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EXTREMUM PROBLEMS FOR THE MOTIONS OF A BILLIARD BALL. III. THE MULTI-DIMENSIONAL CASE OF KÖNIG AND SZÜCS

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LEVEL II

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ABSTRACT

Let

(1) $U_n : 0 \leqq x_\nu \leqq 1$, $(\nu = 1,\dots,n)$

be the unit cube of R^n. Using ideas pioneered in 1913 by König and Szücs in [2], we study the following problem. Let

(2) $L_n^k : x_\nu = \sum_{i=1}^{k} \lambda_\nu^i u_i + a_\nu$ $(k < n)$

be a k-flat, so that the point (a_ν) is interior to U_n, and such that L_n^k is in a general position (G.P.) and write $L_n^k \in$ G.P. By this we mean that any k among the x_ν of (2) may assume preassigned values for appropriate values of the u_i. We interpret L_n^k as an optical signal starting from the point (a_ν) at the time $t = 0$, and spreading uniformly within the k-flat L_n^k. We assume the $2n$ facets $x_\nu = 0$ or 1, of U_n, to be mirrors, so that the reflected path of the signal is a finite or infinite k-dimensional skew polytope $\Pi_n^k \subset U_n$. Using the auxiliary function

$\langle x \rangle = x$ if $0 \leqq x \leqq 1$, $\langle x \rangle = 2-x$ if $1 \leqq x \leqq 2$, and $\langle x+2 \rangle = \langle x \rangle$ if $x \in \mathbb{R}$,

we may represent the reflected path by the parametric equations

(3) $\Pi_n^k : x_\nu = \langle \sum_{i=1}^{k} \lambda_\nu^i u_i + a_\nu \rangle$, $(\nu = 1,\dots,n)$.

For the x_ν defined by (3), we study the quantity

$$\rho_{k,n} = \sup_{L_n^k \in \text{G.P.}} \inf_{(u_i)} (\max_\nu |x_\nu|) ,$$

and wish to determine, or to estimate it.

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Theorem 1. $\rho_{k,n} \geq \frac{1}{2} - \frac{k}{2n}$, $(1 \leq k \leq n-1)$.

Theorem 2. $\rho_{n-1,n} = \frac{1}{2} - \frac{n-1}{2n} = \frac{1}{2n}$.

It is shown that there is an essentially unique $\tilde{\Pi}_n^{n-1}$ which does not penetrate into the cube

$$\max_{\nu} |x_{\nu} - \frac{1}{2}| < \frac{1}{2n} .$$

The polytope $\tilde{\Pi}_3^2$ is identical with the surface of Kepler's regular tetrahedron T inscribed in U_3 , and Theorem 2 gives, for $n = 3$, an apparently new extremum property of T . Finally we state

Conjecture 1 . $\rho_{k,n} = \frac{1}{2} - \frac{k}{2n}$, $(1 \leq k \leq n-1)$.

This was established in [4] for $k = 1$.

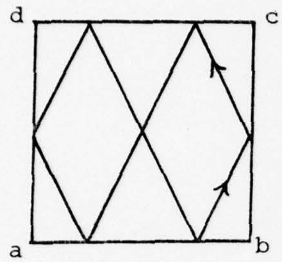
AMS(MOS) Subject Classification: 10K15, 50B99

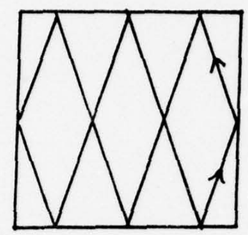
Key Words: Extremum problems
Geometry of numbers
Billiard ball motions

Work Unit Number 2 - Other Mathematical Methods

SIGNIFICANCE AND EXPLANATION

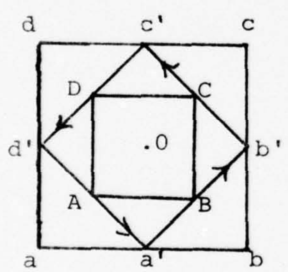
Suppose that we have a square billiard table a b c d, and that we shoot a billiard ball (b.b.) along any of the paths represented by the diagrams





These are very simple closed paths that we see on windows of numerous suburban homes. A famous theorem of the late 19th century mathematician L. Kronecker implies the following: If we shoot a b.b. in a direction having an irrational slope (like $\sqrt{2}$, say) with respect to the sides of the table, then the path of the b.b., if pursued far enough will come as close as we wish, to any point of the table. We then say that the motion is ergodic.

In the opposite direction we ask: What is the largest square A B C D such



that an appropriate b.b. shot, no matter how far pursued, will never penetrate inside the square? (We do mean here that A B C D is concentric with, and parallel to the table a b c d.) The answer is the square A B C D of the diagram, with $AB = \frac{1}{2}\,ab$, and the appropriate shot runs for ever along the boundary of the square a' b' c' d' whose vertices are the midpoints of the sides of the table. This can be shown: The path of any other slanting shot must eventually penetrate inside the square A B C D (See our reference [5]). By slanting shot we mean that the shot is not parallel to any of the side of the table. This represents a characteristic extremum property of the shot a' b' c' d'. The present paper explores the problem in higher dimensions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

EXTREMUM PROBLEMS FOR THE MOTIONS OF A BILLIARD BALL III. THE MULTI-DIMENSIONAL CASE OF KÖNIG AND SZÜCS

I. J. Schoenberg

CONTENTS

EXTREMUM PROBLEMS FOR THE MOTIONS OF A BILLIARD BALL III. THE MULTI-DIMENSIONAL CASE OF KÖNIG AND SZÜCS

I. J. Schoenberg

1. Introduction and main results.

This is the third paper on the subject, but can be read independently of the first two ([3], [4]). Let

$$(1.1)\qquad U_n : 0 \leq x_\nu \leq 1 , \quad (\nu=1,\dots,n)$$

be the unit cube in R^n. Let (a_ν) be a point interior to U_n and

$$(1.2)\qquad L_n^1 : x_\nu = \lambda_\nu u + a_\nu , \quad (\nu=1,\dots,n;-\infty < u < \infty)$$

a rectilinear and uniform motion, where $u = t$ denotes the time. We interpret (1.2) as the motion of a billiard ball (b.b.); as we wish to reflect the b.b. in the usual way on striking the $2n$ facets $x_\nu = 0$ or 1 of U_n, we use the function $\langle x \rangle$ defined by

$$(1.3)\qquad \langle x \rangle = \begin{cases} x & \text{if } 0 \leq x \leq 1 , \\ 2-x & \text{if } 1 \leq x \leq 2. \end{cases} \quad \text{and } \langle x+2 \rangle = \langle x \rangle \text{ for all } x .$$

We have used this function in [3] and [4] in a slightly different normalization. The reflected path of the b.b. within U_n may be described by the equations

$$(1.4)\qquad \Pi_n^1 : x_\nu = \langle \lambda_\nu u + a_\nu \rangle , \quad (\nu=1,\dots,n;-\infty < u < \infty).$$

A classical theorem of Kronecker (See [2]), and its generalization (See [1]), show the following: If the n components (λ_ν) are arithmetically linearly independent, then the motion (1.4) is ergodic, i.e. the path Π_n^1 is dense in U_n. If $1 \leq k \leq n-1$, while the (λ_ν) admit precisely $n-k$ linearly independent linear homogeneous relations with integer coefficients, then the path Π_n^1 is contained in and is dense in a finite *k-dimensional skew polytope* Π_n^k. This was shown by König and Szücs in [2] for $k = 2$ and $n = 3$.

This result shows that the b.b. motions generalize naturally as follows: Let

$$(1.5)\qquad \lambda^i = (\lambda_1^i,\dots,\lambda_n^i), \quad (i=1,\dots,k) \quad (1 \leq k \leq n-1)$$

be k linearly independent vectors. We replace (1.2) by

$$(1.6)\qquad L_n^k : x_\nu = \sum_1^k \lambda_\nu^i u_i + a_\nu , \quad (\nu=1,\dots,n;-\infty < u_i < \infty) ,$$

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which we interpret as a k-dimensional optical signal starting from the point (a_ν) inside U_n at the time $t = 0$, and spreading uniformly within the k-flat L_n^k. As we now think of the 2n facets of U_n as mirrors, the reflected path of the signal is a finite or infinite k-dimensional skew polytope Π_n^k. The function $\langle x \rangle$ may again be used and shows that the reflected path is parametrically represented by the equations

$$\Pi_n^k : x_\nu = \langle \sum_1^k \lambda_\nu^i u_i + a_\nu \rangle , \quad (\nu=1,\ldots,n;-\infty < u_i < \infty) . \tag{1.7}$$

In order to avoid degenerate lower-dimensional problems we shall assume that the original signal (1.6) is in a <u>general position</u>.

