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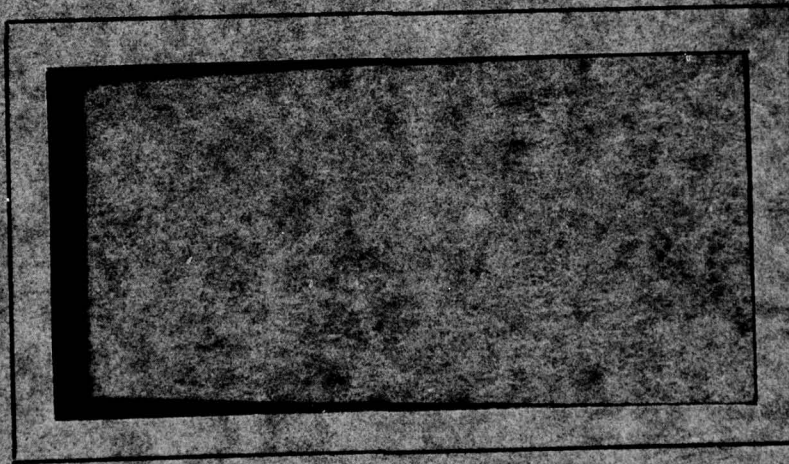DEC 78　H L HASTINGS
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LEVEL II

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ELECTRON-ELECTRON COLLISIONS:

A SIMPLE MODEL FOR

PARTIALLY IONIZED GASES

THESIS

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THESIS

Master's thesis,

Presented to the Faculty of the School of Engineering

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by

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Preface

The effects of electron-electron collisions are important in plasmas of appreciable fractional ionization. These effects can be adequately described by the Fokker-Planck treatment. Unfortunately, the resulting expression greatly complicates the analysis and, for this reason, is often ignored. In this investigation, the Fokker-Planck expression is replaced by a simple relaxation term. Although this approach introduces considerable error, it does serve to indicate several important features which must be included to represent the physical situation adequately. Hopefully, this information will be useful in any further attempts to simplify the Fokker-Planck treatment.

I am grateful to Maj. P. E. Nielsen and Capt. A. M. Hunter for suggesting this problem. Their suggestions and insight were invaluable in overcoming several serious problems encountered during the investigation.

Finally, I would like to thank my family and especially my wife, Debbie, for their understanding and cooperation during this work. Without their support and encouragement, the timely completion of this thesis would have been impossible.

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Abstract

The electron-electron collision term of the steady-state Boltzmann equation is replaced by a simple relaxation term that effectively linearizes the equation. Additional assumptions are made to simplify the equation further. Ionization is ignored and a two-level atom is assumed. The effects of a DC electric field are included. Using the relaxation term to account for electron-electron collisions, an approximate analytic solution is derived. Temporal and steady-state characteristics of the relaxation term are compared to those of the standard Fokker-Planck term. The relaxation term is judged invalid for energies greater than the excitation threshold.

I Introduction

There exists a large class of problems in plasma
physics where important physical quantities are sensitive
to the detailed form of the electron distribution function.
To predict these quantities accurately, the time-independent
Boltzmann equation must be solved.

This equation is very difficult to solve in
general. Some simplifying assumptions must be made, even
to solve the equation numerically. In addition, terms
are often neglected on the basis of complexity. The
effects of electron-electron collisions are particularly
difficult to handle because of the non-linear terms intro-
duced.

For high electron number densities, electron-electron
collisions can be very important. These interactions tend
to drive the distribution toward a Maxwellian and can
appreciably alter the high energy part of the distribution.
Therefore, there exists a need for a simple, but accurate,
term to describe the effects of electron-electron collisions.
In an effort to fulfill this need, a simple Krook-type
relaxation term is developed and its characteristics are
investigated.

The theory leading to the Boltzmann equation and
the Fokker-Planck collision term are reviewed in Section
II. With this background, a simple relaxation term is
developed and an approximate solution to the Boltzmann
equation is derived in Section III. The results of the

relaxation term and the Fokker-Planck term are analyzed
and compared in Section IV. Finally, the conclusions
reached and recommendations for future work are presented
in Section V.

II Theory

In this section, the problem is carefully defined
and the governing equation is derived. The assumptions
necessary to arrive at this equation are stated as they
occur in the derivation. The macroscopic effects and
non-Coulomb interactions are incorporated using the
classical Boltzmann formalism. The Coulomb interactions
are included using the Fokker-Planck approximation.
Finally, the assumptions are justified using order of
magnitude estimates where possible.

The Boltzmann Equation

The plasma considered here is assumed to consist
of electrons, singly-charged ions, and neutral particles.
Internal excitation levels are ignored for the ions
and only one level (above the ground state) is allowed
for the neutrals. Ionization is ignored for both ions
and neutrals. Electron loss mechanisms, such as attach-
ment and recombinations, are also ignored. Therefore,
the number densities of the various particles can be
treated as parameters in what follows.

It is assumed that the equations of kinetic
theory are applicable. This is acceptable provided (Ref 13)

$$\frac{(m_d K T_\alpha)^{1/2}}{n_d^{1/3}} \gg \hbar$$

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where

m_α = mass of species α

T_α = kinetic temperature of species α

n_α = number density of species α

K = Boltzmann's constant

$\hbar$ = Planck's constant/2π

In accordance with kinetic theory, a velocity distribution function $F_\alpha(\underline{r},\underline{v},t)$ is defined for each species such that the average number density of particles of type α with velocities between $\underline{v}$ and $\underline{v} + d\underline{v}$, located in the spatial volume element $d\underline{r}$ centered at $\underline{r}$, at time t is given by $F_\alpha(\underline{r},\underline{v},t)d\underline{r}d\underline{v}$; the average being carried out over a macroscopically infintesimal time interval. Ground state neutrals and excited neutrals are to be regarded as separate species. The distribution functions F_α are determined by the Boltzmann equation. This equation may be written symbolically as

$$\mathcal{D}\left\{ F_\alpha(\underline{r},\underline{v},t) \right\} = \left. \frac{\partial F_\alpha(\underline{r},\underline{v},t)}{\partial t} \right|_{coll} \qquad (1)$$

where

$$\mathcal{D} \equiv \frac{\partial}{\partial t} + \underline{v}\cdot\nabla_r + \frac{F_\alpha}{m_\alpha}\cdot\nabla_v \qquad (2)$$

and

$\underline{F}_A \equiv$ external macroscopic force on species α

$\nabla_v \equiv$ gradient operator in velocity space

$\nabla_r \equiv$ gradient operator in configuration space

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$\dfrac{\partial F_{\alpha}(\underline{r},\underline{v},t)}{\partial t}\Big|_{coll}$ represents the change in F_{α} resulting

from the interaction between particles. Macroscopic
effects, such as external forces, are usually included
in $\mathcal{F}$. Microscopic effects, such as excitation by
electron impact, are usually included in $\dfrac{\partial F_{\alpha}(\underline{r},\underline{v},t)}{\partial t}\Big|_{coll}$.
In general, the change in F_{α} resulting from interactions
with other species depends on the distribution functions
of these species. Therefore, this term couples the
equation for one species to all the other equations.

Fortunately, for many problems a knowledge of the
electron distribution is sufficient to describe the
important features of the system adequately. To simplify
the problem, this assumption is made here. Furthermore,
the distributions of the other species are assumed Maxwellian.

With these assumptions, a complete statistical
solution requires that the electron distribution function
$F(\underline{r},\underline{v},t)$ be determined using Eq (1). Note that $F(\underline{r},\underline{v},t)$,
with no subscript, will be used exclusively for the electron
distribution function.

It is assumed that $F(\underline{r},\underline{v},t)$ is spatially homogeneous,
and the only external forces result from a constant electric
field $\underline{E}$. Since F is now independent of $\underline{r}$, it will be
denoted by $F(\underline{v},t)$. The second term of Eq (2) vanishes
in this case and Eq (1) becomes

$$\frac{\partial F(\underline{v},t)}{\partial t} - \frac{e\underline{E}}{m} \cdot \nabla_{v} F(\underline{v},t) = \frac{\partial F(\underline{v},t)}{\partial t}\Big|_{coll} \qquad (3)$$

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where

      m = electron mass

    -e = electron charge

    n_e = total number density of electrons

The normalization is such that

$$n_e = \int\limits_{all\ \underline{v}} F(\underline{v}, t)\, d\underline{v} \tag{4}$$

In general, n_e should be expressed as a function of the
time, but the neglect of electron sources and sinks requires
that n_e be constant.

    The right side of Eq (3) involves the interactions
of the electrons with themselves and the other plasma
species. It is assumed that these terms impart little net
motion to the electrons. In fact, these interactions should
tend to produce an isotropic velocity distribution. The
second term in Eq (3) accounts for the external field effects.
This term does produce anisotropic effects. If the field is
not too large, the electrons will be accelerated very little
between the randomizing collisions. The principal effect of
the field is a heating of the electrons; the anisotropic
drift in the direction of the field is a relatively minor
effect.

    The above arguments motivate an expansion of $F(\underline{v}, t)$
in spherical harmonics. For small anisotropic effects a
first order expansion is adequate and $F(\underline{v}, t)$ is given by

$$F(\underline{v}, t) = F_o(v, t) + F_1(v, t) \cos \theta \tag{5}$$

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where θ is the angle between $\underline{v}$ and the applied field $\underline{E}$. If Eq (5) is substituted into the left side of Eq (3), the result is

$$\frac{\partial F_0(v,t)}{\partial t} + \cos\theta \frac{\partial F_1(v,t)}{\partial t} - \frac{eE}{m}\cos\theta \frac{\partial F_0(v,t)}{\partial t}$$

$$-\frac{eE}{m}\cos^2\theta \frac{\partial F_1(v,t)}{\partial t} - \frac{eE}{m}\frac{\sin^2\theta}{v} F_1(v,t) = \left.\frac{\partial F(v,t)}{\partial t}\right|_{coll} \tag{6}$$

Equation (6) is now integrated over all solid angles $d\Omega$, where

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

and ϕ is the azimuthal angle about the polar axis. The resulting equation is

$$\frac{\partial F_0(v,t)}{\partial t} - \frac{eE}{3m}\left[\frac{1}{v^2}\frac{\partial}{\partial v} v^2 F_1(v,t)\right] = \frac{1}{4\pi}\int \left.\frac{\partial F(v,t)}{\partial t}\right|_{coll} d\Omega \tag{7}$$

If Eq (6) is multiplied by $\cos\theta$ and again integrated over all solid angles, Eq (8) is obtained:

$$\frac{\partial F_1(v,t)}{\partial t} - \frac{eE}{m}\frac{\partial F_0(v,t)}{\partial v} = \frac{3}{4\pi}\int \cos\theta \left.\frac{\partial F(v,t)}{\partial t}\right|_{coll} d\Omega \tag{8}$$

The collisional integrals in Eqs (7) and (8) remain to be evaluated. The analysis can be simplified by considering each process separately. Since electron sources and

sinks have been ignored, the collision term takes the form:

$$\frac{\partial F(\underline{v},t)}{\partial t}\bigg|_{coll} = \frac{\partial F(\underline{v},t)}{\partial t}\bigg|_{elastic}^{en} + \frac{\partial F(\underline{v},t)}{\partial t}\bigg|_{inelastic}^{en} + \frac{\partial F(\underline{v},t)}{\partial t}\bigg|_{elastic}^{en*}$$

$$+ \frac{\partial F(\underline{v},t)}{\partial t}\bigg|_{inelastic}^{en*} + \frac{\partial F(\underline{v},t)}{\partial t}\bigg|_{ei} + \frac{\partial F(\underline{v},t)}{\partial t}\bigg|_{ee} \qquad (9)$$

The first term on the right side of Eq (9) represents the effect of elastic collisions with neutral particles in the ground state. If the neutral particles are assumed infinitely massive, these collisions will have no effect on an isotropic distribution. Therefore,

$$\frac{\partial F_o(v,t)}{\partial t}\bigg|_{elastic}^{en} = 0$$

For an anisotropic distribution, these collisions will tend to reduce the degree of asymmetry. Using the standard form of the Boltzmann collision integral, it can be shown that (Ref 23:82)

$$\frac{\partial F_1(v,t)}{\partial t}\bigg|_{elastic}^{en} = -\nu_m F_1(v,t)$$

where ν_m is known as the momentum transfer collision frequency for electron-neutral collisions. The collision frequency may be expressed in terms of the momentum transfer cross section $\sigma_m(v)$:

$$\nu_m = n_o v \, \sigma_m(v)$$

where

n_o = total number density of neutral particles

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In general, ν_m is a function of the electron velocity. However, it is assumed here that

$$\sigma_m(v) = \frac{R_m}{v}$$

where R_m is a constant momentum transfer rate. Therefore

$$\nu_m = n_o R_m$$

and (from Appendix D), $R_m = 5.928 \times 10^{-8}$ cm^3-sec^{-1}. With this choice for R_m, the collision frequency used here is approximately the same as that for electrons and ground state Argon atoms.

The second term in Eq (9) represents the effect of inelastic collisions (excitation) with neutral particles in the ground state. These interactions convert high-energy electrons to low-energy electrons. Inelastic collisions have a large effect on the bulk of the distribution, but a minor effect on anisotropies (Ref 12). Therefore

$$\left.\frac{\partial F_1(v,t)}{\partial t}\right|_{\substack{en \\ inelastic}} = 0$$

The effect on $F_o(v,t)$ can also be calculated using the Boltzmann collision integral formalism, but the result is most conveniently expressed in terms of energy. For now, this effect will be denoted by

$$\left.\frac{\partial F_o(v,t)}{\partial t}\right|_{\substack{en \\ inelastic}}$$

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The third and fourth terms in Eq (9) represent the above effects for excited state neutrals as opposed to ground state neutrals. These interactions, known as superelastic collisions, convert low-energy electrons to high-energy electrons. These terms are neglected by assuming negligible excited state populations. This assumption is discussed further at the end of this section.

The fifth term represents the effects of the Coulomb interactions between ions and electrons, and the sixth between electrons and electrons. Since the Boltzmann collision integral is based on short-range, binary collisions, this formulation is not well suited to the long-range Coulomb interactions. The Fokker-Planck equation, presented in the next section, is appropriate in this case. The Fokker-Planck equation will be used only for the electron-electron collision term. The results for the electron-ion collision term are relatively simple and will only be quoted. If the ions are assumed infinitely massive, the argument presented for elastic collisions with neutrals yields

$$\frac{\partial F_o(v,t)}{\partial t}\bigg|_{ei} = 0$$

It can be shown (Ref 23:82)

$$\frac{\partial F_1(v,t)}{\partial t}\bigg|_{ei} = -\nu_{ei}\, F_1(v,t)$$

where ν_{ei} is an effective electron-ion collision frequency. It is shown at the end of this section that $\nu_{ei} \ll \nu_m$ for the fractional ionizations considered here. Therefore,

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electron-ion collisions will be ignored entirely. In addition $\frac{\partial F_1(v,t)}{\partial t}\big|_{ee}$ will be ignored. Electron-electron collisions will tend to reduce the anisotropy of the distribution. However, for the fractional ionizations considered here, the elastic collisions with neutrals are the dominant effect (Ref 21:284). It remains to calculate $\frac{\partial F_0(v,t)}{\partial t}\big|_{ee}$. The next section is devoted exclusively to determining this term.

The collision term may now be expressed as

$$\frac{\partial F(v,t)}{\partial t}\bigg|_{coll} = \frac{\partial F_0(v,t)}{\partial t}\bigg|_{coll} + \cos\theta \frac{\partial F_1(v,t)}{\partial t}\bigg|_{coll} \tag{10}$$

where

$$\frac{\partial F_1(v,t)}{\partial t}\bigg|_{coll} = -\nu_m F_1(v,t) \tag{11}$$

$$\frac{\partial F_0(v,t)}{\partial t}\bigg|_{coll} = \frac{\partial F_0(v,t)}{\partial t}\bigg|_{\substack{en \\ inelastic}} + \frac{\partial F_0(v,t)}{\partial t}\bigg|_{ee} \tag{12}$$

Using Eqs (10) through (12) in Eqs (7) and (8) results in a set of coupled equations independent of θ and ϕ. The steady-state results are

$$-\frac{eE}{3m}\left[\frac{1}{v^2}\frac{d}{dv}v^2 F_1(v)\right] = \frac{\partial F_0(v)}{\partial t}\bigg|_{\substack{en \\ inelastic}} + \frac{\partial F_0(v)}{\partial t}\bigg|_{ee} \tag{13}$$

$$-\frac{eE}{m}\frac{dF_0(v)}{dv} = -2\nu_m F_1(v) \tag{14}$$

Equation (14) is now substituted into Eq (13). The resulting time-independent equation involves only $F_0(v)$,

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and if ν_m is independent of v the desired expression is

$$-\frac{e^2 E^2}{6m^2 \nu_m} \left[\frac{1}{v^2}\frac{d}{dv}\left(v^2 \frac{dF_o(v)}{dv}\right)\right] = \frac{\partial F_o(v)}{\partial t}\Big|_{\substack{en\\inelastic}} + \frac{\partial F_o(v)}{\partial t}\Big|_{ee} \quad (15)$$

Equation (15) may be transformed from velocity space to energy space in a straightforward manner. Let

$$\varepsilon = \frac{1}{2}mv^2$$

and

$$n_e = \int 4\pi v^2 F_o(v)dv = \int n(\varepsilon)d\varepsilon$$

The number density of electrons with energies between ε and $\varepsilon + d\varepsilon$ is $n(\varepsilon)d\varepsilon$. Equation (15), after the appropriate transformations, takes the following form:

$$\frac{2}{3}\dot{\varepsilon}\frac{d}{d\varepsilon}\left[\frac{n(\varepsilon)}{2} - \varepsilon\frac{dn(\varepsilon)}{d\varepsilon}\right] = \frac{\partial n(\varepsilon)}{\partial t}\Big|_{\substack{en\\inelastic}} + \frac{\partial n(\varepsilon)}{\partial t}\Big|_{ee} \quad (16)$$

where

$$\dot{\varepsilon} = \frac{e^2 E^2}{2m\nu_m} \quad (17)$$

The second term in Eq (16) may be deduced from physical arguments. A more rigorous treatment is given by Nielsen (Ref 18). Let x denote the excitation energy of the neutral particles, and $R(\varepsilon)$ the rate at which excitations occur for electrons with energy ε. The excitation rate is related to the total excitation cross section σ_x by

$$R(\varepsilon) = \left(\frac{2\varepsilon}{m}\right)^{1/2}\sigma_x(\varepsilon) \quad (18)$$

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Inelastic collisions can affect the electron distribution
function in two ways. Firstly, electrons with energy ε
can produce excitations at the rate $R(\varepsilon)$ and lose energy
x in the process. This represents a net loss of electrons
with energy ε. Clearly, this loss is proportional to the
number density of electrons with energy ε and the number
density of neutral particles in the ground state. Secondly,
electrons with energy $\varepsilon + x$ can produce excitations at the
rate $R(\varepsilon + x)$ and lose energy x in the process. This
represents a net gain of electrons with energy ε. This
gain is proportional to the number density of electrons with
energy $\varepsilon + x$ and the number density of unexcited neutral
particles. Therefore, the second term in Eq (16) may be
expressed as

$$\frac{\partial n(\varepsilon)}{\partial t}\bigg|_{\substack{en \\ inelastic}} = n_o R(\varepsilon + x) n(\varepsilon + x) - n_o R(\varepsilon) n(\varepsilon) \tag{19}$$

As previously mentioned, superelastic collisions are ignored
in Eq (19). The form of $R(\varepsilon)$ depends on the form of $\sigma_x(\varepsilon)$.
The determination of $\sigma_x(\varepsilon)$ is of great practical interest,
but will not be considered here. It is assumed here that

$$\sigma_x(\varepsilon) = R(\varepsilon) \left(\frac{m}{2\varepsilon}\right)^{1/2} \tag{20}$$

where

$$R(\varepsilon) = \begin{cases} 0 & \varepsilon < x \\ \\ R_o & \varepsilon \geq x \end{cases} \tag{21}$$

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From Appendix D

$$R_o = 1.703 \times 10^{-8} \text{ cm}^3 - \text{sec}^{-1} \qquad (102)$$

With this choice for R_o, the excitation rate used here is approximately equal to that for excitation from the ground state of Argon. Equations (16) and (19) may be combined to yield the following form of the steady-state Boltzmann equation:

$$\frac{2}{3} \dot{\varepsilon} \frac{d}{d\varepsilon} \left[\frac{n}{2} - \varepsilon \frac{dn}{d\varepsilon} \right] = n_o R(\varepsilon + x) n(\varepsilon + x) - n_o R(\varepsilon) n(\varepsilon) + \left. \frac{\partial n}{\partial t} \right|_{ee} \qquad (22)$$

where $\dot{\varepsilon}$ is given by Eq (17) and $R(\varepsilon)$ is given by Eq (21). The explicit form of $\left. \frac{\partial n}{\partial t} \right|_{ee}$ is deduced below.

