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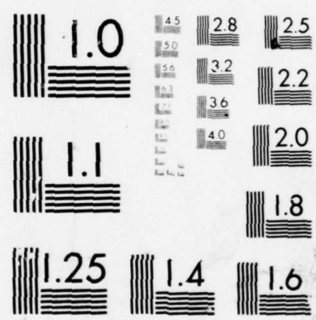
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NUMERICAL SOLUTION OF A SINGULARLY PERTURBED NONLINEAR VOLTERRA EQUATION

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NUMERICAL SOLUTION OF A SINGULARLY PERTURBED NONLINEAR VOLTERRA EQUATION

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ABSTRACT

We discuss the numerical solution of the nonlinear Volterra integrodifferential equation

$$\mu x'(t) + \int_{-\infty}^{t} a(t-s)F(x(t), x(s))ds = 0 \ , \ t \geq 0 \ ,$$

$$x(s) = g(s) \quad , \quad -\infty < s \leq 0 \ .$$

Here $x(t)$ is the unknown function, g is given history, μ is a small positive number, $a(t)$ is a positive, decreasing, logarithmically convex kernel, while F vanishes on the diagonal, and is increasing in the first and decreasing in the second variable. The time discretization is done using backwards differences, and we show that the discretization preserves the qualitative properties of the solutions and give an error bound which is uniform in μ and t for $0 < \mu \leq \mu_0$, and $t \geq t_0 > 0$.

AMS(MOS) Subject Classifications: 65R05, 34E15, 45D05.

Key Words: Singular perturbation, Volterra integrodifferential equation, implicit Euler method

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SIGNIFICANCE AND EXPLANATION

In this paper we discuss the numerical solution of a problem which arises in polymer rheology: A mathematical model was derived ~~in [3]~~ to describe the elastic recovery of molten plastics. We replace this model by a discrete model, for which the solution can immediately be constructed by computer.

The main part of this paper concerns the relation between the solutions to the original and discretized models. The original problem is a Volterra equation; Volterra equations usually occur when one models evolutions which depend on their history. We would like to emphasize that when one solves Volterra equations numerically, it is important that the discrete problem have similar qualitative properties to the original one. In the particular problem considered here this forces us to choose a rather poorly convergent method if we wish to guarantee the results obtained.

The numerical results confirm a discrepancy between theory and experiments indicated previously in [3], namely that when the elongation of a filament is large and rapid, the model predicts somewhat more recovery than is observed experimentally.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

NUMERICAL SOLUTION OF A SINGULARLY PERTURBED NONLINEAR VOLTERRA EQUATION

Olavi Nevanlinna

1. Introduction.

In this paper we consider the numerical solution of some singularly perturbed nonlinear Volterra integrodifferential equations. These were modelled, after a problem arising in polymer rheology, and studied, by Lodge, McLeod and Nohel [3].

The solutions to these equations have a boundary layer for small positive time values but one is actually mainly interested in the behavior as time is large. The solutions are not exponentially stable, so that errors during the time stepping generally affect the estimated behavior at infinity. However, we show that after any positive time t_0, the convergence for the scheme we propose, is of order $O(h^{\delta})$ with some $\delta \in (0,1)$, uniformly on $t \geq t_0$ and $0 < \mu \leq \mu_0$, where μ is the singular perturbation parameter. On any compact $[t_0, T] \subset (0,\infty)$ the convergence order is $O(h)$, uniformly again for $\mu \in (0, \mu_0]$.

In order to obtain such a strong result the discretization has to be chosen to preserve as much properties of the continuous problem as possible. This we achieve using the backwards differences, i.e. we replace $x'(nh)$ by $h^{-1}(x(nh) - x((n-1)h))$ and $\int_{-\infty}^{nh} \varphi(s)ds$ by $h\sum_{-\infty}^{n} \varphi(jh)$. It seems unlikely that similar results could be proved for the second order central difference/ trapezoidal rule discretization. In fact, in order to prove results independently of $\mu \in (0, \mu_0]$ we have to preserve the pointwise monotonicity of the discrete solutions, otherwise the nonlinearity can cause them to run out of control. As a very simple model problem, consider the numerical integration of

$$\mu x' + x = 0,\ t \geq 0,\ x(0) = 1 \tag{1.1}$$

using one-step methods

$$\frac{\mu}{h}(x_n - x_{n-1}) + \theta x_n + (1-\theta)x_{n-1} = 0 \quad .$$

These methods preserve boundedness uniformly in $\mu > 0$ iff $\theta \geq 1/2$, and pointwise monotonicity

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when $\theta \geq 1$. Since for $\theta \geq 1/2$ the local truncation error increases with θ it is reasonable to have $\theta = 1$, i.e. the backwards difference method.

For ordinary differential equations it is possible [6] to derive sharp error bounds if we know for which $\lambda h \in \mathbb{C}$ the method (linear multistep method) yields bounded solutions to the test equation

$$x' = \lambda x \quad . \tag{1.2}$$

The set of such values of λh are customarily said to form the stability region of the method. A similar concept for Volterra integrodifferential equations can be based on a test equation of the form

$$x' = \alpha x + \beta \int_0^t x(s)ds \quad , \tag{1.3}$$

(or similarly for integral equations of the second kind), see [1], [2]. However, this test equation is essentially an autonomous differential equation, and one cannot hope that preserving boundedness of solutions for such a test equation would imply good error behavior for problems where the kernel is truly time dependent. In [4] we proposed a different type of test for discretizing the integral term: often in applications the kernel $k(t,s)$ satisfies some positivity properties of the form:

$$\int_0^T \varphi(t) \int_0^t k(t,s)\varphi(s)ds\, dt \geq 0 \tag{1.4}$$

for all $T > 0$ and all smooth φ. We characterized those discretization schemes which preserve this kind of positivity, and, in [5], we derived error bounds for such methods when applied to some systems of nonlinear Volterra equati ·s. We also applied the technique to ordinary differential equations and showed that for linear multistep methods preserving the positivity was equivalent to A-stability (the left half-plane belonging to the stability region).

