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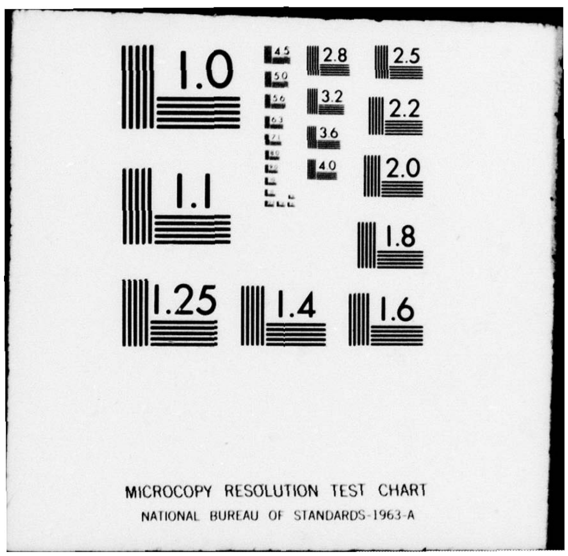
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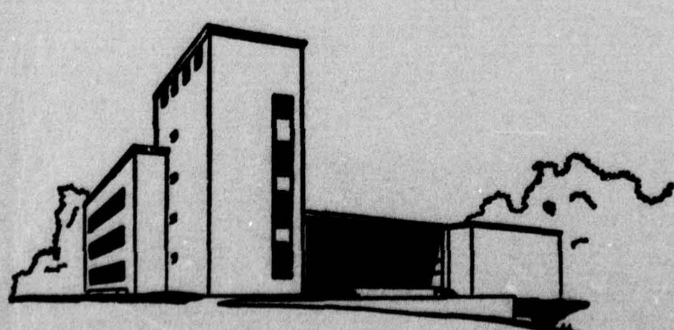

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6 SIMPLE MODELS IN STOCHASTIC

PRODUCTION PLANNING.

by

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SIMPLE MODELS IN STOCHASTIC PRODUCTION PLANNING

by

Suresh P. Sethi and Gerald L. Thompson

ABSTRACT

A simple stochastic production-inventory model with quadratic cost functions is analyzed in detail. The inventory process is assumed to be driven by a white noise process resulting into an Ito stochastic differential equation. Both finite and infinite horizon versions of the problem are treated by a methodology based on the theory of stochastic integrals and differentials. Particular attention is given to illustrate the methodology, which is quite general and capable of dealing with more complicated problems. The paper concludes with some remarks in connection with the relationship of the results of this paper to the results in the deterministic case.

Key Words

Production planning
Stochastic optimal control theory
Ito stochastic differential equation

SIMPLE MODELS IN STOCHASTIC PRODUCTION PLANNING

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1. INTRODUCTION

In an earlier paper [4], we considered a production-inventory model which determines production levels over time to minimize a discounted quadratic loss function. The loss function was defined in terms of the deviations of production and inventory levels from their rated or factory-optimal values. In this paper, we discuss simple stochastic extensions of our earlier paper. We will obtain closed-form solutions for both finite and infinite horizon versions of the stochastic production-inventory model.

2. THE MODEL

Consider a factory producing a homogeneous good and having an inventory warehouse. Define the following quantities:

$x(t)$ = inventory level at time t (state variable)

$u(t)$ = production rate at time t (control variable)

S = the constant demand rate at time t; $S > 0$

T = length of planning period

x_1 = factory-optimal inventory level

u_1 = factory-optimal production rate

x_o = initial inventory level

h = inventory holding cost coefficient

c = production cost coefficient

ρ = the constant discount rate; $\rho \geq 0$, B = salvage value per unit terminal inventory

$z(t)$ = the standard Wiener process; see [1]

σ = the constant diffusion coefficient; see [1].

We now state the conditions of the model. The first condition is the stock-flow equation stated as an Ito stochastic differential equation (see [1,2,3]):

$$dx = (u-S)dt + \sigma dz, \quad x(0) = x_o . \tag{1}$$

We note that process $dz(t)$ can be formally expressed as $\xi(t)dt$, where $\xi(t)$ is considered to be the white noise process [1]. It can be interpreted as "sales returns," "inventory spoilage," etc. which are random in nature. The second is the objective function:

$$\min_u E\{\int_0^T e^{-\rho t}[c(u-u_1)^2 + h(x-x_1)^2]dt + e^{-\rho T} Bx(T)\}. \tag{2}$$

Note that we do not restrict the production rate to be nonnegative as required in our earlier paper [4]. In other words, we permit disposal (i.e. $u < 0$). While this is done for mathematical expediency, we will state conditions under which a disposal is not required. Note further that the inventory level is allowed to be negative, i.e., we permit backlogging of demand.