The Fokker-Planck Electron-Electron Collision Term

To derive the expression for the electron-electron collision term, it is necessary to digress temporarily and consider the electron velocity distribution $F(\underline{v}, t)$. From Appendix A, the effect of electron-electron collisions may be described using the Fokker-Planck equation in the following form:

$$\left. \frac{\partial F(\underline{v}, t)}{\partial t} \right|_{ee} = - \sum_i \frac{\partial}{\partial v_i} F(\underline{v}, t) \langle \Delta v_i \rangle$$

$$+ \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial v_i \partial v_j} F(\underline{v}, t) \langle \Delta v_i \Delta v_j \rangle \qquad (58)$$

where

$$\langle \Delta v_i \rangle = \frac{1}{\Delta t} \int \Delta v_i \, \Psi(\underline{v}, \Delta \underline{v}) \, d(\Delta \underline{v}) \qquad (56)$$

$$\langle \Delta v_i \Delta v_j \rangle = \frac{1}{\Delta t} \int \Delta v_i \Delta v_j \, \Psi(\underline{v}, \Delta \underline{v}) \, d(\Delta \underline{v}) \qquad (57)$$

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and $\Psi(\underline{v}-\Delta\underline{v}, \Delta\underline{v})d(\Delta\underline{v})$ is the probability that an electron changes its velocity from $\underline{v}-\Delta\underline{v}$ to $\underline{v}$ in a time Δt as a result of electron-electron collisions. For an infintesimal time interval Δt, $\frac{\Psi(\underline{v},\Delta\underline{v})d(\Delta\underline{v})}{\Delta t}$ may be expressed in terms of the Coulomb differential scattering cross section σ as (Ref 22)

$$\frac{\Psi(\underline{v},\Delta\underline{v})d(\Delta\underline{v})}{\Delta t} = F(\underline{v}',t)\sigma(\mu,\Omega)\mu\,d\underline{v}'\,d\Omega \qquad (23)$$

where

$$\Delta\underline{v} = \underline{v} - \underline{v}'$$

$$\mu = |\underline{v} - \underline{v}'|$$

$$\sigma(\mu,\Omega) = \left(\frac{e^4}{2m^2\mu^4}\right)\left[\frac{1}{\sin(\theta/2)}\right]^4$$

and θ is the scattering angle in the center-of-mass coordinate system. Using Eq (23), Eqs (56) and (57) can be rewritten as

$$\langle\Delta v_i\rangle = \iint \Delta v_i\, F(\underline{v}',t)\,\sigma(\mu,\Omega)\,\mu\,d\underline{v}'\,d\Omega \qquad (24)$$

$$\langle\Delta v_i \Delta v_j\rangle = \iint \Delta v_i\,\Delta v_j\, F(\underline{v}',t)\sigma(\mu,\Omega)\,\mu\,d\underline{v}'\,d\Omega \qquad (25)$$

The above integrals are easier to calculate in a local coordinate system oriented with respect to the relative velocity $\underline{w}$ between the scattering electrons. It is worth noting that $\underline{w}$ is not the velocity change $\Delta\underline{v}$ of a particular electron upon scattering. However, for elastic scattering between particles of equal mass $\Delta\underline{v} = \frac{1}{2}\underline{w}$.

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A fundamental problem arises when performing the above
integrations; the integrals diverge logarithmically at
small scattering angles. This is due to the long-range
nature of the Coulomb force. It can be shown that the
Debye length is the maximum distance over which mutual
electron interactions are effective (Ref 5). This provides
a natural limit in performing the integrations; scattering
between electrons with impact parameters greater than the
Debye length is ignored.

If the integrals are evaluated in the local coordi-
nate system and transformed back into the fixed coordinate
system, the result is (Ref 22)

$$\langle \Delta v_i \rangle = \Gamma \frac{\partial h}{\partial v_i} \tag{26}$$

$$\langle \Delta v_i \Delta v_j \rangle = \Gamma \frac{\partial^2 g}{\partial v_i \partial v_j} \tag{27}$$

where

$$h = 2 \int F(\underline{v}', t) u^{-1} d\underline{v}' \tag{28}$$

$$g = \int F(\underline{v}', t) u \, d\underline{v}' \tag{29}$$

$$\Gamma = \frac{4\pi e^4}{m^2} \ln \Lambda \tag{30}$$

and

$$\Lambda = \frac{3}{2e^3} \left(\frac{K^3 T^3}{\pi n_e} \right)^{1/2} \tag{31}$$

In Eq (31), KT is proportional to the average electron
energy; for non-Maxwellian distributions, T can be

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regarded as an effective electron temperature. The quantity Λ is actually the ratio of the Debye length to the classical distance of closest approach for two electrons with relative velocity μ. Since Λ appears as the argument of a logarithm, it is customary to neglect this velocity dependence and use an average relative velocity $\mu = \sqrt{3KT/m}$.

In principle, Eqs (26) through (29) can be substituted into Eq (58) to yield an expression for $\left.\frac{\partial F(v,t)}{\partial t}\right|_{ee}$. The above equations, despite their simple form, present considerable computational difficulties for an arbitrary distribution $F(\underline{v},t)$. However, for the isotropic distributions considered here, these results may be further simplified. The result for an isotropic distribution $F_0(v,t)$ is (Ref 19)

$$\left.\frac{\partial F_0(v,t)}{\partial t}\right|_{ee} = \Gamma \left\{ \frac{1}{3} \frac{\partial^2 F(v,t)}{\partial v^2} \left[\int_v^\infty x\, F(x,t)\, dx + \frac{1}{v^3} \int_0^v x^4 F(x,t)\, dx \right] \right.$$
$$+ F^2(x,t) + \frac{2}{3v} \frac{\partial F(v,t)}{\partial v} \left[\int_0^\infty x\, F(x,t)\, dx \right.$$
$$\left. \left. - \int_0^v x\, F(x,t) \left(1 - \frac{x}{v} \right)^2 \left(1 + \frac{x}{2v} \right) dx \right] \right\} \tag{32}$$

Equation (32) may be transformed from velocity space to energy space using the same procedure as for Eq (15). The resulting equation is conveniently expressed in the following "flux-divergent" form (Ref 19):

$$\left.\frac{\partial n(\varepsilon,t)}{\partial t}\right|_{ee} = - \frac{\partial J}{\partial \varepsilon} \tag{33}$$

where

$$J = \alpha \left[P \left(\frac{n}{2\varepsilon} - \frac{\partial n}{\partial \varepsilon} \right) - Q\, n \right] \tag{34}$$

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and

$$P = 2\,\varepsilon^{-1/2} \int_0^{\varepsilon} x\,n(x,t)\,dx \;+\; 2\varepsilon \int_{\varepsilon}^{\infty} x^{-1/2}\,n(x,t)\,dx \qquad (35)$$

$$Q = 3\,\varepsilon^{-1/2} \int_0^{\varepsilon} n(x,t)\,dx \qquad (36)$$

$$\alpha = \frac{2}{3}\pi\,e^{4}\left(\frac{2}{m}\right)^{1/2} \ln\Lambda \qquad (37)$$

Equation (33) is the final form of the electron-electron collision term to be used in Eq (22). Since Eq (22) is a steady-state equation, the time dependence in Eqs (33) through (36) can be suppressed.

A Discussion of the Assumptions

The assumptions made in deriving Eqs (22) and (33) deserve closer examination. These assumptions can be categorized as follows:

(a) Idealizations have been used to model the problem in a simple fashion.

(b) Effects which are small compared with the dominant effects have been ignored.

Assumptions of the first type cannot be expected to be valid in general. The problems considered must be restricted to those with essentially the same physical characteristics as the model. Assumptions of the second type can be rationalized using physical arguments or justified quantitatively if the magnitudes of the various effects can be estimated.

The systems considered here are restricted to DC gas discharges such as those commonly used to pump lasers. The

field is assumed uniform and there is no mass motion of the gas. For these conditions, the electron distribution function can be considered spatially homogeneous. Only one neutral ground-state species, with a number density of $10^{19} cm^{-3}$, is considered. This is closely approximated by a number of actual laser systems, where one species accounts for over 90 per cent of the total gas mixture, and the number density chosen is also representative of many actual laser systems (Ref 15). The applied electric field E may now be expressed in terms of E/n_o , where E is measured in volts/cm and n_o is the number density of the neutral ground-state species. The E/n_o range considered here is restricted to 10^{-18} to 10^{-16} volts-cm^2. This range is well within the stable discharge region for most systems (Ref 15). Detailed numerical calculations for several actual laser systems indicate that a negligible amount of energy goes into ionization for the E/n_o range considered here (Ref 20). Electron attachment and recombination are the most significant electron loss mechanisms. The cross sections for these processes vary depending on the species involved (Ref 29:41). If attachment or recombination is significant, it is necessary to include the effects of ionization to insure an appreciable electron density. The cross sections for multiple-ionization are usually zero until very high energies and then orders of magnitude lower than that for single-ionization. This effect may be safely ignored in almost all calculations.

From the preceding discussion, it is clear that

ionization, recombination, attachment, and multiply-
charged ions may be ignored for the systems considered here.
This does not imply that these processes are unimportant,
but only that their effects can be included in a self-
consistent manner for a given electron density. In this
case, it is justifiable to regard the electron number
density as a parameter. As with the electric field, this
parameter is conveniently expressed in terms of the fraction-
al ionization, n_e/n_o. In keeping with the stable discharge
criterion, the range of n_e/n_o is restricted to 10^{-3} to
10^{-5}. These values are also representative of those found
in numerous real systems.

Based upon the discharge parameters specified above,
the largest fractional excited-state density n^*/n_o
consistent with stable discharge operation is approximately
10^{-5}; and, the effects of superelastic collisions can be
neglected (Ref 15). To strengthen this argument, numerical
calculations have been performed for a variety of excited
state populations. The effect on the equilibrium electron
distribution is found to be quite small (Ref 20).

The next assumption considered is that of constant
Maxwellian distributions for the neutrals and ions. Actually,
the specific distribution of these species is unimportant
since it does not enter into Eq (22). Furthermore, since it
has been shown that the fractional ionization is a parameter
and the excited-state populations can be ignored, the number
densities of these species are constant. What is important
is the energy transfer between electrons and these species.

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Only in this context are the distribution functions for the neutrals and ions important. The same can be said for the assumption that the ions and neutrals are infinitely massive, since this really implies a zero kinetic temperature. Therefore, all of these assumptions are justified provided:

(a) the energy transfer between electrons and other species is negligible,

(b) the kinetic temperature change of the neutrals and ions is negligible during the time the electron distribution reaches steady-state.

Both of these conditions are reasonable in view of the large mass differential between electrons and other species. The neutral particles can gain energy only through collisions with the electrons and ions. Collisions with ions have little effect because the neutrals and ions have comparable temperatures. Collisions with electrons are inefficient because of the large mass difference. The ions can transfer energy with electrons or neutrals and also gain energy from the field. The field is relatively ineffective in heating the ions because of their large mass. The energy transfer with neutrals and electrons has little effect by the same argument used above.

Even though there is a negligible energy transfer between electrons and ions, these collisions do affect the heating rate of the electrons. This effect has been neglected by assuming $\nu_{ei} \ll \nu_m$. Since the number densities of the various species have been determined, these quantities can be calculated. From Appendix D,

$R_m = 5.928 \times 10^{-8}$ and $\nu_m = n_o R_m$. Therefore

$$\nu_m = 5.928 \times 10^{11} \text{ sec}^{-1} \qquad (38)$$

It can be shown that, on the average (Ref 23:258)

$$\nu_{ei} = \frac{4(2\pi)^{1/2}}{3} n_i \left(\frac{e^2}{KT}\right)^2 \left(\frac{KT}{m}\right)^{1/2} \ln \Lambda$$

where n_i is the total number density of ions (cm^{-3}) and T is the kinetic temperature of the electrons. Taking $n_i = 10^{16} \text{cm}^{-3}$ and $KT = 2\text{ev}$ (liberal estimates for the parameters considered here)

$$\nu_{ei} = 6.237 \times 10^{10} \text{ sec}^{-1}$$

Therefore, even under the worst conditions considered here, ν_{ei} is approximately an order of magnitude less than ν_m.

When $F(\underline{v}, t)$ was expanded in spherical harmonics, it was assumed that the anistropic term was small compared to the isotropic part of the distribution for "small" electric fields. To check the validity of this assumption, the electron drift velocity was calculated numerically for all parameter ranges considered here. In all cases, the drift velocity was at least an order of magnitude less than the thermal velocity of the electrons.

The assumptions of a single excitation level, constant momentum transfer rate, and constant excitation rate are gross approximations to reality. These assumptions have been made to simplify the Boltzmann equation. The goal is to derive an approximate analytic solution to the Boltzmann equation under these conditions. Hopefully, this

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solution will predict general characteristics of the system that are insensitive to the detailed cross sections. For numerical solutions to the Boltzmann equation, these assumptions are not necessary. Another problem then arises in that accurate cross sections (especially excitation cross sections) are frequently unavailable.

Lastly, the assumptions used in Appendix A to derive the Fokker-Planck equation should be mentioned. The validity of the "weak-interaction" approximation is established in Appendix A. Correlation effects are neglected in the derivation. Furthermore, the expansions used are carried out to second order only and no estimates are given for the error introduced by higher order terms. The justification of these assumptions is non-trivial. It can be shown, however, that these effects are quite small. A detailed discussion of correlation effects is given by Kaufman (Ref 7:295). Higher order coefficients are estimated by Gasiorowicz, et al (Ref 9).

Summary

The Boltzmann equation, subject to a number of assumptions, has been cast into a relatively simple form. The equation includes the effects of a constant electric field, elastic collisions, electronic excitation and electron-electron collisions. The equation is applicable for E/n_o values from 10^{-18} to 10^{-16} volts-cm^2 and n_e/n_o values from 10^{-3} to 10^{-5}. The relevant equation is Eq (22), with the Fokker-Planck electron-electron collision term given by Eqs (33) through (37) and Eq (31).

III Approach

Equation (22) is a non-linear integro-differential equation for the steady-state electron distribution function $n(\mathcal{E})$. The non-local term, $n(\mathcal{E}+x)$, further complicates matters. This equation cannot be solved exactly by any known techniques. An approximate solution is possible if the electron-electron collision term is ignored. To extend the solution to include these effects, the term must be approximated. It is not obvious from Eqs (33) through (36) how best to approximate this term. Therefore, the general characteristics of the Fokker-Planck term are investigated first. This provides direction in formulating the approximate relaxation term and also an indication of its shortcomings. Finally, Eq (22) is solved approximately using the relaxation term to include electron-electron collisions.

General Characteristics of the Fokker-Planck Term

From physical principles, electron-electron collisions are expected to have the following features:

(a) The total electron number density is conserved.
(b) The average electron energy is conserved.
(c) A Maxwellian distribution is unaltered.

It is straightforward to show that Eq (33) has these features. The change in the total number density, due to electron collisions, is given by

$$\frac{\partial n_e}{\partial t} = \int_0^\infty \frac{\partial n(\mathcal{E})}{\partial t}\Big|_{ee}\, d\mathcal{E} = -\int_0^\infty \frac{\partial J}{\partial \mathcal{E}}\, d\mathcal{E} = J(\mathcal{E})\Big|_\infty^0$$

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where Eq (33) has been used. Since $n(\varepsilon)$ vanishes at zero and infinity, $J(\varepsilon)$ also vanishes at these limits by Eq (34). Therefore, the Fokker-Planck collision term conserves total electron number density.

Similarly, the change in the average energy $\bar{\varepsilon}$ is

$$\frac{\partial \bar{\varepsilon}}{\partial t} = \frac{1}{n_e} \int_0^\infty \varepsilon \left. \frac{\partial n(\varepsilon)}{\partial t}\right|_{ee} d\varepsilon$$

$$= -\frac{1}{n_e} \int_0^\infty \varepsilon \frac{\partial J}{\partial \varepsilon} d\varepsilon = -\frac{1}{n_e} \int_0^\infty \varepsilon \, dJ$$

Using integration by parts and the fact that $J(\varepsilon)$ vanishes at zero and infinity, the above equation becomes

$$\frac{\partial \bar{\varepsilon}}{\partial t} = \frac{1}{n_e} \int_0^\infty J \, d\varepsilon$$

or using Eqs (33) through (36)

$$\frac{\partial \bar{\varepsilon}}{\partial t} = \frac{\alpha}{n_e} \int_0^\infty \left[P\left(\frac{n}{2\varepsilon} - \frac{\partial n}{\partial \varepsilon}\right)\right] d\varepsilon$$

$$= \frac{\alpha}{n_e}\left\{ \int_0^\infty n \left[\varepsilon^{-3/2} \int_0^\varepsilon x\, n(x)\, dx + \int_\varepsilon^\infty x^{-1/2} n(x)\, dx \right] d\varepsilon \right.$$

$$- \int_0^\infty n \left[\varepsilon^{-3/2} \int_0^\varepsilon x\, n(x)\, dx - 2 \int_\varepsilon^\infty x^{-1/2} n(x)\, dx \right] d\varepsilon$$

$$\left. - 3 \int_0^\infty n\, \varepsilon^{-1/2} \left[\int_0^\varepsilon n(x)\, dx \right] d\varepsilon \right\}$$

$$\frac{\partial \bar{\varepsilon}}{\partial t} = \frac{3\alpha}{n_e}\left\{ \int_0^\infty n \left[\int_\varepsilon^\infty x^{-1/2} n(x)\, dx \right] d\varepsilon - \int_0^\infty n\, \varepsilon^{-1/2} \left[\int_0^\varepsilon n(x)\, dx \right] d\varepsilon \right\} \quad (39)$$

The second integral in this equation can be integrated
by parts as follows:

$$\mu = \int_0^{\varepsilon} n(x)\,dx \qquad\qquad d\mu = n(\varepsilon)\,d\varepsilon$$

$$d\nu = n(\varepsilon)\,\varepsilon^{-1/2}\,d\varepsilon \qquad\qquad \nu = \int_0^{\varepsilon} x^{-1/2}\,n(x)\,dx$$

$$\int_0^{\infty} n\,\varepsilon^{-1/2}\left[\int_0^{\varepsilon} n(x)\,dx\right]d\varepsilon = \int_0^{\infty}\mu\,d\nu = \mu\nu\Big|_0^{\infty} - \int_0^{\infty}\nu\,d\mu$$

Therefore

$$\int_0^{\infty} n\,\varepsilon^{-1/2}\left[\int_0^{\varepsilon} n(x)\,dx\right]d\varepsilon = \left[\int_0^{\infty} n(x)\,dx\right]\left[\int_0^{\infty} x^{-1/2}\,n(x)\,dx\right]$$

$$- \int_0^{\infty} n\left[\int_0^{\varepsilon} x^{-1/2}\,n(x)\,dx\right]d\varepsilon$$

The first integral may be written in the following form:

$$\int_0^{\infty} n\left[\int_{\varepsilon}^{\infty} x^{-1/2}\,n(x)\,dx\right]d\varepsilon = \int_0^{\infty} n\left[\int_0^{\infty} x^{-1/2}\,n(x)\,dx\right]d\varepsilon$$

$$- \int_0^{\infty} n\left[\int_0^{\varepsilon} x^{-1/2}\,n(x)\,dx\right]d\varepsilon$$

If these expressions are substituted into Eq (39), the
right side vanishes identically. Therefore, $\frac{\partial\bar{\varepsilon}}{\partial t} = 0$, and
the average electron energy is conserved by the Fokker-Planck
term.