<u>Definition</u> 1. <u>We say that the signal</u> (1.6) <u>is in</u> general position (G.P.), provided that

$$\text{(1.8)} \qquad \underline{\text{the}}\ n\ \underline{\text{by}}\ k\ \underline{\text{matrix}}\ \|\lambda_\nu^i\|\ \underline{\text{has no vanishing minor of order}}\ k .$$

<u>Equivalently</u>: If $1 \le \nu_1 < \nu_2 < \ldots < \nu_k \le n$, then the k linear functions

$$x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k},$$

of (1.6), may assume arbitrarily prescribed values for appropriate u_i.

Let $0 < \rho < \frac{1}{2}$, $x = (x_\nu)$, $c = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, and consider the cube

$$C_\rho^n : \|x-c\|_\infty < \rho , \tag{1.9}$$

where $\|x\|_\infty = \max_\nu (|x_\nu|)$.

<u>Definition</u> 2. <u>We say that the path</u> (1.7) <u>is</u> ρ-admissible, <u>and denote it by</u> $\Pi_n^k(\rho)$, <u>provided that it is in G.P., and that</u> Π_n^k <u>never penetrates into the cube</u> (1.9), <u>hence that</u>

$$\Pi_n^k \cap C_\rho^n = \emptyset . \tag{1.10}$$

As an extreme opposite of the ergodic case, we study the following

<u>Problem</u> 1. <u>To determine, or to estimate, the quantity</u>

$$\rho_{k,n} = \text{supremum } \rho , \tag{1.11}$$

<u>the supremum being taken for all</u> ρ <u>having</u> ρ<u>-admissible path</u> $\Pi_n^k(\rho)$.

Our main result is an estimate.

<u>Theorem</u> 1. <u>We have the inequality</u>

$$\rho_{k,n} \ge \frac{1}{2} - \frac{k}{2n} , \quad (1 \le k \le n-1) . \tag{1.12}$$

In §9 we establish Theorem 1 by constructing a path $\Pi_n^k(\rho)$ for values of ρ which are as close to $\frac{1}{2} - \frac{k}{2n}$ as we wish.

In [4] I have shown that the equality sign holds in (1.12) for the case when $k = 1$. We can now do the same for the other extreme case when $k = n-1$.

Theorem 2. We have that

(1.13) $$\rho_{n-1,n} = \frac{1}{2} - \frac{n-1}{2n} = \frac{1}{2n}, \quad (n \geqq 2)$$

The simplest case when $n = 3$, and therefore

(1.14) $$\rho_{2,3} = \frac{1}{6},$$

leads to what I call Kepler's tetrahedron. J. Kepler was the first to notice that four appropriate vertices of the cube U_3 are the vertices of a regular tetrahedron T . As any two facets of T intersect in a facet of U_3 forming equal angles with that facet, it should be clear that the surface of T carries a reflected signal Π_3^2 . It carries, of course, many such, but let us single out one of them and denote it by $\tilde{\Pi}_3^2$. Actually, this signal $\tilde{\Pi}_3^2$ is readily found to be $\frac{1}{6}$-admissible, and it is essentially the only Π_3^2 which is $\frac{1}{6}$ -admissible. This is an apparently new characteristic extremum property of Kepler's tetrahedron: Any other signal Π_3^2 in general position, must penetrate into the cube C_ρ^3 , with $\rho = \frac{1}{6}$.

Theorem 2 allows us to generalize this extremum property of T : There is an essentially unique signal $\tilde{\Pi}_n^{n-1}$ which is in general position and is $\frac{1}{2n}$ -admissible. It is explicitly given by

(1.15) $$\tilde{\Pi}_n^{n-1} : \begin{array}{l} x_\nu = \langle u_\nu \rangle, \quad (\nu=1,\ldots,n-1), \\ x_n = \langle u_1+u_2+\ldots+u_{n-1} + \frac{n-1}{2} \rangle, \quad (-\infty < u_i < \infty) . \end{array}$$

In our elementary paper [5] we considered the case of $n=2$, when the path of $\tilde{\Pi}_2^1$ is the square with vertices in the midpoints of U_2 .

Theorem 3. We construct explicitly the signal $\tilde{\Pi}_n^k(\frac{1}{2} - \frac{k}{2n})$ for the two cases

(1.16) $$(k,n) = (2,4) \quad \text{and} \quad (k,n) = (2,6).$$

Notice that $(k,n) = (2,5)$ is missing: I could not do it.

In view of Theorems 1,2 and 3, I wish to state

Conjecture 1. The value of (1.11) is

(1.17) $$\rho_{k,n} = \frac{1}{2} - \frac{k}{2n}, \quad (1 \leqq k \leqq n-1).$$

The remainder of this paper is in two parts and an Appendix. Part I deals with monochromos and n-chromos in R^k already used in [4] for $k = 1$. We derive Theorems 1', 2', and 3'; in Part II it will be shown that these theorems are equivalent to the above Theorems 1, 2, and 3, respectively.

There are three outstanding problems that we leave unresolved:

1°. A proof of Conjecture 1.

2°. A general arithmetic-analytic construction of a signal

$$\Pi_n^k\left(\frac{1}{2} - \frac{k}{2n}\right), \tag{1.18}$$

as done by Theorem 3 in two very special cases.

3°. To show that the number of signals (1.18) is finite, as shown in [4] for $k = 1$.

Problems 1° *and* 2° are probably related and all three difficult.

In the Appendix (§§10 and 11) we study the same extremum problems, where the rectilinear reflected k-flats are replaced by k-dimensional Lissajous manifolds. Again, Theorems 1', 2', 3' on n-chromos, allow us to derive immediately three Theorems 1^L, 2^L, 3^L, concerning the new situation.

2. Monochromes.

We consider the function

(2.1) $$\{x\} = \min |x-m| \quad \text{for } m \in \mathbb{Z},$$

which is related to the function (1.3), in fact $\{x\} = \langle 2x \rangle /2$. It seems tailor made for dealing with systems of parallel and equidistant planes in $R^k = \{u = (u_1,\ldots,u_k)\}$. For if $\sum_1^k \lambda^i u_i + a$ is a non-constant linear function of the variables u_i, then the equation

(2.2) $$\{\sum_1^k \lambda^i u_i + a\} = 0$$

represents such a system of planes, it being equivalent with the system of equations

(2.3) $$\sum_1^k \lambda^i u_i + a = j \quad (j \in \mathbb{Z}).$$

Let

(2.4) $$0 < \delta < 1,$$

and let us replace (2.2) by the inequality

(2.5) $$M^k(\delta) : \{\sum_1^k \lambda^i u_i + a\} \leqq \frac{\delta}{2}.$$

This represents a system of congruent, parallel, and equidistant slabs of space. We call the point-set $M^k(\delta)$ a monochrome (M.C.) of R^k, because we like to think of its points as carrying a certain color γ. The most familiar case is $k = 2$, when $M^2(\delta)$ assumes the aspect of an awning, of the kind used to provide shade to storefronts.

We shall refer to the planes (2.3) as the central planes of the monochrome (2.5) (central lines if $k = 2$).

The distance between two consecutive central planes (2.3) is found to be $p = 1/\sum(\lambda^i)^2$, while the width of a slab of (2.5) is seen to be $w = \delta/\sum(\lambda^i)^2$. Therefore

$$\delta = \frac{w}{p},$$

and for this reason we call δ the density of the monochrome $M^k(\delta)$. Clearly δ represents the density of the color γ in the space R^k containing $M^k(\delta)$.

3. n-*chromos*.

Let

(3.1) $$n > k,$$

and let us have in R^k n monochromes

(3.2) $$M_1^k(\delta),\ M_2^k(\delta),\ldots,M_n^k(\delta),$$

all of the same density δ. To make matters more picturesque, we think of $M_\nu^k(\delta)$ as carrying the color γ_ν.

Definition 3. *We say that the* n *monochromes* (3.2) *define an* n-*chromo* $\chi_n^k(\delta)$, *provided that*

(3.3) $$\bigcup_{\nu=1}^{n} M_\nu^k(\delta) = R^k.$$

The characteristic property of an n-chromo is therefore that every point (u_i) of R^k is covered by one or more of the colors γ_ν. Using (2.5) we may represent our monochromes by

(3.4) $$M_\nu^k(\delta) : \{\sum_1^k \lambda_\nu^i u_i + a_\nu\} \le \frac{\delta}{2}, \qquad (\nu=1,\ldots,n).$$

Definition 4. *We say that the* n-*chromo* $\chi_n^k(\delta)$ *is admissible, provided that the set of* n *vectors*

(3.5) $$\vec{\lambda}_\nu = (\lambda_\nu^1, \lambda_\nu^2,\ldots,\lambda_\nu^k), \qquad (\nu=1,\ldots,n),$$

which are the normal vectors of our monochromes, have the following property: Every subset of k *vectors* $\vec{\lambda}_{\nu_1}, \vec{\lambda}_{\nu_2},\ldots,\vec{\lambda}_{\nu_k}$ $(\nu_1 < \ldots < \nu_k)$, *spans the space* R^k.

Equivalently: All $\binom{n}{k}$ kth order minors of the matrix

(3.6) $$\Lambda = \| \lambda_\nu^i \|$$

are different from zero.[†]

The following lemma seems evident and requires no proof.

