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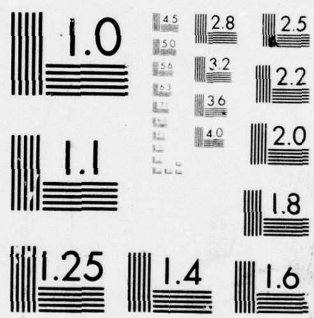

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MRC Technical Summary Report #1884

COMPLEMENTARY ENERGY AND CATASTROPHES.

M. J. Sewell

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Mathematics Research Center
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ABSTRACT

The connection between complementary energy and catastrophes is indicated by a simple example, and by referring to work which shows how Legendre transformations can be continued through their singularities where the Hessian is zero. An experiment is described giving a mapping of the plane onto the plane whose stable singularities appear to be folds and angles, rather than folds and cusps.

The work is directed towards the general problem of how integrands in dual variational principles can pass through their singularities.

The material forms the basis of a lecture to the I.U.T.A.M. Symposium on Variational Methods in the Mechanics of Solids, Northwestern University, U.S.A., September 1978, and will be published by Pergamon Press in Proceedings edited by S. Nemat-Nasser.

AMS (MOS) Subject Classifications: 49.00, 49C99, 58E15, 69.49, 73.49.

Key Words: Legendre transformation; Catastrophes; Singularities; Mappings; Structural stability; Complementary energy.

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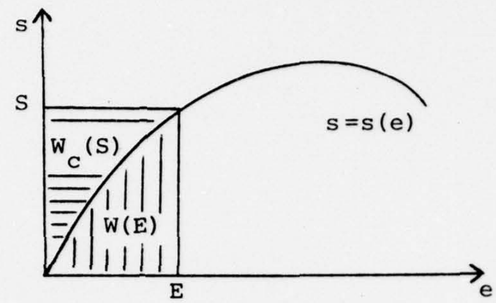
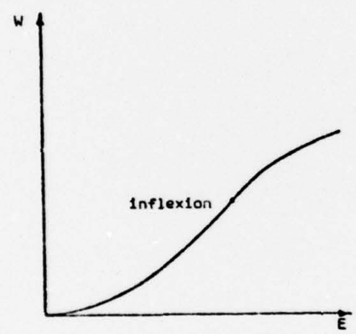
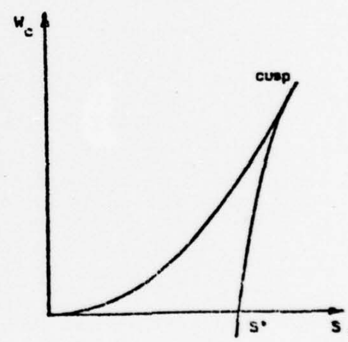
SIGNIFICANCE AND EXPLANATION

When "necking" occurs in a bar under tension, the curve of nominal stress s against strain e looks like:



The quantity $W(E)$ is the energy in the bar when the strain is E, represented by the vertically hatched area; the quantity $W_c(S)$ is the complementary energy in the system, represented by the horizontally hatched area. The graph of $W(E)$ against E looks like the picture on the left below; the plot of $W_c(S)$ against S looks like the plot on the right.





The point that the graph on the right illustrates is that the complementary energy in nonlinear elasticity can be a multi-valued function. It is of some interest to know the precise qualitative way in which the different branches fit together, and this is deduced in a simple example. The result is prompted by ideas in catastrophe theory.

From a different but related point of view, the simplest singularities that are "structurally stable" in the mathematical sense are those that can occur when mapping a plane onto itself. An experiment is described (involving crumpling a plastic sheet on to a plane) that gives some insight into these mathematical ideas.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

COMPLEMENTARY ENERGY AND CATASTROPHES

M. J. Sewell

1 INTRODUCTION

The purpose of these brief remarks is to draw attention to the way in which some rudimentary knowledge of the elementary catastrophes may be used to focus on the problem of finding a complementary energy in situations where the original response curve (for example expressing stress in terms of strain) is not monotonic. The work is only at a preliminary stage, and we shall confine attention to a setting of the problem in terms of what is now known from this author's work [1-3] about continuing a Legendre transformation through the singularity where its Jacobian vanishes.