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The last physical feature to be verified is station-
arity of a Maxwellian distribution $\bar{n}(\varepsilon)$, where

$$\bar{n}(\varepsilon) \propto \varepsilon^{1/2} e^{-\varepsilon/\kappa T} \qquad (40)$$

The verification is straightforward and will not be given in
detail. Substituting $\bar{n}(\varepsilon)$ into Eqs (34) through (36) and
integrating the first integral in Eq (35) by parts, $J(\varepsilon)$
vanishes for all ε. As a result, using Eq (33)

$$\frac{\partial \bar{n}(\varepsilon)}{\partial t}\bigg|_{ee} = 0$$

and a Maxwellian distribution is unaltered.

Therefore, the Fokker-Planck form of the electron-
electron collision term retains the three main physical
features of these interactions. Parenthetically, this gives
further credibility to the assumptions used to derive the
Fokker-Planck equation.

An inspection of Eqs (35) and (36) reveals that the
coefficients, P and Q, involve integrals of $n(\varepsilon)$. This
not only makes the equations non-linear, but also introduces
a non-local effect. Therefore, $\frac{\partial n(\varepsilon)}{\partial t}\big|_{ee}$ depends not only
on n or its derivatives at a given energy, but also on the
shape of the distribution function over the entire energy
range. This is reasonable (and necessary) since electron-
electron collisions cannot change the number density or
average energy. Several techniques have been used to approxi-
mate this effect (Ref 17).

A Simple Relaxation Term

Unfortunately, attempts to include the non-local effects in a simple collision term failed. To retain the physical features discussed above, a Krook-type relaxation term is assumed:

$$\left.\frac{\partial n(\varepsilon)}{\partial t}\right|_{ee} = - \frac{n(\varepsilon) - \bar{n}(\varepsilon)}{\tau(\varepsilon)}$$

where $\tau(\varepsilon)$ is the characteristic time in which the distribution relaxes to a Maxwellian as a result of electron-electron collisions.

In general, τ must be considered a function of energy. There is no reason to expect the distribution to relax uniformly at the same rate. In fact, numerical solutions show that the high energy portion of the distribution relaxes much slower than the low energy portion (Ref 16). A qualitative description is provided by considering similar distributions with different average energies. The result is $\tau(\varepsilon) \propto \left(\bar{\varepsilon}\right)^{3/2}$ (Ref 6:120). It is not clear that this relationship is generally valid for a given distribution. However, this form for $\tau(\varepsilon)$ does insure that the relaxation rate decreases with increasing energy.

Attempts to include an energy-dependent relaxation rate and retain all three physical features have also been unsuccessful. To insure the relaxation term conserves number density and average energy, $\bar{n}(\varepsilon)$ must be altered in a manner determined by $\tau(\varepsilon)$. With this alteration, $\bar{n}(\varepsilon)$ no longer retains its true Maxwellian nature (with the same

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number density and average energy as the actual distribution).
The basic problem is that for an energy-dependent relaxation
rate, the physical features overspecify $\bar{n}(\varepsilon)$. Therefore,
to employ the simple relaxation term a constant τ is
used.

The value of τ used here is the "self-collision
time" τ_{ce} derived by Spitzer (Ref 24:133); it is basically
the average time it takes an electron with average energy to
undergo a 100% change in energy and suffer a 90° deflection
(Ref 16). The self-collision time may be expressed as

$$\tau_{ce} = \frac{0.266\, T^{3/2}}{n_e \ln \Lambda} \quad (\text{sec}) \tag{41}$$

where T is the kinetic temperature of the electrons (°K).
The final form for the relaxation term is

$$\left.\frac{\partial n(\varepsilon)}{\partial t}\right|_{ee} = -\frac{n(\varepsilon) - \bar{n}(\varepsilon)}{\tau_{ce}} \tag{42}$$

It is easily verified that this form of the electron-
electron collision term retains all the physical features
discussed earlier, provided $\bar{n}(\varepsilon)$ has the same number
density and average energy as $n(\varepsilon)$. It is anticipated that
this term will produce a higher than normal relaxation rate
at high energies and a lower than normal rate at low energies.

A problem does arise when using Eq (42) for finite-
differenced numerical solutions. In this case, $\bar{n}(\varepsilon)$ is
not well defined by Eq (40). The method used to circumvent
this problem is given in Appendix C.

Approximate Solution of the Boltzmann Equation

The Boltzmann equation may be solved approximately
if the relaxation term is used for electron-electron colli-
sions. Combining Eqs (22) and (42), the Boltzmann equation
becomes

$$\frac{2}{3}\dot{\varepsilon}\frac{d}{d\varepsilon}\left[\frac{n}{2} - \varepsilon\frac{dn}{d\varepsilon}\right] = n_0 R(\varepsilon + x) n(\varepsilon + x)$$

$$- n_0 R(\varepsilon) n(\varepsilon) - \frac{n(\varepsilon) - \bar{n}(\varepsilon)}{\tau_{ce}} \qquad (43)$$

A fundamental problem arises immediately in that $\bar{n}(\varepsilon)$
depends on the steady-state average energy, which is unknown
at this point. This problem is not peculiar to the relaxation
term; the Fokker-Planck term also depends on the average
energy through the quantity $\ln\Lambda$. However, electron-electron
collisions usually have little effect on quantities averaged
over the entire distribution function (Ref 1:22). Therefore,
Eq (43) could be solved using the techniques mentioned
earlier and ignoring electron-electron collisions. The
average energy obtained from this solution could then be
used to evaluate $\bar{n}(\varepsilon)$.

The solution to Eq (43) will be restricted to the
region $\varepsilon \geq x$. In this region it is reasonable to assume
$n_0 R(\varepsilon + x) n(\varepsilon + x) \ll n_0 R(\varepsilon) n(\varepsilon)$. The numerical solutions presented
in the next section show that this assumption is good until
electron-electron collisions completely dominate inelastic

collisions. The equation may be cast into a more convenient
form by the following change of variable:

$$y = -2 \varepsilon^{-1/2}$$

In addition

$$h(\varepsilon) \equiv n_e \, \varepsilon^{1/2} \, \tilde{n}(\varepsilon)$$

$$\bar{n}(\varepsilon) \equiv n_e \, a \, \varepsilon^{1/2} \, e^{-b\varepsilon}$$

where

$$a = \frac{2}{\sqrt{\pi}} \, b^{3/2}$$

$$b = \frac{1}{KT}$$

$$\int_0^\infty \varepsilon^{1/2} \, \tilde{n}(\varepsilon) \, d\varepsilon = 1$$

With these changes and assumptions, Eq (43) takes the
following form:

$$\frac{d^2 \tilde{n}}{dy^2} - \frac{A}{y^4} \, \tilde{n} = h(y) \tag{44}$$

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where

$$A = \left[\frac{24}{\dot{\varepsilon}} \left(n_o R_o + \frac{1}{\tau_{ce}} \right) \right] \tag{45}$$

$$h(y) = -\frac{24a}{\dot{\varepsilon} \tau_{ce}} \frac{e^{-4b/y^2}}{y^4} \tag{46}$$

Equation (44) can be solved using a combination of the WKB method and variation of parameters. Unfortunately, the complete solution involves integrals which cannot be evaluated in closed form. Since this approach could be useful for future work; the results, though incomplete, are included in Appendix E.

An approximate solution can be obtained by ignoring the second derivative in Eq (44). In this approximation, $\tilde{n}$ is given by

$$\tilde{n}(y) = \frac{24a}{A\dot{\varepsilon}\tau_{ce}} e^{-4b/y^2}$$

Using Eq (45), the solution may be expressed as

$$\tilde{n}(y) = \left[\frac{a}{1 + n_o R_o \tau_{ce}} \right] e^{-4b/y^2} \tag{47}$$

Equation (47) represents a self-consistent solution provided

$$\left| \frac{d^2\tilde{n}}{dy^2} \right| \ll \left| \frac{A}{y^4} \tilde{n} \right| \tag{48}$$

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This requirement follows directly from Eq (44). Differentiating Eq (47) and substituting the result into Eq (48) leads to the following requirement:

$$\left(\frac{8b}{y}\right)^2 - 24b << A$$

In the next section, three numerical solutions to Eq (43) are compared with solutions using the Fokker-Planck collision term. It is easily verified that the above requirement is satisfied for these cases. Transforming Eq (47) back into energy space yields the following approximate solution to Eq (43) for $\varepsilon \geq \chi$:

$$h(\varepsilon) = \frac{n_e \varepsilon^{1/2}}{1 + n_o R_o \tau_{ce}} \, e^{-\varepsilon/kT} \tag{49}$$

Since Eq (49) involves no arbitrary constants, the complete solution to Eq (43) (subject to the imposed physical restraints) will generally be discontinuous at the excitation energy. This behavior and several other features are investigated in the next section. Since a discontinuous distribution function is not physically acceptable, no attempt is made to extend the solution below the excitation energy.

Summary

A simple relaxation term has been formulated that retains the three prominent physical feature of electron-electron collisions. Non-local effects have been disregarded. Using this term for electron-electron collisions,

an approximate solution to the Boltzmann equation is
obtained for energies greater than the excitation energy.
The relaxation term is given by Eq (34); the approximate
solution is given by Eq (49).

IV Analysis and Comparison

In this section, the relaxation form of the electron-electron collision term is compared to the Fokker-Planck collision term. The temporal characteristics of the two terms are analyzed first. Next, steady-state solutions of the Boltzmann equation are analyzed for three sets of E/n_o and n_e/n_o values. The Fokker-Planck and relaxation results are computed numerically for all cases. These computations employ a finite-differenced energy axis with a 20 ev maximum energy and .5 ev energy interval. The analytic results are calculated using Eq (49) for the three steady-state comparisons.

Temporal Characteristics

The initial distribution is chosen to be a Gaussian with a 10 ev average energy and 1 ev standard deviation (Fig. 1). The temporal characteristics of the two terms are compared in Figs. 2 through 5. Only the effects of electron-electron collisions are included in these calculations. The numerical procedure described in Appendix B is used to calculate the Fokker-Planck term; the relaxation term is calculated using the procedure in Appendix C.

As expected, the relaxation term gives a higher than normal rate at high energies (Fig. 2). However, the rate is also higher at low energies. This is just the opposite of the predictions in Section III. This result can be explained by considering the Maxwellian used in the relaxation calculation. It has been verified numerically that this

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10¹

10⁻¹

10⁻³

10⁻⁵

$\tilde{n}$
$(ev^{-3/2})$

10⁻⁷

10⁻⁹

10⁻¹¹

10⁻¹³

10⁻¹⁵

INITIAL
GAUSSIAN
DISTRIBUTION

0   4   8   12   16   20

ε (ev)
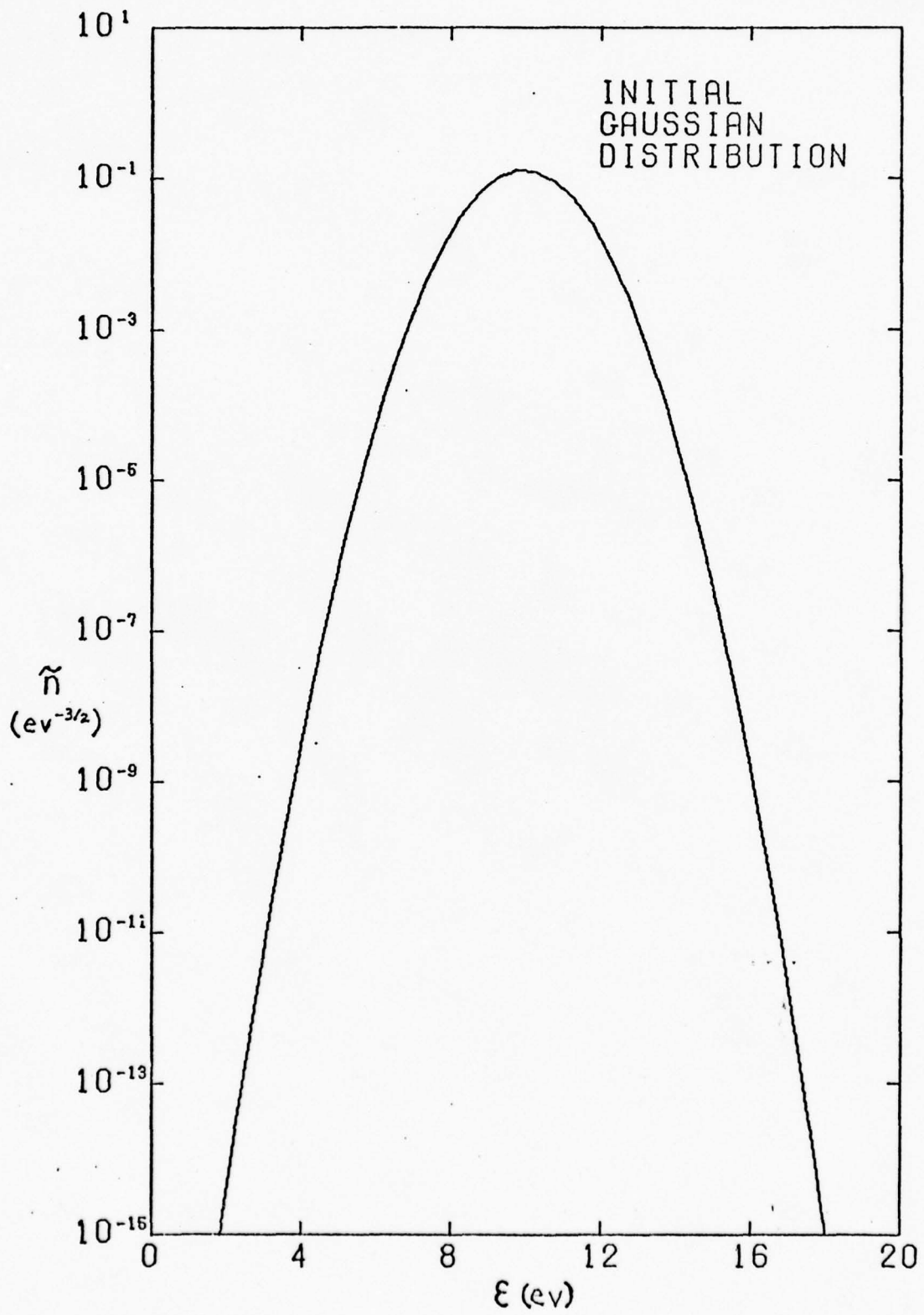
Fig. 1   Initial Gaussian Distribution, t = 0

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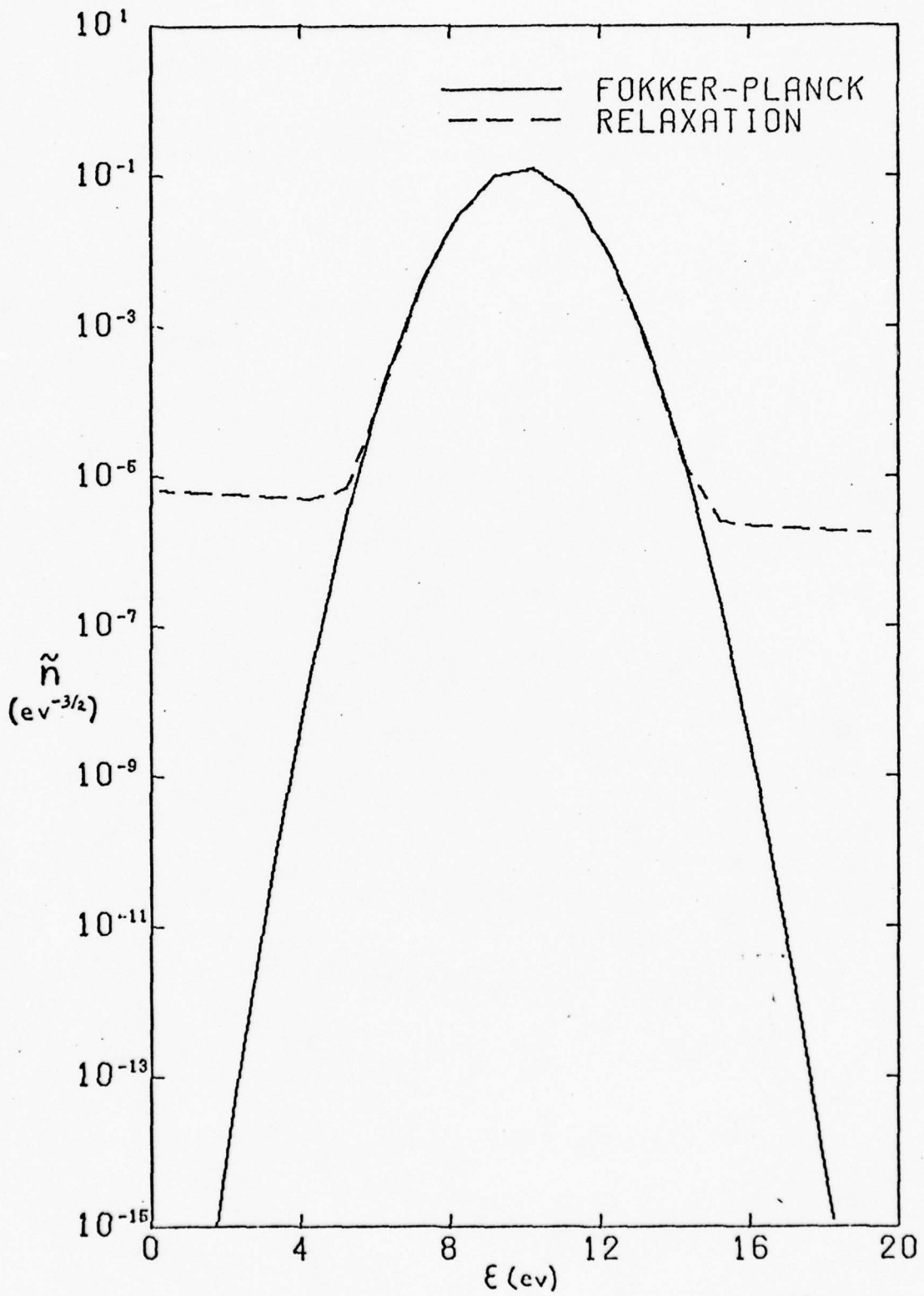
Fig. 2 Temporal Comparison, $t = 10^{-12}$

Maxwellian (time-independent in this case) corresponds exactly to the final steady-state distribution of the Fokker-Planck term. This distribution, shown in Fig. 5, is essentially a horizontal line for the log scale used here. Since the initial Gaussian is symmetrical about the average energy, the relaxation term predicts approximately equal rates for low and high energies.

Therefore, no generally valid relationship exists between the Fokker-Planck rates and the relaxation term rates. Furthermore, an energy-dependent τ_{ce} could eliminate this problem for one specific distribution only. The non-local effects, neglected in formulating the relaxation term, must be included to completely eliminate the problem.

