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ASYMPTOTIC BEHAVIOR OF SEMILINEAR REACTION-DIFFUSION SYSTEMS WITH DIRICHLET BOUNDARY CONDITIONS

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ASYMPTOTIC BEHAVIOR OF SEMILINEAR REACTION-DIFFUSION SYSTEMS WITH DIRICHLET BOUNDARY CONDITIONS*

Robert A. Gardner†

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ABSTRACT

This paper gives conditions which imply the existence of a unique, globally attracting steady state solution of a certain class of reaction-diffusion systems with inhomogeneous Dirichlet conditions. In addition, an example is given which shows that when the above conditions are not satisfied the steady state may bifurcate.

AMS(MOS) Subject Classifications: 35B30, 35B40, 35K50, 35K55.

Key Words: Reaction-Diffusion systems, boundary values, asymptotic behavior, bifurcation.

Work Unit Number 1 - Applied Analysis

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*Portions of this work constitute a part of the author's Ph.D. thesis research, performed under the direction of Professor J. A. Smoller.

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SIGNIFICANCE AND EXPLANATION

Reaction-diffusion systems are systems of partial differential equations which have applications in, for example, chemical reactor theory and in mathematical biology. As a specific example, such systems may be used to model the evolution of the concentrations of two chemical species undergoing simultaneous reaction and diffusion in some container, where the concentration of each reactant is kept at some prescribed level at the boundary of the container.

An important problem is to describe the asymptotic or eventual behavior of such a system for large times, and the relationship between this behavior and the initial conditions. This paper gives conditions which imply (1) that the reaction-diffusion system has a unique time-independent (but spacially heterogeneous) state, and (2) that the system must tend to this equilibrium for larger time, independent of the initial conditions.

An example is also given which shows that if the conditions mentioned above are violated, the steady state may bifurcate; in other words, the reaction diffusion system may then admit more than one equilibrium solution.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ASYMPTOTIC BEHAVIOR OF SEMILINEAR REACTION-DIFFUSION SYSTEMS WITH DIRICHLET BOUNDARY CONDITIONS*

Robert A. Gardner[†]

Section 1.

We shall study certain systems of semilinear reaction-diffusion equations with non-homogeneous Dirichlet boundary conditions of the form

$$(1.1)\qquad \begin{cases} -u_{i_t} + d_i \Delta u_i = f^i(u_1,\dots,u_n) \;, \\ u_i\big|_{t=0} = u_i^0(x), \quad u_i\big|_{\partial\Omega\times\mathbb{R}_+} = \psi_i(x,t), \quad 1 \le i \le n \;, \end{cases}$$

where the d_i's are positive constants, and the data are non-negative. We shall consider systems which include the class where each f^i has the form

$$(1.2)\qquad f^i(u_1,\dots,u_n) = r_i u_1^{p_1^i} u_2^{p_2^i} \cdots u_n^{p_n^i} \;,$$

where $p_j^i \ge 1$ and $r_i > 0$ are constant. Such systems model the behavior of systems of irreversible chemical reactions [3]; each u_i represents a chemical concentration. It is of interest to determine the equilibria of such systems. We will assume that

$$\lim_{t\to\infty} \psi_i(x,t) = \phi_i(x) \;,$$

uniformly for $x \in \partial\Omega$, so that the system will be in equilibrium if $u_i(x,t) = \overline{u}_i(x)$ for all $t \ge 0$, where $\overline{u}_i$ is the solution of

$$(1.3)\qquad \begin{cases} d_i \Delta \overline{u}_i = f^i(\overline{u}_1,\dots,\overline{u}_n) \\ \overline{u}_i\big|_{\partial\Omega} = \phi_i, \quad 1 \le i \le n \;. \end{cases}$$

We shall study the multiplicity of solutions to (1.3), and their asymptotic stability when viewed as stationary solutions of (1.1), (taking $\psi_i(x,t) \equiv \phi_i(x)$, $u_i^0 = \overline{u}_i$).

First, conditions are obtained which imply the uniqueness of the non-negative solution to (1.3). Under these conditions, we show that $(\overline{u}_1,\dots\overline{u}_n)$ is a globally attracting solution of (1.1). Our technique is to perturb a special system where $f^i(u_1,\dots,u_n) = f(u_1,\dots,u_n)$, $1 \le i \le n$ for which solutions of (1.3) are always unique, global attractors, to a system in which $f^i(u_1,\dots,u_n,\delta)$, a smooth function in all arguments, satisfies $f^i(u_1,\dots,u_n,0) = f(u_1,\dots,u_n)$, $1 \le i \le n$. The function $f^i(u_1,\dots,u_n,\delta)$ roughly has the form

$$(1.4)\qquad f^i(u_1,\dots,u_n,\delta) = u_1^{p_1+\delta_1^i} u_2^{p_2+\delta_2^i} \cdots u_n^{p_n+\delta_n^i} \;;$$

the perturbation parameter $\delta \in \mathbb{R}^p$ for some p. We show that if $d_1 = d_2 = \cdots = d_r$, $d_{r+1} = d_{r+2} = \cdots = d_n$, and if

$$(1.5)\qquad \max_{1\le i\le p} |\delta_i| \cdot \gamma(\|\phi\|_\infty, \delta, f^i, d_i) < 1$$

where for given f^i and d_i, $\gamma : \mathbb{R}^{p+1} \to \mathbb{R}_+$, and is continuous with $\gamma(0,\delta) = 0$, then (1.3) has a unique, globally attracting solution.

Next, we consider the particular system

$$(1.6)\qquad \begin{cases} -u_t + d_1 \Delta u = r_1 u^{1+\delta_1} v^{1+\delta_2} \\ -v_t + d_2 \Delta v = r_2 u^{1+\delta_3} v^{1+\delta_4} \;, \end{cases}$$

with initial and boundary conditions as in (1.1). The estimates for (1.6) are more difficult than for (1.1), due to a lack of smoothness in the former system. C. Kahane [6], showed that (1.6) always has a unique globally attracting steady state solution

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in the special case $\delta = 0$. The proof depends on an algebraic cancellation in which the nonlinearity drops out of the calculation entirely. We show that his result is "generic" by proving similar results for the perturbed system under a hypothesis similar to (1.5); when $\delta = 0$, we recover Kahane's result.

We shall state some theorems concerning rates of convergence and the convergence of derivatives.

When condition (1.5) is violated, the steady state solution may bifurcate. In particular, the system

$$\frac{d^2u}{dx^2} = uv^2 \quad , \quad \frac{d^2v}{dx^2} = u^2v \quad , \quad u(\pm 1) = v(\pm 1) = M$$

has more than one solution for sufficiently large M. Hence, the result concerning the uniqueness and global stability of the steady state is in some sense the best possible.

The plan for the rest of the paper is as follows: In Section 2 we describe notation and equations; in Section 3 we discuss existence of solutions to (1.1) and to (1.3); in Section 4 we consider asymptotic behavior and uniqueness, and, in Section 5 we describe the bifurcation of the steady state.

I would like to thank Professor Joel Smoller for suggesting the subject of this paper as a thesis topic. I would also like to thank Professor Jeffrey Rauch for many constructive suggestions.

Section 2.

We shall adhere to the following conventions. Let

$$U = (u_1,\ldots,u_n) \in \mathbb{R}^n, \quad |U| = \max_{1\le i\le n} |u_i| \quad ,$$

$$\delta = (\delta_1,\ldots,\delta_p) \in \mathbb{R}^p, \quad |\delta| = \max_{1\le i\le p} |\delta_i| \quad .$$

Let $\Omega \subseteq \mathbb{R}^m$ be a bounded domain with smooth boundary.

We shall say that $A = (A_1,\ldots,A_n) \ge 0$ if this holds componentwise, and we let

$$\Sigma_A = \{U \in \mathbb{R}^n : 0 \le U \le A\} \quad .$$

The following norms will be used:

$$\|f\|^p_{L^p} = \int_\Omega |f(x)|^p dx, \quad \|f\|_\infty = \sup\{|f(x)| : x \in \Omega\}$$

$$\|f\|^2_q = \sum_{|\beta|\le q} \|D^\beta f\|^2_{L^2}, \quad |f|_q = \sum_{|\beta|\le q} \|D^\beta f\|_\infty \quad ,$$

$$|f|_{q+\alpha} = |f|_q + \sum_{|\beta|=q} \sup_{x,y\in\Omega} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha}, \quad \alpha \in (0,1) \quad .$$

Here, $\beta = (\beta_1,\ldots,\beta_m)$ and $D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\ldots\partial x_m^{\beta_m}}$. If ϕ is a function defined on $\partial\Omega$ we put

$$|\phi|_{m+\alpha} = \inf\{|f|_{m+\alpha} : f|_{\partial\Omega} = \phi\} \quad .$$

For all function space norms, we shall not explicitly refer to the domain (usually either Ω or $\partial\Omega$) on which the function is defined. No ambiguity will result. If $f(x,t)$ is defined on $\Omega \times \mathbb{R}_+$, then we shall, for example, use the notation

$$\|f(t)\|_\infty = \sup\{|f(x,t)| : x \in \Omega\} \quad .$$

Let $L^p(\Omega)$, $H^m(\Omega)$, $BC^m(\Omega)$, $C^{m+\alpha}(\Omega)$, and $C^{m+\alpha}(\partial\Omega)$ be the spaces of functions with finite $\|\ \|_{L^p}$, $\|\ \|_m$, $|\ |_m$, and $|\ |_{m+\alpha}$ norms respectively. Let $C^{2,1}(\Omega\times\mathbb{R}_+)$ be the class of functions with continuous x-derivatives of order ≤ 2, and one continuous t-derivative.

We shall consider equations of the form

(2.1a) $$-U_t + D\Delta U = F(U,\delta) \quad ,$$

or in components $-u_{i_t} + d_i \Delta u_i = f^i(U,\delta)$, $1 \le i \le n$. We shall prescribe the following data for U:

(2.1b) $$U\big|_{t=0} = U^0(x) \quad , \quad U\big|_{\partial\Omega \times \mathbb{R}_+} = \Psi(x,t) \quad .$$

We will also consider systems of the form

(2.2a) $$D\Delta\overline{U} = F(\overline{U},\delta) \quad ,$$

with boundary data

(2.2b) $$\overline{U}\big|_{\partial\Omega} = \phi(x) \quad .$$

The components of U^0, Ψ, and ϕ are always assumed to be non-negative, uniformly bounded, and continuous. Each d_i is a positive constant. We assume that

(2.3) $$d_1 = d_2 = \cdots = d_r, \quad d_{r+1} = d_{r+2} = \cdots = d_n \quad .$$

The following conditions on $F = (f^1,\ldots,f^n)$ will usually be imposed.