For the problem studied in this paper, the global stability of the solution is not a consequence of a positivity property of the form (1.4), but it, instead, follows from a pointwise positivity of the kernel and pointwise monotonicity properties of the nonlinear term and the solution itself. Therefore, we asked that these should be preserved in the discretization. That narrowed the class of method down to the first order backwards difference method, but in return we can prove a fairly strong convergence result.

We state our theorem in section 2, prove it in section 3 and discuss the application to polymer rheology in section 4.

Finally, I would like to thank Professor J. Nohel for introducing me into this problem.

2. Convergence theorem.

The following integral equation was studied in [3]:

$$\mu x'(t) + \int_{-\infty}^{t} a(t-s)F(x(t), x(s))ds = 0, \ t \geq 0 \quad , \tag{2.1}$$

$$x(t) = g(t), \ t \leq 0 \quad ,$$

under the assumptions:

$$(H_a) \begin{cases} a \in C^1[0,\infty) \cap L^1[0,\infty), \ a(t) > 0, \ a'(t) < 0 \quad , \\ a'(t)/a(t) \quad \text{nondecreasing} \quad , \end{cases}$$

$$(H_F) \begin{cases} F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}, \ F(x,x) = 0 \quad \text{for all} \quad x > 0 \quad , \\ F \in C^1(\mathbb{R}_+ \times \mathbb{R}_+) \quad \text{and} \quad F_1(y,z) > 0, \ F_2(y,z) < 0 \end{cases}$$

$(y,z \in \mathbb{R}_+ ;$ subscripts denote partial differentiation)

$$(H_g) \begin{cases} g : (-\infty,0] \to (0,\infty); \ g(-\infty) = 1 \quad , \\ g \quad \text{is nondecreasing and} \quad g(0) > 1 \quad . \end{cases}$$

We state a convergence theorem for the following discretization of (2.1):

$$\frac{\mu}{h}(x_n - x_{n-1}) + h \sum_{j=-\infty}^{n} a(nh-jh)F(x_n,x_j) = 0, \ n > 0 \tag{2.2}$$

$$x_j = g(jh), \ j \leq 0 \quad .$$

We shall assume that the initial function g is also differentiable:

$$(H'_g) \begin{cases} g \quad \text{satisfies} \quad (H_g) \quad \text{and} \quad g \in C^1(-\infty,0] \quad , \\ \sup_{s \in \mathbb{R}_-} |g'(s)| < \infty \quad . \end{cases}$$

<u>Theorem</u>. Assume that (H_a), (H_F) and (H'_g) hold. Then for all $\mu > 0$, $h > 0$, equation (2.2) has a unique solution $\{x_n\}$, such that $x_n \in [1,g(0)]$ and for $n > 0$, $x_n \leq x_{n-1}$. There exist constants $\mu_0 > 0$, $h_0 > 0$, C_1, C_2, C_3, $C_4 > 0$, C_5, such that for any $T < \infty$,

$\mu \in (0,\mu_0]$, $h \in (0,h_0$ and $0 < n \le T/h$ we have

$$
\begin{aligned}
|x(nh) - x_n| &\le C_1 e^{C_2 T}(1 + T)h \\
&\quad + C_3\left(\frac{h}{\mu}\right)\left(1 + \frac{C_4 h}{\mu}\right)^{-n} \\
&\quad + C_5\left(\frac{h}{\mu}\right)\left\{\left(1 + \frac{C_4 h}{\mu}\right)^{1-n} - \left(1 + \frac{2C_4 h}{\mu}\right)^{1-n}\right\} .
\end{aligned}
\tag{2.3}
$$

If additionally $a(t)$ decays exponentially, then there exist $\delta \in (0,1)$ and C_6 such that (2.3) holds for all $n \ge 0$ if we replace the first term $C_1 e^{C_2 T}(1 + T)h$ by $C_6 h^{\delta}$.

□

Remark. Equation (2.1) under hypotheses (H_a), (H_F) and (H_g) contains a large class of problems. For example, let $\|a\|_1 = 1$, and $F(y,z) = \varphi(y) - \varphi(z)$, with $\varphi'(y) > 0$, then we have the following equation

$$
\begin{cases}
\mu x'(t) + \varphi(x(t)) - \int_0^t a(t-s)\varphi(x(s))ds = f(t), \ t \ge 0 \ , \\
x(0) = g(0) \ ,
\end{cases}
$$

where $f(t) = \int_{-\infty}^{0} a(t-s)\varphi(g(s))ds$. In particular, from this equation we see immediately that the limit equation $(\mu = 0)$ is numerically well conditioned, as implied by (2.3).

3. Proof of the theorem.

In order to prove our theorem we have to first show that the discrete equation has essentially all the properties that the continuous problem has and in particular all bounds hold with constants independently of the step length h.

Denote $a(nh) = a_n$, then $\{a_n\} \in \ell^1$, $a_n > 0$, $\nabla a_n = a_n - a_{n-1} < 0$ and $(\frac{\nabla a}{a})_n$ is nondecreasing. We consider the discrete Volterra equation

$$\frac{\mu}{h} \nabla x_n + h \sum_{-\infty}^{n} a_{n-j} F(x_n, x_j) = 0, \; n > 0 \tag{3.1}$$

$$x_j = g_j \quad \text{for} \quad j \leq 0 \quad .$$

Here $g_j = g(jh)$ so that $\lim_{j \to -\infty} g_j = 1$ and $\{g_j\}$ is nondecreasing and $g_0 > 1$.

Lemma 1. There exists a unique $\{x_n\}$ satisfying (3.1). Moreover, $1 < x_n < g_0$ for all $n > 0$.

□

Proof. Write (3.1) into the form

$$x_n + \frac{h^2}{\mu} \sum_{-\infty}^{n} a_{n-j} F(x_n, x_j) = x_{n-1} \quad . \tag{3.2}$$

The left hand side of (3.2) is a continuous increasing function in x_n, mapping $[1, g_0]$ onto $[a,b]$, where $a < 1$, $b > g_0$. Therefore it has a unique solution x_n, in the open interval $(1, g_0)$, whenever $x_{n-1} \in [1, g_0]$.