3. THE HAMILTON-JACOBI EQUATION

The solution of the above model will be carried out via the development of the Hamilton-Jacobi equation satisfied by a certain 'value function.' To simplify the mathematics, we assume that

$$x_1 = u_1 = 0 \quad \text{and} \quad h = c = 1. \tag{3}$$

This assumption results in no loss of generality as the following analysis can be extended in a parallel manner for the case without (3). With (3), we restate the stochastic production planning problem:

$$\max E\left[\int_0^T -(u^2+x^2)\, e^{-\rho t}dt + Be^{-\rho T}x(T)\right], \tag{4}$$

subject to the Ito equation

$$dx = (u-S)dt + \sigma dz, \quad x(0) = x_o. \tag{5}$$

Let $V(t,x)$ denote the expected value of the objective function from time t to the horizon T with $x(t) = x$ and using the optimal policy from t to T. The function $V(t,s)$ is referred to as the value function and it can formally be defined as

$$V(t,x) = \max_u E\left[\int_t^T -(u^2+x^2)e^{-\rho t}dt + Be^{-\rho T}x(T)\right] \tag{6}$$

given $x(t) = x$; clearly

$$V(T,x) = Bxe^{-\rho T}. \tag{7}$$

By the principal of optimality, we can write

$$V(t,x) = \max_u E[-(u^2+x^2)e^{-\rho t}dt + V(t+dt,\ x+dx)]$$

$$= \max_u [-(u^2+x^2)e^{-\rho t}dt + E\ V(t+dt,\ x+dx)]. \tag{8}$$

Note that we have used dx and dt in place of δx and δt for convenience. We can expand $V(t+dt,\ x+dx)$ using Taylor's expansion:

$$V(t+dt,\ x+dx) = V(t,x) + V_t dt + V_x dx + \frac{1}{2} V_{xx}(dx)^2$$

$$+ \text{ higher order terms.} \tag{9}$$

Note that we have retained the term $(dx)^2$. This is an important departure from the deterministic situations. Substituting for dx from (5) and replacing

$$(dz)^2 \text{ by } dt \tag{10}$$

derived in the Ito theory [1,3], we can write (9) as

$$V(t+dt,\ x+dx) = V(t,x) + (V_t + V_x(u-S) + \frac{1}{2}\sigma^2 V_{xx})dt$$

$$+ \sigma V_x dz + \text{ higher order terms.} \tag{11}$$

Taking the expectation, we are left with

$$EV(t+dt,\ x+dx) = V(t,x) + (V_t + V_x(u-S) + \frac{1}{2}\sigma^2 V_{xx})dt$$

$$+ \text{ higher order terms,} \tag{12}$$

Since $E(\sigma V_x dz) = 0.$

Substituting (12) in (8) and cancelling $V(t,x)$ on both sides yields

$$0 = \max_u[-(u^2+x^2)\ e^{-\rho t}dt + (V_t + V_x(u-S) + \frac{1}{2}\sigma^2 V_{xx})dt$$

$$+ \text{ higher order terms}] \tag{13}$$

Dividing by dt and taking the limit as $dt \to 0$, we obtain the Hamilton-Jacobi-Bellman equation

$$0 = \max_u[-(u^2+x^2)e^{-\rho t} + V_t + V_x(u-S) + \frac{1}{2}\sigma^2 V_{xx}]. \tag{14}$$

It is now possible to maximize the expression inside the bracket with respect to u by taking its derivative with respect to u and setting it to zero. This procedure yields

$$u = \frac{V_x e^{\rho t}}{2} . \tag{15}$$

Substituting (15) into (14) yields the following Hamilton-Jacobi equation

$$0 = + \frac{V_x^2 e^{\rho t}}{4} - x^2 e^{-\rho t} + V_t - S V_x + \frac{1}{2} \sigma^2 V_{xx} . \tag{16}$$

This is a partial differential equation which must be satisfied by the value function $V(t,x)$ with the boundary condition (7).

It is important to remark that if production rate were restricted to be nonnegative, then (15) would be changed to

$$u = \max [0, \frac{V_x e^{\rho t}}{2}]. \tag{17}$$

Substituting (17) in (14) would give us a partial differential equation which must be numerically solved for this case. We shall not consider (17) further in this paper.

We now turn to solving (16) in the next section.