Lemma 1. *A non-singular affine transformation of* R^k *into itself maps monochromes and* n-*chromos into like objects of the same density*.

[†] To see examples of n-chromos in R^2, the reader is invited to inspect the 5-chromo $\chi_5^2(2/5)$ of Figure 1 (§5), and the 4-chromo $\chi_4^2(1/2)$ of Figure 2 (§7). The first is *not* admissible, because its monochromes M_3, M_4, and M_5, are parallel; the second *is* *admissible*, since no two of its monochromes are parallel.

4. An extremum problem for admissible n-chromos.

Let

(4.1) $$\chi_n^k(\delta) = \{M_1^k(\delta), M_2^k(\delta), \ldots, M_n^k(\delta)\}$$

denote the n-chromo defined by (3.4). If we keep everything fixed in (3.4), except that we replace the density δ by $\delta' > \delta$, then it is clear that $\chi_n^k(\delta')$ is a fortiori an n-chromo. This is no longer true if we try to diminish the density δ. In fact, keeping only k and n fixed, it will be our main concern to find an admissible n-chromo $\chi_n^k(\delta)$ having as small a density δ as possible. Evidently, δ can not be too small. It is trivial that we must have

(4.2) $$\delta \geqq \frac{1}{n},$$

for if $\delta < \frac{1}{n}$, then our monochromos (3.2) are clearly unable to cover R^k, as required by (3.3): There just isn't enough paint around!

As mentioned above we are interested in

Problem 1'. To determine, or to estimate, the quantity

(4.3) $$\delta_{k,n} = \text{infimum } \delta$$

for all densities δ of admissible n-chromos $\chi_n^k(\delta)$.

The main result of Part I is

Theorem 1'. We have the inequality

(4.4) $$\delta_{k,n} \leqq \frac{k}{n}, \qquad (1 \leqq k \leqq n-1).$$

Remark. The result (4.4) is rather trivial if $k = 1$, in fact

(4.5) $$\delta_{1,n} = \frac{1}{n}.$$

Proof of (4.5): By (4.2) it suffices to exhibit an admissible $\chi_n^1(\frac{1}{n})$ of density $\frac{1}{n}$. Observe first that the requirement that δ_n^1 be admissible drops out because it is automatically fulfilled for $k = 1$: The relations (3.4) reduce to

(4.6) $$M_\nu^1(\delta) : \{\lambda_\nu^1 u_1 + a_\nu\} \leqq \frac{\delta}{2}, \qquad (\nu=1,\ldots,n),$$

where we implicitly assume that $\lambda_\nu^1 \neq 0$ for all ν, or else we could not speak of monochromes: The matrix (3.6) reduces to a column of non-vanishing elements. Secondly, it is clear that the monochromes of R^1 of density $\frac{1}{n}$

(4.7) $$M_\nu^1(\frac{1}{n}) : \{u_1 + \frac{\nu-1}{n}\} \leqq \frac{1}{2n}, \qquad (\nu=1,\ldots,n)$$

do not overlap and cover the real axis R^1. Therefore $\delta_{1,n} \leqq \frac{1}{n}$ and this established (4.5).

5. A proof of Theorem 1'

We shall proceed as follows: We shall exhibit an admissible $\chi_n^k(\delta)$ having a density which is as close to $\frac{k}{n}$ as we wish, thereby establishing the inequality (4.4). This is done in two stages.

A. Construction of a certain non-admissible $\chi_n^k(\delta)$ of density $\delta = \frac{k}{n}$.

Let

(5.1) $$\delta = \frac{k}{n}, \quad q = n-k .$$

We use the freedom afforded by Lemma 1 and may, without loss of generality, assume the central planes of the first k monochromos to be the planes $u_\nu - \frac{1}{2} = j$, hence

(5.2) $$M_\nu^k(\delta) : \{u_\nu - \frac{1}{2}\} \leqq \frac{\delta}{2}, \qquad (\nu=1,\ldots,k) .$$

In Figure 1 we exhibit the case $k = 2$ and $n = 5$ of our construction, but the same construction holds for any k and $n(>k)$.

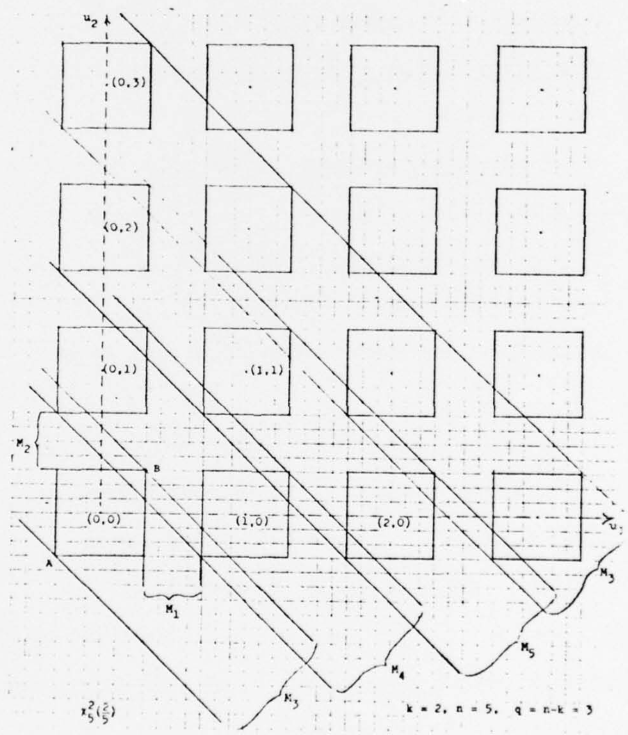

Figure 1.

Notice that monochromes (5.2) already cover all of R^k, with the exception of the lattice of cubes having sides = $1-\delta$

(5.3) $$C(m_1,\ldots,m_k) : \| x - \vec{m}\|_\infty < \frac{1-\delta}{2}, \quad \vec{m} = (m_i) \in \mathbf{Z}^k .$$

The remaining $q = m-k$ monochromes are to cover all those cubes.

I claim that <u>the monochrome</u>

(5.4) $$M^k_{k+1}(\delta) : \left\{ \frac{\sum_1^k u_i}{q} \right\} \leqq \frac{\delta}{2}$$

<u>just covers all cubes</u>

(5.5) $$C(m_1,\ldots,m_2) \ \underline{\text{such that}} \ \sum_1^k m_i \equiv 0 \pmod q .$$

<u>Proof of claim</u>: We look at the cube $C(0,\ldots,0)$ and let

(5.6) $$A = \left(-\frac{1-\delta}{2},\ldots,-\frac{1-\delta}{2}\right), \quad B = \left(\frac{1-\delta}{2},\ldots,\frac{1-\delta}{2}\right)$$

be its vertices such that $\vec{AB}$ has direction numbers $(1,1,\ldots,1)$. The slab of (5.4) containing the origin is defined by

(5.7) $$-\frac{\delta}{2} \leqq \frac{\sum_1^k u_i}{q} \leqq \frac{\delta}{2} .$$

Notice that the right bounding plane $\sum u_i/q = \delta/2$ contains the point B, because at B we have by (5.6)

$$\frac{\sum u_i}{q} = \frac{k}{q}\,\frac{1-\delta}{2} = \frac{k}{q}\,\frac{1-\frac{k}{n}}{2} = \frac{k}{2n}\,\frac{n-k}{q} = \frac{k}{2n} = \frac{\delta}{2}$$

by (5.1). Similarly the left bounding plane $-(\delta/2) = (\sum u_i)/q$ passes through A. The normal to the monochrome (5.4) being the vector $(1,\ldots,1)$, it is clear that (5.4) contains the set

$$\bigcup_{\sum m_i = 0} C(m_1,\ldots,m_k) .$$

However, the central planes of (5.4) are

$$\frac{\sum_1^k u_i}{q} = j \qquad (j \in \mathbf{Z}),$$

and these pass through the centers of all cubes $C(m_1,\ldots,m_n)$ such that $\sum_1^k m_i = q\,j$. This proves our claim.

By parallel translation we now define

(5.8) $$M^k_{k+r}(\delta) : \left\{ \frac{\sum_1^k u_i + r - 1}{q} \right\} \leq \frac{\delta}{2} , \quad (r=1,2,\ldots,q) ,$$

and this covers all cubes

$$C(m_1,\ldots,m_2) \text{ such that } \sum_1^k m_i \equiv -r+1 \pmod q ,$$

because the central planes of (5.8) pass through their centers. The n-chromo

(5.9) $$\chi^k_n(\delta) = \{M^k_1(\delta),\ldots,M^k_n(\delta)\}$$

defined by (5.2) and (5.8) is the inadmissible n-chromo of density $\delta = k/n$ we wish to construct. (5.9) is <u>not</u> admissible because its last $n-k=q$ monochromes are pairwise parallel.