□

Lemma 2. $\{x_n\}$ is nonincreasing for $n \geq 0$.

Proof. By Lemma 1 $x_1 < x_0$. Let $m \geq 2$ be first index such that $\nabla x_m > 0$. We show that then $\nabla^2 x_m \leq 0$, being a contradiction. Since $F(x,x) = 0$, we have

$$\begin{aligned} &\frac{\mu}{h} \nabla^2 x_m + h \sum_{-\infty}^{m-1} a_{m-j} [F(x_m, x_j) - F(x_{m-1}, x_j)] \\ &\qquad = -h \sum_{-\infty}^{m-2} \nabla a_{m-j} F(x_{m-1}, x_j) \quad . \end{aligned} \tag{3.3}$$

By assumption $x_m > x_{m-1}$ so that the second term on left in (3.3) is positive. If we can show that the term in the right is nonpositive then $\nabla^2 x_m$ must be negative and a contradiction follows.

Define J to be the largest index j such that $g_j \leq x_{m-1}$. Set

$$S_m = h \sum_{-\infty}^{m-2} \nabla a_{m-j} F(x_{m-1}, x_j) \quad ,$$

and

$$R_m = h \sum_{-\infty}^{m-2} a_{m-j} F(x_{m-1}, x_j) \quad .$$

Then R_m is negative which follows from the identity

$$R_m = - \frac{\mu}{h} \nabla x_m - h \sum_{-\infty}^{m-1} a_{m-j} \{F(x_m, x_j) - F(x_{m-1}, x_j)\}$$

and from the assumption that $x_m > x_{m-1}$. But

$$\begin{aligned} S_m &\geq S_m - \left(\frac{\nabla a}{a}\right)_{m-J} R_m \\ &= h \sum_{-\infty}^{J} \left\{ \left(\frac{\nabla a}{a}\right)_{m-j} - \left(\frac{\nabla a}{a}\right)_{m-J} \right\} \eta_j \\ &\quad + h \sum_{J+1}^{m-2} \left\{ \left(\frac{\nabla a}{a}\right)_{m-j} - \left(\frac{\nabla a}{a}\right)_{m-J} \right\} \eta_j \quad , \end{aligned}$$

where $\eta_j = a_{m-j} F(x_{m-1}, x_j)$. Since $\left(\frac{\nabla a}{a}\right)$ is nondecreasing the term in brackets in the first sum is nonnegative and in the second sum it is nonpositive. But by construction η_j is nonnegative for $j \leq J$ and negative for $j > J$ and therefore $S_m \geq 0$, and a contradiction follows.

□

Next we formulate some other properties of the solution $\{x_n\}$, all of which have counterparts for the continuous problem, [3].

<u>Proposition 1</u>. Solution depends monotonically on the initial data: if $g_j \leq \hat{g}_j$ for all $j \leq 0$, then $x_n \leq \hat{x}_n$ for all $n > 0$.

<u>Proof</u>. Let m be the first index such that $x_n > \hat{x}_n$. Then

$$\frac{\mu}{h}(\nabla x_m - \nabla \hat{x}_m) = -h \sum_{-\infty}^{m} a_{m-j} \{F(x_m, x_j) - F(\hat{x}_m, x_j)$$

$$+ F(\hat{x}_m, x_j) - F(\hat{x}_m, \hat{x}_j)\} \quad ,$$

and since $F(x_m, x_j) \geq F(\hat{x}_m, x_j)$ and $F(\hat{x}_m, x_j) \geq F(\hat{x}_m, \hat{x}_j)$ we get $\nabla x_m \leq \nabla \hat{x}_m$ which is a contradiction.

□

<u>Proposition 2</u>. Solution depends monotonically on μ: if $\{x_n(\mu_1)\}$, $\{x_n(\mu_2)\}$ are two solutions to (3.1) with the same initial data but different μ's, then $\mu_1 < \mu_2$ implies $x_n(\mu_1) < x_n(\mu_2)$ for all $n > 0$.

□

<u>Proof</u>. One sees easily that $x_1(\mu_1) < x_1(\mu_2)$. Suppose therefore that m is a first index such that $x_m(\mu_1) \geq x_m(\mu_2)$. This would imply $\nabla(x_m(\mu_2) - x_m(\mu_1)) < 0$. But

$$\frac{1}{h} \nabla(x_m(\mu_2) - x_m(\mu_1))$$

$$= -\frac{h}{\mu_2} \sum_{-\infty}^{m} a_{m-j} \{F(x_m(\mu_2), x_j(\mu_2)) - F(x_m(\mu_1), x_j(\mu_1))\} \tag{3.4}$$

$$+ \left[\frac{1}{\mu_1} - \frac{1}{\mu_2}\right] h \sum_{-\infty}^{m} a_{m-j} F(x_m(\mu_1), x_j(\mu_1)) \quad ,$$

which shows that the right hand side of (3.4) is nonnegative. In fact,

$$h \sum_{-\infty}^{m} a_{m-j} F(x_m(\mu_1), x_j(\mu_1)) = -\frac{\mu_1}{h} \nabla x_m(\mu_1) \geq 0 \quad ,$$

and

$$F(x_m(\mu_2), x_j(\mu_2)) - F(x_m(\mu_1), x_j(\mu_1))$$

$$= \{F(x_m(\mu_2), x_j(\mu_2)) - F(x_m(\mu_1), x_j(\mu_2))\}$$

$$+ \{F(x_m(\mu_1), x_j(\mu_2)) - F(x_m(\mu_1), x_j(\mu_1))\} \leq 0 \quad ,$$

since both bracket terms are separately nonpositive. Hence we arrived into a contradiction.