One of the most familiar examples of a Legendre transformation is expressed by the equations

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \iff \dot{q}_i = \frac{\partial H}{\partial p_i}$$

allowing one to pass between the Lagrangian function $L[\dot{q}_i]$ and the Hamiltonian function $H[p_i]$ of a classical discrete system, with n momenta p_i and n velocities $\dot{q}_i$. Position q_i and time t may appear passively in these functions. Standard texts frequently make the tacit assumption

$$\left| \frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_j} \right| \neq 0$$

that the transformation is non-singular without any discussion. By a slight extension of modern terminology we could say that $L[\dot{q}_i]$ is a Morse function at such a non-singular point, but is non-Morse where the determinant vanishes.

Closed chains of four Legendre transformations, linking four generating functions, are found in some subjects. Sewell [3,4] indicates such chains in

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classical mechanics, circuit theory, thermostatics and plastic constitutive equations. The classical thermodynamic potentials of free and internal energy, enthalpy and free enthalpy, are linked by one such chain. In this particular context several authors have independently found catastrophe theory helpful in discussing singularities in the chains, and associated phase transitions, and Poston and Stewart [5, Ch. 14] give one such viewpoint.

2 ONE-DIMENSIONAL EXAMPLE

Suppose we have a context in which there is a response curve $s = s(e)$ relating two variables which fails to be monotonic in the manner illustrated in Fig. 1. For convenience we shall call s stress and e strain. We have in mind that s may be a 'nominal' stress (load per unit of some initial area) instead of 'true' stress (load per unit current area), but we do not at this stage make a precise choice from among the many definitions of stress and 'conjugate' strain contemplated by Hill [6] and others in the modern continuum mechanics literature. There is some evidence, lately reviewed by Hudson, Crouch and Fairhurst [7], that stress-strain curves having the general shape of Fig. 1 occur in rock mechanics, for example.

The 'strain energy' function $W(E)$ is the area under the curve up to the value E of e, i.e.

$$W(E) = \int_0^E s(e)de \ .$$

In the range shown in Fig. 1 this has derivatives

$$\frac{dw}{dE} = s(E) > 0, \qquad \frac{d^2W}{dE^2} = \frac{ds}{dE}, \qquad \frac{d^3W}{dE^3} = \frac{d^2s}{dE^2} < 0$$

and so the strain energy function $W(E)$ is rising but with an inflexion associated with the stationary maximum of $s(e)$, as shown in Fig. 2.

Let S denote the value of $s(e)$ at $e = E$. The 'complementary energy' function $W_c(S)$ is defined to be the area specified again by E but this time