4. SOLUTION FOR THE UNDISCOUNTED FINITE HORIZON CASE

Although (16) with the boundary condition (7) can be solved, it is cumbersome. To simplify matters, we assume that

$$\rho = 0 \tag{18}$$

for the finite horizon case. For this case, we must solve nonlinear second-order partial differential equation

$$0 = \frac{V_x^2}{4} - x^2 + V_t - SV_x + \frac{1}{2}\sigma^2 V_{xx}, \quad V(T,x) = Bx. \tag{19}$$

To solve this equation, we let

$$V(t,x) = Q(t)x^2 + R(t)x + M(t). \tag{20}$$

Then

$$\begin{aligned} V_t &= \dot{Q}x^2 + \dot{R}x + \dot{M} \\ V_x &= 2Qx + R \\ V_{xx} &= 2Q \end{aligned} \tag{21}$$

where $\dot{Y}$ denotes dY/dt. Substituting (21) in (19) and collecting terms gives

$$x^2[\dot{Q} + Q^2 - 1] + x[\dot{R} + RQ - 2SQ] + \dot{M} + \frac{R^2}{4} - RS + \sigma^2 Q = 0. \tag{22}$$

Since (22) must hold for any value of x, we must have

$$\dot{Q} = 1 - Q^2, \qquad Q(T) = 0 \tag{23}$$

$$\dot{R} = 2SQ - RQ, \qquad R(T) = B \tag{24}$$

$$\dot{M} = RS - R^2/4 - \sigma^2 Q, \qquad M(T) = 0, \tag{25}$$

where the boundary conditions for the system of simultaneous differential equations (23), (24) and (25) are obtained by comparing (20) with the boundary condition $B(T,x) = Bx$ of (19).

To solve (23) we expand $\dot{Q}/(1-Q^2)$ by partial fractions to obtain

$$\frac{\dot{Q}}{2}\left[\frac{1}{1-Q} + \frac{1}{1+Q}\right] = 1,$$

which can be easily integrated. The answer is

$$Q = \frac{y-1}{y+1} \tag{26}$$

where,

$$y = e^{2(t-T)} . \tag{27}$$

Since S is assumed to be constant, we can reduce (24) to

$$\dot{R}^o + R^o Q = 0, \quad R^o(T) = B - 2S$$

by the change of variable defined by $R^o = R - 2S$. Clearly the solution is given by

$$\log R^o \Big|_t^T = - \int_t^T Q(\tau) d\tau,$$

which can be further simplified to obtain

$$R = 2S + \frac{2(B-2S)\sqrt{y}}{y+1} . \tag{28}$$

Having obtained solution for R and Q, we can easily express (25) as

$$M(t) = - \int_t^T [RS - R^2/4 - \sigma^2 Q] dt. \tag{29}$$

The optimal control is defined in (15), which under the assumption (18) of $\rho = 0$ and the use of (26) and (28) yields

$$u^* = V_x/2 = Qx + R/2 = S + \frac{(y-1)x + (B-2S)\sqrt{y}}{y + 1} . \tag{30}$$

Remarks

1) The optimal production rate in (30) equals the demand rate plus a correction term which depends on the level of inventory and the distance from horizon. Since $(y-1) < 0$ for $t < T$, it is clear that for the lower values of x,

the optimal production rate is likely to be positive. However, if x is very high, the correction term will become smaller than $-S$ and the optimal control will be negative. In other words, if inventory level is too high, the factory can save money by disposing a part of the inventory resulting in lower holding costs.

2) If the demand rate S were time-dependent, it would have changed the solution of (24). Having computed this new solution in place of (28), we can once again obtain the optimal control as $u^* = Qx + R/2$.

3) Note that when $T \to \infty$, $y \to 0$ and

$$u^* \to S - x. \tag{31}$$

But the undiscounted objective function value (4) in this case becomes $(-\infty)$. Clearly, any other policy will render the objective function value to be $-\infty$. In a sense, the optimal control problem becomes ill-defined. One way to get out of this difficulty is to impose a nonzero discount rate. This will be carried out in the next section.

5. SOLUTION OF THE DISCOUNTED INFINITE HORIZON CASE

When $\rho > 0$, it is convenient to express the value function $V(t,x)$ in time-t dollars, i.e.,

$$W(t,x) = V(t,x)e^{\rho t}. \tag{32}$$

With this definition of $W(t,x)$, we can easily convert (14), (15) and (16), respectively, as follows:

$$0 = \max_u[-(u^2+x^2) - \rho W + W_t + W_x(u-S) + \frac{1}{2}\sigma^2 W_{xx}], \tag{33}$$

$$u = W_x/2$$

$$0 = W_x^2/4 - x^2 - \rho W + W_t - SW_x + \frac{1}{2}\sigma^2 W_{xx}. \tag{34}$$