□

Proposition 3. For $\mu_0 > 0$, $h_0 > 0$ small enough there are positive constants C_1, C_2, G such that for all $\mu \in (0,\mu_0]$, $h \in (0,h_0]$ we have

(3.5) $$0 \le h^{-1}\nabla x_n \le C_1\mu^{-1}(1 + hG/\mu)^{1-n} + C_2 h \sum_{j=n}^{\infty} a_j, \ n > 0 \quad .$$

□

Proof. In order to bound ∇x_n one has to solve a second order difference equation. By mean value theorem we have for some $\xi_j \in (x_n, x_{n-1})$,

$$\frac{\mu}{h}\nabla^2 x_n + h\sum_{-\infty}^{n-1} a_{n-j}F_1(\xi_j, x_j)\nabla x_n = -h\sum_{-\infty}^{n-1} \nabla a_{n-j}F(x_{n-1}, x_j) \quad .$$

This is of the form

(3.6) $$\frac{\mu}{h}\nabla(\nabla x)_n + G_n \cdot \nabla x_n = f_n \quad ,$$

where G_n is uniformly bounded from below:

$$G_n \ge \inf_{y,z\in[1,g_0]} F_1(y,z)\cdot h\sum_{1}^{\infty} a_j$$

$$= \gamma\{\int_0^{\infty} a(s)ds + O(h)\} \quad ,$$

and f satisfies

(3.7) $$|f_n| \le C|(\frac{\nabla a}{a})_1|\{h\sum_{j=n}^{\infty} a_j + (1 + O(h))|\nabla x_n|\}$$

where C only depends on g_0. In fact, write

$$-f_n = h\sum_{-\infty}^{0} \nabla a_{n-j}F(x_{n-1}, g_j) + h\sum_{1}^{n-1} \nabla a_{n-j}F(x_{n-1}, x_j) = S_1 + S_2 \quad .$$

Then
$$0 \le S_2 = h\sum_{1}^{n-1}\left(\frac{\nabla a}{a}\right)_{n-j} a_{n-j} F(x_{n-1}, x_j)$$

$$\le \left(\frac{\nabla a}{a}\right)_1 h \sum_{1}^{n-1} a_{n-j} F(x_{n-1}, x_j) \le C\left|\left(\frac{\nabla a}{a}\right)_1\right| h \sum_{n}^{\infty} a_j$$

$$+ \left|\left(\frac{\nabla a}{a}\right)_1\right| \gamma\left\{\int_0^{\infty} a(s)ds + O(h)\right\} |\nabla x_n| \quad ,$$

since $\frac{\nabla a}{a}$ is negative and nondecreasing,

$$F(x_{n-1}, x_j) \le 0 \quad , \text{ and}$$

$$-h\sum_{1}^{n-1} a_{n-j}F(x_{n-1}, x_j) = \frac{\mu}{h}\nabla x_n + h\sum_{-\infty}^{0} a_{n-j}F(x_n, g_j)$$

$$-h\sum_{1}^{n-1} a_{n-j}F_1(\xi_j, x_j)\nabla x_n \quad ,$$

so that $C = \sup\limits_{y,z\in[1,g_0]} F(y,z)$.

For the other sum we get

$$|S_1| \le h\sum_{-\infty}^{0} -\left(\frac{\nabla a}{a}\right)_{n-j} a_{n-j} |F(x_{n-1}, g_j)|$$

$$\le C\left|\left(\frac{\nabla a}{a}\right)_n\right| h \sum_{n}^{\infty} a_j \quad ,$$

and together they imply (3.7).

Set $\nabla x_n = y_n$. Then we obtain from (3.6) that $\frac{\mu}{h} y_n + G_n y_n = \frac{\mu}{h} y_{n-1} + f_n$

and further

$$\left(\frac{\mu}{h} + G_n\right)|y_n| \le \frac{\mu}{h}|y_{n-1}| + |f_n| \quad .$$

Suppose from here on that $h \le h_1$, where h_1 is so chosen that $1 - \left|\left(\frac{\nabla a}{a}\right)_1\right| \ge 1/2$. Then we see from the bounds for G_n and f_n that for some constant $G > 0$

$$\left(\frac{\mu}{h} + G\right)|y_n| \le \frac{\mu}{h}|y_{n-1}| + g_n \quad , \tag{3.8}$$

where $g_n \le C|(\frac{\nabla a}{a})_1|h\sum_n^\infty a_j$. Since $(\frac{\nabla a}{a})_1 = \frac{a'(0)}{a(0)}h + O(h^2)$ we obtain from (3.8)

$$|y_n| \le (1+\frac{hG}{\mu})^{1-n}|y_1| + \frac{h/\mu}{1+\frac{hG}{\mu}}\cdot Ch\sum_{j=2}^{n}(1+\frac{hG}{\mu})^{j-n}h\sum_{k=j}^{\infty}a_k \ . \tag{3.9}$$

Consider the second term in the right hand side of (3.9):

$$S_n = \sum_{j=2}^{n}(1+\frac{hG}{\mu})^{j}h\sum_{k=j}^{\infty}a_k \ .$$

By partial summation we obtain

$$S_n = (1+\frac{hG}{\mu})^2\left\{h\sum_{j=2}^{n-1}\frac{(1+\frac{hG}{\mu})^{j-1}-1}{hG/\mu}a_j + \frac{(1+\frac{hG}{\mu})^{n-1}-1}{hG/\mu}h\sum_{k=n}^{\infty}a_k\right\} .$$

But $a_j = -\sum_{k=j+1}^{\infty}\nabla a_k \le -(\frac{\nabla a}{a})_1\sum_{k=j+1}^{\infty}a_k$

so that

$$(1+\frac{hG}{\mu})^2\sum_{j=2}^{n-1}\frac{(1+\frac{hG}{\mu})^{j-1}-1}{hG/\mu}a_j \le \frac{\mu}{hG}|(\frac{\nabla a}{a})_1|S_n$$

and hence, if μ_0, and $h_0 \le h_1$ are such that $1-\frac{\mu}{G}\frac{1}{h}|(\frac{\nabla a}{a})_1| \ge 1/2$ for $\mu \in (0,\mu_0]$, $h \in (0,h_0]$ (which is possible since $(\frac{\nabla a}{a})_1 = \frac{a'(0)}{a(0)}h + O(h^2)$), we finally have

$$S_n \le 2\frac{\mu}{hG}[(1+\frac{hG}{\mu})^{n+1} - (1+\frac{hG}{\mu})^2]h\sum_{k=n}^{\infty}a_k \ . \tag{3.10}$$

Since

$$|y_1| = |x_1 - g_0| \le C\frac{h}{\mu}h\sum_1^\infty a_j \ ,$$

(3.5) follows by substituting the bound (3.10) for S_n into (3.9).