(2.4a) For each i, $f^i(U,\delta) \ge 0$ for all $U \ge 0$,

(2.4b) $$f^i(u_1,\ldots,u_{i-1}, 0, u_{i+1},\ldots,u_n, \delta) = 0 \quad \text{for all } i \ ,$$

(2.4c) $$f^i(U,0) = f^j(U,0) \quad \text{for all } i, j \text{ and } U \ ,$$

(2.4d) $$\partial f^i/\partial u_k(U,0) \ge 0 \quad \text{for all } i, k, \text{ and } U \ge 0 \ ,$$

(2.4e) $\partial f^i/\partial u_k$, $\partial^2 f^i/\partial u_k \partial\delta_j$ are continuous for all i, j, k :

The functions f^i defined in (1.4) satisfy conditions (2.4a) - (2.4e) provided that we perturb only those p_i's for which $p_i > 1$.

The following quantities will be used frequently.

(2.5) $$C(A,\delta) = \max\{|\partial^2 f^i/\partial u_k \partial\delta_j(U,s)| : U \in \Sigma_A \ ,$$
$$|s| \le |\delta|,\ 1 \le i,\ k \le n, \text{ and } 1 \le j \le p\} \ ,$$

(2.6) $$M(A) = \max(\partial f^i/u_k(U,0) : U \in \Sigma_A,\ 1 \le i,\ k \le n)$$

(2.7) $$d = \min_{1 \le i \le n} (d_i) \quad .$$

We shall denote $\overline{\lim}_{t\to\infty}$, (resp. $\lim_{t\to\infty}$) by lim sup, (resp. lim).

Section 3.

The question of existence of solutions to problem (2.1) has been reduced to showing that the vector field F has certain geometric properties. Following Chueh, Conley, and Smoller, [1], we introduce the following definition.

Definition: A set $\Sigma \subseteq \mathbb{R}^n$ is (positively) invariant for (2.1) if $U^0(x)$, $\Psi(x,t) \in \Sigma$ imply that the local flow of (2.1) has values in Σ for all $x \in \Omega$ and $t \ge 0$ for which $U(x,t)$ is defined.

Theorem 3.1. Let Σ be a parallelepiped, $\Sigma = \Pi_{i=1}^n [a_i,b_i]$, $a_i < b_i$; then Σ is invariant for (2.1a) if and only if $\nu_p \cdot F(p,\delta) \ge 0$ for all $p \in \partial\Sigma$, where ν_p is an outward normal to $\partial\Sigma$ at p.

See [1] for details. Geometrically, Σ is invariant if $-F$ doesn't point out of Σ along $\partial\Sigma$, provided that Σ is a parallelepiped as in Theorem 3.1. This restriction on the geometry of Σ is necessary if the diffusion rates are unequal.

Corollary 3.2. Assume (2.4a), (2.4b) hold. Then each Σ_A is invariant for (2.1).

Proof: Let $e_i = (0,\ldots,0,1,0,\cdots 0)$, with 1 in the ith slot. Note that $\partial\Sigma_A$ is a union of portions of $(n-1)$ planes, with $\nu_p = e_i$ if p is in the face $p_i = A_i$, and $\nu_p = -e_i$ if p is in the face $p_i = 0$. In the first case, $\nu_p \cdot F(p,\delta) = f^i(p,\delta) \ge 0$, and in the second, $\nu_p \cdot F(p,\delta) = -f^i(p_1,\ldots,0,\ldots p_n, \delta) = 0$, by (2.1a) and (2.1b) respectively. ■

The existence of a bounded invariant region for (2.1a) yields a-priori bounds for solutions of (2.1). Once such bounds are obtained it is easy to construct global solutions to problem (2.1). This may be accomplished by rewriting (2.1a) as an equivalent integral equation, using Green's function for the operator $-\partial/\partial t + d_i\Delta$. Local solutions to (2.1) are obtained as fixed points of a map on an appropriate Banach space by applying the Banach fixed point theorem. The interval of existence of this solution, $0 \le t \le \delta$, depends only on A where A is chosen such that $U^0(x)$, $\forall(x,t) \in \Sigma_A$. Since Σ_A is invariant, we have that $U(x,t) \in \Sigma_A$ for $(x,t) \in \Omega \times [0,\delta]$. We then repeat this procedure, using $U(x,\delta)$ as "initial" data to obtain a solution, again with values in Σ_A since Σ_A is invariant, and hence, existing throughout the interval $\delta \le t \le 2\delta$. In this manner, we obtain a solution $U(x,t)$ to the integral equation which exists for all $(x,t) \in \Omega \times \mathbb{R}_+$. This solution is Hölder continuous with values in Σ_A. Standard Schauder estimates yield the continuity of the derivatives U_t, U_{x_i}, and $U_{x_i x_j}$.

<u>Theorem 3.3</u>. There exists a unique non-negative solution to (2.1) $U(x,t) \in BC^0(\Omega\times\mathbb{R}_+) \cap C^{2,1}(\Omega\times\mathbb{R}_+)$.

The reader is referred to [1] for the details of the proof of existence, as well as for the proof of uniqueness.

Now consider problem (2.2).

<u>Theorem 3.4</u>. Suppose that $\phi \in C^{2+\alpha}(\partial\Omega)$ and that (2.4a), (2.4b) hold. Then there exists a non-negative solution $\overline{U}(x) \in C^{2+\alpha}(\Omega)$ to (2.2).

<u>Proof</u>: Consider the mapping $\Phi_\pi(W) = Z$ where $Z = (z_1,\ldots,z_n)$, $W = (w_1,\ldots,w_n)$, and where Z is the solution of

$$d_i(\Delta-\pi)z_i = f^i(W,\delta) - d_i\pi w_i,\ z_i\big|_{\partial\Omega} = \phi_i,\ 1 \le i \le n \ .$$

Choose $A > 0$ such that $\phi(x) \in \Sigma_A$ for all $x \in \partial\Omega$, and let

$$\pi = \max\{d_i^{-1}|\partial f^i/\partial w_i(W,\delta)| : W \in \Sigma_A,\ 1 \le i \le n\} \ .$$

We shall show that $0 \le z_i \le A_i$ provided that $0 \le w_i \le A_i$, $1 \le i \le n$. For some $W^* \in \Sigma_A$ we have that

$$d_i(\Delta-\pi)(z_i-A_i) = f^i(W,\delta) - d_i\pi(w_i-A_i)$$

$$(3.1) \qquad \ge f^i(W,\delta) - f^i(w_1,\ldots,A_i,\ldots,w_n,\delta) - d_i\pi(w_i-A_i)$$

$$= [\partial f^i/\partial w_i(W^*,\delta) - d_i\pi](w_i-A_i) \ge 0 \ ,$$

since $w_i-A_i \le 0$ and $\partial f^i/w_i - d_i\pi \le 0$. Since $(z_i-A_i)\big|_{\partial\Omega} \le 0$, we have that $z_i-A_i \le 0$ by the maximum principle for the operator $d_i(\Delta-\pi)$. Similarly,

$$d_i(\Delta-\pi)z_i = f^i(W,\delta) - d_i\pi w_i$$

$$(3.2) \qquad = f^i(w,\delta) - f^i(w_1,\ldots,0,\ldots,w_n,\delta) - d_i\pi w_i$$

$$= \partial f^i/\partial w_i(W^*,\delta) - d_i\pi)w_i \le 0 \ ,$$

since $w_i \ge 0$. Since $z_i\big|_{\partial\Omega} \ge 0$, we have that $z_i \ge 0$ in Ω. The assumptions (2.4a) and (2.4b) were used in (3.1) and (3.2) respectively.

Now let $\Gamma = \{W \in BC^0(\Omega)^n : W(x) \in \Sigma_A\ \forall x \in \Omega,\ W\big|_{\partial\Omega} = \phi(x)\}$. The calculation given above shows that $\Phi_\pi : \Gamma \to \Gamma$. Note that Γ is a closed convex set in $BC^0(\Omega)$. Finally, standard Schauder estimates imply that $|z_i|_{1+\alpha} \le C$, $1 \le i \le n$, where C depends only on A, F, D, Ω, and $|\phi|_{2+\alpha}$. Hence $\Phi_\pi(\Gamma)$ is compact in the $BC^0(\Omega)$ topology; we now apply Schauder's fixed point theorem to obtain $U \in \Gamma$ with $\Phi_\pi(U) = U$. Then U is also a solution of (2.1a), (2.1b). We apply the $|\cdot|_{1+\alpha}$- Schauder estimates to (2.1) to bound $|\overline{U}|_{1+\delta}$. We next apply the $|\cdot|_{2+\alpha}$ - Schauder estimates and the smoothness of F to bound $|\overline{U}|_{2+\alpha}$; hence $\overline{U}$ is a classical solution. ∎

We remark that Theorems 3.3 and 3.4 remain valid if we replace (2.4a), (2.4b) with the weaker hypothesis $\nu_p \cdot F(p,\delta) \ge 0$ for all $p \in \partial\Sigma_A$, where A is chosen such that $\phi(x) \in \Sigma_A$.

Section 4.

We now consider the asymptotic behavior of the solution to problem (2.1). Assume for the moment that $\lim \Psi(x,t) = \phi(x)$ uniformly for $x \in \partial\Omega$. We shall show under hypothesis (1.5) that $\lim U(x,t) = \overline{U}(x)$, uniformly for $x \in \overline{\Omega}$. Our approach will be from a slightly more general framework. We assume that (2.3) and (2.4a) - (2.4e) hold.

Suppose that $U^k = (u_1^k,\dots,u_n^k)$, $k = 1,2$, are two solutions of (2.1a) with data

$$U^k\big|_{t=0} = U^{0k}(x), \qquad U^k\big|_{\partial\Omega\times\mathbb{R}_+} = \Psi^k(x,t), \qquad k = 1,2,$$

respectively.