The form of the distribution at $t = 10^{-10}$ sec is shown in Fig. 3. The relaxation term exhibits the same features as before; however, the diffusion characteristics of the Fokker-Planck term are now evident. At this point, the distribution has spread (slightly weighted toward the higher energies) about the average energy and the number of electrons with the average energy has decreased slightly.

Figure 4 shows the form of the distribution at $t = 10^{-9}$ sec. This time closely approximates τ_{ce} for the parameters assumed here. Obviously, the Fokker-Planck and relaxation distributions are considerably different. However, Fig. 4 does indicate that in either case, τ_{ce} is a reasonable time scale for electron-electron collisions to relax a distribution to a Maxwellian.
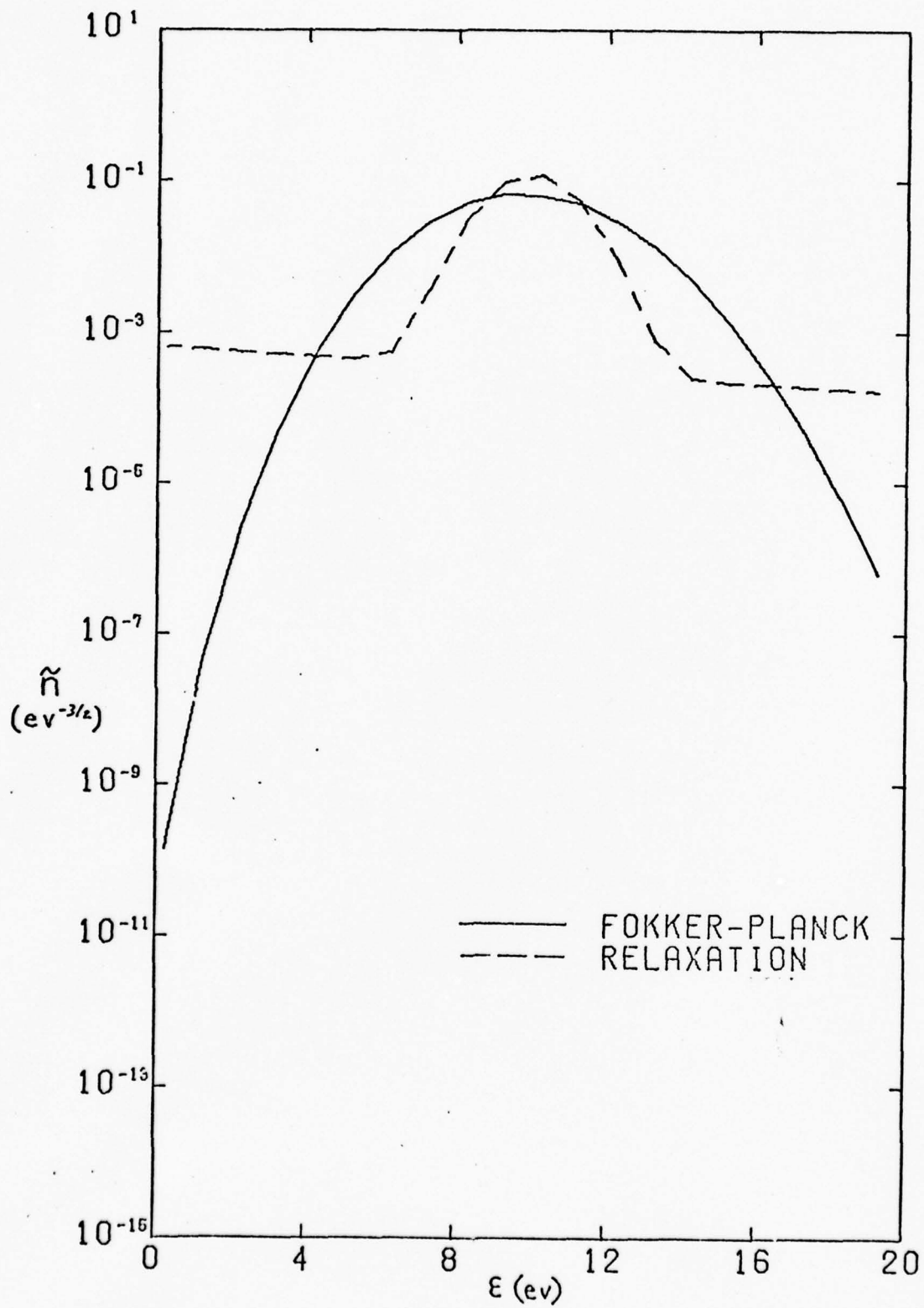
Fig. 3 Temporal Comparison, $t = 10^{-10}$
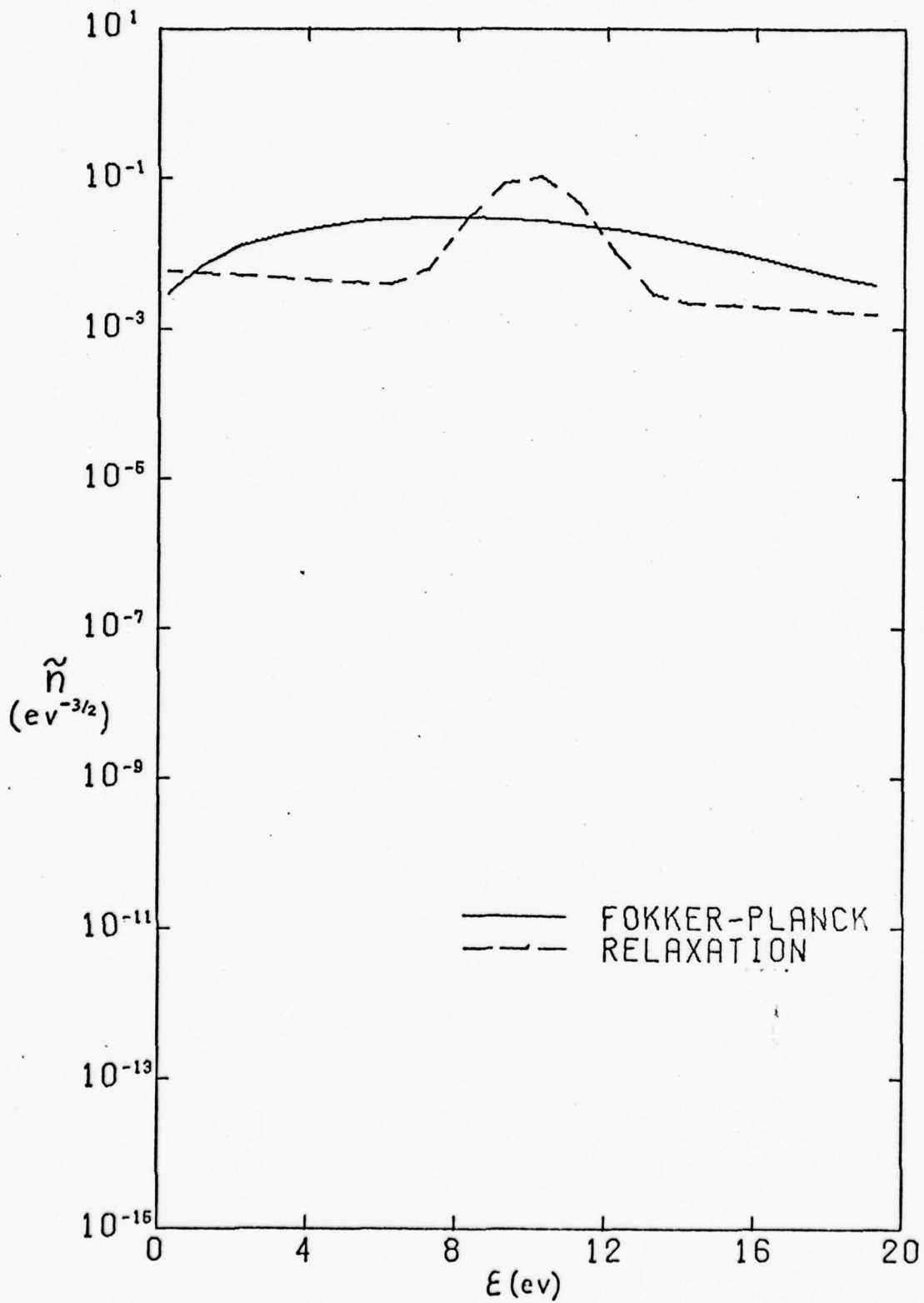
Fig. 4 Temporal Comparison, t = 10^{-9}
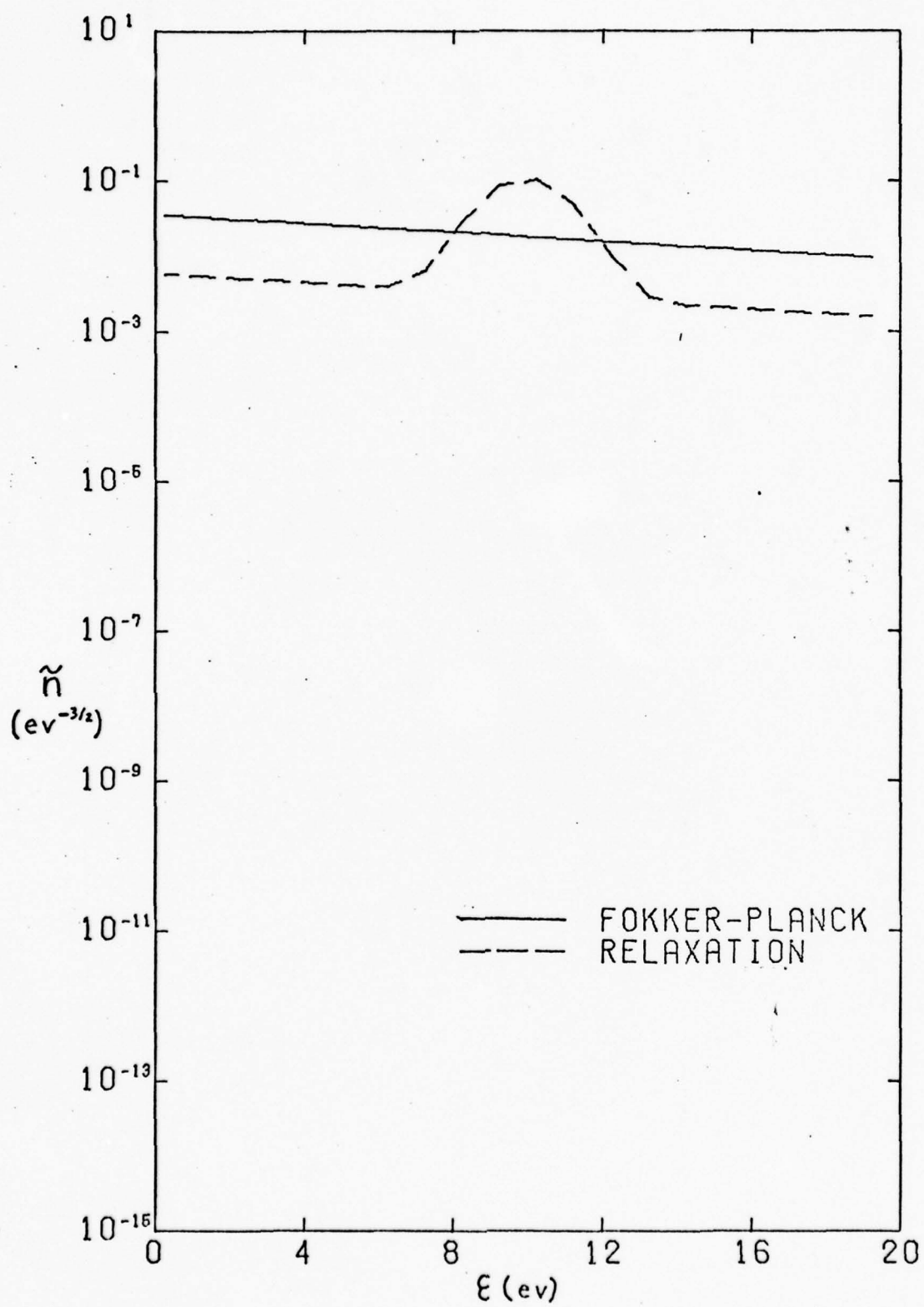
Fig. 5 Temporal Comparison, $t = 10^{-8}$

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The steady-state condition is shown in Fig. 5. No later times are shown because the numerical integration procedure was unable to advance the relaxation term any further. It appears that the relaxation term is not well suited to numerical integration. As mentioned previously, the Maxwellian used in the relaxation calculation is identical to the Fokker-Planck steady-state distribution. Therefore, both approaches should yield identical steady-state results. This problem was not encountered when the initial distribution differed only slightly from a Maxwellian. In this case, both methods result in identical steady-state distributions. The large average energy, coarse energy zoning and relatively small maximum energy could be a factor. Time did not permit a closer investigation of this problem.

Steady-State Comparisons

Steady-state distributions for three sets of E/n_o and n_e/n_o values are shown in Figs. 6 through 8. The Fokker-Planck and relaxation curves represent numerical calculations. The analytic curve represents the evaluation of Eq (49) for the appropriate parameters. The excitation threshold energy is 8 ev in all cases. For completeness, the results of the numerical calculations are tabulated in Appendix F.

There are two characteristic features of all three cases. First, the Fokker-Planck and relaxation approaches agree well at energies below the excitation threshold. This

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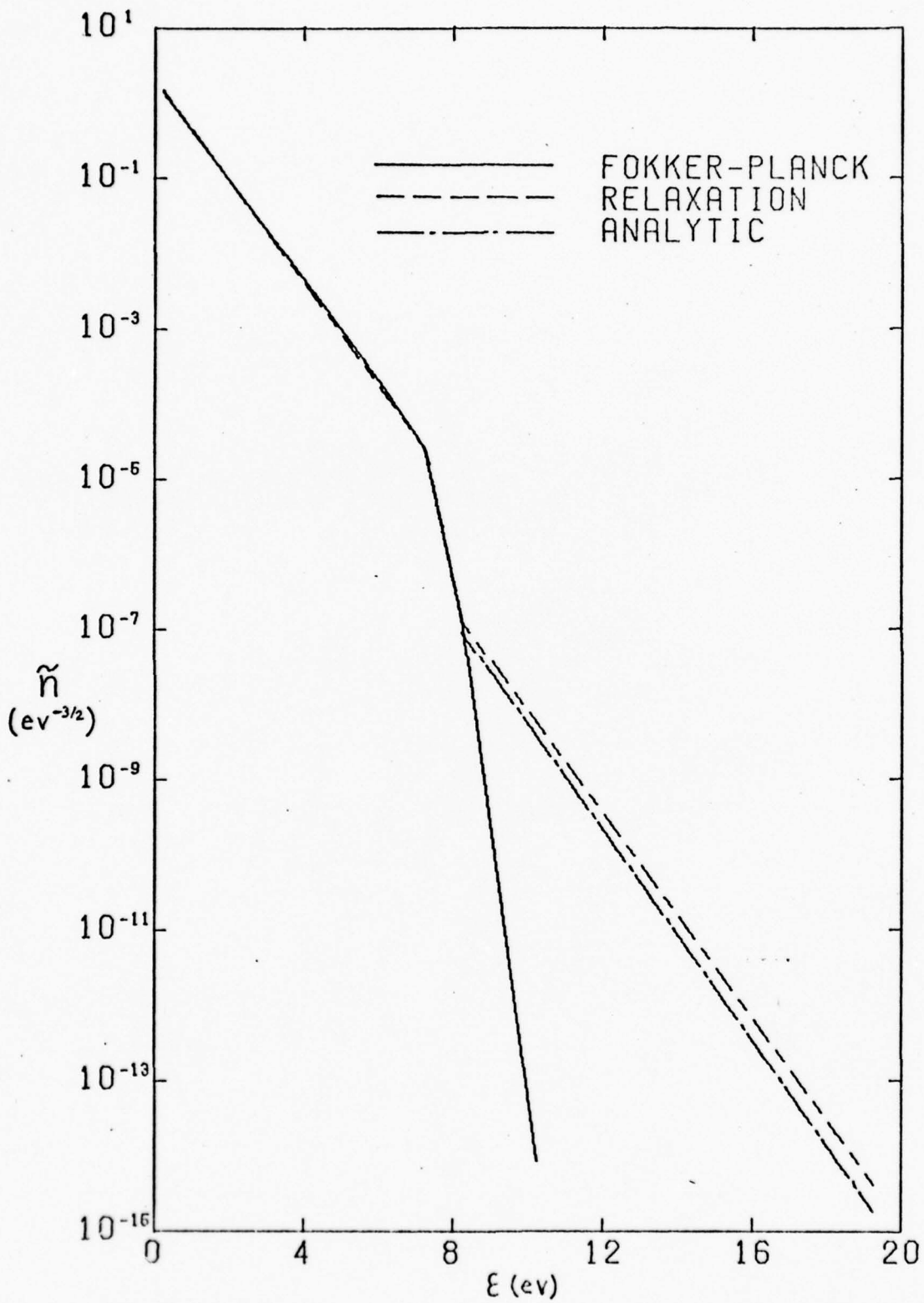
Fig. 6 Steady-State Dist., $n_c/n_o = 10^{-5}$, $\mathcal{E}/n_o = 10^{-18}$

43
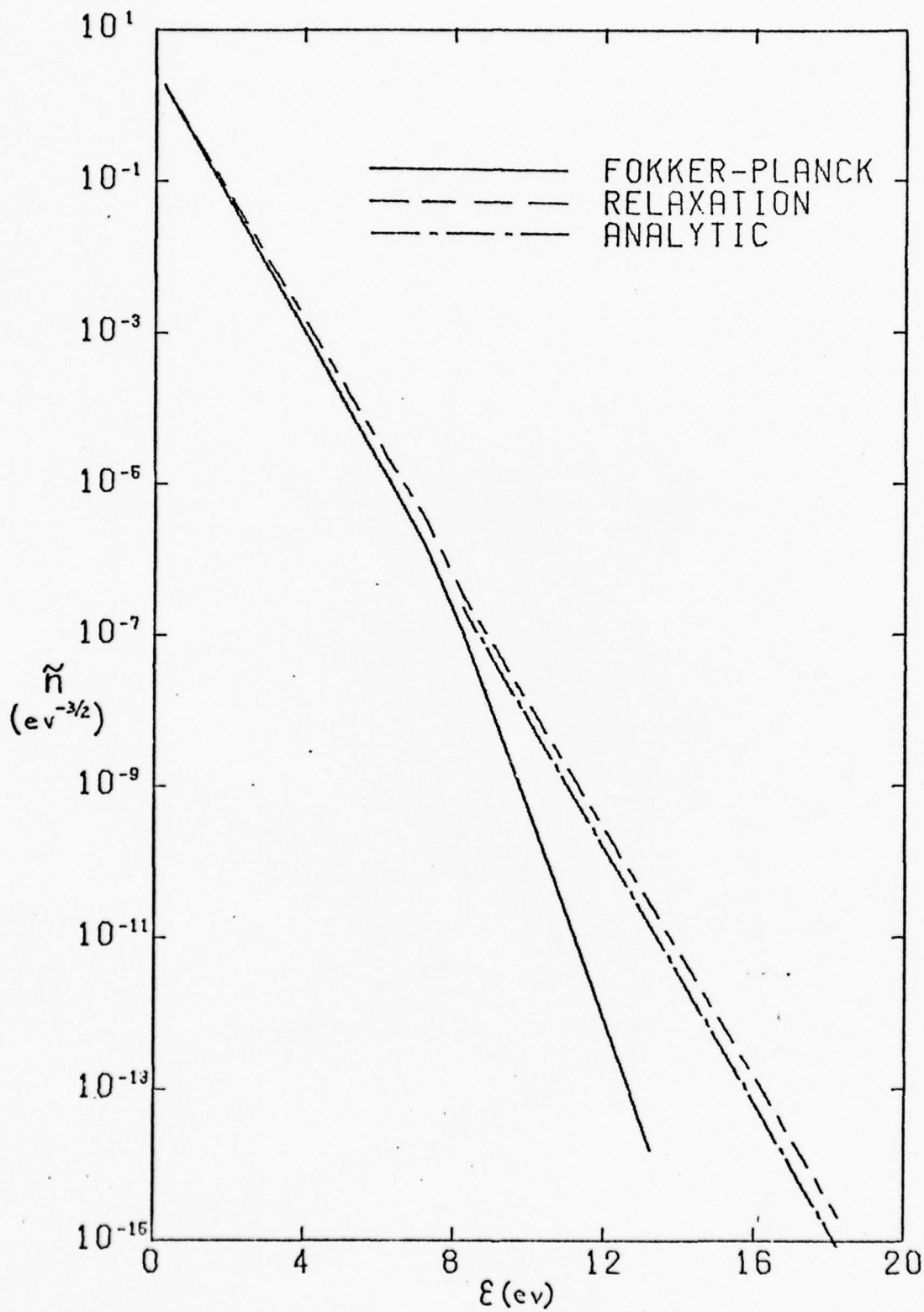
Fig. 7 Steady-State Dist., $n_e^o/n_o = 10^{-3}$, $E/n_o = 10^{-18}$