□

The next result will be used to prove the uniform convergence of the approximations on the half line $t \geq 0$.

Corollary 1. If additionally $\{ja_j\} \in \ell^1$, then

$$x_n - x_\infty \leq C_1\left(1 + \frac{hG}{\mu}\right)^{-n} + C_2 h \sum_{k=n+1}^{\infty} (k-n)h\, a_k \quad . \tag{3.11}$$

□

Proof. (3.11) follows immediately from (3.5).

□

Up to this point we have more or less followed the treatment of the continuous problem in [3]. In order to prove convergence we need to investigate the perturbation properties of the difference equation. When proving the convergence we would like to substitute the local truncation error in the place of the perturbation.

However, we can control the solutions only when we allow nonnegative perturbations, and the local truncation error does generally change signs during time stepping, so that some extra consideration is in order.

Lemma 3. Assume $u_j = v_j = g_j$, $j \leq 0$ and $\{u_n\}$, $\{v_n\}$ satisfy

$$\frac{\mu}{h} \nabla u_n + h \sum_{-\infty}^{n} a_{n-j} F(u_n, u_j) = \sigma_n \tag{3.12}$$

$$\frac{\mu}{h} \nabla v_n + h \sum_{-\infty}^{n} a_{n-j} F(v_n, v_j) \geq q_n \quad . \tag{3.13}$$

Then $v_n \geq u_n$ for all $n \geq 0$.

Proof. Let m be the first index such that $v_m < u_m$. Then $\nabla(v-u)_m < 0$ and we get

$$\frac{\mu}{h} \nabla(v-u)_m + h \sum_{-\infty}^{m} a_{m-j}\{F(v_m, v_j) - F(u_m, u_j)\} < 0 \quad ,$$

contradicting (3.12), (3.13).

□

Let now $\{y_n\}$ be a sequence from which it is known that for all n $y_n \in [1, g_0]$ (we know that $x(nh) \in [1, g_0]$) and that it satisfies

$$\frac{\mu}{h} \nabla y_n + h \sum_{-\infty}^{n} a_{n-j} F(y_n, y_j) = p_n, \ n > 0 \quad ,$$

and $y_j = g_j$, for $j \leq 0$. Construct two sequences $\{w_n\}$ and $\{z_n\}$ as follows: w_n satisfies

$$\frac{\mu}{h} \nabla w_n + h \sum_{-\infty}^{n} a_{n-j} F(w_n, w_j) = q_n$$

where $w_j = g_j$ for $j \leq 0$ and $q_n = |p_n|$ if this will imply $w_n \leq g_0$, otherwise set $w_n = g_0$ and compute the corresponding value for q_n. The other sequence $\{z_n\}$ will serve as a lower bound. Therefore define it by the same equation, now with perturbation $r_n = -|p_n|$ whenever you would have $z_n \geq 1$, otherwise set $z_n = 1$ and define r_n correspondingly.

Lemma 4. If $\{x_n\}$ satisfies (3.1) and $\{y_n\}$, $\{w_n\}$, $\{z_n\}$ are given as above, then $|x_n - y_n| \leq |w_n - z_n|$. □

Proof. By construction, $q_n \geq 0$ and $r_n \leq 0$ so that by Lemma 3 we have $z_n \leq x_n \leq w_n$. The conclusion will follow since we also have $z_n \leq y_n \leq w_n$. To see this, assume for example that m would be the first index for which $y_m > w_m$. Then, since $y_m \leq g_0$ we must have $q_n = |p_n|$, and therefore we are back in the situation of Lemma 3 and a contradiction would follow.

□

According to Lemma 4 we can proceed as follows. Suppose that $x(t)$ is the solution to the continuous problem. Define the local truncation error sequence $\{\tau_n^h\}$ by

$$\frac{\mu}{h}(x(nh) - x((n-1)) + h \sum_{-\infty}^{n} a_{n-j} F(x(nh), x(jh)) = \tau_n^h \ . \tag{3.14}$$

If x_n satisfies (3.1) with $g_j = g(jh)$, then in order to bound $|x_n - x(nh)|$ it is sufficient to bound $|w_n - z_n|$, where $\{w_n - z_n\}$ satisfies

$$0 \leq \frac{\mu}{h} \nabla (w-z)_n + h \sum_{-\infty}^{n} a_{n-j} \{F(w_n, w_j) - F(z_n, z_j)\} \leq 2|\tau_n^h| \quad , \tag{3.15}$$

with $w_j = z_j = g_j$, $j \leq 0$, and by the above construction we can assume that $1 \leq z_n \leq w_n \leq g_0$ for all $n \geq 1$.