B. <u>Construction of an admissible</u> n-<u>chromo of density</u> δ <u>close to</u> k/n.

This will be achieved by an appropriate slight perturbation of (5.9). We start by selecting a fixed matrix

(5.10) $$A = \| a_{ri} \| \quad (r=1,\ldots,q; i=1,\ldots,k)$$

having the following properties:

(5.11) <u>The elements</u> a_{ri} <u>are integers</u>,

(5.12) <u>All minors of</u> A, <u>hence of orders from</u> 1 <u>to</u> $\min(q,k)$ <u>are</u> $\neq 0$.

From the known <u>total positivity</u> properties of the binomial coefficients, both conditions are verified if we select

(5.13) $$a_{\nu i} = \binom{k+\nu}{i} .$$

We are now going to modify the n-chromo (5.9) as follows. We will select for it a density $\tilde{\delta}$ to be determined later. We replace the first monochromes (5.2) by

(5.14) $$M^k_\nu(\tilde{\delta}) : \{u_\nu - \tfrac{1}{2}\} \leq \frac{\tilde{\delta}}{2} \quad (\nu=1,\ldots,k).$$

For the last $n-k=q$ monochromes we prescribe their central planes to be

(5.15) $$\pi_{r,j} : N \frac{\sum_1^k u_i + (r-1)}{q} + a_{r1}u_1 + a_{r2}u_2 + \ldots + a_{rk}u_k = j , \quad (j \in \mathbb{Z}),$$

$$(r = 1,\ldots,q) .$$

Here N is a positive integer to be made large later. We claim that <u>every lattice point</u> $(m_i) \in \mathbb{Z}^k$ <u>is in one of these planes</u> $\pi_{r,j}$.

For if $(m_i) \in \mathbf{Z}^k$ is given, we determine the unique r such that

(5.16) $$r \equiv -\sum_1^k m_i + 1 \pmod q$$

and then

(5.17) $$(m_i) \in \pi_{r,j'} \quad \text{for some } j' \in \mathbf{Z} .$$

Thus

(5.18) $$\mathbf{Z}^k \subset \bigcup_{r=1}^{q} \bigcup_{j=-\infty}^{\infty} \pi_{r,j} .$$

Let us now look at the geometric aspect of the planes $\pi_{r,j}$. (5.15) may be written as

(5.19) $$\pi_{r,j} : \sum_1^k u_i + \frac{q}{N}(a_{r1}u_1 + a_{r2}u_2 + \ldots + a_{rk}u_k) = \frac{q}{N} j - r+1 ,$$

and this shows that

(5.20) <u>all the planes</u> $\pi_{r,j}$ <u>are nearly parallel to the plane</u> $\sum_1^k u_i = 0$, <u>provided that</u> N <u>is sufficiently large.</u>

Let $\overrightarrow{A(m_i)B(m_i)}$ be the diagonal of the cube $C(m_i)$, which is parallel to the old diagonal $\overrightarrow{AB}$ of $C(0,\ldots,0)$. Let r be fixed such that (5.17) holds. We construct a monochrome $M^k_{k+r}(\delta_r)$, parallel to $\pi_{r,j'}$, which <u>just covers</u> the cube $C(m_i)$. It is obtained by bounding its slab of color (containing $C(m_i)$) by the two planes parallel to $\pi_{r,j'}$, and passing through the points $A(m_i)$ and $B(m_i)$, respectively. This monochrome will also cover all cubes $C(m_i)$ such that (5.16) holds, or

(5.21) $$\sum_1^k m_i \equiv -r+1 \pmod q .$$

We may write

(5.22) $$M^k_{k+r}(\delta_r) : \left\{ N \frac{\sum_1^k u_i + r-1}{q} + \sum_1^k a_{ri}u_i \right\} \leq \frac{\delta_r}{2} .$$

In view of (5.19) we conclude that its density δ_r will be as close as we wish to the old density k/n of (5.9).

For the final selection of our monochrome $\tilde{M}^k_\nu$, we keep the inequalities (5.14) and (5.22), only modifying the density, by selecting for both groups the common density $\tilde{\delta}$ defined by

(5.23) $$\tilde{\delta} = \max\left(\frac{k}{n}, \delta_1, \delta_2, \ldots, \delta_q\right) .$$

Thus $\tilde{\delta} \geq \frac{k}{n}$. If $\tilde{\delta} > \frac{k}{n}$, then (5.14) shows that our old cubes $C(m_i)$ have shrunk, and are therefore a fortiori covered by the $M^k_{k+r}(\tilde{\delta})$.

Since $\tilde{\delta} \to \frac{k}{n}$ as $N \to \infty$, the n-chromo

(5.24) $$\tilde{X}^k_n(\tilde{\delta}) = \{\tilde{M}_1(\tilde{\delta}),\ldots,\tilde{M}_n(\tilde{\delta})\}$$

will have a density $\tilde{\delta}$ as close to $\frac{k}{n}$, provided that we select N sufficiently large.

The question: <u>Is the</u> n-<u>chromo</u> (5.24) <u>admissible</u>?

By (5.14) and (5.19) we see that the matrix (3.6) for its central planes is

(5.25) $$\Lambda = \|\lambda^i_\nu\| = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & \\ 0 & \ddots & 1 \\ 1+\frac{q}{N}a_{11},\ldots & , & 1+\frac{q}{N}a_{1k} \\ \vdots & & \\ 1+\frac{q}{N}a_{q1},\ldots & , & 1+\frac{q}{N}a_{qk} \end{matrix} \right\| = \left\| \begin{matrix} I_k \\ \\ 1+\frac{q}{N}a_{ri} \end{matrix} \right\| .$$

We claim that <u>all its</u> kth <u>order minors are</u> $\neq 0$ <u>if</u> N <u>is sufficiently large.</u>

This will be the case if and only if

(5.26) <u>for sufficiently large</u> N <u>the matrix</u> $\|1 + \frac{q}{N}a_{ri}\|$ <u>has no vanishing minor of any order from</u> 1 <u>to</u> $\min(q,k)$.

To verify this statement let us look at an sth order minor of the matrix of (5.26).[+] We inspect the leading minor $\det|1 + \frac{q}{N}a_{ri}|$ for $r = 1,\ldots,s$, $i = 1,\ldots,s$. Splitting each of its columns into two columns, we find

(5.27) $$\det|1 + \frac{q}{N}a_{ri}|_{1,s} = (\frac{q}{N})^s \det|a_{ri}|_{1,s} + (\frac{q}{N})^{s-1}\cdot S ,$$

where S is the sum of s determinants obtained from $\det|a_{ri}|_{1,s}$ by replacing each of its columns successively by a column of 1's. We distinguish two cases:

1. If $S \neq 0$, then the right hand side of (5.27) will surely be $\neq 0$ if N is sufficiently large.
2. If $S = 0$, we reach the same conclusion in view of the property (5.12) which implies that $\det|a_{ri}|_{1,s} \neq 0$. We have shown that the n-chromo (5.24) is admissible, which completes our proof of Theorem 1'.

[+]The same reasoning will apply to any other minor.

6. Solution of Problem 1' if $k = n-1$.

Among the n-chromos (5.9) for $k = 2,3,\dots,n-1$ we single out the case

(6.1) $$k = n-1 ,$$

this being the only one which is admissible. Its density is

(6.2) $$\delta = \frac{n-1}{n} .$$

By (5.2) and (5.8), its monochromes are described by

(6.3) $$M_\nu\left(\frac{n-1}{n}\right) : \{u_\nu - \frac{1}{2}\} \leqq \frac{n-1}{2n} \qquad (\nu=1,\dots,n-1)$$

and

(6.4) $$M_n\left(\frac{n-1}{n}\right) : \{\sum_1^k u_i\} \leqq \frac{n-1}{2n} ,$$

since $q = 1$.

We wish to prove the

Theorem 2' We have that

(6.5) $$\delta_{n-1,n} = \frac{n-1}{n} , \qquad (n \geqq 2) .$$

Proof: We know from §5 that the monochromes (6.3) cover all of R^{n-1} with the exception of the lattice of cubes

(6.6) $$C(m_1,\dots,m_{n-1}), \quad (m_1,\dots,m_{n-1}) \in \mathbf{Z}^{n-1} ,$$

centered at the lattice points and having sides $= 2.\ \frac{1-\delta}{2} = 1-\delta = 1 - \frac{n-1}{n} = \frac{1}{n}$.

We also know that the last monochrome (6.4) just covers all these cubes.

For convenience we say that a monochrome $M^{n-1}(\delta')$ of R^{n-1} is slanting, provided that all n-1 components of its normal vector are positive.