Theorem 4.1. Assume that δ and A are chosen such that

$$\frac{2|\delta|C(A,\delta)pn}{\lambda_1 d}\left[1 + \frac{2M(A)(n-2)}{\lambda_1 d}\right] < 1, \tag{4.1}$$

where $C(A,\delta)$, $M(A)$, and d are defined in (2.5) - (2.7), and λ_1 is the smallest eigenvalue of $-\Delta$ on Ω with homogeneous Dirichlet boundary conditions. Assume also that

(i) $\limsup \Psi^k(x,t) := (\limsup \phi_1^k(x,t),\dots,\limsup \phi_n^k(x,t)) \in \Sigma_A$, uniformly for $x \in \partial\Omega$.

(ii) $\lim|\Psi^1(x,t) - \Psi^2(x,t)| = 0$, uniformly for $x \in \partial\Omega$.

Then $\lim|U^1(x,t) - U^2(x,t)| = 0$, uniformly for $x \in \Omega$.

Remarks. (1) Assumption (4.1) relates the perturbation parameter δ to $\limsup \Psi(x,t)$, and hence to A. In most examples, $C(A,\delta)$ and $M(A)$ tend to zero as $A \to 0$, and hence these terms give some measure of $\limsup \Psi$. Condition (4.1) is roughly that

$$|\delta|\gamma(|A|, \delta, F, d) < 1$$

where γ is as in (1.4). Given δ, this condition is satisfied by choosing $|A|$ small, and conversely, for fixed A, (4.1) holds for small δ. Hence, we obtain

a stronger theorem than we would have obtained had we perturbed the data or non-linear term separately. This idea was introduced by Nishida and Smoller [7] in the context of gas dynamics. (2) We will actually prove a little more than was stated at the beginning of this section. We do not require that Ψ^1 and Ψ^2 converge to a limit as $t \to \infty$. We require only that these functions coalesce; Ψ^k may continue to oscillate for large t. However, if the oscillations are asymptotically confined to Σ_A, theorem 4.1 shows that the solutions U^1 and U^2 coalesce as $t \to \infty$.

Corollary 4.2. Suppose that $\phi(x) \in \Sigma_A$ and that (4.1) holds. Then the non-negative solution $\overline{U}(x)$ to (2.2) is unique

Proof: Let $\overline{U}^i(x)$, $i = 1, 2$ be two non-negative solutions to (2.2). Then each $\overline{U}^i$ may be viewed as a solution of (2.1a) with $\Psi^i(x,t) = \phi(x)$, $U^{i0}(x) = \overline{U}^i(x)$, $i = 1, 2$. Theorem 4.1 then yields $\lim\|\overline{U}^1(x) - \overline{U}^2(x)\|_\infty = 0$. ∎

We shall begin by proving several lemmas.

Lemma 4.3. Let $m(t)$ be $C^1(t \geq 0)$ and let $b(t)$ be uniformly bounded. Suppose that $\gamma > 0$, and that

$$m' + \gamma m \leq b\ . \tag{4.2}$$

Then $\limsup m(t) \leq \gamma^{-1} \limsup b(t)$.

Proof: Multiply (4.2) by $e^{\gamma t}$ and integrate to obtain

$$m(t) \leq m(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t e^{\gamma s} b(s)ds\ .$$

We split the integral up into the sum of an integral from 0 to $t/2$ plus an integral from $t/2$ to t. If we take the sup of the integrand in the first integral, and the sup of the factor $b(s)$ in the second, and then integrate the remaining terms, we obtain

$$m(t) \leq (m(0) + \tfrac{t}{2}\|b\|_\infty + \|b\|_\infty)e^{-\frac{\gamma}{2}t} + \gamma^{-1} \max_{t/2\leq s\leq t} b(s)\ . \qquad ∎$$

Lemma 4.4. Under the hypotheses of Theorem 4.1, we have that $\limsup U^k(x,t) \in \Sigma_A$, $k = 1, 2$, uniformly for $x \in \Omega$.

Proof: Let $b = \limsup \max\{U^k(x,t) : x \in \partial\Omega\}$. Then $b \in \Sigma_A$. Let $\chi = (\chi_1, \ldots, \chi_n)$ be continuous on $\partial\Omega \times \mathbb{R}_+$ such that $\lim \chi = b$ uniformly for $x \in \partial\Omega$, and $\chi \geq U^k$ in $\partial\Omega \times \mathbb{R}_+$. If $W = (w_1, \ldots, w_n)$ is the solution of

$$-W_t + D\Delta W = 0, \quad W\big|_{t=0} = U^{0k}(x), \quad W\big|_{\partial\Omega \times \mathbb{R}_+} = \chi ,$$

then by a standard result for linear equations, (see [4, p. 158]), we have that $\lim W(x,t) = b$ uniformly for $x \in \Omega$. Moreover, since $f^i(U^k, \delta)$ is non-negative, we have that $-(U^k - W)_t + D\Delta(U^k - W) \geq 0$. Since $U^k - W \leq 0$ on the initial and lateral boundary of $\Omega \times \mathbb{R}_+$, we have that $0 \leq U^k \leq W$ in $\Omega \times \mathbb{R}_+$, by the maximum principle for the heat equation. ∎

Proof of Theorem 4.1: Let $\sigma = (\sigma_1, \ldots, \sigma_n) = U^1 - U^2$. We shall find a linear system satisfied by σ; the conditions (2.4) will give us information about the coefficients of this system. Note that

$$-\sigma_{i_t} + d_i \Delta\sigma_i = f^i(U^1, \delta) - f^i(U^2, \delta) ,$$

and that

$$\begin{aligned}
f^i(U^1, \delta) - f^i(U^2, \delta)
&= \sum_{j=1}^{p} f^i(U^1, \delta_1, \ldots, \delta_j, 0\cdots0) - f^i(U^1, \delta_1, \ldots, \delta_{j-1}, 0\cdots0) \\
&+ \sum_{k=1}^{n} f^i(u_1^1, \ldots, u_k^1, u_{k+1}^2, \ldots, u_n^2, \underline{0}) - f^i(u_1^1, \ldots, u_{k-1}^1, u_k^2 \cdots u_n^2, \underline{0}) \\
&+ \sum_{j=1}^{p} f^i(U^2, \delta_1, \ldots \delta_{j-1}, 0\cdots0) - f^i(U^2, \delta_1, \ldots, \delta_j, 0\cdots0) \\
&= \sum_{j=1}^{p} \int_0^{\delta_j} [f^i_{\delta_j}(U^1, \delta_1, \ldots, \delta_{j-1}, s, 0\cdots0) - f^i_{\delta_j}(U^2, \delta_1, \ldots, \delta_{j-1}, \\
&\qquad\qquad s, 0\cdots0)]ds
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{k=1}^{n} \int_{u_k^1}^{u_k^2} f^i_{u_k}(u_1^1, \ldots, u_{k-1}^1, \xi, u_{k+1}^2, \ldots, u_n^2, \underline{0}) d\xi \\
&= \sum_{j=1}^{p} \sum_{k=1}^{n} \int_0^{\delta_j} \int_{u_k^1}^{u_k^2} f^i_{\delta_j u_k}(u_1^1, \ldots, u_{k-1}^1, \xi, u_{k+1}^2, \ldots, u_n^2, \delta_1, \ldots, \delta_{j-1}, \\
&\qquad\qquad s, 0\cdots0) d\xi ds \\
&+ \sum_{k=1}^{n} \int_{u_k^1}^{u_k^2} f^i_{u_k}(u_1^1, \ldots, u_{k-1}^1, \xi, u_{k+1}^2, \ldots, u_n^2, \underline{0}) d\xi .
\end{aligned}$$

We now define for $\delta_j \sigma_k \neq 0$

$$\alpha^j_{ik}(x,t) = (\delta_j \sigma_k)^{-1} \int_0^{\delta_j} \int_{u_k^1}^{u_k^2} f^i_{u_k \delta_j}(u_1^1, \ldots, \xi, \ldots, u_n^2, \delta_1, \ldots, \delta_{j-1}, \\
s, 0\cdots0) d\xi ds$$

$$\beta_k(x,t) = (\sigma_k)^{-1} \int_{u_k^1}^{u_k^2} f^i_{u_k}(u_1^1, \ldots, \xi, \ldots, u_n^2, \underline{0}) d\xi .$$

We remark that α^j_{ik} and β_k are continuous functions of (x,t) by (2.4e). Moreover, β_k doesn't depend on i since by (2.4c) $f^i(U, \underline{0}) = f^j(U, \underline{0})$ for all i, j, and hence, $f^i_{u_k}(U, \underline{0}) = f^j_{u_k}(U, \underline{0})$ for all i, j, k. Also note that by (2.4d), we have that $\beta_k \geq 0$ for all k. Thus σ_i satisfies the linear system

$$-\sigma_{i_t} + d_i \Delta\sigma_i = \sum_{k,j} \delta_j \alpha^j_{ik} \sigma_k + \sum_k \beta_k \sigma_k, \quad 1 \leq i \leq n . \tag{4.3}$$

Recall that $d_1 = d_2 = \cdots = d_r$, $d_{r+1} = \cdots = d_n$. If either $1 \leq i, q \leq r$ or $r + 1 \leq i, q \leq n$, we have that

$$-(\sigma_i - \sigma_q)_t + d_i \Delta(\sigma_i - \sigma_q) = \sum_{j,k} \delta_j (\alpha^j_{ik} - \alpha^j_{qk}) \sigma_k . \tag{4.4}$$

Let $e(x)$ be an eigenfunction of $-\Delta$ with homogeneous Dirichlet conditions corresponding to the eigenvalue λ_1. We may assume that $e(x) > 0$ in Ω.