Lemma 5. Assume that $\{w_n\}$, $\{z_n\} \subset [1,g_0]$ and satisfy

$$0 \le \frac{\mu}{h} \nabla(w-z)_n + h \sum_{-\infty}^{n} a_{n-j}\{F(w_n,w_j) - F(z_n,z_j)\} \le p_n \quad , \tag{3.16}$$

and $w_j = z_j = g_j$ for $j \le 0$. If μ_0, h_0 are small enough, $\mu \in (0,\mu_0]$, and $h \in (0,h_0]$, then for some constants A, B, C, depending on μ_0 and h_0 but not on μ and h we have

$$|w_n - z_n| \le Ch \sum_{\nu=0}^{n-1} p_{n-\nu} e^{\frac{Bh_\nu}{A}} + \frac{h/\mu}{1+\frac{hA}{\mu}} \sum_{\nu=0}^{n-1} p_{n-\nu}\left(1 + \frac{Ah}{\mu}\right)^{-\nu} \quad . \tag{3.17}$$

□

Proof. As in the proof of Lemma 3 we observe that (3.16) implies $w_n \ge z_n$ for all $n \ge 0$. Therefore we can devide the sum in (3.16) into two parts:

$$\frac{\mu}{h} \nabla(w-z)_n + h \sum_{-\infty}^{n-1} a_{n-j}\{F(w_n,w_j) - F(z_n,w_j)\} \tag{3.18}$$

$$\le p_n + h \sum_{1}^{n-1} a_{n-j}\{F(z_n,z_j) - F(z_n,w_j)\} \quad .$$

Let $\alpha = \min_{\zeta,\eta\in[1,g_0]} F_1(\xi,\eta)$ and $\beta = \max_{\zeta,\eta\in[1,g_0]} -F_2(\xi,\eta)$, then (3.18) and $w_j \ge z_j$ imply

$$\frac{\mu}{h} \nabla(w-z)_n + \alpha h \sum_{-\infty}^{n-1} a_{n-j} \cdot (w_n - z_n) \tag{3.19}$$

$$\le p_n + \beta h \sum_{1}^{n-1} a_{n-j}(w_j - z_j) \quad .$$

Set $w-z = \zeta$ and $\beta a(0) = B$. Choose $A < \alpha\int_0^\infty a(s)ds$ and h so small that $\alpha h \sum_1^\infty a(jh) > A$. Then (3.19) implies

$$\left(1 + \frac{Ah}{\mu}\right)\zeta_n \le \zeta_{n-1} + \frac{Bh}{\mu} h \sum_{1}^{n-1} \zeta_j + \frac{h}{\mu} p_n, \; n > 1 \tag{3.20}$$

where

$$\zeta_1 \le \left(1 + \frac{Ah}{\mu}\right)^{-1} \frac{h}{\mu} p_1 \quad .$$

To solve (3.20) we first consider the roots of the polynomial

$$(1 + \frac{Ah}{\mu})\lambda^2 - (2 + \frac{Ah}{\mu} + \frac{Bh^2}{\mu})\lambda + 1 \quad .$$

When μ_0 and h_0 are small then we have after a straightforward estimation two positive roots, λ_1, λ_2 say, such that for $\mu \in (0,\mu_0]$, $h \in (0,h_0]$

$$\lambda_1 \leq \frac{1 + \frac{h}{\mu} \cdot Kh^2}{1 + \frac{Ah}{\mu}} < 1$$

and

$$\lambda_2 \leq e^{\frac{Bh}{A}} \quad ,$$

where K depends on μ_0 and h_0. The solution ζ_n then satisfies

$$(3.21) \qquad \zeta_n \leq \frac{h}{\mu} \frac{1}{1 + \frac{Ah}{\mu}} \frac{1}{\lambda_2 - \lambda_1} \sum_{\nu=0}^{n-1} p_{n-\nu} \{\lambda_2^{\nu+1} - \lambda_2^{\nu} - \lambda_1^{\nu+1} + \lambda_1^{\nu}\} \quad .$$

Now $\lambda_2 - \lambda_1 \geq \frac{Ah/\mu}{1 + Ah/\mu}$ and $(\lambda_2 - 1)\lambda_2^{\nu} \leq \frac{Bh}{A}(1 + \frac{Bh}{A})e^{\frac{Bh}{A}\nu}$ and $-(\lambda_1 - 1)\lambda_1^{\nu} \leq \frac{Ah}{\mu + Ah}(1 + \frac{Ah}{\mu})^{-\nu}$ (after possibly choosing a smaller A and h_0). Substituting these bounds into (3.21) yields

$$\begin{aligned}
\zeta_n &\leq \frac{1}{A} \sum_{\nu=0}^{n-1} p_{n-\nu} \{\frac{Bh}{A}(1 + \frac{Bh}{A})e^{\frac{Bh}{A}\nu} + \frac{Ah/\mu}{1+Ah/\mu}(1 + \frac{Ah}{\mu})^{-\nu}\} \\
&\leq Ch \sum_{\nu=0}^{n-1} e^{\frac{Bh}{A}\nu} p_{n-\nu} \\
&\quad + \frac{h/\mu}{1+Ah/\mu} \sum_{\nu=0}^{n-1} (1 + \frac{Ah}{\mu})^{-\nu} p_{n-\nu} \quad ,
\end{aligned}$$

for some constant C, and we hence obtained (3.17).

$\square$

Next Lemma bounds the local truncation error.

Lemma 6. Let $\{\tau_n^h\}$ denote the local truncation error (see (3.14)). Then there exists $C = C(\mu_0, h_0, g_0)$ and $K = K(\mu_0, g_0)$ such that for $\mu \in (0,\mu_0]$, $h \in (0,h_0]$ we have

$$|\tau_n^h| \le Ch\left\{ \int_{nh-h}^{\infty} a(s)\,ds + \frac{1}{\mu} e^{-K(n-1)h/\mu} \right.$$

(3.22)

$$\left. + \int_0^{nh} a(nh-s)\left[\int_s^{\infty} a(s)\,ds + \frac{1}{\mu} e^{-Ks/\mu}\right] ds \right\} .$$

□

Proof. If $x(t)$ is the solution to the continuous problem, then

$$\mu x'(nh) + \sum_{-\infty}^{n} \int_{(j-1)h}^{jh} a(nh-s)F(x(nh),x(s))\,ds = 0 \quad . \tag{3.23}$$