Lemma 2. If the slanting monochrome

(6.7) $$M^{n-1}(\delta') : \{u_1 + \gamma_2 u_2 + \dots + \gamma_{n-1} u_n + b\} \leqq \frac{\delta'}{2} ,$$

where

(6.8) $$\gamma_2 > 0, \dots, \gamma_{n-1} > 0 ,$$

covers the set

(6.9) $$\Gamma = \bigcup_{(m_\nu) \in \mathbf{Z}^{n-1}} C(m_1,\dots,m_{n-1}) ,$$

then we must have that

(6.10) $$\gamma_2 = \gamma_3 = \dots = \gamma_{n-1} = 1 .$$

Proof of Lemma 2: Let S denote the set of planes π which are parallel to (6.7) and intersect the set Γ, hence

(6.11) $$S = \{ \pi : u_1 + \sum_2^{n-1} \gamma_i u_i = \text{const.} ; \pi \cap \Gamma \neq \phi \} .$$

Crucial in our discussion is the nature of the set

(6.12) $$\Omega = S \cap R^1 ,$$

where

(6.13) $$R^1 = \{(u_i) ; u_2 = \dots = u_{n-1} = 0\}$$

is the u_1-axis.

We claim: If

(6.14) $$(\gamma_2, \gamma_3, \dots , \gamma_{n-1}) \neq (1,1,\dots,1) ,$$

then

(6.15) $$\Omega = R^1 .$$

Proof of Claim: The set S is the union of those planes π which intersect the individual cubes $C(m_1,\dots,m_{n-1})$. At this point it is more convenient to shift the origin of R^{n-1} to the "lower left hand corner" of the cube $C(0,\dots,0)$. This is the point A of (5.6), for $k = n-1$. Since $1-\delta = \frac{1}{n}$, we see that after this shift of origin

(6.16) $$C(m_1,\dots,m_{n-1}) = \{(u_\nu) : m_i \leq u_i \leq m_i + \frac{1}{n} , \quad (i=1,\dots,n-1)\}.$$

Let us project this cube onto R^1 by planes parallel to our monochrome. The two extreme planes are two planes of support of C and their equations are

$$(u_1-m_1) + \sum_2^{n-1} \gamma_i (u_i-m_i) = 0 \quad \text{and} \quad (u_1-m_1-\frac{1}{n}) + \sum_2^{n-1} \gamma_i (u_i-m_i-\frac{1}{n}) = 0 ,$$

respectively. To intersect them with R^1, we set $u_2 = \dots = u_{n-1} = 0$ in these equations and solve them for u_1. In this way we find that the cube (6.16) is projected into the interval

(6.17) $$I(m_1,\dots,m_{n-1}) = [m_1 + \sum_2^{n-1} \gamma_i m_i , m_1 + \sum_2^{n-1} \gamma_i m_i + \frac{1}{n} (1 + \gamma_2 + \dots + \gamma_{n-1})] .$$

For the set (6.12) we now find that

(6.18) $$\Omega = \bigcup I(m_1,\dots,m_{n-1}) \quad \text{for} \quad (m_1,\dots,m_{n-1}) \in \mathbf{Z}^{n-1} .$$

We distinguish two cases.

1. Among the γ_i there is an irrational one, γ_2 say. Setting $m_i = 0$ for $i > 2$, we find the lower endpoint of $I(m_1,m_2,0,\dots,0)$ to be

(6.19) $$m_1 + \gamma_2 m_2 \qquad (\gamma_2 \text{ is irrational})$$

and Kronecker's theorem shows that these lower endpoints are dense in R^1. By (6.18) our conclusion (6.15) clearly follows.

2. All γ_ν are rational. Writing them in simplest terms with a common denominator we have

$$(6.20) \qquad \gamma_\nu = \frac{a_\nu}{b}, \quad (\nu=2,\ldots,n-1), \quad (b,a_2,\ldots,a_{n-1}) = 1 .$$

As our assumption (6.14) excludes the case when $b = a_2 = \ldots = a_{n-1} = 1$, we have $b + a_2 + \ldots + a_{n-1} \geqq n$ and therefore

$$(6.21) \qquad \frac{1}{n}(1 + \gamma_2 + \ldots + \gamma_{n-1}) \geqq \frac{1}{b} .$$

However, the lower endpoints of the intervals (6.17) form the arithmetic progression j/b ($j \in \mathbf{Z}$). Since (6.21) shows that the common length of our intervals (6.17) is $\geqq 1/b$, again we have by (6.18) that (6.15) holds.

Completing a proof of Theorem 2'. By Lemma 2 we learn that a monochrome (6.7) covering the set (6.9), must be of the form

$$(6.22) \qquad M^{n-1}(\delta') : \{\sum_1^{n-1} u_i + b\} \leq \frac{\delta'}{2} .$$

As this must also cover (6.4), we conclude that $\delta' \geqq \frac{n-1}{n}$. This establishes Theorem 2': For if we diminish the common density of the (6.3), then this would increase the common side of the cubes (6.6), and then these could only be covered by a slanting monochrome of density $> \frac{n-1}{n}$, as we have seen.

In view of Theorems 1', 2', and the examples of Theorem 3', I wish to state

Conjecture 1'. The value of (4.3) is

$$(6.23) \qquad \delta_{k,n} = \frac{k}{n} \qquad (1 \leqq k \leqq n-1) .$$

Remark. Just a comment on the monochromes (4.7) of R^1, of density $\delta = \frac{1}{n}$. Clearly, the inequalities

$$M_\nu^k\left(\frac{1}{n}\right): \{u_1 + \frac{\nu-1}{n}\} \leqq \frac{1}{2n}, \qquad (\nu=1,\ldots,n)$$

also define an n-chromo $\chi_n^k(\frac{1}{n})$ in R^k, having the density $\frac{1}{n}$. This does not contradict the above conjectured relation (6.23): The quantity $\delta_{k,m}$ was defined as the infimum of δ for n-chromos $\chi_n^k(\delta)$ in R^k, which are admissible, while the above n-chromos $\chi_n^k(\frac{1}{n})$ is far from satisfying that essential requirement. In fact all of its n monochromes are parallel.

7. Two special explicit n-chromos.

Theorem 1' was not established by exhibiting an n-chromo

(7.1) $$\chi_n^k\left(\frac{k}{n}\right)$$

which is both admissible and of density k/n. Rather in §5 we construct admissible $\chi_n^k(\delta)$, with δ as close to k/n as we wished. In view of our Conjecture 1' of §6, the construction of an admissible n-chromo (7.1), for prescribed k and n $(k < n)$, is a most desirable but as yet unsolved problem. Even for low values of k and n, the success depends, so far, on luck and visual inspection. Needed is a general arithmetic-analytic construction.

As a guide to the nature of this problem, the following two specific examples might be useful.

Theorem 3'. We give explicit constructions of the n-chromo (7.1) for the following two cases

(7.2) $$(k,n) = (2,4) \quad \text{and} \quad (k,n) = (2.6).$$

1. $k = 2,\ n = 4$. Here the density is

(7.3) $$\delta = \frac{1}{2}.$$

The four monochromes of $\chi_4^2(\frac{1}{2})$ are

(7.4)
$$M_1^2\left(\frac{1}{2}\right): \{u_1 - \tfrac{1}{2}\} \leq \frac{1}{4}, \qquad M_2^2\left(\frac{1}{2}\right): \{u_2 - \tfrac{1}{2}\} \leq \frac{1}{4},$$
$$M_3^2\left(\frac{1}{2}\right): \left\{\frac{u_1 + u_2}{2}\right\} \leq \frac{1}{4}, \qquad M_4^2\left(\frac{1}{2}\right): \left\{\frac{u_1 - u_2 + 1}{2}\right\} \leq \frac{1}{4}.$$

These are easily derived from Figure 2 which shows that we have an admissible 4-chromo of R^2.

The first two monochromes (7.4) cover the plane with the exception of the lattice of squares $C(m_1, m_2)$ having sides = 1/2. The third monochromo M_3 covers all those squares such that $m_1 + m_2$ is even, while M_4 covers those with an odd sum $m_1 + m_2$.