To solve (34), we let

$$W(t,x) = Q(t)x^2 + R(t)x + M(t) \tag{35}$$

and with a procedure similar to that used in deriving (23)-(25), we can show that that $Q(t)$, $R(t)$, and $M(t)$ satisfy the following system:

$$\left.\begin{aligned} \dot{Q} &= 1 + \rho Q - Q^2 , & Q(T) &= 0 \\ \dot{R} &= \rho R + 2SQ - RQ , & R(T) &= B \\ \dot{M} &= \rho M + RS - R^2/4 - \sigma^2 Q, & M(T) &= 0 . \end{aligned}\right\} \tag{36}$$

It can be shown that the solution for $Q(t)$ is given by

$$Q(t) = \frac{m_2 e^{(m_1-m_2)T} - m_2 e^{(m_1-m_2)t}}{e^{(m_1-m_2)T} - \frac{m_2}{m_1} e^{(m_1-m_2)t}} , \tag{37}$$

where,

$$m_1 = \frac{\rho - \sqrt{\rho^2+4}}{2} \quad \text{and} \quad m_2 = \frac{\rho + \sqrt{\rho^2+4}}{2} \tag{38}$$

are the roots of $1 + \rho Q - Q^2 = 0$. It is clear that as $T \to \infty$,

$$Q \to m_1 , \tag{39}$$

since $(m_1-m_2) < 0$. This is what is to be expected since m_1 is the stable root.

We note some simple identities that m_1 and m_2 satisfy which will be used later. First the equation

$$m^2 - \rho m = 1 \tag{40}$$

holds for m being either m_1 or m_2. Writing (40) for m_2 and dividing by m_2^2 gives

$$1 - \frac{\rho}{m_2} = \frac{1}{m_2^2} > 0. \tag{41}$$

Writing (40) for m_1 and dividing both sides by $m_1 - \rho$ gives

$$m_1 = \frac{1}{m_1 - \rho} \ . \tag{42}$$

Then, since $m_2 > 0$, and the product of m_1 and m_2 is -1 we have

$$m_1 = -\frac{1}{m_2} < 0. \tag{43}$$

The last identity needed is

$$m_1^2 = 1 + \rho m_1 = 1 - \frac{\rho}{m_2} \tag{44}$$

which can be obtained by writing (40) for m_1, solving for m_1^2 and using (43).

We now make an important observation. When $T = \infty$, it is obvious that

$$W(t,x) = W(x) \tag{45}$$

is explicitly independent of t. This is because W is expressed in time-t dollars. This means $W_t = 0$, which reduces (34) to

$$0 = W_x^2/4 - x^2 - SW_x + \frac{1}{2}\sigma^2 W_{xx}. \tag{46}$$

The solution of (41) takes the form

$$W(x) = Qx^2 + Rx + M \tag{47}$$

where Q, R and M are constants to be determined. It is not difficult to conclude from (36) that Q, R, and M satisfy

$$1 + \rho Q - Q^2 = 0 \tag{48}$$

$$\rho R + 2SQ - RQ = 0 \tag{49}$$

$$\rho M + RS - R^2/4 - \sigma^2 Q = 0. \tag{50}$$

Of the two roots of Q for the quadratic equation (48), we have already shown, by a limiting argument that

$$Q = m_1. \tag{51}$$

It is easy to show using (42) that

$$R = 2Sm_1/(m_1-\rho) = 2Sm_1^2 = 2S(\rho m_1+1) \tag{52}$$

and

$$M = [R^2/4 - RS + \sigma^2 Q]/\rho = [m_1^2 S^2(m_1^2-2) + \sigma^2 m_1]/\rho \tag{53}$$

From (33), we can obtain the optimal production rate

$$u^* = m_1 x + \frac{m_1 S}{m_1-\rho} = m_1^2 S + m_1 x = (1-\frac{\rho}{m_2})S - \frac{1}{m_2}x \tag{54}$$

Here the optimal policy is to produce a positive fraction $(1 - \frac{\rho}{m_2})$ times the current demand plus a correction term $-(1/m_2)x$ involving the inventory level x. Thus it is optimal to produce a high level when inventory is low, and to produce at a low level when inventory is high.

<u>Condition for Disposal</u>

From (54), it is possible to derive necessary and sufficient conditions for disposal or negative production to be optimal. We have

$$u^* < 0 \Leftrightarrow m_1^2 S + m_1 x < 0$$
$$\Leftrightarrow \frac{x}{S} > -m_1 = \frac{1}{m_2} > 0 \tag{55}$$

We can also use (44) to write the condition (55) for a disposal to be optimal as:

$$u^* < 0 \Leftrightarrow \frac{x}{S} > -(\rho + \frac{1}{m_1}) . \tag{56}$$

Condition (52) may be compared with the one derived for the deterministic case in [4]. Obviously, a disposal would be optimal if the inventory level is too high compared to the demand rate. This makes intuitive sense.