Multiply (4.4) by e, integrate over Ω, and integrate by parts to obtain

$$-\frac{\partial}{\partial t}\int_\Omega (\sigma_i - \sigma_q)e\,dx - \lambda_1 d_i \int_\Omega (\sigma_i - \sigma_q)e\,dx$$

$$= \sum_{j,k}\int_\Omega \delta_j(\alpha^j_{ik} - \alpha^j_{qk})\sigma_k e\,dx + \varepsilon(t) \quad ,$$

where $\varepsilon(t)$ contains the boundary terms. By hypothesis (ii) of Theorem 4.1, $\lim \varepsilon(t) = 0$. If we let $m(t) = \int_\Omega (\sigma_i - \sigma_q)e\,dx$, and

$$b(t) = \sum_{j,k}|\delta_j| \int_\Omega |\alpha^j_{ik} - \alpha^j_{qk}|\,|\sigma_k|e\,dx + |\varepsilon(t)| \quad ,$$

we have that $m' + \lambda_1 dm \le b(t)$. By Lemma 4.3 and hypothesis (i),

(4.5) $$\limsup|\alpha^j_{ik}| \le C(A,\delta),\ \limsup|\beta_k| \le M(A) \quad ,$$

and thus, by Lemma 4.2

$$\limsup \int_\Omega (\sigma_i - \sigma_q)\phi_1 dx \le \alpha|\delta|C(A,\delta)p(\lambda_1 d)^{-1}\sum_k \limsup \int_\Omega |\sigma_k|e\,dx \quad .$$

Reversing i and q shows that

(4.6) $$\limsup \int_\Omega |\sigma^i - \sigma^q|e\,dx \le \alpha|\delta|C(A,\delta)p(\lambda_1 d)^{-1}\sum_k \limsup \int_\Omega |\sigma_k|e\,dx \quad .$$

We now put (4.3) into a more convenient form. Suppose now that $1 \le i \le r$ and $r+1 \le q \le n$. Then

$$-\sigma_{i_t} + d_i\Delta\sigma_i = \sum_{j,k}\delta_j\alpha^j_{ik}\sigma_k + \sum_{k\le r}\beta_k(\sigma_k - \sigma_i) + \Big(\sum_{k\le r}\beta_k\Big)\sigma_i$$

$$+\Big(\sum_{k\ge r+1}\beta_k\Big)\sigma_q + \sum_{k\ge r+1}\beta_k(\sigma_k - \sigma_q) \quad .$$

Let

(4.7a) $$Q = \sum_{j,k}\delta_j\alpha^j_{ik}\sigma_k + \sum_{k\le r}\beta_k(\sigma_k - \sigma_i) + \sum_{k\ge r+1}\beta_k(\sigma_k - \sigma_q)$$

(4.7b) $$R = \sum_{j,k}\delta_j\alpha^j_{qk}\sigma_k + \sum_{k\le r}\beta_k(\sigma_k - \sigma_i) + \sum_{k\ge r+1}\beta_k(\sigma_k - \sigma_q)$$

$$a = \sum_{k\le r}\beta_k \quad , \quad b = \sum_{k\ge r+1}\beta_k \quad .$$

Note that a and b are non-negative. We may then write the ith and qth equations of (4.3) as

$$-\sigma_{i_t} + d_i\Delta\sigma_i = Q + a\sigma_i + b\sigma_q$$

$$-\sigma_{q_t} + d_q\Delta\sigma_q = R + a\sigma_i + b\sigma_q \quad .$$

We introduce comparison systems (g^+, h^-) and (g^-, h^+) as follows: let $Q^-(x,t) = \min(Q(x,t), 0)$, $R^+(x,t) = \max(R(x,t), 0)$; let

$$-g^+_t + d_i\Delta g^+ = Q^- + ag^+ + bh^-$$

(4.8) $$-h^- + d_q\Delta h^- = R^+ + ag^+ + bh^-$$

$$g^+(x,t) = \max(\sigma_i(x,t), 0),\ h^-(x,t) = \min(\sigma_q(x,t), 0) \quad ,$$

for all $(x,t) \in P$, where

(4.9) $$P = \partial\Omega \times \mathbb{R}_+ \cup \Omega \times \{0\} \quad .$$

Similarly, we define (g^-, h^+) to be the solution of

$$-g^-_t + d_i\Delta g^- = Q^+ + ag^- + bh^+$$

$$-h^+_t + d_q\Delta h^+ = R^- + ag^- + bh^+ \quad ,$$

$$g^-|_P = \left(\sigma_i|_P\right)^- \quad , \quad h^+|_P = \left(\sigma_q|_P\right)^+ \, .$$

We shall show that the following inequalities hold in $\Omega \times \mathbb{R}_+$.

(1) $g^- \le 0 \le g^+ \quad , \quad h^- \le 0 \le h^+$

(2) $g^- \le \sigma_i \le g^+ \quad , \quad h^- \le \sigma_q \le h^+ \quad .$

We begin by showing that $g^+ \geq 0$, $h^- \leq 0$. Let $h = -h^-$. Then g^+ and h are non-negative on P and

$$(4.11)\qquad \begin{aligned} -g_t^+ + d_i\Delta g^+ &= Q^- + ag^+ - bh \leq ag^+ - bh \\ -h_t + d_i\Delta h &= -R^+ - ag^+ + bh \leq -ag^+ + bh \ ; \end{aligned}$$

we apply a maximum principle for weakly coupled parabolic systems, [8], to see that the non-negativity of g^+ and h on P together with the differential inequalities (4.11), imply the non-negativity of g^+ and h in $\bar{\Omega} \times \mathbb{R}_+$. The proof that $h^- \leq 0$ and $h^+ \geq 0$ is similar.

We next show that $g^+ \geq \sigma_i$, $h^- \leq \sigma_q$.

$$\begin{aligned} -(g^+ - \sigma_i)_t + d_i\Delta(g^+ - \sigma_i) &= Q^- - Q + a(g^+ - \sigma_i) - b(\sigma_q - h^-) \\ &\leq a(g^+ - \sigma_i) - b(\sigma_q - h^-) \ ; \\ -(\sigma_q - h^-)_t + d_q\Delta(\sigma_q - h^-) &= -R^+ + R - a(g^+ - \sigma_i) + b(\sigma_q - h^-) \\ &\leq -a(g^+ - \sigma_i) + b(\sigma_q - h^-) \ . \end{aligned}$$

The non-negativity of $g^+ - \sigma_i$ and $\sigma_q - h^-$ on P together with these two differential inequalities imply the *non-negativity of* $g^+ - \sigma_i$, $\sigma_q - h^-$ in $\Omega \times \mathbb{R}_+$, by the maximum principle mentioned above. The inequalities $h^+ - \sigma_q \geq 0$, $\sigma_i - g^- \geq 0$ are proved in a similar manner.

Inequalities (1) and (2) imply that

$$|\sigma_i| \leq g^+ - g^- \quad , \quad |\sigma_q| \leq h^+ - h^- \ .$$

Let $r = g^+ - g^-$, $s = h^+ - h^-$; then (r,s) satisfies

$$(4.12)\qquad \begin{aligned} -r_t + d_i\Delta r &= -|Q| + ar - bs \quad , \quad r|_P = |\sigma_i|\,|_P \\ -s_t + d_q\Delta s &= -|R| - ar + bs \quad , \quad s|_P = |\sigma_q|\,|_P \ . \end{aligned}$$

Multiply each equation by $-e$, add them together, integrate over Ω, and integrate by parts to obtain

$$\frac{d}{dt}\int_\Omega (r+s)e\,dx + \lambda_1\int_\Omega (d_i r + d_q s)e\,ds = \int_\Omega [|Q| + |R|]e\,dx + \varepsilon(t) \ ,$$

where $\varepsilon(t)$ contains the boundary terms; $\lim \varepsilon(t) = 0$. If $m(t) = \int_\Omega (r+s)\phi_1$ and $b(t) = \int_\Omega [|Q| + |R|]e\,dx + |\varepsilon(t)|$, we have that $m' + \lambda_1 dm \leq b(t)$, so that by Lemma 4.2

$$(4.13)\qquad \begin{aligned} \limsup \int_\Omega (|\sigma_i| + |\sigma_q|)e\,dx &\leq \limsup \int_\Omega (r+s)e\,dx \\ &\leq \limsup\, (\lambda_1 d)^{-1}\int_\Omega [|Q| + |R|]e\,dx \ . \end{aligned}$$

Let $\gamma = \sum_{k=1}^{n} \limsup \int_\Omega |\sigma_k| e\,dx$. From (4.5), (4.6), and (4.7), we see that

$$\begin{aligned} \limsup \int_\Omega |Q|\,e\,dx &\leq |\delta|pC(A,\delta)\gamma + M(A)\sum_{k=1}^{r} \limsup \int_\Omega |\sigma_k - \sigma_i| e\,dx \\ &\quad + M(A)\sum_{k=r+1}^{n} \limsup \int_\Omega |\sigma_k - \sigma_q| e\,dx \\ &\leq |\delta|pC(A,\delta)\gamma + 2p(n-2)|\delta|C(A,\delta)M(A)(\lambda_1 d)^{-1}\gamma \\ &= |\delta|pC(A,\delta)\left[1 + \frac{2(n-2)M(A)}{\lambda_1 d}\right]\gamma \ . \end{aligned}$$

A similar estimate holds for $\limsup \int_\Omega |R|e\,dx$. Hence from (4.13), we see that

$$\limsup \int_\Omega |\sigma^i| e\,dx \leq \frac{2|\delta|pC(A,\delta)}{\lambda_1 d}\left[1 + \frac{2(n-2)M(A)}{\lambda_1 d}\right]\gamma \ .$$

If we sum each side over i, $1 \leq i \leq n$, we obtain

$$\gamma \leq \frac{2|\delta|pC(A,\delta)n}{\lambda_1 d}\left[1 + \frac{2(n-2)M(A)}{\lambda_1 d}\right]\gamma \ .$$

By assumption (4.1), $\gamma = 0$, so that $\lim \int_\Omega |\sigma_i| e\,dx = 0$, $1 \leq i \leq n$. From this estimate, we obtain L^1, L^p, and finally uniform decay of σ_i, $1 \leq i \leq n$, by arguments appearing in [6, Th. 3.2]. ■

We remark that the assumption $d_1 = d_2 = \cdots = d_r,\ d_{r+1} = \cdots = d_n$ seems unnaturally restrictive. However, the part of the argument which employs the maximum principle for weakly coupled systems does not seem to apply to the case where three or more (d_i)'s are distinct. This is somewhat surprising since a simple modification of the proof of Theorem 4.1 applied to the steady state equations shows that the steady state, under the hypotheses of Corollary 4.4, is unique for arbitrary $d_i \geq d$; (see [5] for the details of this argument). In addition, numerical experiments involving three equations with three distinct diffusion rates were performed. The solution to a related system of difference equations was observed to converge to the unique steady state as $t \to \infty$; no sustained oscillations were observed. Hence, it seems likely that Theorem 4.1 is true for arbitrary $d_i \geq d$, $1 \leq i \leq n$.

