\ell:n} - Y_{j:n}}{Y_{k:n} - Y_{i:n}} \geq a\right],$$

and the conclusion follows.

If $F \underset{cv}{<} G$ the proof is similar. ||

Let $a_{\ell jki}$ be a positive constant for each ℓ, j, k, and i such that $1 \leq i < j \leq k < \ell$ and let $V_n(\underline{X}) \equiv \sum I[X_{\ell:n} - X_{j:n} \geq a_{\ell jki}(X_{k:n} - X_{i:n})]$, where the summation ranges over all i, j, k and ℓ such that $1 \leq i \leq k < \ell$. Equivalently, $V_n(\underline{X})$ is the number of comparisons for which the inequality $X_{\ell:n} - X_{j:n} \geq a_{\ell jki}(X_{k:n} - X_{i:n})$ holds as i, j, k and ℓ range over the appropriate domain. We can now state a corollary of Theorem 4.9 which can be used for nonparametric tests for $F \underset{c}{<}(\underset{cv}{<})G$; in particular, tests for IFR, SDFR, and decreasing density.

Corollary 4.10. Let $F \underset{c}{<}(\underset{cv}{<})G$ and the support of F be an interval. Then

$$V_n(\underline{Y}) \overset{st}{\geq} (\overset{st}{\leq})\ V_n(\underline{X}) .$$

5. Starshaped (Antistarshaped) Ordering and Order Statistics.

In this section we consider another ordering, namely starshaped (antistarshaped) ordering. The first result gives a necessary and sufficient condition in terms of the order statistics for two life distributions to be related under the starshaped (antistarshaped) ordering.

Theorem 5.1. Let F and G be continuous life distributions with finite means. Assume that the supports of both F and G are intervals and that $G(0) = F(0) = 0$. Then $F \leq_{*} (\leq_{a*}) G$ if and only if $EX_{i:n}/EY_{i:n}$ is decreasing (increasing) in $i(i = 1, 2, \ldots, n)$ for infinitely many n.

Proof. We prove $F \leq_{*} G$ if and only if $EX_{i:n}/EY_{i:n}$ is decreasing in $i(i = 1, 2, \ldots, n)$ for infinitely many n. The counterpart result for $F \leq_{a*} G$ has a similar proof. The "only if" part is Theorem 3.6 of Barlow and Proschan (1966).

To show the "if" part recall that in the proof of Theorem 4.4 we showed that $EX_{[nt]:n} \to F^{-1}(t)$ and $EY_{[nt]:n} \to G^{-1}(t)$ as $n \to \infty$. Thus if $EX_{[nt]:n}/EY_{[nt]:n}$ is decreasing in t $(0 < t < 1)$, then $F^{-1}(t)/G^{-1}(t)$ is decreasing in t $(0 < t < 1)$. Equivalently, $F^{-1}(F(x))/G^{-1}(F(x)) = x/G^{-1}F(x)$ is decreasing (increasing) in $x(0 < x < F^{-1}(1))$. The "if" part follows. ||

Corollary 5.2. (Theorem 5.6 of Langberg, León, and Proschan). Let F be a continuous life distribution with finite mean. Assume that the support of F is an interval and that $F(0) = 0$. Then F is IFRA(DFRA) if and only if $EX_{i:n}/\sum_{k=1}^{i}(n-k+1)^{-1}$ is decreasing (increasing) in i $(i = 1, 2, \ldots, n)$ for infinitely many n.

Proof. With $G(x) = 1 - e^x$ in Theorem 5.1, $EY_{i:n} = \sum_{k=1}^{i} (n - k + 1)^{-1}$ for $i = 1, 2, \ldots, n$ (see Barlow and Proschan, 1975, p. 60). The conclusion follows. ||

Corollary 5.3. Let F be as in Corollary 5.2. Then F is a life distribution with decreasing density if and only if $(n/i)\ EX_{i:n}$ is increasing in i ($i = 1, 2, \ldots, n$) for infinitely many n.

Proof. With G the uniform distribution on (0, 1), $EY_{i:n} = i/(n + 1)$ for $i = 1, 2, \ldots, n$. ||

As in the case $F \underset{c}{<} (\underset{cv}{<}) G$, if $F \underset{*}{<} (\underset{a*}{<}) G$, then information about the order statistics, $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$ yields information about the order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. The next two theorems show two ways to make the above statement precise.