44
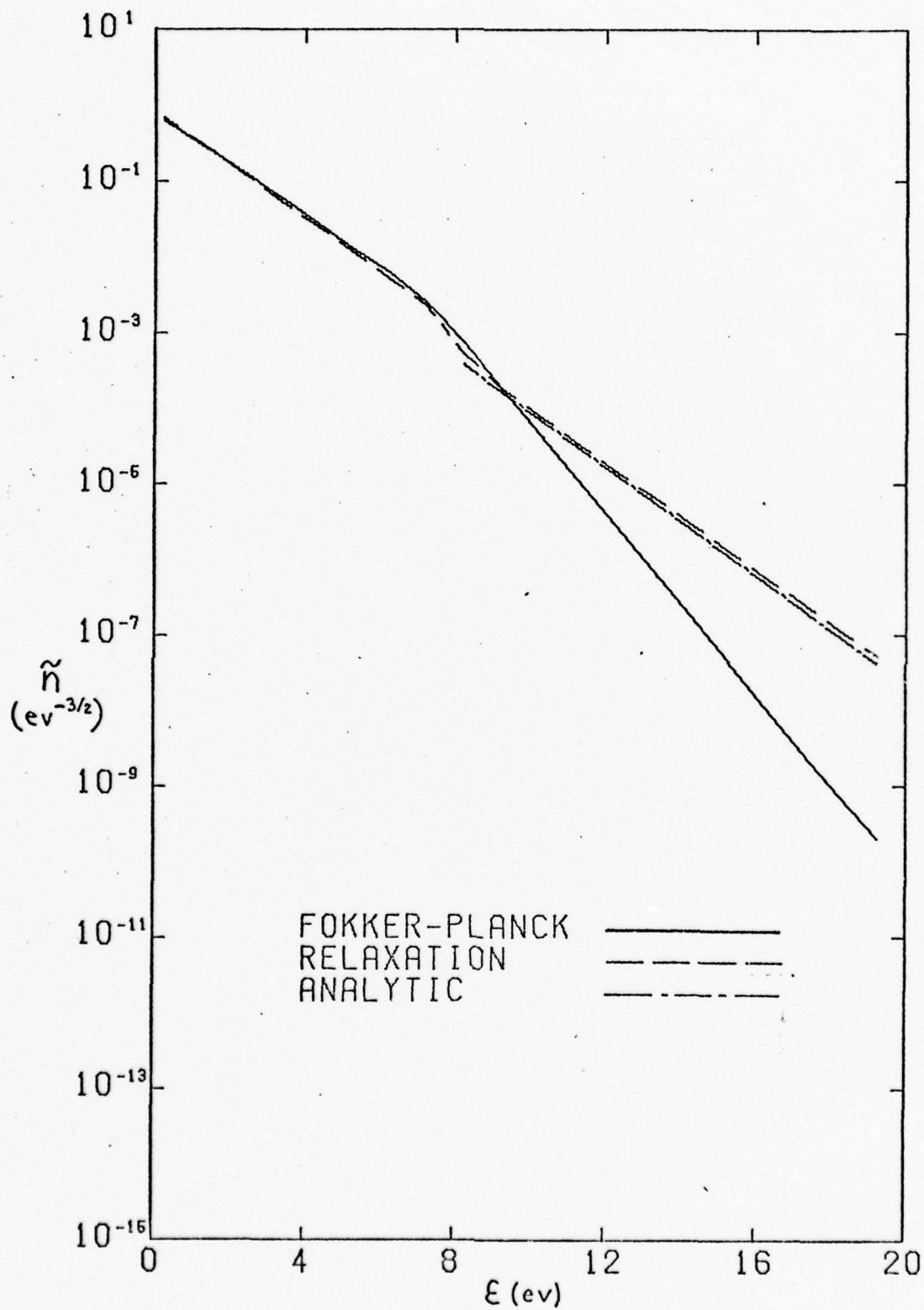
Fig. 8 Steady-State Dist., $n_e/n_o = 10^{-3}$, $E/n_o = 10^{-16}$

is to be expected because there are no energy loss mechanisms in this range; energy gained from the field must be redistributed by electron-electron collisions. Inelastic and superelastic processes are treated identically in both numerical procedures. Therefore, the precise rate of electron-electron collisions should have little effect on the steady-state solutions for energies below the excitation threshold. Secondly, the distributions are essentially Maxwellian over the entire energy range. The Fokker-Planck distributions exhibit a lower temperature above threshold. The relaxation distributions can be characterized by a single temperature over the whole energy range. This important and discouraging feature will now be investigated more thoroughly.

The effect of the large relaxation rate, characteristic of the relaxation term at high energies, is evident in all three cases. In fact, the inelastic collision process is completely overwhelmed. The only departure from a true Maxwellian is a discontinuity in the neighborhood of the excitation threshold. The largest discontinuity occurs in Fig. 6. This case corresponds to the smallest field and fractional ionization considered. The smallest discontinuity occurs in Fig. 7. This case corresponds to a small field and large fractional ionization. Therefore, the approaches agree better as the number density increases and electron-electron collisions become more important.

These results may be explained as follows. First, the relaxation term tends to overemphasize electron-electron

collisions in all cases. Since there is an upper limit to
this effect (a true Maxwellian distribution), the relaxation
theory and Fokker-Planck theory agree better at higher
fractional ionizations. Secondly, the relaxation theory
has been forced to conserve number density and average
energy. This requirement, together with the rapid relaxa-
tion rate at low and high energies, produces a discontinuity
in the neighborhood of the excitation threshold. Therefore,
the magnitude of this discontinuity can decrease as the
actual distribution approaches a Maxwellian.

The average energies of the distributions are given
in Tables III through VIII. These values indicated that
the two approaches agree to within five per cent. The
largest difference occurs for $n_e/n_o = 10^{-3}$ and $\mathcal{E}/n_o = 10^{-18}$.
These are also the conditions for which the low energy portion
of the distribution is affected most. Nevertheless, the
assumption made concerning averages over the entire distri-
bution function seems to be justified.

The analytic solution derived in Section III is also
plotted in Fig. 6 through 8. Obviously, the agreement
between the analytic and numerical solutions is quite good.
In addition, it can be concluded that the problems encountered
with numerical solutions using the relaxation model are not
numerical in origin but result from the discontinuity at
the excitation threshold. It is surprising that the numeri-
cal integration procedure is able to handle the problem as
well as it does. The slight difference in slope between

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the numerical and analytic solutions could result from neglecting the second derivative in Eq (44). Another possible explanation is that the approximation made in solving this equation is not good for very large energies.

Furthermore, Eq (49) reveals that changing the magnitude of τ_{ce} would not alter the form of the relaxation solution; the magnitude of the discontinuity would be affected but the overall form would be unchanged. This argument is valid provided the assumptions made in deriving Eq (49) are justified. Since an energy-dependent τ_{ce} would change the nature of the analytic solution, it is possible that such an approach would lead to better agreement with the Fokker-Planck theory.

V   Conclusion

The analysis and comparisons presented in Section IV lead to several logical conclusions concerning the relaxation approach to electron-electron collisions. The approach, as presented here, is not applicable for the electric fields and fractional ionizations commonly used in discharge lasers. The high energy portion of the distribution function is of primary importance in these devices. The relaxation collision term yields distributions too large by several orders of magnitude in this range.

The three most serious defects in the relaxation approach are the following:

(a) The high relaxation rate at high energies completely dominates the more important inelastic processes.

(b) Energy diffusion, an important characteristic, is ignored.

(c) The steady-state distribution is in general discontinuous at the excitation threshold.

The analytic solution using the relaxation approach indicates that a different relaxation rate would not significantly improve the situation. However, an energy-dependent rate could possibly lead to better agreement with the Fokker-Planck theory.

Regardless of the approach, diffusion effects appear to play a key role in representing the physical situation. It is not clear how to include this feature

in a collision term with reasonable simplicity. Finally,
a simple but accurate, numerical procedure might be more
feasible and valuable than an approach based on drastic
assumptions.

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Appendix A

Derivation of the Fokker-Planck Equation

The purpose of this appendix is to derive an expression that adequately describes the effects of Coulomb collisions on the electron velocity distribution function. The Boltzmann collision term is based on the assumption that particle interactions are short-range, binary encounters. For electron-neutral interactions, this assumption is usually justified. To describe Coulomb collisions, the effect of simultaneous interactions between many particles must be considered.

It is not possible to solve this many-body problem exactly, taking all the particle correlation effects into account. However, if the interactions are assumed "weak", these simultaneous interactions may be regarded as the sum of many uncorrelated binary collisions. The following considerations clarify and lend credibility to this approach.

A weak interaction is one in which the momentum change of a particle is small compared to the total particle momentum. In highly ionized plasmas, where Coulomb collisions must be considered, the cumulative effect of many weak interactions is more effective in deflecting an electron than a single strong interaction. It can be shown (Ref 14: 294) that weak interactions dominate strong interactions by a factor of $8 \ln \Lambda$, where Λ is the Coulomb factor defined in Section II. Since $\ln \Lambda$ usually lies between 10 and 20, weak interactions represent the overall Coulomb collisional effect to several orders of magnitude.

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The simultaneous interactions may now be regarded as simultaneous small perturbations and the total effect obtained by adding the individual perturbations linearly. Using these ideas, the form of the Coulomb collision term can be derived by several independent methods. The most general procedure starts with the Liouville equation (Ref 11:115). Another method involves expanding the Boltzmann collision integral in powers of the momentum interchange (Ref 8). The method used here, similar to that used in the study of Brownian motion (Ref 23:31), results in the Fokker-Planck equation. This equation correctly describes, to second order, the temporal evolution of the electron velocity distribution function resulting from many weak, uncorrelated binary encounters. Let $F(\underline{v}, t)$ represent the spatially homogeneous electron velocity distribution function. $\Psi(\underline{v} - \Delta \underline{v}, \Delta \underline{v}) d(\Delta \underline{v})$ is defined as the probability that an electron changes its velocity from $\underline{v} - \Delta \underline{v}$ to $\underline{v}$ in a time Δt as a result of Coulomb interactions. Since $\Psi(\underline{v} - \Delta \underline{v}, \Delta \underline{v}) d(\Delta \underline{v})$ is a probability, the following relation must hold:

$$\int \Psi(\underline{v} - \Delta \underline{v}, \Delta \underline{v}) d(\Delta \underline{v}) = 1 \qquad (50)$$

The distribution function at time $t + \Delta t$ is related to the distribution at time t by

$$F(\underline{v}, t + \Delta t) = \int F(\underline{v} - \Delta \underline{v}, t) \Psi(\underline{v} - \Delta \underline{v}, \Delta \underline{v}) d(\Delta \underline{v}) \qquad (51)$$

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Since weak interactions are being considered, $\Delta \underline{v}$ is small in a small time interval Δt. The terms in Eq (51) can be expanded as follows:

$$F(\underline{v}, t+\Delta t) = F(\underline{v},t) + \frac{\partial F(\underline{v},t)}{\partial t} \Delta t \qquad (52)$$

$$F(\underline{v}-\Delta \underline{v}, t) = F(\underline{v},t) - \sum_i \Delta v_i \frac{\partial F(\underline{v},t)}{\partial v_i}$$
$$+ \sum_{i,j} \frac{1}{2} \Delta v_i \Delta v_j \frac{\partial^2 F(\underline{v},t)}{\partial v_i \partial v_j} + \cdots \qquad (53)$$

$$\Psi(\underline{v}-\Delta \underline{v}, \Delta \underline{v}) = \Psi(\underline{v}, \Delta \underline{v}) - \sum_i \Delta v_i \frac{\partial \Psi(\underline{v}, \Delta \underline{v})}{\partial v_i}$$
$$+ \sum_{i,j} \frac{1}{2} \Delta v_i \Delta v_j \frac{\partial^2 \Psi(\underline{v}, \Delta \underline{v})}{\partial v_i \partial v_j} + \cdots \qquad (54)$$

where Δv_i is the ith component of $\Delta \underline{v}$. If Eqs (52) through (54) are substituted into Eq (51), the result is

$$F(\underline{v},t) + \frac{\partial F(\underline{v},t)}{\partial t} \Delta t = F(\underline{v},t) \int \Psi(\underline{v}, \Delta \underline{v}) \, d(\Delta \underline{v})$$

$$- \sum_i F(\underline{v},t) \frac{\partial}{\partial v_i} \int \Delta v_i \Psi(\underline{v}, \Delta \underline{v}) \, d(\Delta \underline{v})$$

$$- \sum_i \frac{\partial F(\underline{v},t)}{\partial v_i} \int \Delta v_i \Psi(\underline{v}, \Delta \underline{v}) \, d(\Delta \underline{v})$$

$$+ \frac{1}{2} \sum_{i,j} F(\underline{v},t) \frac{\partial^2}{\partial v_i \partial v_j} \int \Delta v_i \Delta v_j \Psi(\underline{v}, \Delta \underline{v}) \, d(\Delta \underline{v})$$

$$+ \frac{1}{2} \sum_{i,j} \frac{\partial^2 F(\underline{v},t)}{\partial v_i \partial v_j} \int \Delta v_i \Delta v_j \Psi(\underline{v}, \Delta \underline{v}) \, d(\Delta \underline{v}) \qquad (55)$$

The first terms on the left and right sides of Eq (55)
cancel identically by Eq (50). Defining

$$\langle \Delta v_i \rangle = \frac{1}{\Delta t} \int \Delta v_i \, \Psi(\underline{v}, \Delta \underline{v}) \, d(\Delta \underline{v}) \qquad (56)$$

$$\langle \Delta v_i \Delta v_j \rangle = \frac{1}{\Delta t} \int \Delta v_i \, \Delta v_j \, \Psi(\underline{v}, \Delta \underline{v}) \, d(\Delta \underline{v}) \qquad (57)$$

Eq (55) may be rewritten as

$$\frac{\partial F(\underline{v}, t)}{\partial t} = -\sum_i \frac{\partial}{\partial v_i} F(\underline{v}, t) \langle \Delta v_i \rangle$$

$$+ \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial v_i \partial v_j} F(\underline{v}, t) \langle \Delta v_i \Delta v_j \rangle \qquad (58)$$

Equation (58) is the Fokker-Planck equation, and is
phenomenological until explicit expressions are derived
for Eqs (56) and (57). The form of these equations is
considered in Section II and will not be derived here.
Short of this, several general remarks can be made concerning
the Fokker-Planck equation.

Equation (58) is basically a conservation equation in
velocity space. The terms $\langle \Delta v_i \rangle$ and $\langle \Delta v_i \Delta v_j \rangle$ may be
interpreted as diffusion coefficients (Ref 24:125). It is
important to note that these coefficients are time-ensemble
average quantities and differ dimensionally from the corres-
ponding ensemble average quantities. The quantity $\langle \Delta \underline{v} \rangle / v$
has been termed the coefficient of dynamical friction

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(Ref 2). It represents the average deceleration experienced

by an electron of velocity $\underline{v}$ due to Coulomb interactions

(Ref 8). Similarly, the quantity $\langle \Delta v_i \Delta v_j \rangle$ is called the

coefficient of diffusion tensor. This term describes the

random fluctuations about the average velocity (Ref 23:249).

In conclusion, it should be mentioned that applica-

tions of the Fokker-Planck equation are not restricted to

Coulomb interactions in plasmas. In fact, the theory has

been extensively used to analyze problems in stellar

dynamics (Ref 2,3,4). A mathematically rigorous discussion

is given by Haken (Ref 10).

Appendix B

Numerical Solution of the Fokker-Planck

Electron-Electron Collision Term

(Subroutine Funee)

A significant part of this thesis effort has been the development of a computer code to solve the Boltzmann equation under a variety of conditions. This work, in cooperation with three other students, resulted in a computer program called NGB. The operating procedure for NGB is to be documented in an AFIT Technical Report. By mutual agreement, the details of the various segments of NGB are presented as Appendices in each individual's thesis. This appendix fulfills this requirement for the electron-electron collision term.

The equations used to numerically evaluate the electron-electron collision term are derived and presented in a form suitable for direct computer coding. Particular attention is given to units and numerical factors. The method used here was developed by Proctor and Canavan (Ref 19), although the notation parallels that of Rockwood (Ref 20). Subroutine Funee, the coding used for electron-electron collisions in NGB, is included at the end of this appendix.

From Section II, the electron-electron collision term may be expressed as

$$\frac{\partial n(\varepsilon, t)}{\partial t}\bigg|_{ee} = -\frac{\partial J}{\partial \varepsilon} \tag{33}$$

where $n(\varepsilon,t)d\varepsilon$ is the number density of electrons with energies between ε and $\varepsilon+d\varepsilon$ at time t, and

$$J = \alpha \left[P\left(\frac{n}{2\varepsilon} - \frac{\partial n}{\partial \varepsilon}\right) - Q n \right] \tag{34}$$

$$P = 2\varepsilon^{-1/2} \int_0^\varepsilon x \, n(x,t) \, dx + 2\varepsilon \int_\varepsilon^\infty x^{-1/2} n(x,t) \, dx \tag{35}$$

$$Q = 3\varepsilon^{-1/2} \int_0^\varepsilon n(x,t) \, dx \tag{36}$$

$$\alpha = \frac{2}{3} \pi e^4 \left(\frac{2}{m}\right)^{1/2} \ln \Lambda \tag{37}$$

$$\Lambda = \frac{3}{2e^3} \left(\frac{K^3 T^3}{\pi n_e}\right)^{1/2} \tag{31}$$

and n_e = total number density of electrons
Units and numerical factors will be considered first.
Energies are assumed to be in electron volts (ev's) and
number densities per cubic centimeter (cm^{-3}). The other
quantities of interest must have the indicated units to
insure consistency:

$$n(\varepsilon,t) \sim cm^{-3} - ev^{-1}$$

$$\left.\frac{\partial n(\varepsilon,t)}{\partial t}\right|_{ee} \sim cm^{-3} - ev^{-1} - sec^{-1}$$

$$\alpha \sim cm^{-3} - ev^{3/2} - sec^{-1}$$

$$\Lambda \sim dimensionless$$

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From Eq (31), Λ can be rendered dimensionless by entering the electron charge in esu's, n_e in cm^{-3} and KT in ergs. Denoting the average energy by $\bar{\mathcal{E}}$, the result is

$$\Lambda = 4.15764 \times 10^{27} \left[\frac{\bar{\mathcal{E}}^{3/2} (ergs)^{3/2}}{n_e^{1/2} (cm^{-3})^{1/2}} \right]$$

Since 1 ev = 1.60210×10^{-12} ergs

$$\Lambda = 8.43105 \times 10^{9} \left[\frac{\bar{\mathcal{E}}^{3/2} (ev)^{3/2}}{n_e^{1/2} (cm^{-3})^{1/2}} \right]$$

and

$$\ln \Lambda = 22.85519 + \ln \left(\frac{\bar{\mathcal{E}}^{3/2}}{n_e^{1/2}} \right) \tag{59}$$

From Eq (37), with the electron mass entered in grams and the charge in esu's, the units for α are $cm^3-ergs^{3/2}-sec^{-1}$. After converting from ergs to ev's, the following result is obtained:

$$\alpha = (2.57540 \times 10^{-6}) \ln \Lambda \; (cm^3-ev^{3/2}-sec^{-1}) \tag{60}$$

Equation (33) represents only one term in the Boltzmann equation. Therefore, the numerical procedure used to evaluate this term must be consistent with the method used to solve the Boltzmann equation. A brief description of the method used in NGB is in order.