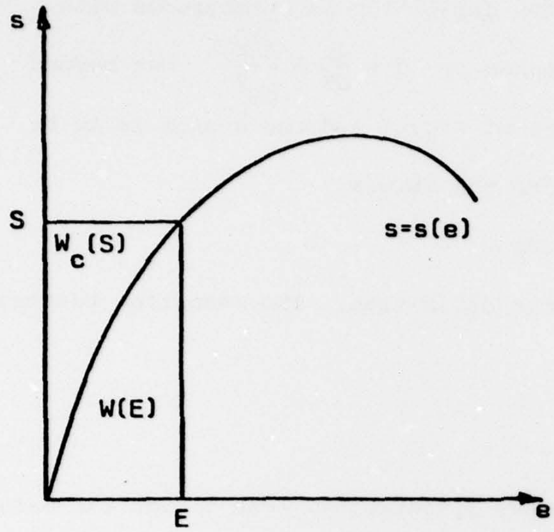

Fig. 1

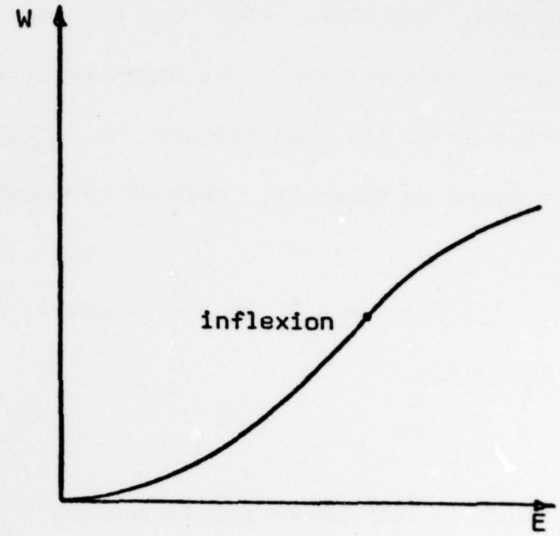

Fig. 2

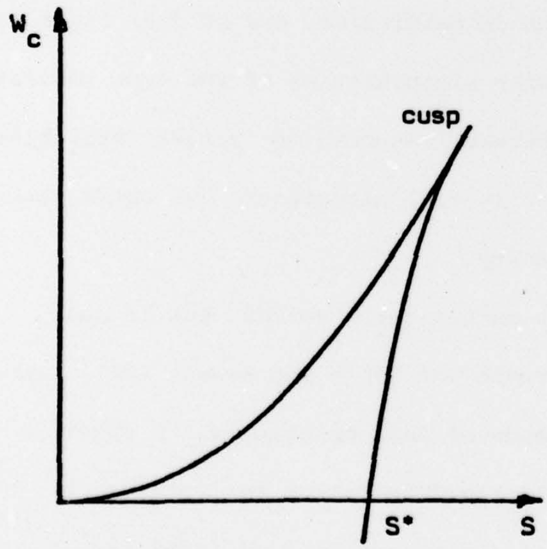

Fig. 3

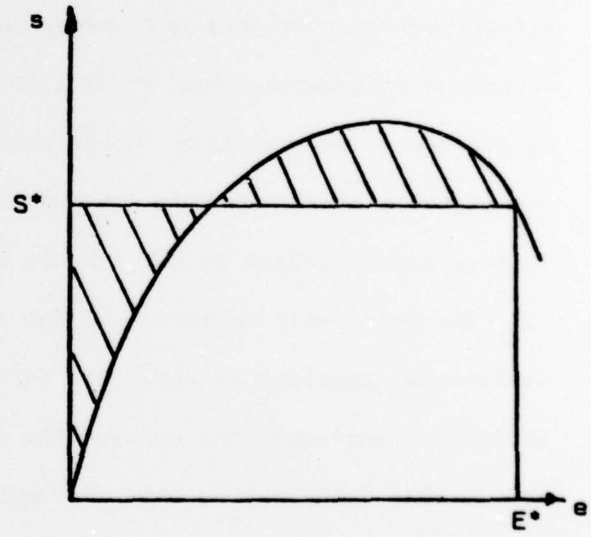

Fig. 4

between the curve $s(e)$ and the s-axis. The definition is unambiguous with positive areas until the singularity is reached at $0 = \frac{ds}{dE} = \frac{d^2W}{dE^2}$, but beyond that point the area between the falling part of Fig. 1 and the s-axis is to be counted as negative. This is consistent with the values

$$W_c = SE - W(E)$$

assigned to $W_c(S)$ by the Legendre transform definition. The resulting function has slope

$$\frac{dW_c}{dS} = E$$

which is continuous and increasing through the singularity, even though its value regresses from there like that of S. It follows from these facts that the complementary energy function $W_c(S)$ is cusp-shaped as shown in Fig. 3.

The value S^* where $W_c(S^*) = 0$ corresponds to the particular E^* for which the two shaded areas of Fig. 4 are equal in magnitude.