We also note that a simple modification of the proof of Theorem 4.1 shows that if we replace $f^i(U,\delta)$ with $r_i f^i(U,\delta)$, where r_i is a positive constant, then the conclusion of Theorem 4.1 holds if in place of (4.1), we assume that

$$\frac{2|\delta|C(A,\delta)pnr}{\lambda_1 d}\left[1 + \frac{2M(A)(n-2)r}{\lambda_1 d}\right] < 1 ,$$

where $r = \max r_i$.

We can relax condition (2.4a) as follows. Assume that $f^i(U_1,0) \geq 0$ for all i and for all $U \geq 0$, and that there exists an $A_0 > 0$ such that for all $\delta \neq 0$ and $A \geq A_0$ we have that $\nu_p \cdot F(P,\delta) > 0$ for all $P \in \partial\Sigma_A$, where ν_p is an outward normal to $\partial\Sigma_A$ at P. We must then replace the expressions $C(A,\delta)$ and $M(A)$ in (4.1) with $C_0 = \max((A_0,\delta),\ C(A,\delta))$ and $M_0 = \max(M(A_0),\ M(A))$. If A_0 and A are both small, we may choose δ to be large, whereas if either A_0 or A is large, then δ must be chosen to be small. The proof is the same as that of Theorem 4.1, with the exception of Lemma 4.4. See [5] for the details of this argument.

As an example, consider the system of two equations where $F = (u_1 M_1,\ u_2 M_2)$

where $M_i = u_j - \delta(a_i - u_i)(u_i - b_i)$, where $i \neq j$ and $0 \leq a_i < b_i$, $i = 1,2$. M_1 and M_2 may be thought of as the local per capita growth rates of two competing species with population densities u_1 and u_2 respectively. For this example, $A_0 = (b_1, b_2)$. Figure 4.1(a) shows the vector field $-F$ with A_0 small and δ large; Figure 4.1(b) shows $-F$ with A_0 large and δ small.

(a) (b)

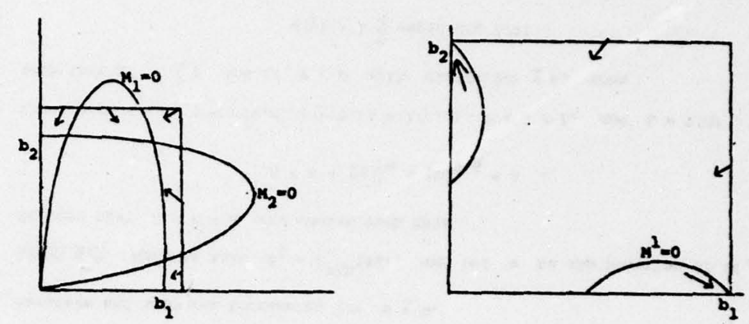

Figure 4.1

We shall now state several theorems concerning rates of convergence and convergence of derivatives; their proofs are given in [5].

<u>Theorem 4.5</u>. Suppose that $d_1 = d_2 = \cdots = d_n$, and that

$$|\Psi^1(x,t) - \Psi^2(x,t)| \leq Lt^{-\mu} ,$$

$$\Psi_i^k(x,t) \leq A_i + Lt^{-\mu}$$

for some constants $L, \mu > 0$. Suppose that $\partial^2 f^i/\partial u_k \partial\delta_j$ are Lipschitz continuous, and let

$$\gamma = |\delta|C(A,\delta)np\,d^{-1}[1 + 2M(A)d^{-1}] ,$$

where J is a constant depending only on Ω, and let $a = \log \gamma/2 \log 4$. Then there exist constants $L_i > 0$, $i = 1,2$ depending only on Ω, $\|U_k\|_{L^\infty(\Omega\times\mathbb{R}_+)}$, $k = 1,2$, ν, and the Lipschitz constants of $f^i_{u_k}$, $f^i_{\delta_j u_k}$ such that

$$|U^1(x,t) - U^2(x,t)| \le L_1 t^{-\nu/2} + L_2 t^a .$$

<u>Theorem 4.6</u>. Suppose that $\gamma = 0$, $d_1 = \cdots = d_n$, and that

$$|V^1(x,t) - V^2(x,t)| \le Le^{-ft} ,$$

for some constants $L, f > 0$. Then there exist constants L_i, $i = 1,2$, as in Theorem 4.5, and a constant C which depends only on Ω and D such that

$$|U^1(x,t) - U^2(x,t)| \le L_2 e^{-ft} + L_2 e^{-Ct} .$$

<u>Theorem 4.7</u>. Suppose that $V^k(x,t) = \phi(x)$, $k = 1,2$, for all $t \ge 0$, and that $\phi \in C^{2+\alpha}(\partial\Omega)$. Let $U(x,t)$, $\overline{U}(x)$ be the solutions of (2.1), (2.2) respectively. Then under the hypotheses of Theorem 4.1,

$$\lim \|\nabla_{(x)} U(t) - \nabla\overline{U}\|_\infty = 0 .$$

We shall now consider system (1.6). This system is of particular interest due to its physical applications when $\delta = 0$. However, it doesn't satisfy the hypotheses of Theorem 4.1. In particular, (2.4e) fails to hold; $\partial^2/\partial u\partial\delta(u^{1+\delta}) = (1+\delta)u^\delta \log u$ is singular in a neighborhood of $u = 0$, $\delta = 0$. In order to handle this singularity, we must estimate the solution of (1.6) from below. We shall use a comparison function $w(x)$ which is the solution to

$$\Delta w = \alpha w \quad , \quad w\big|_{\partial\Omega} = \phi_1(x) \quad , \tag{4.9}$$

where α is a positive constant, chosen such that the property

$$w(x) \le \liminf u(x,t)$$

holds for $x \in \Omega$.

We shall assume without loss of generality that $d_1 = d_2 = r_1 = r_2 = 1$, and that $\delta_i = 0$, $i \ge 2$.

<u>Lemma 4.8</u>. (Mean Value Theorem). Let

$$f(r) = \sum_{\nu=0}^{\infty} \frac{\alpha^\nu r^{2\nu}}{a_\nu} \quad , \quad a_\nu = \Pi_{j=1}^{\nu}\, 2j(2j+m-2),\ a_0 = 1 .$$

If $|x - x_0| = r$, then

$$\frac{1}{s^{m-1} r^{m-1}} \int_{|x-x_0|=r} w(x)\,d\sigma = w(x_0) f(r) ,$$

where s^{m-1} is the surface area of the $(m-1)$ sphere of unit radius, and where w is the solution of (4.9).

The proof of this lemma for $m=3$ may be found in [2, p. 288]; a different proof for all m due to J. Rauch may be found in [5]. Note that f is non-negative and monotone increasing for $r \ge 0$.

<u>Lemma 4.9</u>. Suppose that $\phi_1 \in C^{2+\beta}(\partial\Omega)$, and let w be the solution of (4.9). Suppose that $0 < H < A$ are chosen such that

$$0 < H < \|\phi\|_\infty < |\phi|_{2+\beta} < A .$$

Then there exist $\rho = \rho(A,H,\Omega) > 0$, $\lambda = \lambda(\alpha,\Omega)$, $0 < \lambda < 1$, and $L = L(\Omega) > 0$ such that if $\epsilon \le \rho$ and if $y \in \Omega$ with $\mathrm{dist}(y,\partial\Omega) \le \epsilon$, then

$$w(y) \ge \lambda \frac{H}{4} \exp(L \log \lambda/\epsilon)$$

<u>Remark</u>: It may be the case that $\phi_1 > 0$ on a set $\Gamma \subseteq \partial\Omega$ of small measure. If we let $|\Gamma| \to 0$ and $\|\phi_1\|_\infty = 1$, then $\|w\|_{L^\infty(\Omega_c)} \to 0$ for any Ω_c compactly contained in Ω. To see this, let $e > 0$ be an eigenfunction of $-\Delta$ on Ω corresponding to λ_1; then

$$\lambda_1 \int_\Omega ew\,dx = -\int_\Omega \alpha ew\,dx + \int_{\partial\Omega} \frac{de}{dn}\phi_1 d\sigma .$$

Hence $\int_{\Omega_c} w\,dx \le D\|\phi_1\|_\infty |\Gamma|$, where $D = D(\Omega,\Omega_c)$. We may then apply the interior $|\ |_{1+\delta}$ Schauder estimates to obtain uniform bounds for $\|\nabla w\|_\infty$ depending on $\|\phi_1\|_\infty$, Ω_c and Ω. The derivative estimate may then be combined with the L^1 decay to prove uniform decay of w on Ω_c as $|\Gamma| \to 0$. This calculation shows that our global lower estimate must depend on the derivatives of ϕ_1 as well as $\|\phi_1\|_\infty$. In other words, maximum principle arguments alone cannot yield such an inequality.

<u>Proof:</u> Note that $w \ge 0$ by the maximum principle. Also, $w > 0$ in Ω since w is analytic in Ω. If $w(x_0) = 0$ then by Lemma 4.8, $w = 0$ a.e. on $|x - x_0| = r$ for all small r, contradicting the analyticity of w.

Standard Schauder estimates imply that

$$|w|_{2+\beta} \le C|\phi|_{2+\beta} ,$$

where C is a constant depending only on Ω. Hence $\|\nabla w\|_\infty \le CA$. Take a ball B of radius $H/2CA$ about a point $\bar{x} \in \partial\Omega$ where $\phi(\bar{x}) = H$. For each $x_1 \in \partial\Omega \cap B$, $|w(x_1) - w(\bar{x})| \le \|\nabla w\|_\infty |x_1 - \bar{x}| \le H/2$, so that $w(x_1) > H/2$. Now let $\varepsilon_1 > 0$ be the largest $\varepsilon > 0$ such that the map

$$\Phi : \partial\Omega \times [0,\varepsilon] \to \Omega ,$$

$$\Phi(y,s) = y + sn ,$$

where n is the unit inward normal to $\partial\Omega$, is a diffeomorphism. Let $\rho = \min(H/4CA, \varepsilon_1/2)$, and let $N = \Phi(B \cap \partial\Omega \times [0,\rho])$; for each $x \in N$, we have that $w(x) \ge H/4$. Let $x_0 = \Phi(\bar{x},\rho)$, and suppose that $y \in \Omega$ with $\mathrm{dist}(y,\partial\Omega) = \varepsilon \le \rho$. Take a smooth curve γ_y which satisfies

(i) $\gamma_y(0) = \bar{x}$, $\gamma_y(1) = y$,

(ii) $\{z \in \mathbb{R}^n \mid \mathrm{dist}(z,\gamma_y) \le \varepsilon\} \subseteq \Omega$

(iii) length $(\gamma_y) \le$ length $(\bar{\gamma}_y)$, where $\bar{\gamma}_y$ is any smooth curve satisfying (i) and (ii).