The Boltzmann equation is finite-differenced into a set of N discrete energy intervals of width $\Delta \mathcal{E}$. The resulting time dependent equation is then integrated using a forward-marching sixth order Gear-type algorithm. The initial distribution is Maxwellian with reasonable, but

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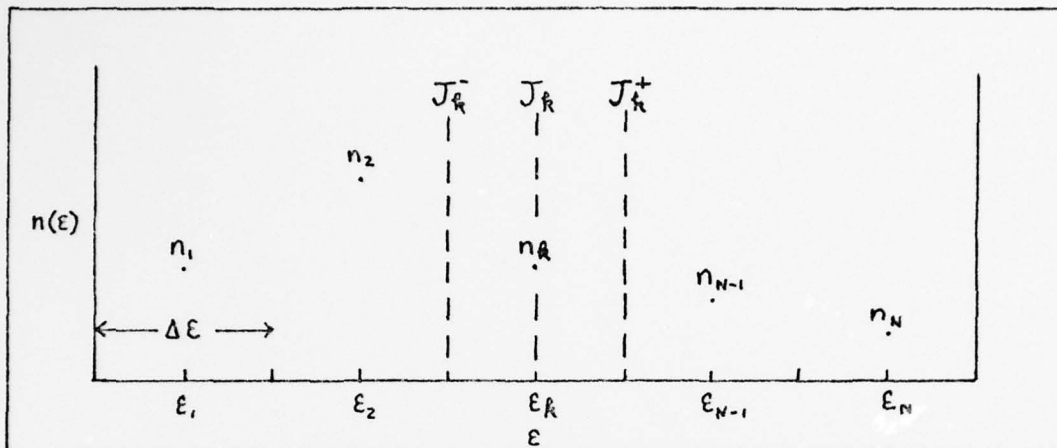
Fig. 9   Finite-Differenced Energy Axis

arbitrary, parameters. For further details, the Technical
Report describing NGB should be consulted.

Equation (33) must now be expressed in a form consis-
tent with the above differencing scheme. A diagram of the
energy axis is given in Fig. 9. J_k represents the electron
number density flux (along the energy axis) resulting from
electron-electron collisions.

The differenced forms of Eqs (33) through (36) are
given below:

$$\frac{\partial n_k}{\partial t} = - \frac{J_k^+ - J_k^-}{\Delta \varepsilon} \tag{61}$$

$$J_k^+ = \alpha \left[\left(\frac{P_{k+1} + P_k}{2} \right) \left(\frac{n_{k+1} + n_k}{4 \varepsilon_{k+1/2}} - \frac{n_{k+1} - n_k}{\Delta \varepsilon} \right) \right.$$
$$\left. - \left(\frac{Q_{k+1} + Q_k}{2} \right) \left(\frac{n_{k+1} + n_k}{2} \right) \right] \tag{62}$$

$$P_k = 2 \varepsilon_k^{-1/2} \sum_{i=1}^{k} \varepsilon_i n_i \Delta \varepsilon + 2 \varepsilon_k \sum_{i=k}^{N} \varepsilon_i^{-1/2} n_i \Delta \varepsilon \tag{63}$$

$$Q_k = 3 \varepsilon_k^{-1/2} \sum_{i=1}^{k} n_i \Delta \varepsilon \tag{64}$$

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To insure a constant total electron number density, the following relations are imposed:

$$J_1^- = J_N^+ = 0 \tag{65}$$

$$J_k^- = J_{k-1}^+ \tag{66}$$

Substituting Eq (62) into Eq (66) yields

$$J_k^- = \alpha \left[\left(\frac{P_k + P_{k-1}}{2} \right) \left(\frac{n_k + n_{k-1}}{4\,\varepsilon_{k-1/2}} - \frac{n_k - n_{k-1}}{\Delta\varepsilon} \right) \right.$$
$$\left. - \left(\frac{Q_k + Q_{k-1}}{2} \right) \left(\frac{n_k + n_{k-1}}{2} \right) \right] \tag{67}$$

Equations (62) and (67) are now substituted into Eq (61) to give

$$\dot{n}_1 = -\frac{\alpha}{2\Delta\varepsilon} \left[\left(P_2 + P_1 \right) \left(\frac{n_2 + n_1}{4\,\varepsilon_{3/2}} - \frac{n_2 - n_1}{\Delta\varepsilon} \right) - \left(Q_2 + Q_1 \right) \left(\frac{n_2 + n_1}{2} \right) \right] \qquad k = 1$$

$$\dot{n}_k = \frac{\alpha}{2\Delta\varepsilon} \left\{ \left[\left(P_k + P_{k-1} \right) \left(\frac{n_k + n_{k-1}}{4\,\varepsilon_{k-1/2}} - \frac{n_k - n_{k-1}}{\Delta\varepsilon} \right) \right. \right.$$
$$\left. - \left(Q_k + Q_{k+1} \right) \left(\frac{n_k + n_{k-1}}{2} \right) \right] - \left[\left(P_{k+1} + P_k \right) \left(\frac{n_{k+1} + n_k}{4\,\varepsilon_{k+1/2}} - \frac{n_{k+1} - n_k}{\Delta\varepsilon} \right) \right.$$
$$\left. \left. - \left(Q_{k+1} + Q_k \right) \left(\frac{n_{k+1} + n_k}{2} \right) \right] \right\} \qquad k = 2, N-1$$

$$\dot{n}_N = \frac{\alpha}{2\Delta\varepsilon} \left[\left(P_N + P_{N-1} \right) \left(\frac{n_N + n_{N-1}}{4\,\varepsilon_{N-1/2}} - \frac{n_N - n_{N-1}}{\Delta\varepsilon} \right) - \left(Q_N + Q_{N-1} \right) \left(\frac{n_N + n_{N-1}}{2} \right) \right] \quad k = N$$

where Eq (65) has been used for k=1 and k=N. If this
set of equations is expanded and the coefficients of the
various n_k collected, the following set of equations is
obtained:

$$\dot{n}_1 = -\frac{\alpha}{2\Delta\varepsilon}\left\{\left[(P_2+P_1)\left(\frac{1}{4\varepsilon_{3/2}}+\frac{1}{\Delta\varepsilon}\right)-\frac{Q_2+Q_1}{2}\right]n_1\right.$$
$$\left.+\left[(P_2+P_1)\left(\frac{1}{4\varepsilon_{3/2}}-\frac{1}{\Delta\varepsilon}\right)-\frac{Q_2+Q_1}{2}\right]n_2\right\}$$

$$\dot{n}_k = \frac{\alpha}{2\Delta\varepsilon}\left\{\left[(P_k+P_{k-1})\left(\frac{1}{4\varepsilon_{k-1/2}}+\frac{1}{\Delta\varepsilon}\right)-\frac{Q_k+Q_{k-1}}{2}\right]n_{k-1}\right.$$

$$+\left[(P_k+P_{k-1})\left(\frac{1}{4\varepsilon_{k-1/2}}-\frac{1}{\Delta\varepsilon}\right)-\frac{Q_k+Q_{k-1}}{2}\right.$$

$$\left.-(P_k+P_{k+1})\left(\frac{1}{4\varepsilon_{k+1/2}}+\frac{1}{\Delta\varepsilon}\right)+\frac{Q_k+Q_{k+1}}{2}\right]n_k$$

$$+\left[-(P_k+P_{k+1})\left(\frac{1}{4\varepsilon_{k+1/2}}-\frac{1}{\Delta\varepsilon}\right)+\frac{Q_k+Q_{k+1}}{2}\right]n_{k+1}\right\}$$

$$\dot{n}_N = \frac{\alpha}{2\Delta\varepsilon}\left\{\left[(P_N+P_{N-1})\left(\frac{1}{4\varepsilon_{N-1/2}}+\frac{1}{\Delta\varepsilon}\right)-\frac{Q_N+Q_{N-1}}{2}\right]n_{N-1}\right.$$

$$\left.+\left[(P_N+P_{N-1})\left(\frac{1}{4\varepsilon_{N-1/2}}-\frac{1}{\Delta\varepsilon}\right)-\frac{Q_N+Q_{N-1}}{2}\right]n_N\right\} \qquad (68)$$

This may be written more compactly by defining the following
quantities:

$$a'_k = \frac{\alpha}{2\Delta\varepsilon}\left\{(P_k+P_{k+1})\left(\frac{1}{4\varepsilon_{k+1/2}}+\frac{1}{\Delta\varepsilon}\right)-\frac{Q_k+Q_{k+1}}{2}\right\}$$

$$b'_k = -\frac{\alpha}{2\Delta\varepsilon}\left\{(P_k+P_{k-1})\left(\frac{1}{4\varepsilon_{k-1/2}}-\frac{1}{\Delta\varepsilon}\right)-\frac{Q_k+Q_{k-1}}{2}\right\} \qquad (69)$$

Using Eqs (69), Eqs (68) can be expressed in the following simple form:

$$\dot{n}_1 = b'_2 \, n_2 - a'_1 \, n_1$$

$$\dot{n}_k = a'_{k-1} \, n_{k-1} - (a'_k + b'_k) \, n_k + b'_{k+1} \, n_{k+1}$$

$$\dot{n}_N = a'_{N-1} \, n_{N-1} - b'_N \, n_N \tag{70}$$

Equation (70) is used to evaluate $\dot{n}_k$ in Funee. It remains to find explicit expressions for a'_k and b'_k. The following definitions considerably simplify the derivation:

$$\mu_k^{\pm} = \left(\frac{1}{\Delta \varepsilon} \pm \frac{0.25}{\varepsilon_{k+1/2}} \right) \tag{71}$$

$$H_{k,\ell} = \begin{cases} 0 & k < \ell \\ 1 & k \ge \ell \end{cases} \tag{72}$$

Using Eq (71), Eqs (69) may be written in the following form:

$$a'_k = \frac{\alpha}{2\Delta \varepsilon} \left\{ (P_k + P_{k+1}) \mu_k^+ - \frac{Q_k + Q_{k+1}}{2} \right\}$$

$$b'_k = \frac{\alpha}{2\Delta \varepsilon} \left\{ (P_k + P_{k-1}) \mu_{k-1}^- + \frac{Q_k + Q_{k-1}}{2} \right\} \tag{73}$$

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Using Eq (72), Eqs (63) and (64) become

$$P_k = 2\,\varepsilon_k^{-1/2}\Delta\varepsilon\sum_{\ell=1}^{N}\varepsilon_\ell\,n_\ell\,H_{k,\ell} + 2\,\varepsilon_k\,\Delta\varepsilon\sum_{\ell=1}^{N}\varepsilon_\ell^{-1/2}\,n_\ell\,(1-H_{k-1,\ell})$$

$$Q_k = 3\,\varepsilon_k^{-1/2}\,\Delta\varepsilon\sum_{\ell=1}^{N}n_\ell\,H_{k,\ell} \qquad (74)$$

Equations (73) can now be written in the following convenient form:

$$a'_k = \sum_{\ell=1}^{N}\alpha\left\{\left[\varepsilon_k^{-1/2}\varepsilon_\ell\,H_{k,\ell} + \varepsilon_k\,\varepsilon_\ell^{-1/2}(1-H_{k-1,\ell}) + \varepsilon_{k+1}^{-1/2}\varepsilon_\ell\,H_{k+1,\ell}\right.\right.$$

$$\left.\left. + \varepsilon_{k+1}\varepsilon_\ell^{-1/2}(1-H_{k,\ell})\right]\mu_k^{+} - 0.75\left(\varepsilon_k^{-1/2}H_{k,\ell} + \varepsilon_{k+1}^{-1/2}H_{k+1,\ell}\right)\right\}n_\ell \qquad (75)$$

$$b'_k = \sum_{\ell=1}^{N}\alpha\left\{\left[\varepsilon_k^{-1/2}\varepsilon_\ell\,H_{k,\ell} + \varepsilon_k\,\varepsilon_\ell^{-1/2}(1-H_{k-1,\ell}) + \varepsilon_{k-1}^{-1/2}\varepsilon_\ell\,H_{k-1,\ell}\right.\right.$$

$$\left.\left. + \varepsilon_{k-1}\varepsilon_\ell^{-1/2}(1-H_{k-2,\ell})\right]\mu_{k-1}^{-} + 0.75\left(\varepsilon_k^{-1/2}H_{k,\ell} + \varepsilon_{k-1}^{-1/2}H_{k-1,\ell}\right)\right\}n_\ell \qquad (76)$$

Defining two matrices $A_{k,\ell}$ and $B_{k,\ell}$, Eqs (75) and (76) become

$$a'_k = \sum_{\ell=1}^{N}A_{k,\ell}\,n_\ell \qquad\qquad b'_k = \sum_{\ell=1}^{N}B_{k,\ell}\,n_\ell \qquad (77)$$

where

$$A_{k,\ell} = \alpha \left\{ \left[\varepsilon_k^{-1/2} H_{k,\ell} + \varepsilon_{k+1}^{-1/2} H_{k+1,\ell} \right] \left[\varepsilon_\ell \mu_k^+ - 0.75 \right] \right.$$

$$\left. + \left[\varepsilon_k (1 - H_{k-1,\ell}) + \varepsilon_{k+1} (1 - H_{k,\ell}) \right] \varepsilon_\ell^{-1/2} \mu_k^+ \right\} \tag{78}$$

The corresponding relation for $B_{k,\ell}$ is not needed explicitly. This will become clear in what follows. If Eqs (77) are inserted into Eqs (70) and summed over all k, the resulting expression vanishes for any $A_{k,\ell}$ and $B_{k,\ell}$ whatsoever. Therefore

$$\frac{\partial n_e}{\partial t} = \frac{\partial}{\partial t} \sum_{k=1}^{N} n_k \Delta \varepsilon = \Delta \varepsilon \sum_{k=1}^{N} \dot{n}_k = 0$$

and the total electron number density remains constant. This important feature is retained by the differenced expressions, and is basically a result of the conditions imposed by Eqs (65) and (66).

Unfortunately, when Eqs (70) are multiplied by ε_k and summed over all k (with Eqs (77) inserted as before) the result is

$$\sum_{k=1}^{N} \varepsilon_k \dot{n}_k = \Delta \varepsilon \sum_{\ell=1}^{N} \left\{ \sum_{j=1}^{N} (A_{j,\ell} - B_{j,\ell}) - A_{N,\ell} n_N + B_{1,\ell} n_1 \right\} \tag{79}$$

Since

$$\frac{\partial \bar{\varepsilon}}{\partial t} = \frac{\partial}{\partial t} \sum_{k=1}^{N} \varepsilon_k n_k \Delta \varepsilon = \Delta \varepsilon \sum_{k=1}^{N} \varepsilon_k \dot{n}_k$$

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the average energy remains constant only if the right side
of Eq (79) vanishes. To insure that this feature is
retained by the differenced expressions, the following
conditions must be imposed:

$$B_{j,\ell} = A_{\ell,j} \tag{80}$$

$$B_{l,\ell} = 0 \quad \text{for all } \ell \tag{81}$$

$$A_{N,\ell} = 0 \quad \text{for all } \ell \tag{82}$$

Combining Eq (80) and Eq (81) leads to the additional
requirement:

$$A_{\ell,1} = 0 \quad \text{for all } \ell \tag{83}$$

The differenced expressions may now be reformulated in terms
of $A_{k,\ell}$, a'_k and b'_k. For convenience, the results so far
are summerized below:

$$\dot{n}_1 = b'_2 n_2 - a'_1 n_1$$

$$\dot{n}_k = a'_{k-1} n_{k-1} - (a'_k + b'_k) n_k + b'_{k+1} n_{k+1}$$

$$\dot{n}_N = a'_{N-1} n_{N-1} - b'_N n_N \tag{70}$$

where

$$a'_k = \sum_{\ell=1}^{N} A_{k,\ell}\, n_\ell \qquad b'_k = \sum_{\ell=1}^{N} A_{\ell,k}\, n_\ell \tag{84}$$

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and

$$A_{k,\ell} = \alpha \left\{ \left[\varepsilon_k^{-1/2} H_{k,\ell} + \varepsilon_{k+1}^{-1/2} H_{k+1,\ell} \right] \left[\varepsilon_\ell \mu_k^+ - 0.75 \right] \right.$$

$$\left. + \left[\varepsilon_k (1 - H_{k-1,\ell}) + \varepsilon_{k+1} (1 - H_{k,\ell}) \varepsilon_\ell^{-1/2} \mu_k^+ \right] \right\} \tag{78}$$

It was shown in Section III that

$$\frac{\partial \bar{n}(\varepsilon,t)}{\partial t} \bigg|_{ee} = 0$$

when $\bar{n}(\varepsilon,t)$ is Maxwellian. The differenced expressions developed thus far do not retain this property. Consider a Maxwellian distribution $\bar{n}_k$ where

$$\bar{n}_k \propto \varepsilon_k^{1/2} e^{-\varepsilon_k / KT}$$

and K is Boltzmann's constant. It is easily verified that

$$\bar{n}_\ell \bar{n}_k = \left(\frac{\varepsilon_\ell \varepsilon_k}{\varepsilon_{\ell-1} \varepsilon_{k+1}} \right)^{1/2} \bar{n}_{\ell-1} \bar{n}_{k+1} \tag{85}$$

for all ℓ and k. From Eqs (70) and (84)

$$\dot{\bar{n}}_1 = \sum_{\ell=1}^{N} A_{\ell,2} \bar{n}_\ell \bar{n}_2 - \sum_{\ell=1}^{N} A_{1,\ell} \bar{n}_\ell \bar{n}_1 \tag{86}$$

$$\dot{\bar{n}}_k = \sum_{\ell=1}^{N} A_{k-1,\ell} \bar{n}_\ell \bar{n}_{k-1} + \sum_{\ell=1}^{N} A_{\ell,k+1} \bar{n}_\ell \bar{n}_{k+1}$$

$$- \sum_{\ell=1}^{N} (A_{k,\ell} + A_{\ell,k}) \bar{n}_\ell \bar{n}_k \tag{87}$$

$$\dot{\bar{n}}_N = \sum_{\ell=1}^{N} A_{N-1,\ell} \bar{n}_\ell \bar{n}_{N-1} - \sum_{\ell=1}^{N} A_{\ell,N} \bar{n}_\ell \bar{n}_N \tag{88}$$

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The first sum in Eq (86) may be rewritten using Eq (85):

$$\sum_{\ell=1}^{N} A_{\ell,2}\, \bar{n}_\ell\, \bar{n}_2 = \sum_{\ell=1}^{N} A_{\ell,2} \left(\frac{\varepsilon_2\, \varepsilon_\ell}{\varepsilon_1\, \varepsilon_{\ell+1}} \right)^{1/2} \bar{n}_{\ell+1}\, \bar{n}_1$$

$$= \sum_{\ell=1}^{N-1} A_{\ell,2} \left(\frac{\varepsilon_2\, \varepsilon_\ell}{\varepsilon_1\, \varepsilon_{\ell+1}} \right)^{1/2} \bar{n}_{\ell+1}\, \bar{n}_1$$

$$= \sum_{\ell=2}^{N} A_{\ell-1,2} \left(\frac{\varepsilon_2\, \varepsilon_{\ell-1}}{\varepsilon_1\, \varepsilon_\ell} \right)^{1/2} \bar{n}_\ell\, \bar{n}_1$$

Since $A_{11} = 0$, the second sum in Eq (86) can be rewritten also:

$$\sum_{\ell=1}^{N} A_{1,\ell}\, \bar{n}_\ell\, \bar{n}_1 = \sum_{\ell=2}^{N} A_{1,\ell}\, \bar{n}_\ell\, \bar{n}_1$$

It follows that $\dot{\bar{n}}_1$ vanishes if:

$$A_{1,\ell} = A_{\ell-1,2} \left(\frac{\varepsilon_2\, \varepsilon_{\ell-1}}{\varepsilon_1\, \varepsilon_\ell} \right)^{1/2} \tag{89}$$

The same procedure may be applied to Eqs (87) and (88). The general expression, obtained from Eq (87), is

$$A_{k,\ell} = A_{\ell-1,\,k+1} \left(\frac{\varepsilon_{\ell-1}\, \varepsilon_{k+1}}{\varepsilon_\ell\, \varepsilon_k} \right)^{1/2} \tag{90}$$

Equation (90) is sufficient to guarantee that, when $\bar{n}_k$ is Maxwellian, $\dot{\bar{n}}_k = 0$ for all k. It is evident from Eq (90) that the $A_{k,\ell}$ are not independent. In fact, only those above the main diagonal need be calculated from Eq (78); the remaining terms are given by Eq (90). It should be

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mentioned that, while the preceeding arguments have been
strictly mathematical, identical results can be obtained
from more physical arguments (Ref 19).