This elementary analysis suggests that the presence of a cusp may be an intrinsic feature of complementary energy functions in particular, and of dual Legendre functions in general, when we have to deal with singularities of the type indicated by failure of monotonicity in the relation between response or 'active' variables. The Legendre transformation does not necessarily fail altogether, but extra care is required in taking it through the singularity.

The point will be pertinent not only in mechanics of solids, but in dual variational problems in any field (cf. the review of Noble and Sewell [8]), since Legendre transformations connect the integrands of dual principles. A specific calculation illustrating this in compressible fluid mechanics is described by Sewell [2]. In that context $-W$ and E are fluid pressure and speed respectively, and the thermodynamics leads to a function $W[E]$ which has the inflexion of Fig. 2 at the sonic speed. The dual active variable S takes the values of mass flow = density × speed, while the 'complementary energy' function $W_c(S)$ takes the values of pressure + density × $(\text{speed})^2$ and has the cusp of Fig. 3 at the sonic singularity. A comprehensive statement of the associated variational principles was given in 1963 by Sewell [9].

3 STRUCTURAL STABILITY OF A PLANE MAPPING

To anticipate what might be the essence of generalizations of the elementary calculation of the Section 2 to the case of, for example, constitutive equations expressed invariantly for three space dimensions in terms of tensor functions, the Legendre duals of certain of the elementary catastrophes were calculated by Sewell [1,2]. The reason for emphasizing the catastrophes is connected with the idea of structural stability, and the purpose of this Section is to give a novel explanation of that idea in a particular case.

We shall consider only smooth mappings of the plane onto the plane, which we write as

$$u = u[x, y] \qquad v = v[x, y] \; .$$

There is a theorem of Whitney [10] stating that the only stable singularities of such mappings occur along what are called 'folds' (which appear as curved lines in the x, y plane) or at 'cusps' (which appear at points where two such lines meet tangentially). A singularity is where the Jacobian vanishes. An explanation of these ideas convenient for applied mathematicians is to be found in an article by Thorndike, Cooley and Nye [12]. These authors go on to consider the example of the hodograph transformation in which x, y are place coordinates and u, v are velocity components, and they exhibit experimental evidence of the singularities in the velocity field of the polar ice cap.

Here we offer new experimental evidence from nearer home. Take a thin sheet of clear flexible polythene or plastic, which can be cut out from an ordinary household plastic bag obtainable from grocer or supermarket. Imagine the u, v coordinate system drawn onto the plastic sheet. Regard the working plate of a viewgraph or overhead projector as the x, y plane. Switch on the projector, crumple the plastic arbitrarily in the hand and put it down on the plate of the projector. Press it down flat with a sheet of clear glass. This procedure defines a typical mapping of the x, y system onto the u, v system, which we can conveniently examine in a lecture by looking at the image projected onto the screen. Two examples are shown in Figs. 5 and 6 (obtained with an initially rectangular sheet whose edges show up in certain places).

What do we see? Above each x, y point of the plate there lies one or more u, v points of the plastic, or none at all, depending on the number of local layers into which it happens to have been folded. The number of solutions of the mapping could be counted by calibrating with the varying shades of grey in the photograph - more solutions mean less light transmitted. The shade changes at the lines which, unless they are edges of the finite sheet, represent folds in the plastic and singularities in the mapping where the number of solutions changes by (typically) two. The actual number of solutions depends, of course, on what assumption is made about how the plastic sheet extends to infinity after the mapping process.

It appears that (apart from edge effects) a fold terminates only where it meets another fold. Such meeting points are also singularities of the mapping, but it appears that they are typically shaped like finite angles and <u>not</u> cusps (zero angles). In some cases it is hard to be sure when we look on the finest local scale, but a qualitative theory which is only verifiable under a magnifying glass may be misleading when a natural scale is semi-global or macroscopic. Frequently angles appear in pairs at the same point, and sometimes in complementary pairs, as in the classical 'folded handkerchief' illustrated in Fig. 8.