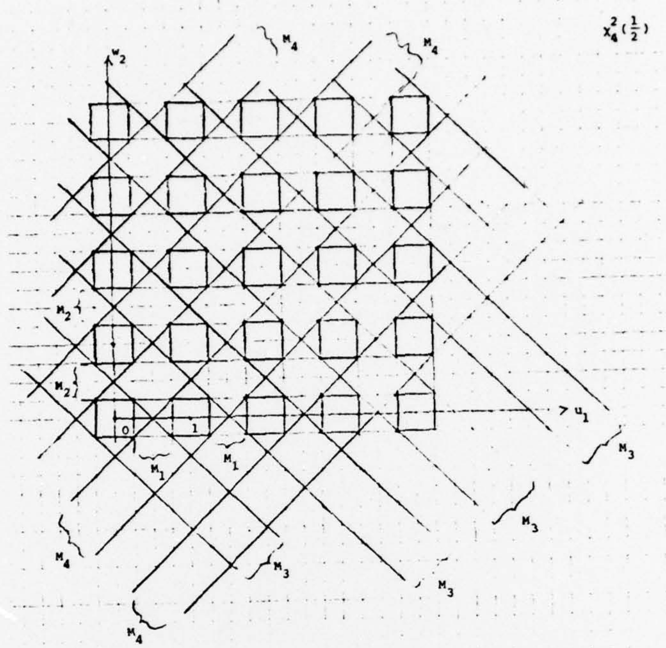

Figure 2

2. <u>$k = 2$, $n = 6$.</u> Now

(7.5) $$\delta = \frac{1}{3} .$$

The six monochromes of $\chi_6^2(\frac{1}{3})$ are

(7.6) $$M_1^2(\tfrac{1}{3}) \ : \{u_1 - \tfrac{1}{2}\} \leqq \tfrac{1}{6} \ , \quad M_2^2(\tfrac{1}{3}) : \{u_2 - \tfrac{1}{2}\} \leqq \tfrac{1}{6}$$

(7.7) $$M_3^2(\tfrac{1}{3}) \ : \{u_1 + u_2\} \leqq \tfrac{1}{6} \ , \quad M_4^2(\tfrac{1}{3}) \ : \left\{\frac{u_1 - u_2}{3}\right\} \leqq \frac{1}{6}$$

(7.8) $$M_5^2(\tfrac{1}{3}) \ : \left\{\frac{2u_1 + u_2}{3} + \frac{1}{2}\right\} \leqq \frac{1}{6} \ , \quad M_6^2(\tfrac{1}{3}) \ : \left\{\frac{u_1 + 2u_2}{3} + \frac{1}{2}\right\} \leqq \frac{1}{6} .$$

These are easily derived from Figure 3 which shows that we have an admissible 6-chromo of R^2 .

A guiding word in this maze of lines seems appropriate. The two monochromes (7.6) cover R^2, except for the lattice of squares $C(m_1,m_2)$ having sides $= 2/3$. The monochrome M_3, having central lines $u_1+u_2 = j$ $(j \in \mathbb{Z})$, is seen to slice each of the squares into two congruent isosceles triangles; we denote the lower one by $T_1(m_1,m_2)$ and the upper one by $T_2(m_1,m_2)$. The monochrome M_4, having central lines $u_1-u_2 = 3j$ $(j \in \mathbb{Z})$, is seen to cover all pairs

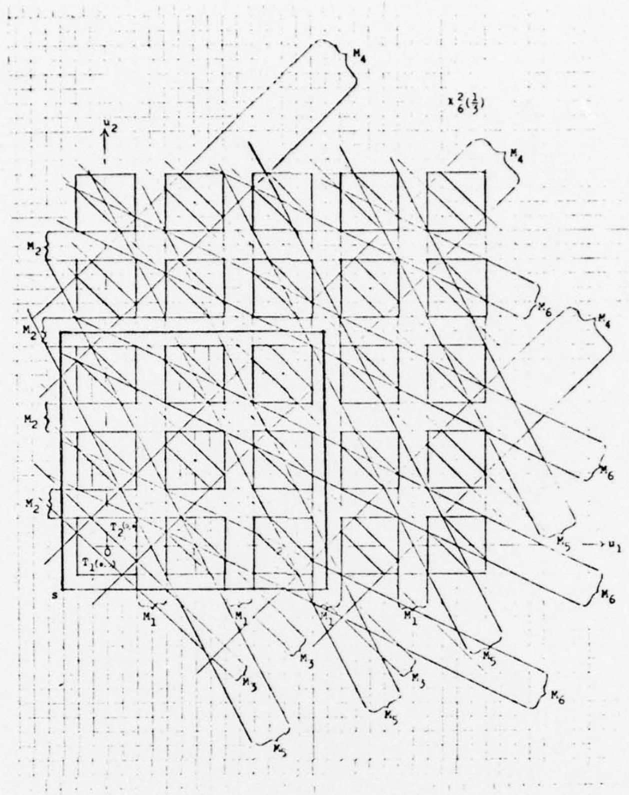

Figure 3

$$T_1(m_1,m_2), \quad T_2(m_1,m_2)$$

such that $m_1-m_2 \equiv 0 \pmod 3$. The last two monochromos M_5 and M_6 are to cover all remaining triangles.

At this point we observe, by (7.6), (7.7), (7.8), that each of the six M_ν admits a (double) periodicity of period 3 in each of the variables u_1 and u_2. It follows that it suffices to inspect our Figure 3 only in the square

$$S: -\frac{1}{2} \leq u_1 < 2 + \frac{1}{2}, \quad -\frac{1}{2} \leq u_2 < 2 + \frac{1}{2},$$

which in Figure 3 is indicated by a solid frame. In that square we are only left with the following triangles as yet uncovered:

$$T_1(1,0),\ T_2(1,0),\ T_1(2,1),\ T_2(2,1),\ T_1(2,0),\ T_2(2,0),$$

and the symmetric set

$$T_1(0,1),\ T_2(0,1),\ T_1(1,2),\ T_2(1,2),\ T_1(0,2),\ T_2(0,2).$$

However, M_5 covers

$$T_1(1,0),\ T_2(0,1),\ T_1(0,2) \quad \text{and} \quad T_2(2,0),\ T_1(2,1),\ T_2(1,2)\ ,$$

while M_6 covers their symmetric images

$$T_1(0,1),\ T_2(1,0),\ T_1(2,0) \quad \text{and} \quad T_2(0,2),\ T_1(1,2),\ T_2(2,1)\ .$$

This proves that we have a 6-chromo; it is admissible because no two monochromes are parallel.

Let me add that I could not discover a $\chi_5^2(2/5)$.

II. Applications of n-chromos to billiard ball motions.

8. The equivalence of Problems 1 and 1'.

This equivalence appears immediately as soon as we switch the problems (or "action") from the space $R^n = \{(x_\nu)\}$ to the lower dimensional space $R^k = \{(u_i)\}$. *Indeed, let*

(8.1) $$\Pi_n^k : x_\nu = \langle \sum_{i=1}^{k} \lambda_\nu^i u_i + a_\nu \rangle , \quad (\nu=1,\dots,n),$$

be a ρ-admissible reflected signal. Let ρ and δ be related by

(8.2) $$\rho = \frac{1}{2} - \frac{\delta}{2} \quad \text{or} \quad \delta = 1-2\rho.$$

That (8.1) is ρ-admissible means that it is contained in the cubical shell

(8.3) $$B_\rho^n = U_n - C_\rho^n ,$$

having the width $\frac{1}{2} - \rho = \delta/2$. The structure of the function $\langle x \rangle$ implies the following: The point of R^n

$$\left(\sum_1^k \lambda_\nu^i u_i + a_\nu \right) \quad (\nu=1,\dots,n)$$

has the property that for every (u_ν) and for some ν the number $\sum \lambda_\nu^i u_i + a_\nu$ differs from an integer by $\leq \delta/2$. However, this last property can be expressed thus:

(8.4) For every (u_ν) and for some ν we have $\{\sum \lambda_\nu^i u^i + a_\nu\} \leq \frac{\delta}{2}$.

In terms of the monochromes

(8.5) $$M_\nu^k(\delta) : \{ \sum_1^k \lambda_\nu^i u_i + a_\nu \} \leq \frac{\delta}{2} , \quad (\nu=1,\dots,n),$$

the property (8.4) is equivalent to the set relation

(8.6) $$R^k = \bigcup_{\nu=1}^{n} M_\nu^k ,$$

which is the definition (3.3) of an n-chromo. The steps can be reversed and establish

Lemma 4. Let the relations (8.2) hold. The reflected signal (8.1) is ρ-admissible if and only if

(8.7) $$X_n^k(\delta) = \{M_1^k(\delta),\dots,M_n^k(\delta)\} ,$$

defined by (8.5), is an n-chromo. That (8.1) is in general position if and only if (8.5) is admissible is obvious, because they are expressed by the same condition on the matrix $\Lambda = \| \lambda_\nu^i \|$.

9. Applications of Lemma 4: Proofs of Theorems 1, 2, and 3.

The relation (8.2), Lemma 4, and the definitions (1.11) of $\rho_{k,n}$, and (4.3) of $\delta_{k,n}$, show that

$$\rho_{k,n} = \frac{1}{2} - \frac{\delta_{k,n}}{2}, \tag{9.1}$$

or

$$\delta_{k,n} = 1 - 2\rho_{k,n}. \tag{9.2}$$

By Theorem 1' $\delta_{k,n} \leqq \frac{k}{n}$ and (9.1) implies that $\rho_{k,n} \geqq \frac{1}{2} - \frac{k}{2n}$ and Theorem 1 is established.

By Theorem 2' $\delta_{n-1,n} = \frac{n-1}{n}$ and (9.1) implies that $\rho_{n-1,n} = \frac{1}{2} - \frac{n-1}{2n} = \frac{1}{2n}$ and Theorem 2 is proved.