Relationship with the Deterministic Turnpike

From our earlier paper [4] we know that $\bar{x} = -\rho S$ is the turnpike level for the inventory level and $\bar{u} = S$ is the turnpike production rate for the special problem of this section with $\sigma = 0$. We now show that even with $\sigma \neq 0$, the optimal production rate is S when the observed inventory level is $x = -\rho S$. This is done by substituting $x = -\rho S$ in (54) and (40) to obtain

$$u^* = (m_1^2 - \rho m_1)S = S .$$

Summarizing we have

$$u^* = S \text{ when } x = -\rho S . \tag{57}$$

This means that the trajectory for the optimal inventory level would be a diffusion process about $x = -\rho S$ (see Figure 1). In this sense, $x = -\rho S$ can be considered to be the turnpike inventory level for the stochastic production planning problem dealt with in this section.

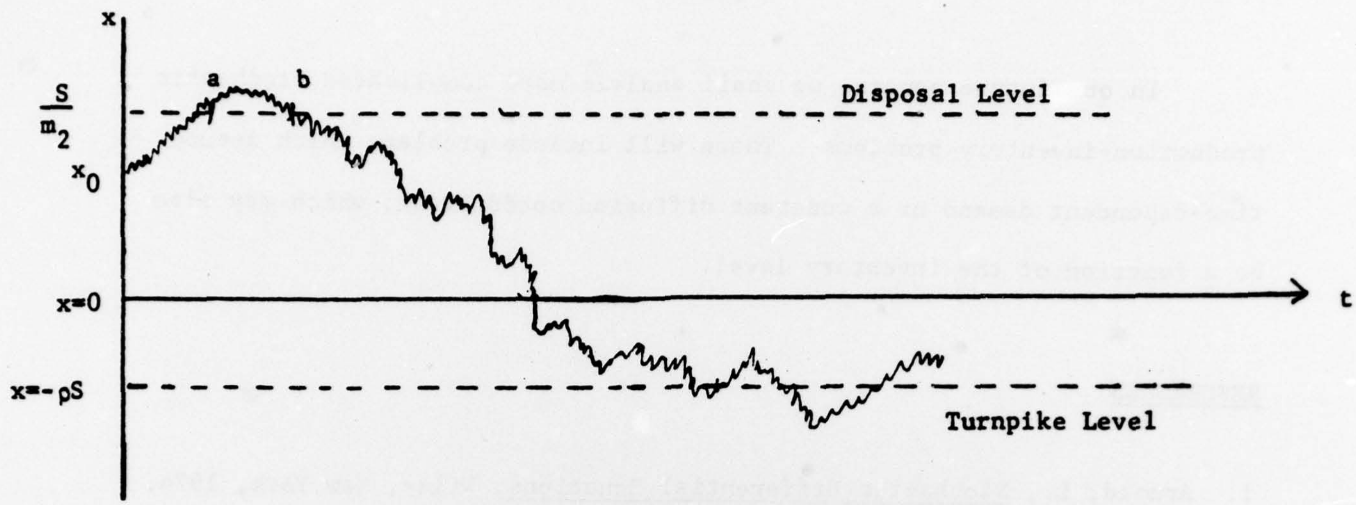

Fig. 1. A Sample Path for Optimal Inventory Level

The sample path in Figure 1 shows a path with high initial inventory, which increases (due to sales returns) up to the disposal level. Between points a and b, production is negative, which means we dispose of inventory. However, the inventory continues to increase until it drops back at point (b). Disposal activities help bring back the inventory to lower levels. Further on the path the inventory declines randomly to the turnpike level $-\rho S$ and remains in that vicinity with high probability.

The fact that the turnpike level is negative is because $x_1 = 0$ was assumed in (3). When x_1 is sufficiently positive the turnpike level, $x_1 - \rho S$, will become positive. Details are omitted here. The deterministic case is treated in [4].

6. EXTENSIONS AND CONCLUDING REMARKS

We have analyzed a simple stochastic production planning in detail. We have deliberately chosen a simple problem to introduce the stochastic control theory methodology, which may not be generally familiar. The methodology is based on the theory of stochastic differential equations developed by Ito.

In our future papers, we shall analyze more complicated stochastic production-inventory problems. These will include problems which assume time-dependent demand or a constant diffusion coefficient, which may also be a function of the inventory level.

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