The primary observational conclusion is therefore that the only stable singularities of such a mapping of plane onto plane are folds and angles, not

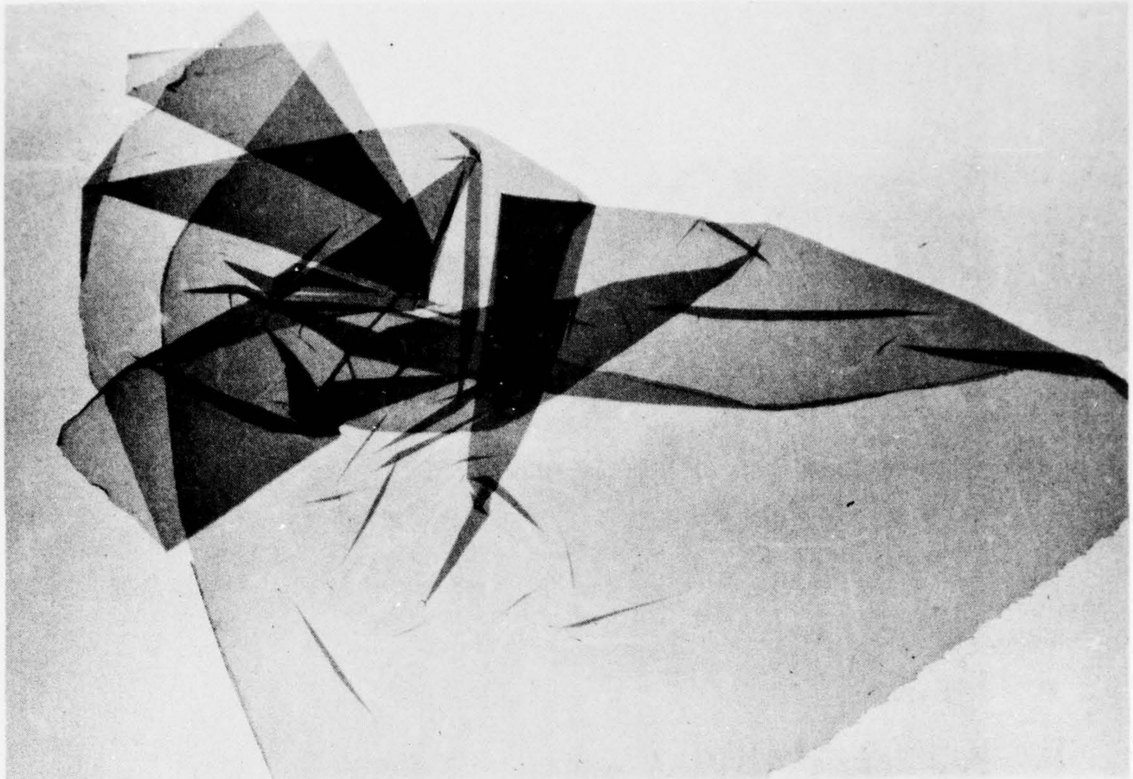

Fig. 5

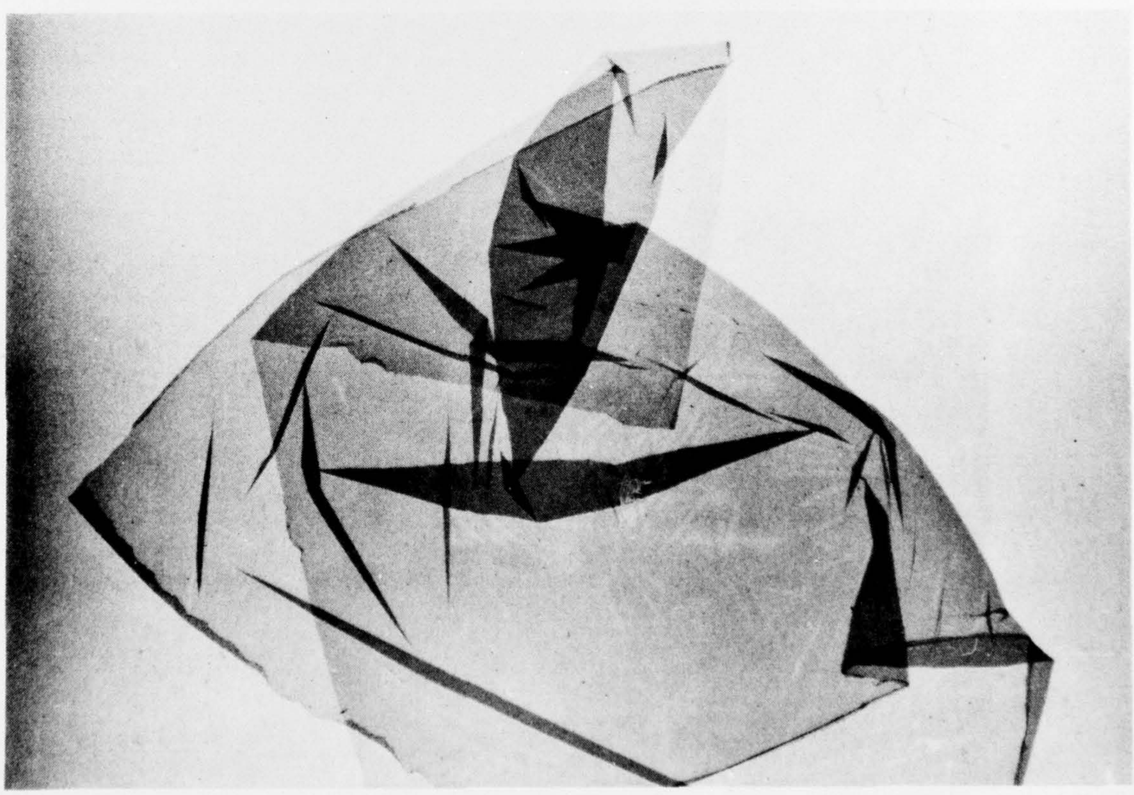

Fig. 6

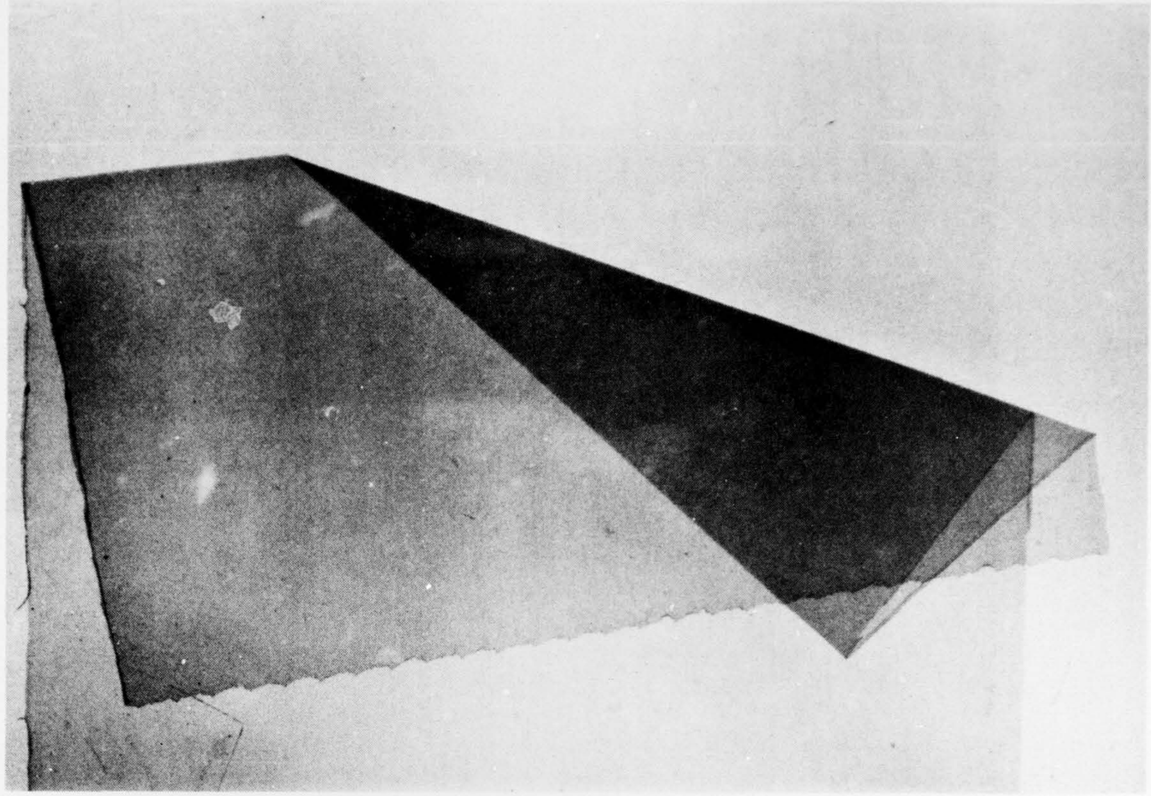