In summary, given a set of n_k, $\dot{n}_k$ due to electron-
electron collisions is calculated using the following
equations:

$$\dot{n}_1 = b'_2 n_2 - a'_1 n_1 \qquad\qquad k=1$$

$$\dot{n}_k = a'_{k-1} n_{k-1} - (a'_k + b'_k) n_k + b_{k+1} n_{k+1} \quad k=2, N-1$$

$$\dot{n}_N = a'_{N-1} n_{N-1} - b'_N n_N \qquad\qquad k=N \qquad (70)$$

where

$$a'_k = \sum_{\ell=1}^{N} A_{k,\ell}\, n_\ell \qquad\qquad b'_k = \sum_{\ell=1}^{N} A_{\ell,k}\, n_\ell \qquad (84)$$

$$A_{k,\ell} = \alpha \left\{ \left[\varepsilon_k^{-1/2} H_{k,\ell} + \varepsilon_{k+1}^{-1/2} H_{k+1,\ell} \right] \left[\varepsilon_\ell \mu_k^+ - 0.75 \right] \right.$$

$$\left. + \left[\varepsilon_k (1 - H_{k-1,\ell}) + \varepsilon_{k+1} (1 - H_{k,\ell}) \right] \left[\varepsilon_\ell^{-1/2} \mu_k^+ \right] \right\} \qquad (78)$$

for $\ell > k$

$$A_{k,\ell} = A_{\ell-1, k+1} \left(\frac{\varepsilon_{\ell-1}\, \varepsilon_{k+1}}{\varepsilon_\ell\, \varepsilon_k} \right)^{1/2} \qquad (90)$$

for $\ell \leq k$

$$\mu_k^+ = \left(\frac{1}{\Delta \varepsilon} + \frac{0.25}{\varepsilon_{k+\frac{1}{2}}} \right) \tag{71}$$

$$H_{k,\ell} = \begin{cases} 0 & k < \ell \\ 1 & k \geq \ell \end{cases} \tag{72}$$

$$\alpha = (2.57540 \times 10^{-6}) \ln \Lambda \quad (\text{cm}^3\text{-ev}^{3/2}\text{-sec}^{-1}) \tag{60}$$

and

$$\ln \Lambda = 22.85519 + \ln \left(\frac{\bar{\varepsilon}^{3/2}}{n_e^{1/2}} \right) \tag{59}$$

This procedure strictly conserves total electron number density and average energy. For a Maxwellian, $\dot{\bar{n}}_k = 0$ for all k. Since the A matrix depends only on the differencing scheme and not on the distribution function, A is calculated only once and stored for subsequent calculations. As a result, execution time is minimal.

There are several disadvantages to this procedure. The size of the A matrix is given by the square of the number of energy intervals used. Thus, storage requirements may be a limiting factor. Secondly, it is difficult to estimate the error introduced by imposing conservation of average energy and stationarity of a Maxwellian.

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```
C********************************************************************
      SUBROUTINE FUNEE(FE,DFEDT2)
C********************************************************************
C     FUNEE COMPUTES THE TIME DERIVATIVE OF THE ELECTRON DIST FUNCTION
C     FE=ELECTRON DISTRIBUTION FUNCTION
C     DFEDT2=TIME DERIVATIVE OF FE DUE TO E-E COLLISIONS
C********************************************************************
      DIMENSION FE(1),DFEDT2(1)
      COMMON AP(40),BP(40),A(40,40)
      COMMON /MISC/ NBIN,DE
      UP(I)=(1+.25/I)/DE
      E(I)=(I-.5)*DE
      DATA INITIAL /0/
      ETOT=0.
      TOTN=0.
      NM=NBIN-1
      DO 100 I=1,NBIN
      ETOT=ETOT+FE(I)*(I-.5)*DE*DE
      TOTN=TOTN+FE(I)*DE
      DFEDT2(I)=0.
  100 CONTINUE
      IF(TOTN.LT.1.E10)RETURN
      EBAR=ETOT/TOTN
      ALAMDA=ALOG(EBAR**1.5/SQRT(TOTN))+22.855
      ALPHA=2.57956E-6*ALAMDA
      IF(INITIAL.GT.0)GO TO 500
      DO 200 L=2,NBIN
      LM=L-1
      DO 200 K=1,LM
      A(K,L)=((H(K+1,L)/SQRT(E(K+1))+H(K,L)/SQRT(E(K)))*(E(L)*UF(K))-.75
     1)+((1-H(K,L))*E(K+1)+(1-H(K-1,L))*E(K))*(UF(K)*(UF(K)/SQRT(E(L))))
  200 CONTINUE
      DO 300 I=1,NBIN
      A(NBIN,I)=0.
      A(I,1)=0.
  300 CONTINUE
```

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```
      DO 400 K=2,NM
      DO 400 L=2,K
      A(K,L)=A(L-1,K+1)*SQRT(((E(K+1)/E(K))*(E(L-1)/E(L)))
  400 CONTINUE
      WRITE 3
    3 FORMAT(//1HO,"*** ELECTRON-ELECTRON DIAGNOSTICS ***",//,1X)
      WRITE 1,EBAR,TOTN,ALAMDA,ALPHA
    1 FORMAT(1HO,10X,"EBAR=",E12.5,5X,"TOTN=",E12.5,5X,"LOG LAMDA=",E1
     12.5,5X,"ALPHA=",E12.5,//,60X,"A(K,L)")
      WRITE 2,((A(K,L),L=1,10),K=1,10)
    2 FORMAT((10E12.5),//)
      INITIAL=1
  500 DO 700 K=1,NBIN
      AP(K)=0.
      BP(K)=0.
      DO 600 L=1,NBIN
      AP(K)=AP(K)+A(K,L)*FE(L)
      BP(K)=BP(K)+A(L,K)*FE(L)
  600 CONTINUE
      AP(K)=AP(K)*ALPHA
      BP(K)=BP(K)*ALPHA
  700 CONTINUE
      DFEDT2(1)=BP(2)*FE(2)-AP(1)*FE(1)
      DFEDT2(NBIN)=AP(NM)*FE(NM)-BP(NBIN)*FE(NBIN)
      DO 800 I=2,NM
      DFEDT2(I)=AP(I-1)*FE(I-1)+BP(I+1)*FE(I+1)-(AP(I)+BP(I))*FE(I)
  800 CONTINUE
      RETURN
      END

      FUNCTION H(K,L)
      IF(K.LT.L)H=0
      IF(K.GE.L)H=1
      RETURN
      END
```

Appendix C

Numerical Solution of the Relaxation

Electron-Electron Collision Term

(Subroutine Funeer)

In this appendix, the numerical method used to evaluate the relaxation electron-electron collision term is presented. The basic problem is defining a Maxwellian such that the term retains the desired conservation properties.

The relaxation collision term is given by:

$$\frac{\partial n(\varepsilon)}{\partial t} = - \frac{n(\varepsilon) - \bar{n}(\varepsilon)}{\tau_{ce}} \tag{42}$$

where $\bar{n}(\varepsilon)$ is the Maxwellian distribution with the same average energy and number density as $n(\varepsilon)$. Equation (42) is easily cast into a finite-difference form compatible with the other terms of the Boltzmann equation. The result is

$$\frac{\partial n_i}{\partial t} = - \frac{n_i - \bar{n}_i}{\tau_{ce}} \tag{91}$$

An explicit expression must now be assumed for $\bar{n}_i$. This is not necessary in evaluating the Fokker-Planck term where only the ratio n_i/n_j is required. By analogy with the continuous Maxwellian

$$\bar{n}_i \equiv A \, \varepsilon_i^{1/2} \, e^{-\varepsilon_i/T} \tag{92}$$

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where A and T are constants to be determined.

To conserve total number density and average energy, the following requirements must be imposed:

$$\sum_{i=1}^{N} n_i = \sum_{i=1}^{N} \bar{n}_i$$

$$\sum_{i=1}^{N} \varepsilon_i n_i = \sum_{i=1}^{N} \varepsilon_i \bar{n}_i$$

The sums on the left of these equations are recognized as the total number density and average energy (to within a factor of $\Delta\varepsilon$). Therefore

$$\sum_{i=1}^{N} \bar{n}_i = {}^{n}e/_{\Delta\varepsilon} \qquad (93)$$

$$\sum_{i=1}^{N} \varepsilon_i \bar{n}_i = {}^{n_e}\bar{\varepsilon}/_{\Delta\varepsilon} \qquad (94)$$

From Fig. 8

$$\varepsilon_i = (i - \tfrac{1}{2})\Delta\varepsilon \qquad (95)$$

where $\Delta\varepsilon$ is the energy interval used. Using Eqs (92) and (95), Eqs (93) and (94) become

$$A \Delta\varepsilon^{1/2} e^{\Delta\varepsilon/2T} \sum_{i=1}^{N} (i-\tfrac{1}{2})^{1/2} e^{-i\Delta\varepsilon/T} = {}^{n}e/_{\Delta\varepsilon} \qquad (96)$$

$$A \Delta\varepsilon^{3/2} e^{\Delta\varepsilon/2T} \sum_{i=1}^{N} (i-\tfrac{1}{2})^{3/2} e^{-i\Delta\varepsilon/T} = {}^{n_e}\bar{\varepsilon}/_{\Delta\varepsilon} \qquad (97)$$

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Dividing Eq (97) by Eq (96), the result is

$$\bar{\varepsilon} = \frac{S_1}{S_2} \Delta \varepsilon \tag{98}$$

where

$$S_1 = \sum_{i=1}^{N} \left(i - \tfrac{1}{2}\right)^{3/2} e^{-i\Delta\varepsilon/T} \tag{99}$$

$$S_2 = \sum_{i=1}^{N} \left(i - \tfrac{1}{2}\right)^{1/2} e^{-i\Delta\varepsilon/T} \tag{100}$$

To conserve average energy, the value of T must be such that Eq (98) is satisfied. This is accomplished using an iterative procedure with the first estimate for T being approximately two-thirds the average energy. Subsequently, the last calculated value of T is used as a first estimate. For all calculations here, the value of T is accepted when the difference between the actual and calculated average energy is less than or equal to 10^{-10}. Knowing T, the value of A is easily calculated from either Eq (96) or Eq (97).

The finite-differenced Maxwellian is now given by Eq (92) and the relaxation term evaluated using Eq (91).

```
C*********************************************************************
      SUBROUTINE FUNEER(FE,DFEDT2)
C*********************************************************************
C     FUNEE COMPUTES THE TIME DERIVATIVE OF THE ELECTRON DIST FUNCTION
C     FE=ELECTRON DISTRIBUTION FUNCTION
C     DFEDT2=TIME DERIVATIVE OF FE DUE TO E-E COLLISIONS
C*********************************************************************
      DIMENSION FE(1),DFEDT2(1)
      COMMON AF(40)
      COMMON /MISC/ NBIN,DE
      E(I)=(I-.5)*DE
      BX(I)=SQRT(I-.5)
      EX(I)=BX(I)*(I-.5)
      DX(I)=EXP(-I*DE/T)
      DATA INITIAL /O/
      ETOT=0.
      TOTN=0.
      DO 100 I=1,NBIN
      ETOT=ETOT+FE(I)*E(I)*DE
      TOTN=TOTN+FE(I)*DE
      DFEDT2(I)=0.
100   CONTINUE
      EBAR=ETOT/TOTN
      IF(ABS(EBAR-ETEST).LE.1.E-10)GO TO 550
      ALAMDA=ALOG(EBAR**1.5/SQRT(TOTN))+22.855
      TCE=1.898E5*EBAR**1.5/(TOTN*ALAMDA)
      IF (INITIAL.EQ.O) T=EBAR*2./2.9
200   S1=0.
      S2=0.
      DO 300 I=1,NBIN
      S1=S1+EX(I)*DX(I)
      S2=S2+BX(I)*DX(I)
300   CONTINUE
      ETEST=DE*S1/S2
      IF(ABS(ETEST-EBAR).LE.1.E-10) GO TO 400
      T=T+EBAR-ETEST
```