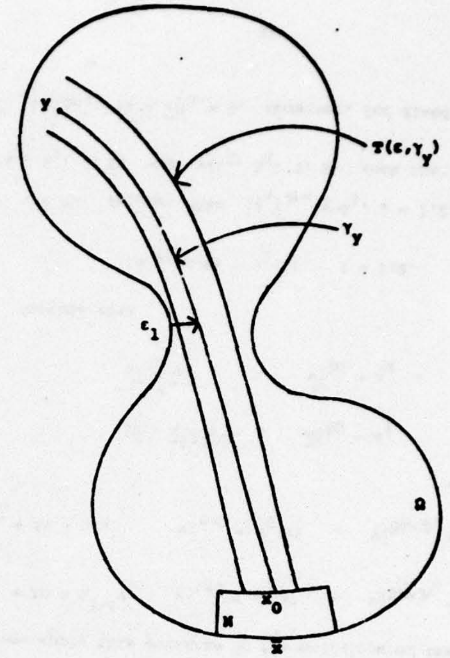

Figure 4.2

Now let $T(a,\gamma_y) = \{z \in \Omega \mid \mathrm{dist}(z,\gamma_y) \le a\}$, and let $T_1 = \{z \in T(\varepsilon/2,\gamma_y)\setminus N \mid \mathrm{dist}(z,\partial N) \le \varepsilon/4\}$. (In Figure 4.3, z is in the most extreme position.) Let S_z, $z \in T_1$, be a sphere of radius $\varepsilon/2$ and center z, and let $G = \{x \in \partial\Omega \mid \Phi(x,\rho) \in T(\varepsilon/2,x_0)\}$. Let

$$\beta_1(y) = \min_{z \in T_1} \{\mathrm{area}(\Phi(G \times [0,\rho]) \cap S_z)/\mathrm{area}(S_z)\} ;$$

Clearly $0 < \beta_1(y) < 1$, since $\partial\Omega$ is smooth. By Lemma 4.8,

$$w(z) = [f(\varepsilon/2)\,\mathrm{area}(S_z)]^{-1} \int_{S_z} w\,d\sigma \ge \frac{\beta_1(y)}{f(\varepsilon/2)} \frac{H}{4} .$$

Let $\lambda_1(y) = \beta_1(y)f(\varepsilon/2)^{-1}$; note that $0 < \lambda_1(y) < 1$.

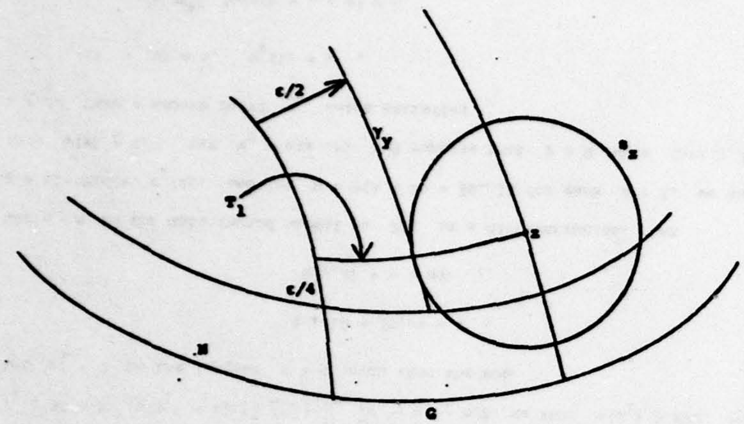

Figure 4.3

Now let $T_2 = \{z \in T(\varepsilon/2,\gamma_y) \setminus (T_1 \cup N) \mid \mathrm{dist}(z,\partial T_1) \le \varepsilon/4\}$. If $z \in T_2$ take S_z as above, and let

$$\beta_2(y) = \min_{z \in T_2} \{\mathrm{area}(T_1 \cap S_z)/\mathrm{area}(S_z)\} .$$

Since $w(x) \ge \lambda_1(y)\frac{H}{4}$ in T_1, we have by Lemma 4.8 that $w(z) \ge \lambda_1(y)\lambda_2(y)\frac{H}{4}$ where $\lambda_2(y) = \beta_2(y)/f(\varepsilon/2)$. By induction we obtain $\lambda_k(y)$ and T_k, with $0 < \lambda_k(y) < 1$

$$w(z) \ge \lambda_1(y)\lambda_2(y)\cdots\lambda_k(y)\frac{H}{4} , \quad z \in T_k .$$

Let $k(y)$ be such that $y \in T_k$, and let

$$\lambda = \inf\{\lambda_i(y) \mid y \in \Omega,\ 1 \le i \le k(y)\} .$$

Since $\partial\Omega$ is smooth, it is clear that $\lambda > 0$. We will not explicitly determine

how λ depends on the curvature of $\partial\Omega$. We have that $w(y) \ge \lambda^{k(y)}H/4$. Let

$$L = 4 \sup\{\mathrm{length}(\gamma_y) \mid \mathrm{dist}(y,\partial\Omega) \le \rho\} .$$

Note that as $y \to \partial\Omega$, $\mathrm{length}(\gamma_y)$ increases, so that L doesn't depend on ρ. If $\varepsilon = \mathrm{dist}(y,\partial\Omega) \le \rho$, let $p(y)$ be an integer such that $\varepsilon p(y) \ge L > \varepsilon(p(y) - 1)$. We must then perform at most $p(y)$ iterations to reach y; hence

$$w(y) \ge \frac{H}{4}\lambda^{p} \ge \frac{H}{4}\lambda\lambda^{p-1}$$

$$\ge \frac{H}{4}\lambda\lambda^{L/\varepsilon} = (H/4)\lambda \exp(L \log \lambda/\varepsilon) . \quad \blacksquare$$

We remark that if $y \in \Omega$ with $\mathrm{dist}(y,\partial\Omega) \ge \rho$, then Lemma 4.9 yields the estimate

$$w(y) \ge (\lambda H/4)\exp(L \log \lambda/\rho) .$$

We shall now apply this estimate to the solutions of the systems

$$\text{(4.10)} \quad \begin{aligned} -u_t + \Delta u &= u^{1+\delta}v, & u|_{t=0} &= u_0(x) , & u|_{\partial\Omega\times\mathbb{R}_+} &= \phi_1(x,t) \\ -v_t + \Delta v &= uv, & v|_{t=0} &= v_0(x) , & v|_{\partial\Omega\times\mathbb{R}_+} &= \phi_2(x,t) , \end{aligned}$$

and

$$\text{(4.11)} \quad \begin{aligned} \Delta\bar{u} &= \bar{u}^{1+\delta}\bar{v} , & \bar{u}|_{\partial\Omega} &= \phi_1 \\ \Delta\bar{v} &= \bar{u}\bar{v} , & \bar{v}|_{\partial\Omega} &= \phi_2 . \end{aligned}$$

<u>Theorem 4.10</u>. Suppose that

$$\lim \phi_i(x,t) = \phi_i(x), \quad i = 1,2,$$

uniformly for $x \in \partial\Omega$. Suppose that $|\phi_i|_{2+\alpha} \le \Lambda_i$, $i = 1,2$. There exist functions $R(\Lambda_1, \Lambda_2, \delta, \Omega)$ and $\gamma(\Lambda_1, \Lambda_2, \delta, \Omega)$ such that if $R\gamma < 1$, then $\lim\|u(t) - \bar{u}\|_\infty = \lim\|v(t) - \bar{v}\|_\infty = 0$. Moreover, for fixed δ, R approaches

Hence $\int_{\Omega_c} w\,dx \le D\|\phi_1\|_\infty |\Gamma|$, where $D = D(\Omega,\Omega_c)$. We may then apply the interior $|\ |_{1+\delta}$ Schauder estimates to obtain uniform bounds for $\|\nabla w\|_\infty$ depending on $\|\phi_1\|_\infty$, Ω_c and Ω. The derivative estimate may then be combined with the L^1 decay to prove uniform decay of w on Ω_c as $|\Gamma| \to 0$. This calculation shows that our global lower estimate must depend on the derivatives of ϕ_1 as well as $\|\phi_1\|_\infty$. In other words, maximum principle arguments alone *cannot* yield such an inequality.

<u>Proof:</u> Note that $w \ge 0$ by the maximum principle. Also, $w > 0$ in Ω since w is analytic in Ω. If $w(x_0) = 0$ then by Lemma 4.8, $w = 0$ a.e. on $|x - x_0| = r$ for all small r, contradicting the analyticity of w.

Standard Schauder estimates imply that

$$|w|_{2+\beta} \le C|\phi|_{2+\beta} \quad ,$$

where C is a constant depending only on Ω. Hence $\|\nabla w\|_\infty \le CA$. Take a ball B of radius $H/2CA$ about a point $\bar{x} \in \partial\Omega$ where $\phi(\bar{x}) = H$. For each $x_1 \in \partial\Omega \cap B$, $|w(x_1) - w(\bar{x})| \le \|\nabla w\|_\infty |x_1 - \bar{x}| \le H/2$, so that $w(x_1) > H/2$. Now let $\epsilon_1 > 0$ be the largest $\epsilon > 0$ such that the map

$$\Phi : \partial\Omega \times [0,\epsilon] \to \Omega \quad ,$$

$$\Phi(y,s) = y + sn \quad ,$$

where n is the unit inward normal to $\partial\Omega$, is a diffeomorphism. Let $\rho = \min(H/4CA, \epsilon_1/2)$, and let $N = \Phi(B \cap \partial\Omega \times [0,\rho])$; for each $x \in N$, we have that $w(x) \ge H/4$. Let $x_0 = \Phi(\bar{x},\rho)$, and suppose that $y \in \Omega$ with $\mathrm{dist}(y,\partial\Omega) = \epsilon \le \rho$. Take a smooth curve γ_y which satisfies

(i) $\gamma_y(0) = \bar{x}$, $\gamma_y(1) = y$,

(ii) $\{z \in \mathbb{R}^n \mid \mathrm{dist}(z,\gamma_y) \le \epsilon\} \subseteq \Omega$

(iii) length $(\gamma_y) \le$ length $(\bar{\gamma}_y)$, where $\bar{\gamma}_y$ is any smooth curve satisfying (i) and (ii).