Fig. 7

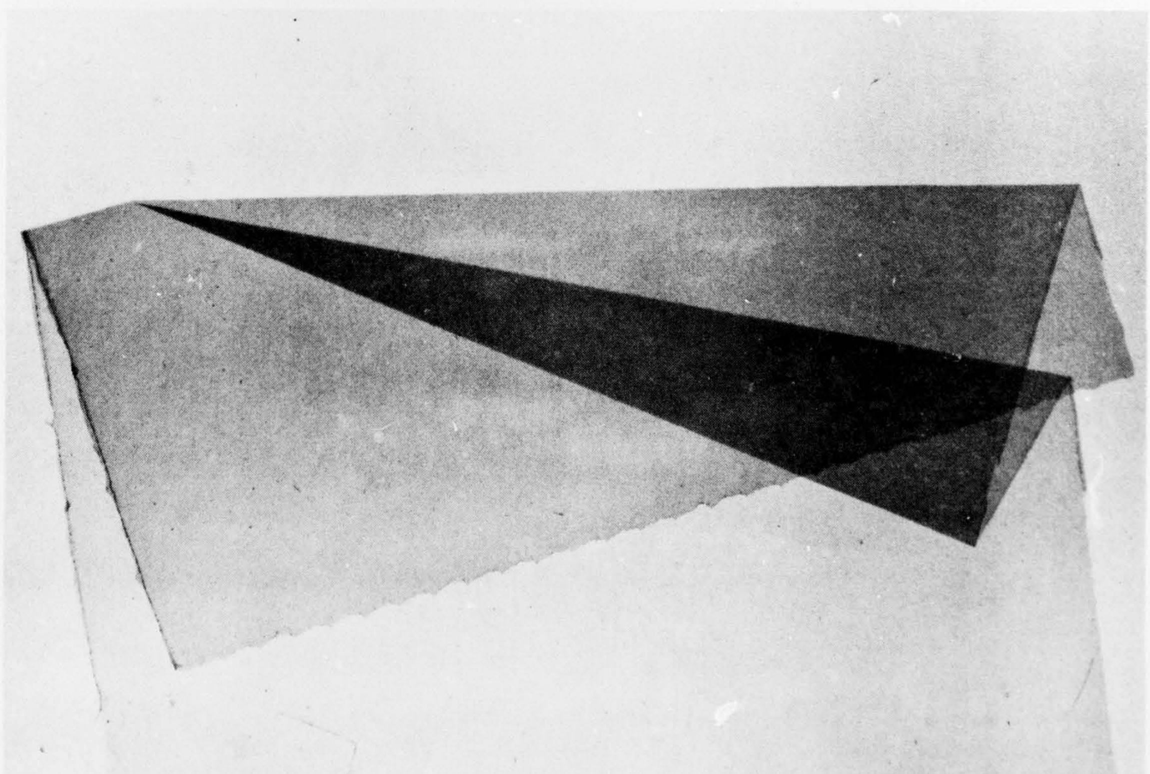

Fig. 8

folds and cusps as Whitney's theorem requires. The discrepancy may be arising because the experiment is not constructing general mappings, but only those constrained by the fact that very little or no extensional or shearing distortion tangentially in the sheet is being imposed. It would be very convenient for pedagogic purposes if one of our pure mathematical colleagues would write out a self-contained proof of a theorem stating under what circumstances folds and angles are the only stable singularities of plane mappings.