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```
      GO TO 200
400   AX=TOTN/(S2*DE**1.5*EXP(DE/(2.*T)))
      DO 500 I=1,NBIN
      AF(I)=AX*SQRT(E(I))*EXP(-E(I)/T)
      DFEDT2(I)=(AF(I)-FE(I))/TCE
500   CONTINUE
      IF (INITIAL.NE.O) RETURN
      GO TO 700
550   DO 600 I=1,NBIN
      DFEDT2(I)=(AF(I)-FE(I))/TCE
600   CONTINUE
      RETURN
700   WRITE 1,TCE
    1 FORMAT(//1HO,"*** EL-EL DIAGNOSTICS***",//,1HO,
     110X,"SELF COLLISION TIME=",1PE12.5)
      INITIAL=1
      RETURN
      END
```

Appendix D

Cross Sections and Rates

The momentum transfer rate R_m can be expressed in terms of the momentum transfer cross section $\sigma_m(\varepsilon)$ as

$$R_m(\varepsilon) = \left(\frac{2\varepsilon}{m}\right)^{1/2} \sigma_m(\varepsilon)$$

where ε and m are the electron energy and mass respectively. Table I gives the momentum transfer cross sections used throughout this work. These cross sections have been constructed to yield a constant rate. Therefore, to compute this rate it is sufficient to consider only one value of the energy. For $\varepsilon = 1.25$ ev

$$R_m = \left(\frac{2 \times 1.25 \times 1.602 \times 10^{-12}}{9.109 \times 10^{-28}}\right)^{1/2} \left(8.940 \times 10^{-16}\right)$$

and

$$R_m = 5.928 \times 10^{-8} \text{cm}^3\text{-sec}^{-1} \tag{101}$$

Similarly, the excitation rate R can be expressed as

$$R(\varepsilon) = \left(\frac{2\varepsilon}{m}\right)^{1/2} \sigma_x(\varepsilon)$$

Table II gives the excitation cross sections used here. These have been constructed to give a zero rate below threshold and a constant rate R_0 above threshold. The threshold energy for all calculations is 8 ev. Using the

same procedure as above, for $\varepsilon = 8.25$ ev

$$R_o = \left(\frac{2 \times 8.25 \times 1.602 \times 10^{-12}}{9.109 \times 10^{-28}} \right)^{1/2} \left(1.0 \times 10^{-16} \right)$$

and

$$R_o = 1.703 \times 10^{-8} \text{ cm}^3 \text{ sec}^{-1} \tag{102}$$

Table I

Momentum-Transfer Cross Sections

ε(ev)	σ_m(cm^2)	ε(ev)	σ_m(cm^2)
.25	2.00×10^{-15}	10.25	3.12×10^{-16}
.75	2.16×10^{-15}	10.75	3.05×10^{-16}
1.25	8.94×10^{-16}	11.25	2.98×10^{-16}
1.75	7.56×10^{-16}	11.75	2.92×10^{-16}
2.25	6.67×10^{-16}	12.25	2.86×10^{-16}
2.75	6.03×10^{-16}	12.75	2.80×10^{-16}
3.25	5.55×10^{-16}	13.25	2.75×10^{-16}
3.75	5.15×10^{-16}	13.75	2.70×10^{-16}
4.25	4.85×10^{-16}	14.25	2.65×10^{-16}
4.75	4.59×10^{-16}	14.75	2.60×10^{-16}
5.25	4.36×10^{-16}	15.25	2.56×10^{-16}
5.75	4.17×10^{-16}	15.75	2.52×10^{-16}
6.25	4.00×10^{-16}	16.25	2.48×10^{-16}
6.75	3.85×10^{-16}	16.75	2.44×10^{-16}
7.25	3.71×10^{-16}	17.25	2.41×10^{-16}
7.75	3.59×10^{-16}	17.75	2.37×10^{-16}
8.25	3.48×10^{-16}	18.25	2.34×10^{-16}
8.75	3.38×10^{-16}	18.75	2.31×10^{-16}
9.25	3.29×10^{-16}	19.25	2.28×10^{-16}
9.75	3.20×10^{-16}	19.75	2.25×10^{-16}

Table II

Excitation Cross Sections

ε (ev)	$\sigma_x (cm^2)$	ε (ev)	$\sigma_x (cm^2)$
.25	0.00	10.25	9.00×10^{-17}
.75	0.00	10.75	8.80×10^{-17}
1.25	0.00	11.25	8.60×10^{-17}
1.75	0.00	11.75	8.40×10^{-17}
2.25	0.00	12.25	8.20×10^{-17}
2.75	0.00	12.75	8.00×10^{-17}
3.25	0.00	13.25	7.90×10^{-17}
3.75	0.00	13.75	7.70×10^{-17}
4.25	0.00	14.25	7.60×10^{-17}
4.75	0.00	14.75	7.50×10^{-17}
5.25	0.00	15.25	7.40×10^{-17}
5.75	0.00	15.75	7.20×10^{-17}
6.25	0.00	16.25	7.10×10^{-17}
6.75	0.00	16.75	7.00×10^{-17}
7.25	0.00	17.25	6.90×10^{-17}
7.75	0.00	17.75	6.80×10^{-17}
8.25	1.00×10^{-16}	18.25	6.70×10^{-17}
8.75	9.70×10^{-17}	18.75	6.60×10^{-17}
9.25	9.40×10^{-17}	19.25	6.50×10^{-17}
9.75	9.20×10^{-17}	19.75	6.50×10^{-17}

Appendix E

Analytic Solution Using the WKB Method

Equation (44) can be solved using a combination of the WKB method and variation of parameters. The WKB method yields the following solution to the associated homogeneous equation:

$$\tilde{n}_H(y) = \frac{1}{y}\left[c_1 e^{-A^{1/2}/y} + c_2 e^{A^{1/2}/y} \right]$$

where c_1 and c_2 are arbitrary constants of integration. Using variation of parameters, a particular solution is given by

$$\tilde{n}_P(y) = \frac{1}{y}\left[V_1(y) e^{-A^{1/2}/y} + V_2(y) e^{A^{1/2}/y} \right]$$

where

$$V_1(y) = -\int \frac{f_2(y)h(y)}{W[f_1, f_2]}\, dy$$

$$V_2(y) = \int \frac{f_1(y)h(y)}{W[f_1, f_2]}\, dy$$

and

$$f_1(y) = \frac{1}{y} e^{-A^{1/2}/y}$$

$$f_2(y) = \frac{1}{y} e^{A^{1/2}/y}$$

$$W[f_1, f_2] = f_1 f_2' - f_2 f_1'$$

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Making the appropriate substitutions, $V_1(y)$ and $V_2(y)$ may be expressed as

$$V_1(y) = -\frac{12\,a}{A^{1/2}\,\dot{\varepsilon}\,\tau_{ce}} \int \frac{1}{y}\, e^{A^{1/2}/y - 4b^2/y^2}\, dy \qquad (103)$$

$$V_2(y) = \frac{12\,a}{A^{1/2}\,\dot{\varepsilon}\,\tau_{ce}} \int \frac{1}{y}\, e^{-A^{1/2}/y - 4b^2/y^2}\, dy$$

Clearly $V_2(y) = -V_1(-y)$, and the particular solution may be expressed as

$$\tilde{n}_p(y) = \frac{1}{y}\left[V_1(y)\, e^{-A^{1/2}/y} - V_1(-y)\, e^{A^{1/2}/y} \right]$$

Therefore, the general solution to Eq (44) is

$$\tilde{n}(y) = \frac{1}{y}\left[C_2\, e^{A^{1/2}/y} + V_1(y)\, e^{-A^{1/2}/y} - V_1(-y)\, e^{A^{1/2}/y} \right] \qquad (104)$$

where the term involving C_1 has been disregarded. This term is unbounded and physically unacceptable.

Attempts to perform the integration in Eq (103) have been unsuccessful. Furthermore, if the terms involving $V_1(y)$ are ignored, the constant in Eq (104) cannot be evaluated. Even if this obstacle could be overcome, a severe computational difficulty is encountered; the constant A in the exponent is generally a very large number. Therefore, the individual terms in Eq (104) are at best difficult to calculate. For these reasons, a more straightforward method is used to solve Eq (44) in Section III.

Appendix F

Steady-State Distributions (Numerical Results)

This appendix contains the numerically calculated steady-state distribution functions. Appendices B and C contain details of the numerical procedures. Plots of the normalized distributions, $\tilde{n}(\varepsilon)$, are presented in Figs. 6 through 8.

Table III

Fokker-Planck Steady-State Dist. $-n_e/n_o = 10^{-5}$, $E/n_o = 10^{-18}$, $\bar{\epsilon} = .961$

$\epsilon(ev)$	$n(cm^{-3}ev^{-1})$	$\tilde{n}(ev^{-3/2})$	$\epsilon(ev)$	$n(cm^{-3}ev^{-1})$	$\tilde{n}(ev^{-3/2})$
.25	6.98×10^{13}	1.40×10^{0}	10.25	2.73×10^{0}	8.54×10^{-15}
.75	5.66×10^{13}	6.53×10^{-1}	10.75	4.11×10^{-2}	1.25×10^{-16}
1.25	3.42×10^{13}	3.06×10^{-1}	11.25	6.04×10^{-4}	1.80×10^{-18}
1.75	1.89×10^{13}	1.43×10^{-1}	11.75	8.69×10^{-6}	2.54×10^{-20}
2.25	1.00×10^{13}	6.69×10^{-2}	12.25	1.23×10^{-7}	3.52×10^{-22}
2.75	5.19×10^{12}	3.13×10^{-2}	12.75	1.72×10^{-9}	4.80×10^{-24}
3.25	2.64×10^{12}	1.46×10^{-2}	13.25	2.33×10^{-11}	6.40×10^{-26}
3.75	1.33×10^{12}	6.85×10^{-3}	13.75	3.13×10^{-13}	8.45×10^{-28}
4.25	6.60×10^{11}	3.20×10^{-3}	14.25	4.12×10^{-15}	1.09×10^{-29}
4.75	3.26×10^{11}	1.49×10^{-3}	14.75	5.30×10^{-17}	1.38×10^{-31}
5.25	1.59×10^{11}	6.96×10^{-4}	15.25	6.69×10^{-19}	1.71×10^{-33}
5.75	7.72×10^{10}	3.22×10^{-4}	15.75	8.40×10^{-21}	2.12×10^{-35}
6.25	3.67×10^{10}	1.47×10^{-4}	16.25	1.04×10^{-22}	2.58×10^{-37}
6.75	1.68×10^{10}	6.48×10^{-5}	16.75	1.26×10^{-24}	3.08×10^{-39}
7.25	7.13×10^{9}	2.65×10^{-5}	17.25	1.51×10^{-26}	3.64×10^{-41}
7.75	2.38×10^{9}	8.54×10^{-6}	17.75	1.79×10^{-28}	4.24×10^{-43}
8.25	4.10×10^{7}	1.43×10^{-7}	18.25	2.09×10^{-30}	4.88×10^{-45}
8.75	6.86×10^{5}	2.32×10^{-9}	18.75	2.41×10^{-32}	5.56×10^{-47}
9.25	1.12×10^{4}	3.68×10^{-11}	19.25	2.75×10^{-34}	6.26×10^{-49}
9.75	1.77×10^{2}	5.68×10^{-13}	19.75	3.09×10^{-36}	6.96×10^{-51}

Table IV

Relaxation Steady-State Dist. — $n_e/n_o = 10^{-5}$, $E/n_o = 10^{-18}$, $\bar{\varepsilon} = .934$

$\varepsilon(ev)$	$n(cm^{-3}ev^{-1})$	$\tilde{n}(ev^{-3/2})$
.25	7.20×10^{13}	1.44×10^{0}
.75	5.71×10^{13}	6.59×10^{-1}
1.25	3.37×10^{13}	3.02×10^{-1}
1.75	1.83×10^{13}	1.38×10^{-1}
2.25	9.48×10^{12}	6.32×10^{-2}
2.75	4.80×10^{12}	2.89×10^{-2}
3.25	2.39×10^{12}	1.32×10^{-2}
3.75	1.17×10^{12}	6.06×10^{-3}
4.25	5.72×10^{11}	2.77×10^{-3}
4.75	2.77×10^{11}	1.27×10^{-3}
5.25	1.33×10^{11}	5.81×10^{-4}
5.75	6.38×10^{10}	2.66×10^{-4}
6.25	3.04×10^{10}	1.22×10^{-4}
6.75	1.45×10^{10}	5.57×10^{-5}
7.25	6.86×10^{9}	2.55×10^{-5}
7.75	3.25×10^{9}	1.17×10^{-5}
8.25	3.53×10^{7}	1.25×10^{-7}
8.75	1.68×10^{7}	5.70×10^{-8}
9.25	7.95×10^{6}	2.62×10^{-8}
9.75	3.72×10^{6}	1.19×10^{-8}

$\varepsilon(ev)$	$n(cm^{-3}ev^{-1})$	$\tilde{n}(ev^{-3/2})$
10.25	1.74×10^{6}	5.44×10^{-9}
10.75	8.15×10^{5}	2.48×10^{-9}
11.25	3.82×10^{5}	1.14×10^{-9}
11.75	1.79×10^{5}	5.21×10^{-10}
12.25	8.38×10^{4}	2.39×10^{-10}
12.75	3.93×10^{4}	1.10×10^{-10}
13.25	1.82×10^{4}	5.00×10^{-11}
13.75	8.55×10^{3}	2.31×10^{-11}
14.25	3.97×10^{3}	1.05×10^{-11}
14.75	1.84×10^{3}	4.79×10^{-12}
15.25	8.53×10^{2}	2.19×10^{-12}
15.75	4.01×10^{2}	1.01×10^{-12}
16.25	1.86×10^{2}	4.62×10^{-13}
16.75	8.64×10^{1}	2.11×10^{-13}
17.25	4.01×10^{1}	9.66×10^{-14}
17.75	1.86×10^{1}	4.42×10^{-14}
18.25	8.66×10^{0}	2.03×10^{-14}
18.75	4.02×10^{0}	9.29×10^{-15}
19.25	1.87×10^{0}	4.26×10^{-15}
19.75	8.56×10^{-1}	1.93×10^{-15}

Table V

Fokker-Planck Steady-State Dist. $-n_e/n_o = 10^{-3}$, $E/n_o = 10^{-18}$, $\bar{\epsilon} = .732$

$\epsilon(ev)$	$n(cm^{-3}ev^{-1})$	$\tilde{n}(ev^{-3/2})$	$\epsilon(ev)$	$n(cm^{-3}ev^{-1})$	$\tilde{n}(ev^{-3/2})$
.25	9.19×10^{15}	1.84×10^{0}	10.25	7.19×10^{6}	2.25×10^{-10}
.75	5.89×10^{15}	6.80×10^{-1}	10.75	1.53×10^{6}	4.66×10^{-11}
1.25	2.82×10^{15}	2.52×10^{-1}	11.25	3.22×10^{5}	9.59×10^{-12}
1.75	1.23×10^{15}	9.33×10^{-2}	11.75	6.71×10^{4}	1.96×10^{-12}
2.25	5.13×10^{14}	3.45×10^{-2}	12.25	1.39×10^{4}	3.97×10^{-13}
2.75	2.12×10^{14}	1.26×10^{-2}	12.75	2.85×10^{3}	7.99×10^{-14}
3.25	8.54×10^{13}	4.74×10^{-3}	13.25	5.79×10^{2}	1.59×10^{-14}
3.75	3.40×10^{13}	1.75×10^{-3}	13.75	1.17×10^{2}	3.16×10^{-15}
4.25	1.34×10^{13}	6.49×10^{-4}	14.25	2.34×10^{1}	6.20×10^{-16}
4.75	5.24×10^{12}	2.40×10^{-4}	14.75	4.64×10^{0}	1.21×10^{-16}
5.25	2.04×10^{12}	8.89×10^{-5}	15.25	9.11×10^{-1}	2.33×10^{-17}
5.75	7.88×10^{11}	3.29×10^{-5}	15.75	1.78×10^{-1}	4.49×10^{-18}
6.25	3.02×10^{11}	1.21×10^{-5}	16.25	3.47×10^{-2}	8.60×10^{-19}
6.75	1.14×10^{11}	4.40×10^{-6}	16.75	6.68×10^{-3}	1.63×10^{-19}
7.25	4.18×10^{10}	1.55×10^{-6}	17.25	1.28×10^{-3}	3.08×10^{-20}
7.75	1.39×10^{10}	5.00×10^{-7}	17.75	2.43×10^{-4}	5.78×10^{-21}
8.25	3.14×10^{9}	1.09×10^{-7}	18.25	4.60×10^{-5}	1.03×10^{-21}
8.75	6.98×10^{8}	2.36×10^{-8}	18.75	8.66×10^{-6}	2.00×10^{-22}
9.25	1.54×10^{8}	5.06×10^{-9}	19.25	1.63×10^{-6}	3.73×10^{-23}
9.75	3.35×10^{7}	1.07×10^{-9}	19.75	3.33×10^{-7}	7.49×10^{-24}

Table VI

Relaxation Steady-State Dist. $-n_e/n_o = 10^{-3}$, $\varepsilon/n_o = 10^{-18}$, $\bar{\varepsilon} = .772$

ε(ev)	n(cm^{-3}ev^{-1})	$\tilde{n}$(ev$^{-3/2}$)	ε(ev)	n(cm^{-3}ev^{-1})	$\tilde{n}$(ev$^{-3/2}$)
.25	8.73×10^{15}	1.75×10^{0}	10.25	2.38×10^{8}	7.43×10^{-9}
.75	5.89×10^{15}	6.80×10^{-1}	10.75	9.49×10^{7}	2.89×10^{-9}
1.25	2.96×10^{15}	2.65×10^{-1}	11.25	3.78×10^{7}	1.13×10^{-9}
1.75	1.37×10^{15}	1.03×10^{-1}	11.75	1.51×10^{7}	4.39×10^{-10}
2.25	6.03×10^{14}	4.02×10^{-2}	12.25	5.99×10^{6}	1.71×10^{-10}
2.75	2.60×10^{14}	1.57×10^{-2}	12.75	2.39×10^{6}	6.68×10^{-11}
3.25	1.10×10^{14}	6.10×10^{-3}	13.25	9.45×10^{5}	2.60×10^{-11}
3.75	4.60×10^{13}	2.38×10^{-3}	13.75	3.76×10^{5}	1.01×10^{-11}
4.25	1.91×10^{13}	9.25×10^{-4}	14.25	1.49×10^{5}	3.94×10^{-12}
4.75	7.85×10^{12}	3.60×10^{-4}	14.75	5.89×10^{4}	1.53×10^{-12}
5.25	3.22×10^{12}	1.40×10^{-4}	15. 5	2.33×10^{4}	5.97×10^{-13}
5.75	1.31×10^{12}	5.47×10^{-5}	15.75	9.26×10^{3}	2.33×10^{-13}
6.25	5.32×10^{11}	2.13×10^{-5}	16.25	3.66×10^{3}	9.08×10^{-14}
6.75	2.15×10^{11}	8.29×10^{-6}	16.75	1.45×10^{3}	3.54×10^{-14}
7.25	8.70×10^{10}	3.23×10^{-6}	17.25	5.72×10^{2}	1.38×10^{-14}
7.75	3.50×10^{10}	1.26×10^{-6}	17.75	2.26×10^{2}	5.37×10^{-15}
8.25	9.28×10^{9}	3.23×10^{-7}	18.25	8.93×10^{1}	2.09×10^{-15}
8.75	3.73×10^{9}	1.26×10^{-7}	18.75	3.53×10^{1}	8.15×10^{-16}
9.25	1.49×10^{9}	4.91×10^{-8}	19.25	1.39×10^{1}	3.18×10^{-16}
9.75	5.96×10^{8}	1.91×10^{-8}	19.75	5.47×10^{0}	1.23×10^{-16}

Table VII

Fokker-Planck Steady-State Dist. $-n_e/n_o = 10^{-3}$, $\mathcal{E}/n_o = 10^{-16}$, $\bar{\mathcal{E}} = 1.89$

$\varepsilon(ev)$	$n(cm^{-3}ev^{-1})$	$\tilde{n}(ev^{-3/2})$	$\varepsilon(ev)$	$n(cm^{-3}ev^{-1})$	$\tilde{n}(ev^{-3/2})$
.25	3.08×10^{15}	6.15×10^{-1}	10.25	1.47×10^{12}	4.58×10^{-5}
.75	3.64×10^{15}	4.20×10^{-1}	10.75	7.53×10^{11}	2.30×10^{-5}
1.25	3.21×10^{15}	2.87×10^{-1}	11.25	3.85×10^{11}	1.15×10^{-5}
1.75	2.60×10^{15}	1.97×10^{-1}	11.75	1.97×10^{11}	5.75×10^{-6}
2.25	2.02×10^{15}	1.35×10^{-1}	12.25	1.01×10^{11}	2.88×10^{-6}
2.75	1.53×10^{15}	9.20×10^{-2}	12.75	5.14×10^{10}	1.44×10^{-6}
3.25	1.13×10^{15}	6.29×10^{-2}	13.25	2.62×10^{10}	7.19×10^{-7}
3.75	8.32×10^{14}	4.30×10^{-2}	13.75	1.33×10^{10}	3.59×10^{-7}
4.25	6.03×10^{14}	2.93×10^{-2}	14.25	6.77×10^{9}	1.79×10^{-7}
4.75	4.33×10^{14}	1.99×10^{-2}	14.75	3.44×10^{9}	8.95×10^{-8}
5.25	3.08×10^{14}	1.34×10^{-2}	15.25	1.75×10^{9}	4.47×10^{-8}
5.75	2.16×10^{14}	9.00×10^{-3}	15.75	8.87×10^{8}	2.24×10^{-8}
6.25	1.49×10^{14}	5.95×10^{-3}	16.25	4.51×10^{8}	1.12×10^{-8}
6.75	1.00×10^{14}	3.85×10^{-3}	16.75	2.29×10^{8}	5.60×10^{-9}
7.25	6.47×10^{13}	2.40×10^{-3}	17.25	1.16×10^{8}	2.80×10^{-9}
7.75	3.91×10^{13}	1.40×10^{-3}	17.75	5.94×10^{7}	1.41×10^{-9}
8.25	2.04×10^{13}	7.11×10^{-4}	18.25	3.05×10^{7}	7.13×10^{-10}
8.75	1.06×10^{13}	3.60×10^{-4}	18.75	1.59×10^{7}	3.67×10^{-10}
9.25	5.52×10^{12}	1.81×10^{-4}	19.25	8.69×10^{6}	1.98×10^{-10}
9.75	2.85×10^{12}	9.12×10^{-5}	19.75	5.30×10^{6}	1.19×10^{-10}

Table VIII

Relaxation Steady-State Dist. $n_e/n_o = 10^{-5}$, $\mathcal{E}/n_o = 10^{-16}$, $\bar{\mathcal{E}} = 1.80$

$\mathcal{E}(ev)$	$n(cm^{-3}ev^{-1})$	$\tilde{n}(ev^{-3/2})$	$\mathcal{E}(ev)$	$n(cm^{-3}ev^{-1})$	$\tilde{n}(ev^{-3/2})$
.25	3.35×10^{15}	6.70×10^{-1}	10.25	2.52×10^{12}	7.87×10^{-5}
.75	3.83×10^{15}	4.43×10^{-1}	10.75	1.71×10^{12}	5.22×10^{-5}
1.25	3.30×10^{15}	2.95×10^{-1}	11.25	1.16×10^{12}	3.47×10^{-5}
1.75	2.60×10^{15}	1.97×10^{-1}	11.75	7.91×10^{11}	2.31×10^{-5}
2.25	1.96×10^{15}	1.31×10^{-1}	12.25	5.37×10^{11}	1.53×10^{-5}
2.75	1.45×10^{15}	8.73×10^{-2}	12.75	3.65×10^{11}	1.02×10^{-5}
3.25	1.05×10^{15}	5.82×10^{-2}	13.25	2.47×10^{11}	6.78×10^{-6}
3.75	7.51×10^{14}	3.88×10^{-2}	13.75	1.67×10^{11}	4.51×10^{-6}
4.25	5.33×10^{14}	2.58×10^{-2}	14.25	1.13×10^{11}	3.00×10^{-6}
4.75	3.75×10^{14}	1.72×10^{-2}	14.75	7.63×10^{10}	1.99×10^{-6}
5.25	2.63×10^{14}	1.15×10^{-2}	15.25	5.15×10^{10}	1.32×10^{-6}
5.75	1.84×10^{14}	7.65×10^{-3}	15.75	3.49×10^{10}	8.80×10^{-7}
6.25	1.27×10^{14}	5.10×10^{-3}	16.25	2.36×10^{10}	5.85×10^{-7}
6.75	8.81×10^{13}	3.39×10^{-3}	16.75	1.59×10^{10}	3.89×10^{-7}
7.25	6.00×10^{13}	2.23×10^{-3}	17.25	1.07×10^{10}	2.58×10^{-7}
7.75	3.83×10^{13}	1.37×10^{-3}	17.75	7.23×10^{9}	1.72×10^{-7}
8.25	1.49×10^{13}	5.20×10^{-4}	18.25	4.88×10^{9}	1.14×10^{-7}
8.75	8.49×10^{12}	2.87×10^{-4}	18.75	3.30×10^{9}	7.62×10^{-8}
9.25	5.53×10^{12}	1.82×10^{-4}	19.25	2.27×10^{9}	5.17×10^{-8}
9.75	3.72×10^{12}	1.19×10^{-4}	19.75	1.68×10^{9}	3.78×10^{-8}

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The electron-electron collision term of the steady-state Boltzmann equation is replaced by a simple relaxation term that effectively linearizes the equation. Additional assumptions are made to simplify the equation further. Ionization is ignored and a two-level atom is assumed. The effects of a DC electric field are included. Using the relaxation term to account for electron-electron collisions, an approximate analytic solution is derived. Temporal and steady-state characteristics of the relaxation term

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are compared to those of the standard Fokker-Planck term. The relaxation term is judged invalid for energies greater than the excitation threshold.