Let us use Lemma 4 to derive for $k = n-1$ the equations for the signal $\tilde{\Pi}_n^{n-1}(\frac{1}{2n})$. From the relations (6.3), (6.4), we find by Lemma 4 for this signal the equations

$$\tilde{\Pi}_n^{n-1}\left(\frac{1}{2n}\right): \quad \begin{aligned} x_\nu &= \langle u_\nu - \tfrac{1}{2} \rangle, \qquad (\nu=1,\ldots,n-1). \\ x_n &= \langle \textstyle\sum_1^{n-1} u_i \rangle . \end{aligned} \tag{9.3}$$

Replacing here u_ν by $u_\nu + \frac{1}{2}$, we obtain

$$\tilde{\Pi}_n^{n-1}\left(\frac{1}{2n}\right): \quad \begin{aligned} x_\nu &= \langle u_\nu \rangle, \qquad (\nu=1,\ldots,n) \\ x_n &= \langle \textstyle\sum_1^{n-1} u_i + \frac{n-1}{2} \rangle , \end{aligned} \tag{9.4}$$

which are identical with (1.15). The essential unicity of the n-chromo (6.3), (6.4), established in §6, implies the essential unicity of the signal (9.4).

In the special case that $n = 3$, we obtain Kepler's tetrahedron T mentioned in connection with the relation (1.14). By (9.4) its parametric equations are

$$\tilde{\Pi}_3^2\left(\frac{1}{6}\right): x_1 = \langle u_1 \rangle, \quad x_2 = \langle u_2 \rangle, \quad x_3 = \langle u_1+u_2+1 \rangle. \tag{9.5}$$

The vertices of $T = ABCD$ are

$$A = (0,0,1), \; B = (1,0,0), \; C = (0,1,0), \; D = (1,1,1).$$

An even simpler case is $n = 2$ when

$$\tilde{\Pi}_2^1\left(\frac{1}{4}\right): x_1 = \langle u_1 \rangle, \quad x_2 = \langle u_1 + \tfrac{1}{2} \rangle . \tag{9.6}$$

This is the square having as vertices the midpoints of the sides of U_2 (See [5]).

As a last application of Lemma 4 we use Theorem 3' to give the explicit constructions of the two signals for the cases (1.16) of Theorem 3. From (7.4), and Lemma 4, we find immediately

$$\Pi_4^2(\tfrac{1}{4}) : \quad x_1 = \langle u_1 - \tfrac{1}{2} \rangle , \quad x_2 = \langle u_2 - \tfrac{1}{2} \rangle , \quad x_3 = \langle \frac{u_1 + u_2}{2} \rangle , \quad x_4 = \langle \frac{u_1 - u_2 + 1}{2} \rangle$$

Replacing u_i by $u_i + \frac{1}{2}$, we obtain

$$\Pi_4^2(\tfrac{1}{4}) : \quad x_1 = \langle u_1 \rangle , \quad x_2 = \langle u_2 \rangle \quad x_3 = \langle \frac{u_1 + u_2 + 1}{2} \rangle , \quad x_4 = \langle \frac{u_1 - u_2 + 1}{2} \rangle$$

Likewise, (7.6), (7.7), (7.8), and Lemma 4, show that

$$\Pi_6^2(\tfrac{1}{3}) : \quad x_1 = \langle u_1 \rangle , \quad x_2 = \langle u_2 \rangle$$
$$x_3 = \langle u_1 + u_2 + 1 \rangle , \quad x_4 = \langle \frac{u_1 - u_2}{3} \rangle$$
$$x_5 = \langle \frac{2u_1 + u_2}{3} + 1 \rangle , \quad x_6 = \langle \frac{u_1 + 2u_2}{3} + 1 \rangle .$$

It is to be expected that these explicit parametric equations, as well as (9.4), should reveal pertinent geometric aspects of the polytopes that they represent.

Our approach via n-chromos suggests that a promising attack on the three problems stated at the end of §1, should be to solve the corresponding problems for n-chromos in R^k. These are:

1'°. A proof of Conjecture 1' ,

2'°. A general arithmetic-analytic construction of the admissible n-chromo

$$\chi_n^k(\tfrac{k}{n}) . \qquad (9.7)$$

3'°. A proof that the number of n-chromos (9.7), no two of which are affinely equivalent, is finite. This was done in [4] for $k = 1$.

Appendix. Extremum problems for Lissajous-type manifolds

10. Applications of n-chromos to Lissajous-type manifolds.

In [3,§6] we discussed our extremum problem for Lissajous curves in the unit cube $-\frac{1}{2} \le x_\nu \le \frac{1}{2}$, $(\nu=1,\ldots,n)$, the underlying norm being the Euclidean one. Here two changes alter the situation:

1. We replace the above cube by our cube U_n of (1.1). This requires replacing the basic function $w(x) = \cos x$ of [3,§6] by the function

$$L(x) = \sin^2 \frac{\pi x}{2}, \quad \text{(See Figure 4)}. \tag{10.1}$$

Observe that $L(x)$ interpolates at the integers the zigzag curve of $\langle x \rangle$ defined by (1.3). The absence of corners assures the smoothness of the resulting motions within U_n. However, the 1-dimensional Lissajous motions

$$x_\nu = L(\lambda_\nu t + a_\nu), \quad (\nu=1,\ldots,n), \tag{10.2}$$

of [3, relation (6.3)] again exhibit the ergodic (or denseness) property described for b.b. motions in the second paragraph of our Introduction. For this reason, and following again the lead of König and Szücs, we replace the motion (10.2) by the k-dimensional Lissajous-type manifold

$$\Lambda_n^k : x_\nu = L\left(\sum_{i=1}^{k} \lambda_\nu^i u_i + a_\nu \right), \quad (\nu=1,\ldots,n). \tag{10.3}$$

2. We replace the Euclidean norm of [3] by the L_∞ norm of the present paper.

The Definitions 1 and 2, of §1, concerning the reflected path (1.7) carry over without any changes to the L-manifold (10.3). We may therefore safely assume that we know what is meant by "a Λ_n^k in general position", and by "a Λ_n^k that is ρ^L- admissible", The latter will again be denoted by $\Lambda_n^k(\rho^L)$.

As in the Introduction we propose

Problem 1^L. To determine, or estimate, the quantity

$$\rho_{k,n}^L = \text{supremum } \rho^L, \tag{10.4}$$

the supremum being taken for all ρ^L having ρ^L-admissible L-type manifolds $\Lambda_n^k(\rho^L)$.

It does seem remarkable that our results of Part I, on n-chromos in R^k, apply equally well to establish Theorems 1^L, 2^L, and 3^L, below, that correspond to Theorems 1, 2, and 3, on b.b. motions. In particular the $\delta_{k,n}$ below, is again the old constant (4.3) for n-chromos. These theorems are as follows.

Theorem 1^L. We have the inequality

(10.5) $$\rho^L_{k,n} \geq \frac{1}{2} - \sin^2\left(\frac{\pi}{2}\cdot\frac{k}{2n}\right), \quad (1 \leq k \leq n-1).$$

Theorem 2^L. We have that

(10.6) $$\rho^L_{n-1,n} = \frac{1}{2} - \sin^2\left(\frac{\pi}{2}\cdot\frac{n-1}{2n}\right).$$

Theorem 3^L. We construct explicitly the L-type manifold

(10.7) $$\Lambda^k_n(\rho^L), \text{ where } \rho^L = \frac{1}{2} - \sin^2\left(\frac{\pi}{2}\cdot\frac{k}{2n}\right),$$

for the two cases

(10.8) $$(k,n) = (2,4) \text{ and } (k,n) = (2,6).$$

At this point we need an analogue of Lemma 4, that we shall call Lemma 4^L, which will relate L-type manifolds to n-chromos. Let (10.3) be ρ^L-admissible. This means that for every $(u_i) \in R^k$, the point of R^n

(10.9) $$L\left(\sum_1^k \lambda^i_\nu u_i + a_\nu\right), \quad (\nu=1,\ldots,n),$$

should belong to the closed cubical shell

(10.10) $$B^n_{\rho^L} = U_n - C^n_{\rho^L}, \text{ where } C^n_{\rho^L} = \{\|x - c\|_\infty < \rho^L\}.$$

Equivalently:

(10.11) For some ν, the number $L(\sum \lambda^i_\nu u_i + a_\nu)$ should differ from an integer by $\leq \delta^L/2$, where

(10.12) $$\frac{\delta^L}{2} = \frac{1}{2} - \rho^L.$$

How is this condition expressed in terms of $\sum \lambda^i_\nu u^i + a_\nu$? If we define $\delta/2$ as a solution of the equation

(10.13) $$\frac{\delta^L}{2} = L\left(\frac{\delta}{2}\right),$$

then the symmetries of the graph of $L(x)$ show (Figure 4) that (10.11) will hold if and only if

(10.14) $$\left\{\sum_1^k \lambda^i_\nu u_i + a_\nu\right\} \leq \frac{\delta}{2}.$$