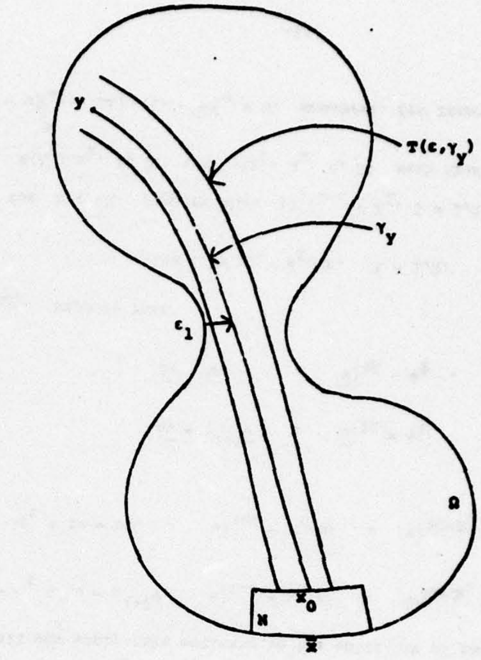

Figure 4.2

Now let $T(a,\gamma_y) = \{z \in \Omega \mid \mathrm{dist}(z,\gamma_y) \le a\}$, and let $T_1 = \{z \in T(\epsilon/2,\gamma_y)\backslash N \mid \mathrm{dist}(z,\partial N) \le \epsilon/4\}$. (In Figure 4.3, z is in the most extreme position.) Let S_z, $z \in T_1$, be a sphere of radius $\epsilon/2$ and center z, and let $G = \{x \in \partial\Omega \mid \Phi(x,\rho) \in T(\epsilon/2,x_0)\}$. Let

$$\beta_1(y) = \min_{z \in T_1} \{\mathrm{area}(\Phi(G \times [0,\rho]) \cap S_z)/\mathrm{area}(S_z)\} \quad ;$$

Clearly $0 < \beta_1(y) < 1$, since $\partial\Omega$ is smooth. By Lemma 4.8,

$$w(z) = [f(\epsilon/2)\mathrm{area}(S_z)]^{-1} \int_{S_z} w\,d\sigma \ge \frac{\beta_1(y)}{f(\epsilon/2)} \frac{H}{4} \ .$$

Let $\lambda_1(y) = \beta_1(y)f(\varepsilon/2)^{-1}$; note that $0 < \lambda_1(y) < 1$.

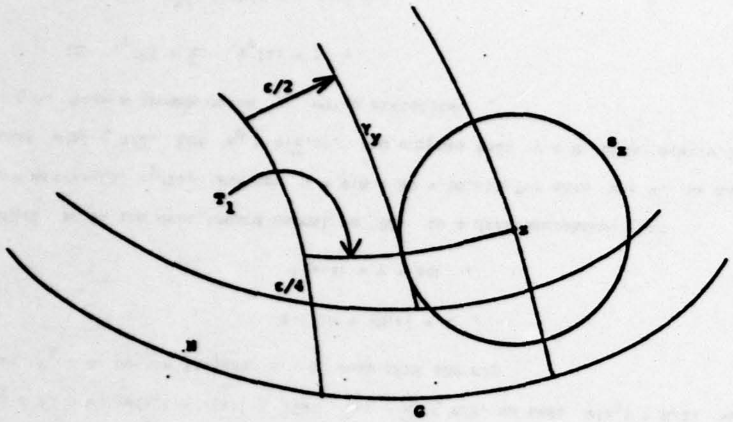

Figure 4.3

Now let $T_2 = \{z \in T(\varepsilon/2,\gamma_y) \setminus (T_1 \cup N) \,|\, \mathrm{dist}(z,\partial T_1) \leq \varepsilon/4\}$. If $z \in T_2$ take S_z as above, and let

$$\beta_2(y) = \min_{z\in T_2}\{\mathrm{area}(T_1 \cap S_z)/\mathrm{area}(S_z)\} .$$

Since $w(x) \geq \lambda_1(y)\frac{H}{4}$ in T_1, we have by Lemma 4.8 that $w(z) \geq \lambda_1(y)\lambda_2(y)\frac{H}{4}$ where $\lambda_2(y) = \beta_2(y)/f(\varepsilon/2)$. By induction we obtain $\lambda_k(y)$ and T_k, with $0 < \lambda_k(y) < 1$

$$w(z) \geq \lambda_1(y)\lambda_2(y)\cdots\lambda_k(y)\frac{H}{4} , \quad z \in T_k .$$

Let $k(y)$ be such that $y \in T_k$, and let

$$\lambda = \inf\{\lambda_i(y) \,|\, y \in \Omega,\ 1 \leq i \leq k(y)\} .$$

Since $\partial\Omega$ is smooth, it is clear that $\lambda > 0$. We will not explicitly determine

how λ depends on the curvature of $\partial\Omega$. We have that $w(y) \geq \lambda^{k(y)}H/4$. Let

$$L = 4\sup\{\mathrm{length}(\gamma_y) \,|\, \mathrm{dist}(y,\partial\Omega) \leq \rho\} .$$

Note that as $y \to \partial\Omega$, $\mathrm{length}(\gamma_y)$ increases, so that L doesn't depend on ρ. If $\varepsilon = \mathrm{dist}(y,\partial\Omega) \leq \rho$, let $p(y)$ be an integer such that $\varepsilon p(y) \geq L > \varepsilon(p(y) - 1)$. We must then perform at most $p(y)$ iterations to reach y; hence

$$w(y) \geq \frac{H}{4}\lambda^p \geq \frac{H}{4}\lambda\lambda^{p-1}$$

$$\geq \frac{H}{4}\lambda\lambda^{L/\varepsilon} = (H/4)\lambda\exp(L\log\lambda/\varepsilon) . \quad \blacksquare$$

We remark that if $y \in \Omega$ with $\mathrm{dist}(y,\partial\Omega) \geq \rho$, then Lemma 4.9 yields the estimate

$$w(y) \geq (\lambda H/4)\exp(L\log\lambda/\rho) .$$

We shall now apply this estimate to the solutions of the systems

$$(4.10)\qquad \begin{aligned} -u_t + \Delta u &= u^{1+\delta}v, & u|_{t=0} &= u_0(x) , & u|_{\partial\Omega\times\mathbb{R}_+} &= \psi_1(x,t) \\ -v_t + \Delta v &= uv, & v|_{t=0} &= v_0(x) , & v|_{\partial\Omega\times\mathbb{R}_+} &= \psi_2(x,t) . \end{aligned}$$

and

$$(4.11)\qquad \begin{aligned} \Delta\bar{u} &= \bar{u}^{1+\delta}\bar{v} , & \bar{u}|_{\partial\Omega} &= \phi_1 \\ \Delta\bar{v} &= \bar{u}\bar{v} , & \bar{v}|_{\partial\Omega} &= \phi_2 . \end{aligned}$$

<u>Theorem 4.10</u>. Suppose that

$$\lim \psi_i(x,t) = \phi_i(x), \quad i = 1,2,$$

uniformly for $x \in \partial\Omega$. Suppose that $|\phi_i|_{2+\alpha} \leq A_i$, $i = 1,2$. There exist functions $R(A_1, A_2, \delta, \Omega)$ and $\gamma(A_1, A_2, \delta, \Omega)$ such that if $R\gamma < 1$, then $\lim\|u(t) - \bar{u}\|_\infty = \lim\|v(t) - \bar{v}\|_\infty = 0$. Moreover, for fixed δ, R approaches zero as $A \to 0$, and for fixed A, γ approaches zero as $\delta \to 0$, roughly at the rate $|\log \delta|^{-1}$.

We shall begin by stating two lemmas which will be used in the proof of Theorem 4.10. For details, see [5].

<u>Lemma 4.11</u>. Let $f(x,t)$, $\alpha(x,t) \in BC^0(\Omega \times \mathbb{R}_+)$ with $\alpha \geq 0$, and let g be the solution of

$$-g_t + \Delta g = \alpha g + f(x,t), \; g|_{t=0} = 0, \; g|_{\partial\Omega \times \mathbb{R}_+} = 0 .$$

Then there exists a constant $K = K(\Omega)$ such that

$$\limsup \|g(t)\|_\infty \leq K \limsup \|f(t)\|_\infty .$$

<u>Lemma 4.12</u>. Suppose that f and g are as in Lemma 4.11, and that $\alpha \equiv 0$. Then there exists a constant $J = J(\Omega)$ such that

$$\limsup \|\nabla_{(x)} g(t)\|_\infty \leq J \limsup \| f(t)\|_\infty .$$

<u>Proof of Theorem 4.10</u>: We define comparison functions z and w to be the solutions of the following equations

$$-z_t + \Delta z = \alpha(t) z, \quad z|_{t=0} = u_0(x), \quad z|_{\partial\Omega \times \mathbb{R}_+} = \phi_1(x,t) ,$$

$$\Delta w = a w, \quad w|_{\partial\Omega} = \phi_1(x) ,$$

where $a = \|\phi_1\|_\infty^\delta \|\phi_2\|_\infty$ and $\alpha(t) = \max\{\|u(t)\|_\infty^\delta \|v(t)\|_\infty, a\}$. From Lemma 4.4 we see that $\lim \alpha(t) = a$, so that $\lim\|z(t) - w\|_\infty = 0$; (for example, see [4, p. 158]). Moreover, we have that $0 \leq z(x,t) \leq u(x,t)$ in $\Omega \times \mathbb{R}_+$, and that $0 \leq w(x) \leq \bar{u}(x)$ in Ω, by the maximum principles for the operators $-\partial/\partial t + \Delta - \alpha(t)$ and $\Delta - a$ respectively. Hence, the inequality $0 \leq w(x) \leq \liminf u(x,t)$ holds uniformly for $x \in \Omega$.