Let us consider the sum first:

$$\sum_{-\infty}^{n} \int_{(j-1)h}^{jh} a(nh-s)F(x(nh),x(s))\,ds$$

$$= h \sum_{-\infty}^{n} a(nh-jh)F(x(nh),x(jh)) + S_1 + S_2$$

where

$$S_1 = \sum_{-\infty}^{n} \int_{(j-1)h}^{jh} [(j-1)h-s][-a'(nh-s)F(x(nh),x(s))\,ds$$

and

$$S_2 = \sum_{-\infty}^{n} \int_{(j-1)h}^{jh} [(j-1)h-s][a(nh-s)F_2(x(nh),x(s))]x'(s)\,ds \quad .$$

We know that $x(t)$ is decreasing and $x(t) \in (1,g_0]$ for $t \ge 0$ [Theorem 1;3]. Therefore, set $C = \sup_{y,z \in [1,g_0]} |F(y,z)|$ and $D = \sup_{y,z \in [1,g_0]} |F_2(y,z)|$. We split S_i's into two parts and estimate:

$$|S_{11}| = \left| \sum_{-\infty}^{0} \int_{(j-1)h}^{jh} [(j-n)h-s][-a'(nh-s)F(x(nh),x(s))]\,ds \right|$$

$$\leq Ch \sum_{-\infty}^{0} \int_{(j-1)h}^{jh} |a'(nh-s)| ds$$

$$\leq C \left|\frac{a'(0)}{a(0)}\right| h \cdot \int_{nh}^{\infty} a(s) ds \quad ,$$

where we again used the logarithmic convexity of $a(s)$. Similarly we get, since $x(t)$ is decreasing,

$$|S_{12}| \leq h \left|\frac{a'(0)}{a(0)}\right| \int_{0}^{nh} a(nh-s)(-F(x(nh,x(s)))ds \quad .$$

But $-\int_{0}^{nh} a(nh-s)F(x(nh),x(s))ds = \mu x'(nh) + \int_{-\infty}^{0} a(nh-s)F(x(nh),g(s))ds$, and since $x'(nh) < 0$, we obtain

$$|S_{12}| \leq C \left|\frac{a'(0)}{a(0)}\right| h \int_{nh}^{\infty} a(s) ds \quad .$$

Hence

$$|S_1| \leq 2C \left|\frac{a'(0)}{a(0)}\right| h \int_{nh}^{\infty} a(s) ds \quad .$$

Also S_{21} is of this form:

$$|S_{21}| = \left| \sum_{-\infty}^{0} \int_{(j-1)h}^{jh} [(j-1)h-s] a(nh-s) F_2(x(nh),g(s)) g'(s) ds \right|$$

$$\leq D \sup_{s \in (-\infty,0]} g'(s) \cdot h \int_{nh}^{\infty} a(s) ds \quad ,$$

while S_{22} carries the effect of the boundary layer: By [Theorem 2;3] (which should be compared to Proposition 3) there exists constants K_1 and K_2 such that

$$0 \leq -x'(t) \leq \frac{K_2}{\mu} e^{-K_1 t/\mu} + K_2 \int_{t}^{\infty} a(s) ds \quad . \tag{3.24}$$

Substituting this into S_{22} yields

$$|S_{22}| \leq Dh \int_{0}^{nh} a(nh-s) \left(\frac{K_2}{\mu} e^{-K_1 s/\mu} + K_2 \int_{s}^{\infty} a(r) dr\right) ds \quad .$$

We have to still consider the error introduced by replacing $\mu x'$ by $\frac{\mu}{h}\nabla x$. By mean value theorem we have for some $\xi_n \in ((n-1)h, nh)$

$$\mu \frac{x(nh) - x((n-1)h)}{h} = \mu x'(nh) - \mu \frac{1}{2} hx''(\xi_n) \quad . \tag{3.25}$$

Using line (4.1) in [3] and (3.24) we obtain

$$\mu \frac{h}{2} |x''(\xi_n)| \leq Ch \left\{ \frac{e^{-K_1\xi_n/\mu}}{\mu} + \int_{\xi_n}^{\infty} a(s)ds \right\} \quad .$$

Collecting all terms yields (3.22).

□

Note that (3.22) implies that $|\tau_n^h|$ is uniformly bounded for $n \geq 2$, $\mu \in (0,\mu_0]$, $h \in (0,h_0]$. However, the same is true also for $|\tau_1^h|$, which we now demonstrate. We see from the proof above that the unbounded term in the bound for τ_1^h: $C\frac{h}{\mu}$, comes from (3.25). However, the solution has a boundary layer for small $t > 0$, and this implies the uniform boundedness of this term. In fact, there exists a continuously differentiable function ξ on $[0,\infty)$, such that $\xi(t) \in [1,g_0]$ and $\sup_{t\in[0,\infty)} |\xi'(t)| < \infty$, and a constant $K = K(h_0)$, such that

$$|x(h) - \xi(h/\mu)| \leq Kh, \quad \text{for } h \in (0,h_0] \quad , \tag{3.26}$$

(see [Theorem 7;3] and notice that since $\xi(t)$ is decreasing, so is $|\xi'(t)|$). But (3.26) implies that

$$\left|\mu \frac{x(h)-g_0}{h} - \mu x'(h)\right| \leq \left|\frac{\xi(h/\mu)-g_0}{h/\mu}\right| + |\xi'(h/\mu)| + 2K\mu \quad ,$$

which is bounded as soon as $\mu \leq \mu_0$, and therefore there exists $M = M(\mu_0,h_0)$ such that $|\tau_1^h| \leq M$ for $\mu \leq \mu_0$, $h \leq h_0$.