Theorem 5.4. Let $F \underset{*}{<} (\underset{a*}{<})\ G$ and the support of F be an interval. Then $Y_{i:n} \overset{st}{\geq} (\overset{st}{\leq}) a\ Y_{j:n}$ implies $X_{i:n} \overset{st}{\geq} (\overset{st}{\leq}) a\ X_{j:n}$, where $0 < a < 1$ and $1 \leq i < j < n$, $n \geq 2$.

Proof. Let $Y'_{1:n}, Y'_{2:n}, \ldots, Y'_{n:n}$ be as in the proof of Theorem 4.8. Then for $i < j$,

$$\begin{aligned} X_{i:n} &= F\ G^{-1}(Y'_{i:n}) \\ &\overset{st}{=} F\ G^{-1}(Y_{i:n}) \\ &\overset{st}{\geq} (\overset{st}{\leq}) F\ G^{-1}(a\ Y_{j:n}) \\ &\geq (\leq) a\ F\ G^{-1}(Y_{j:n}) \end{aligned}$$

[since $F\ G^{-1}$ is antistarshaped (starshaped)]

$$\begin{aligned} &\overset{st}{=} a\ F\ G^{-1}(Y'_{j:n}) \\ &= a\ X_{j:n}. \end{aligned}$$

Hence $X_{i:n} \overset{st}{\geq} (\overset{st}{\leq}) a\, X_{j:n}$. ||

<u>Theorem 5.5</u>. Let $F \underset{*}{\leq}({}_{a}\underset{*}{\leq})G$ and the support of F be an interval. Let $1 \leq i < j \leq n$ and $a > 0$. Then

$$P(Y_{j:n} \geq aY_{i:n}) \geq (\leq)\ P(X_{j:n} \geq a\, X_{i:n}).$$

<u>Proof</u>. Let $F \underset{*}{\leq} G$ and $Y'_{1:n}, Y'_{2:n}, \ldots, Y'_{n:n}$ be as in the proof of Theorem 4.8. Let $\phi(y)$ be the antistarshaped function $F^{-1}G$. Then for $1 \leq i \leq j \leq n$,

$$\frac{\phi(Y'_{i:n})}{Y'_{i:n}} \geq \frac{\phi(Y'_{j:n})}{Y'_{j:n}} .$$

Hence

$$\frac{Y'_{j:n}}{Y'_{i:n}} \geq \frac{X_{j:n}}{X_{i:n}} .$$

The conclusion follows as in the proof of Theorem 4.9.

If $F\ {}_{a}\underset{*}{\leq}\ G$, the proof is similar. ||

It is clear that a corollary to Theorem 5.5 can be fashioned along the lines of Corollary 4.10. This corollary can be used for nonparametric tests for $F \underset{*}{\leq}({}_{a}\underset{*}{\leq})\ G$; in particular, for tests for IFRA and DFRA.

We prove a converse of Theorem 5.4.

<u>Theorem 5.6</u>. Let the support of F be an interval. Suppose $EY_{i:n} \geq (\leq) a\ EY_{j:n}$ implies $EX_{i:n} \geq (\leq) a\ EX_{j:n}$ for all $a (0 < a < 1)$ and all $i, j, n\ (1 \leq i < j \leq n)$. Then $F \underset{*}{\leq}({}_{a}\underset{*}{\leq})G$.

<u>Proof</u>. Suppose $F \underset{*}{\leq} G$ is not true. Then there exist an $a\ (0 < a < 1)$ and an $x \geq 0$ such that $G^{-1}F(ax) \geq a\ G^{-1}F(x)$. Therefore there exists a $y > x$ such that $G^{-1}F(ax) > a\ G^{-1}F(y)$. Hence for n sufficiently large, $EY_{[nF(ax)]:n} > a\ EY_{[nF(y)]:n}$. By hypothesis, this implies that for n sufficiently large, $EX_{[nF(ax)]:n} > a\ EX_{[nF(y)]:n}$. Consequently $F^{-1}(F(ax)) \geq a\ F^{-1}(F(y))$; that is, $ax \geq ay$ - a contradiction. ||

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In this paper we obtain characterizations of life distributions F such that (a) $G^{-1}(F)$ is convex (concave) and alternatively (b) $G^{-1}(F)$ is starshaped (antishaped), where G is an absolutely continuous life distribution with positive, bounded, right continuous density. These characterizations generalize earlier results for the IFR(DFR) and IFRA(DFRA) classes, and should prove useful in unifying the study of the class distributions with decreasing density, comparing Weibull (Gamma) distributions with different shape parameters, etc.