An unstable singularity is illustrated in Fig. 7. First make a single fold (thus giving the plastic a V-shaped cross-section across the single line singularity). Starting at the angle in Fig. 7, this has been converted into an enlarging treble fold with W-shaped cross-section.

The two bottom folds in the W overlay each other exactly in Fig. 7, but this is only achieved by special precautions and is therefore untypical and an unstable singularity of the mapping. If we disturb the configuration of Fig. 7 by rotational sliding around the angle, we obtain two complementary angles at the same point, as shown in Fig. 8. The relative locations of the angles and folds in this configuration persist under this particular circumferential kind of disturbance (even though the sizes of the angles and the positions of the folds do not). The angles and folds of Fig. 8 are therefore stable singularities in that sense. A radial disturbance, seeking to separate the corners of the two angles, sometimes seems to create the suggestion of a cusp on the finest scale, as if by a particular section of the hyperbolic umbilic.

The idea of illustrating singularities of plane maps by folding pieces of paper is of course very familiar, and a convenient review has been given recently by Callahan [12]. I have not seen it used before with clear flexible plastic on a viewgraph for lecture demonstration purposes (Sewell [13]) in the way described here.

We can also introduce an index to represent some of the properties of topological degree, as follows. Distinguish between the two sides of the plastic sheet, by designating one as the 'upper' side when the sheet is laid flat on the plate of the viewgraph. After the mapping has been performed, e.g. as in Figs. 5-8, assign in index i to each point of the sheet as follows: $i = +1$ wherever the 'upper' side still faces upwards; $i = -1$ if the upper side has been turned over to face downward; $i = 0$ along every fold except where it terminates in an angle pointing into a $+1$ or a -1 region; $i = +1$ or -1 respectively at angles of the latter type. The last three properties are rather more specific than the analogous 'property 5' employed by Benjamin [14] in applying Leray-Schauder degree theory to bifurcation phenomena associated with the Navier-Stokes equations, in particular to the Taylor column problem. They suggest certain other viewpoints for that problem which there is not space to describe here. We note that, in the present situation, over any x, y point the sum of the indices attached to all the associated u, v solutions is a constant, for example zero in the case of a simple fold or its perturbations in Figs. 7 and 8. The constant may also be $+1$ or -1, as when a single angle is present in a sheet otherwise extending to infinity in all directions. The case of $+1$ is analogous to Benjamin's theorem (op. cit. 'property 4') for fluid flows which can evolve in principle from a unique stable flow.

It is worth noting that the Whitney theorem and the degree theory are not restricted to circumstances in which a variational principle exists.

4 LEGENDRE TRANSFORMS OF ELEMENTARY CATASTROPHES

When a variational principle does exist however, we expect there to be more than two stable singularities of associated projection mappings, and in finite-dimensional theory without symmetry constraints these are described by the eleven elementary catastrophes (5 cuspoids including fold and cusp, and 6 umbilics). The remainder of this lecture described the 'ladder for the cuspoids' established in [1] for relating Legendre transforms and cuspoid catastrophes, and the Legendre transforms found in [2] for the elliptic and hyperbolic umbilics. We do not repeat that information here.

The problem which remains in deriving specific complementary energies, as in some other catastrophe theory investigations where quantitative results are really needed, is to reintroduce the detailed diffeomorphisms which a purely qualitative theory is entitled to discard.

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ABSTRACT (continued)

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