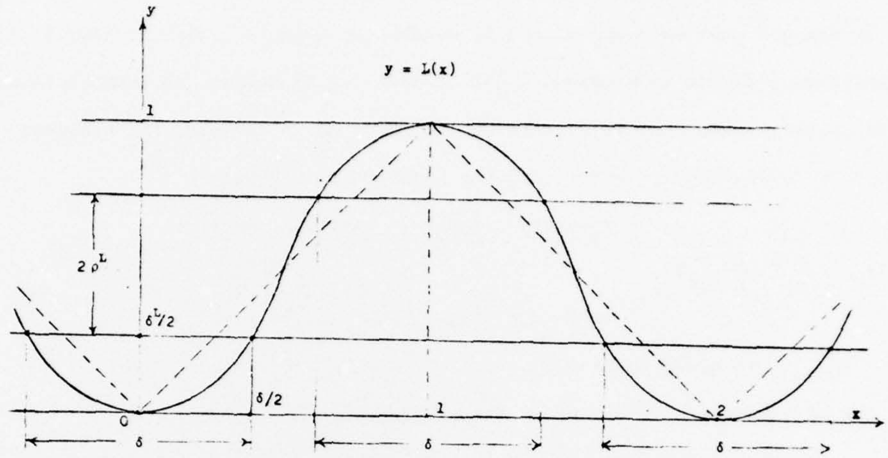

Figure 4

This establishes

Lemma 4^L. Let ρ^L, $0 < \rho^L < \frac{1}{2}$, be prescribed, then δ^L be defined by (10.12), and finally δ such that (10.13) holds. The L-manifold (10.3) is ρ^L-admissible if and only if the n monochromes

$$M_\nu^k(\delta) : \{ \sum_1^k \lambda_\nu^i u_i + a_\nu \} \leq \frac{\delta}{2}, \quad (\nu=1,\dots,n), \tag{10.15}$$

define an n-chromo in R^k.

Eliminating δ^L between (10.12) and (10.13), we find that

$$\rho^L = \frac{1}{2} - L(\frac{\delta}{2}) . \tag{10.16}$$

If ρ^L tends to its supremum $\rho^L_{k,n}$, then δ tends to its infimum $\delta_{k,n}$, and we obtain

$$\rho^L_{k,n} = \frac{1}{2} - L(\frac{\delta_{k,n}}{2}) , \tag{10.17}$$

which is the analogue of (9.1).

Theorem 1', hence that $\delta_{k,n} \leq k/n$, and (10.17), immediately establishes (10.5), hence Theorem 1^L. Likewise Theorem 2', hence that $\delta_{n-1,n} = (n-1)/n$, gives (10.6), hence Theorem 2^L, again in view of (10.17). Finally, Theorem 3' implies Theorem 3^L.

11. Examples of extremal Lissajous manifolds.

It was not mentioned above, but is evident by Lemma 4^L, that if (10.15) are the inequalities defining an n-chromo $\chi_n^k(\delta)$, then (10.3) defines an L-manifold which is ρ^L-admissible, where ρ^L is defined by (10.16). As an example, the n-chromo $\chi_n^{n-1}(\delta)$, defined by (6.2), (6.3), (6.4), give the L-manifold of Theorem 2^L

$$(11.1) \qquad \Lambda_n^{n-1}(\rho_{n-1,n}^L): \quad \begin{aligned} x_\nu &= \sin^2(\tfrac{\pi}{2} u_\nu), \quad (\nu=1,\ldots,n-1), \\ x_n &= \sin^2 \tfrac{\pi}{2}(u_1+\ldots+u_{n-1}+\tfrac{n-1}{2}) \end{aligned}$$

where $\rho_{n-1,n}^L$ is given by (10.6).

Let us look at this for the smallest values of n.

1. <u>$k = 1$, $n = 2$</u>. Here $\rho_{1,2}^L = \frac{1}{2} - \sin^2(\pi/8) = (2\sqrt{2})^{-1}$. The extremizing L-motion

$$x_1 = \sin^2 \frac{\pi u_1}{2} = \frac{1}{2}(1-\cos\pi u_1)$$

$$x_2 = \sin^2(\frac{\pi}{2} u_1 + \frac{\pi}{4}) = \frac{1}{2}(1 + \sin\pi u_1),$$

is seen to be a circular motion along the circle inscribed in U_2. This is the analogue of the b.b. motion (9.6).

2. <u>$k = 2$, $n = 3$</u>. The extremizing L-surface is found to be

$$(11.2) \qquad \begin{aligned} x_1 &= \sin^2 \frac{\pi u_1}{2} = \frac{1}{2}(1 - \cos\pi u_1) \\ x_2 &= \sin^2 \frac{\pi u_2}{2} = \frac{1}{2}(1 - \cos\pi u_2), \\ x_3 &= \cos^2 \frac{\pi}{2}(u_1+u_2) = \frac{1}{2}(1 + \cos\pi(u_1+u_2)). \end{aligned}$$

This is the L-analogue of Kepler's tetrahedron T parametrically given by (9.5). The largest cube inscribed in T was found to have its side $= \frac{1}{3}$. For our Λ_3^2 we find a <u>larger</u> cube $\|x-c\|_\infty < \frac{1}{4}$ of side $= \frac{1}{2}$, because

$$\rho_{2,3}^L = \frac{1}{2} - \sin^2 \frac{\pi}{6} = \frac{1}{4}.$$

The intersections of (11.2) with the planes $x_\nu = c$ $(0 \leq c \leq 1)$, $(\nu=1,2,3)$ are ellipses, inscribed in the unit square, with axes parallel to the diagonals of the square. The surface is convex.

From (10.17), for $k = n-1$, we find that

(11.3) $$\lim_{n\to\infty} \rho^L_{n-1,n} = 0 .$$

Most likely the above Λ^2_3, given by (11.2), is the last Λ^{n-1}_n which is the boundary of a convex set in R^n.

3. k = 1, general n. With this last example we come close to the subject studied in [4]. The n-chromo (4.7) and Lemma 4^L show that

(11.4) $$x_\nu = \sin^2 \frac{\pi}{2}(u_1 + \frac{\nu-1}{n}) , \quad (\nu=1,\dots,n;\ 0 \leq u_1 \leq 2) ,$$

describe an extremal curve Λ^1_n. From (10.17), for $k = 1$, we obtain that

(11.5) $$\lim_{n\to\infty} \rho^L_{1,n} = \frac{1}{2} .$$

The curve (11.4) is the Lissajous-analogue of the "lucky" billiard ball shot Γ^*_n of [4, relation (10.2) for $n = 3$, and Figure 2].

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Let

(1) $U_n : 0 \leq x_\nu \leq 1$, $(\nu=1,\dots,n)$

be the unit cube of R^n. Using ideas pioneered in 1913 by König and Szücs in [2], we study the following problem. Let

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Abstract continued

(2) $$L_n^k : x_\nu = \sum_{i=1}^{k} \lambda_\nu^i u_i + a_\nu \quad (k < n)$$

be a k-flat, so that the point (a_ν) is interior to U_n, and such that L_n^k is in a general position (G.P.) and write $L_n^k \in$ G.P. By this we mean that any k among the x_ν of (2) may assume preassigned values for appropriate values of the u_i. We interpret L_n^k as an optical signal starting from the point (a_ν) at the time $t = 0$, and spreading uniformly within the k-flat L_n^k. We assume the $2n$ facets $x_\nu = 0$ or 1, of U_n, to be mirrors, so that the reflected path of the signal is a finite or infinite k-dimensional skew polytope $\Pi_n^k \subset U_n$. Using the auxiliary function

$$\langle x \rangle = x \text{ if } 0 \leq x \leq 1, \quad \langle x \rangle = 2-x \text{ if } 1 \leq x \leq 2, \text{ and } \langle x+2 \rangle = \langle x \rangle \text{ if } x \in \mathbb{R},$$

we may represent the reflected path by the parametric equations

(3) $$\Pi_n^k : x_\nu = \langle \sum_{i=1}^{k} \lambda_\nu^i u_i + a_\nu \rangle, \quad (\nu=1,\ldots,n).$$

For the x_ν defined by (3), we study the quantity

$$\rho_{k,n} = \sup_{L_n^k \in \text{G.P.}} \inf_{(u_i)} (\max_\nu |x_\nu|),$$

and wish to determine, or to estimate it.

Theorem 1. $\rho_{k,n} \geq \frac{1}{2} - \frac{k}{2n}, \quad (1 \leq k \leq n-1).$

Theorem 2. $\rho_{n-1,n} = \frac{1}{2} - \frac{n-1}{2n} = \frac{1}{2n}.$

It is shown that there is an essentially unique $\tilde{\Pi}_n^{n-1}$ which does not penetrate into the cube

$$\max_\nu |x_\nu - \tfrac{1}{2}| < \frac{1}{2n}.$$

The polytope $\tilde{\Pi}_3^2$ is identical with the surface of Kepler's regular tetrahedron T inscribed in U_3, and Theorem 2 gives, for $n = 3$, an apparently new extremum property of T. Finally we state

Conjecture 1. $\rho_{k,n} = \frac{1}{2} - \frac{k}{2n}, \quad (1 \leq k \leq n-1).$

This was established in [4] for $k = 1$.