Next we simplify (3.22) somewhat by assuming that $a(t) \leq \alpha e^{-\beta t}$ for some $\alpha > 0$, $\beta \geq 0$. Then we have for some $C < \infty$ and $\gamma \geq 0$, with $\gamma > 0$ if $\beta > 0$,

$$|\tau_n^h| \leq Ch(e^{-\gamma(n-1)h} + \frac{1}{\mu} e^{-K(n-1)h/\mu}), \; n > 1$$

and

$$|\tau_1^h| \leq C.$$

Substitute these into (3.17). The first term in the right of (3.17) yields,

(3.27) $$C_1 e^{BT/A} h, \quad \text{if} \quad \gamma > 0, \quad \text{and}$$

(3.28) $$C_1 (1+T) e^{BT/A} h, \quad \text{if} \quad \gamma = 0 \quad .$$

The other term yields

$$\frac{h}{\mu}\left(1 + \frac{Ah}{\mu}\right)^{-1}\left\{\left(1 + \frac{Ah}{\mu}\right)^{-n+1}\left|\tau_1^h\right| + \sum_0^{n-2}\left|\tau_{n-\nu}^h\right|\left(1 + \frac{Ah}{\mu}\right)^{-\nu}\right\}$$

$$= C\,\frac{h}{\mu}\left(1 + \frac{Ah}{\mu}\right)^{-n} + S, \quad \text{where} \quad S = S_1 + S_2 \quad ,$$

and,

$$|S_1| \le \frac{h}{\mu}\left(1 + \frac{Ah}{\mu}\right)^{-1} \cdot Ch \sum_0^{n-2} 1 \cdot \left(1 + \frac{Ah}{\mu}\right)^{-\nu} \le \frac{C}{A}\,h \quad ,$$

and

$$|S_2| = \left(\frac{h}{\mu}\right)^2 \frac{\alpha^{1-n} - \beta^{1-n}}{\alpha - \beta} \quad ,$$

where $\alpha = 1 + \frac{Ah}{\mu}$ and $\beta = e^{Kh/\mu}$. Taking, if necessary, a smaller A, we see easily that $|S_2|$ is bounded in the form

$$|S_2| \le C\,\frac{h}{\mu}\left\{\left(1 + \frac{Ah}{\mu}\right)^{1-n} - \left(1 + \frac{2Ah}{\mu}\right)^{1-n}\right\} \quad .$$

Summarizing the above bounds yields (2.3). To complete the proof assume that $a(t)$ decays exponentially so that (3.27) holds. From Corollary 1 we see that for $m \ge n$, and for some $\gamma > 0$,

$$|x_n - x_m| \le Ce^{-\gamma nh} + C\left(1 + \frac{Ah}{\mu}\right)^{-n} \quad ,$$

and for the *continuous problem we* have similarly (see [Corollary 2.1;3])

$$|x(nh) - x(mh)| \le Ce^{-\gamma nh} \quad .$$

Assume that $mh > T$ and $nh \le T$.

Then

$$|x(mh) - x_m| \le |x(mh) - x(nh)| + |x(nh) - x_n| + |x_n - x_m| \quad ,$$

where the right hand side is bounded by terms of the form

$$C_1 h e^{C_2 T} , \; C_3 e^{-\gamma T}$$

(and by the boundary layer terms). Allowing T to grow with h one can bound these both terms by Kh^{δ}, with $\delta = (1 + C_2/\gamma)^{-1}$, which completes the proof of the theorem.

□

4. <u>Example from polymer rheology</u>.

For background material see [3]. In this example $x(t)$ denotes the length of a molten plastics filament: it is originally of length 1, then it is elongated up to time 0 when it is allowed to recover freely. Hence $x(t) = g(t)$, $t \leq 0$ describes the elongation history and $x(t)$ for $t > 0$ the free recovery.

Functions $a(t)$, $F(y,z)$ and $g(t)$ are as follows:

$$a(t) = \sum_{i=1}^{8} a_i e^{-t/\tau_i} ,$$

where

i	τ_i	a_i
1	10^3	10^{-3}
2	10^2	1.8
3	10	$1.89 \cdot 10^{+2}$
4	1	$9.8 \cdot 10^3$
5	10^{-1}	$2.67 \cdot 10^5$
6	10^{-2}	$5.86 \cdot 10^6$
7	10^{-3}	$9.48 \cdot 10^7$
8	10^{-4}	$1.29 \cdot 10^9$

(these numbers were provided by Professor Lodge and based on results of H. M. Laun),

$$F(y,z) = \frac{y^3}{z^2} - z ,$$

and

$$g(t) = \begin{cases} 1 & , t \leq -t_0 \\ e^{K(t+t_0)} & , -t_0 \leq t \leq 0 . \end{cases}$$

Since a, F, and g are all of relatively simple form, one can write the integral

$$\int_{-\infty}^{0} a(t-s)F(x, g(s))ds$$

down exactly and only discretize the other part

$$\int_0^t a(t-s)F(x, x(s))ds \quad .$$

Furthermore, although the backwards differences lead to an implicit method, the specific form of F implies that at each step one has to find a real root to a cubic polynomial, and that can be written down in terms of square and cubic roots. Therefore the final recursion formula is essentially an explicit method, which generates from the finite information $\{x_1,\ldots,x_n\}$ a new value x_{n+1}.

In an earlier work (see [Appendix A;3]) equation (2.1) was considered with $\mu = 0$, and it was found that the model predicted more recovery than was observed experimentally. Since the solution is increasing with μ, introducing $\mu > 0$ leads to a reduction in the predicted recovery. Unfortunately, the numerical experiments show that the final recovery is effected very little by μ (for $\mu \in [10^{-5}, 1]$), although the solutions behave very differently for small $t > 0$, and so, for high strains there still is a disagreement between theory and data.

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$$\mu x'(t) + \int_{-\infty}^{t} a(t-s)F(x(t), x(s))ds = 0, \ t \geq 0, \ x(s) = g(s), \ -\infty < s \leq 0.$$

Here $x(t)$ is the unknown function, g is given history, μ is a small positive number, $a(t)$ is a positive, decreasing, logarithmically convex kernel, while F vanishes on the diagonal, and is increasing in the first and decreasing in the second variable. The time discretization is done using backwards differences, and we show that the discretization preserves the qualitative properties of

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ABSTRACT (continued)

the solutions and give an error bound which is uniform in μ and t for $0 < \mu \leq \mu_0$, and $t \geq t_0 > 0$.