Now let $\sigma_1 = u - \bar{u}$, $\sigma_2 = v - \bar{v}$, and let

$$H = \min\{(K^2 A_1^{1+\delta})^{-1}, (K^2 A_2)^{-(1/1+\delta)}\} , \tag{4.12}$$

where K is as in Lemma 4.11. We first show that if either $\|\phi_1\|_\infty < H$ or $\|\phi_2\|_\infty < H$ then $\lim\|\sigma_i(t)\|_\infty = 0$, $i = 1,2$.

We have that

$$\begin{aligned} -\sigma_{1_t} + \Delta\sigma_1 &= (1+\delta)\xi^\delta \bar{v}\sigma_1 + u^{1+\delta}\sigma_2 \\ -\sigma_{2_t} + \Delta\sigma_2 &= \bar{v}\sigma_1 + u\sigma_2 , \end{aligned} \tag{4.13}$$

where $\xi = \xi(x,t)$ is between $\bar{u}(x)$ and $u(x,t)$. Let b_i be the solution of $-b_{i_t} + \Delta b_i = 0$, $b_i|_P = \sigma_i|_P$ $i = 1,2$, where P is as in (4.9). Since $\lim|\sigma_i(x,t)| = 0$ uniformly for $x \in \partial\Omega$, we have that $\lim|b_i(x,t)| = 0$, $i = 1,2$, uniformly for $x \in \Omega$. If we let $\tau_i = \sigma_i - b_i$, $i = 1,2$, we see that τ_i satisfies a system similar to (4.13) with an additional term on the right hand side which approaches zero uniformly as $t \to \infty$. From Lemma 4.11, we see that

$$\limsup\|\tau_1(t)\|_\infty \leq K \limsup\|u(t)\|_\infty^{1+\delta} \limsup\|\tau_2(t)\|_\infty ,$$

$$\limsup\|\tau_2(t)\|_\infty \leq K \|\bar{v}\|_\infty \limsup\|\tau_1(t)\|_\infty ;$$

since $\limsup|\tau_i(x,t)| = \limsup|\sigma_i(x,t)|$, $i = 1,2$, we see that σ_1 and σ_2 satisfy a similar set of inequalities. Combining these estimates, we obtain

$$\limsup\|\sigma_i(t)\|_\infty \leq K^2\|\bar{v}\|_\infty \limsup\|u(t)\|_\infty^{1+\delta} \limsup\|\sigma_i(t)\|_\infty ,$$

$i = 1,2$. Let $D = K^2\|\bar{v}\|_\infty \limsup\|u(t)\|_\infty^{1+\delta}$. If $D < 1$, then $\limsup\|\sigma_i(t)\|_\infty = 0$, $i = 1,2$. Note that $\|\bar{v}\|_\infty \leq \|\phi_2\|_\infty$ by the maximum principle, and that $\limsup u(x,t) \leq \|\phi_1\|_\infty$ by Lemma 4.4. Hence $D \leq \min(K^2 A_1^{1+\delta}\|\phi_2\|_\infty, K^2\|\phi_1\|_\infty^{1+\delta} A_2)$, so that if either $\|\phi_1\|_\infty < H$ or $\|\phi_2\|_\infty < H$, with H defined as in (4.12), we see that $D < 1$.

We now suppose that $\|\phi_i\|_\infty > H$, $i = 1,2$. Let (g^+, h^-) be the solution of the system

$$- g_t^+ + \Delta g^+ = (1+\delta)\xi^\delta \overline{v} g^+ + u^{1+\delta} h^-, \quad g^+\big|_P = (\sigma_1)^+\big|_P$$

$$- h_t^- + \Delta h^- = \overline{v} g^+ + u h^-, \quad h^-\big|_P = (\sigma_2)^-\big|_P ,$$

and let (g^-, h^+) be the solution of an analogously defined system. As in the proof of Theorem 4.1, we see that if $r = g^+ - g^-$, $s = h^+ - h^-$, then (r,s) is the solution of

$$- r_t + \Delta r = (1+\delta)\xi^\delta \overline{v} r - u^{1+\delta} s, \quad r\big|_P = |\sigma_1|_P$$

(4.14)

$$- s_t + \Delta s = -\overline{v} r + us, \quad s\big|_P = |\sigma_2|_P ,$$

and that r and s are non-negative. Let λ_1 and e be as in the proof of Theorem 4.1. Add equations (4.14) together, multiply by $-e$, integrate over Ω, and integrate by parts to obtain

$$\frac{\partial}{\partial t}\int_\Omega e(r+s)dx + \lambda_1 \int_\Omega e(r+s)dx \le$$

$$c(t) + \int_\Omega e[\overline{v}(\xi^\delta - 1 + \delta\xi^\delta)r + \delta u^{1+\delta}(\log u)s]dx ,$$

where $c(t)$ contains the boundary terms, and $d = d(x,t)$ is chosen between 0 and δ. By lemmas 4.3 and 4.4 and Fatou's Lemma, we see that

(4.15)
$$\limsup \int_\Omega e(r+s) \le \lambda_1^{-1}\Lambda_2 \int_\Omega \limsup|\xi^\delta - 1| er\, dx +$$
$$\delta R_1 \limsup \int_\Omega e(r+s)dx ,$$

where we define

$$R_1(\Lambda, \delta, \Omega) = \lambda_1^{-1}\max(\Lambda_1|\log \Lambda_1|, \ \Lambda_1^{1+\delta}|\log \Lambda_1|, \ \Lambda_2\Lambda_1^{1+\delta}) .$$

We now estimate $\int_\Omega \limsup|\xi^\delta - 1|dx$. Let $\Omega_\delta = \{x \in \Omega | w(x) < \delta\}$. Assume that δ and Λ are chosen such that

(4.16)
$$(4\delta/H\lambda)\exp(- L \log(\lambda/\rho)) < 1 ,$$

where ρ, L, and λ are as in Lemma 4.9. Note that $H^{-1} = \max(K^2\Lambda_1^{1+\delta}, (K^2\Lambda_2)^{1/1+\delta})$, and hence H^{-1} approaches zero with Λ. Let

(4.17)
$$R_2 = 4/(H\lambda)\exp(- L \log \lambda/\rho) .$$

Since $\rho^{-1} \le 4\, C\Lambda_1 H^{-1}$, we see that ρ^{-1} approaches zero with Λ, and hence, this is also true of R_2. Note also that if $y \in \partial\Omega_\delta$ with $\mathrm{dist}(y,\partial\Omega) \ge \rho$, then the remark at the end of Lemma 4.9 implies that

$$\delta = w(y) \ge (\lambda H/4)\exp(L \log \lambda/\rho) ;$$

however, this contradicts (4.16). Hence (4.16) implies that if $y \in \Omega_\delta$, then $\mathrm{dist}(y,\partial\Omega) \le \rho$, so that Lemma 4.9 can be applied to obtain lower bounds for $w(y)$.

We have that

$$\int_\Omega \limsup|\xi^\delta - 1|dx \le \int_{\Omega_\delta} \limsup|\xi^\delta - 1|dx + \int_{\Omega\setminus\Omega_\delta} \limsup|\xi^\delta - 1|dx .$$

To estimate the last integral, we note that $g(\xi) = \xi^\delta - 1$ is an increasing function of ξ, so that if $\delta \le \xi \le \Lambda_1$, we have that $|g(\xi)| \le \max(|g(\delta)|, |g(\Lambda_1)|)$. In view of the inequalities

$$\delta \le w(y) \le \liminf \xi(y,t) \le \limsup \xi(y,t) \le \Lambda_1 ,$$

for $y \in \Omega\setminus\Omega_\delta$, we have that

$$\int_{\Omega\setminus\Omega_\delta} \limsup|\xi^\delta - 1|dx \le \gamma_1(\Lambda_1,\Lambda_2,\delta,\Omega) ,$$

where we define

(4.18)
$$\gamma_1 = \max(|\delta^\delta - 1|, |\Lambda_1^\delta - 1|)\,\mathrm{vol}(\Omega) .$$

For Λ fixed, $\gamma_1 = O(|\delta \log \delta|)$ as $\delta \to 0$.

Next, we estimate $\int_{\Omega_\delta} |\xi^\delta - 1|dx$. Let $y \in \partial\Omega_\delta\setminus\partial\Omega$. We have already noted that $\varepsilon = \mathrm{dist}(y,\partial\Omega) \le \rho$, so that by Lemma 4.9, we have that

$$\delta = w(y) \ge (\lambda M/4)\exp(L \log \lambda/\varepsilon) \ .$$

By assumption (4.16), we can invert this inequality to see that

$$\varepsilon \le L \log \lambda/\log(4\delta/M\lambda) \ ,$$

and hence, that

$$\int_{\Omega_\delta} \limsup |\xi^\delta - 1| dx \le (\Lambda_1^\delta + 1)\mathrm{vol}(\Omega_\delta) \le \gamma_2 \ ,$$

where we define

$$\gamma_2(\Lambda_1,\Lambda_2,\delta,\Omega) = (\Lambda_1^\delta + 1)\, E\, L \log\lambda/\log(4\delta/M\lambda) \ , \tag{4.19}$$

where $E = E(\Omega)$. For fixed Λ, $\gamma_2 = O(|\log \delta|^{-1})$ as $\delta \to 0$. If we let

$$R_3 = \max(R_1, \mathrm{vol}(\Omega)R_1, \lambda_1^{-1}\Lambda_2) \ , \tag{4.20}$$

we see from (4.15) that

$$\limsup \int_\Omega er dx \le R_3(\gamma_1 + \gamma_2 + \delta)\cdot(\limsup \|er(t)\|_\infty + \limsup \int_\Omega es dx) \ , \tag{4.21}$$

with a similar estimate holding for $\limsup \int_\Omega es dx$.

To complete the proof, we must show for some $R_4 = R_4(\Lambda,\delta,\Omega)$ that

$$\limsup \|er(t)\|_\infty \le R_4 \limsup \int_\Omega er dx \ . \tag{4.22}$$

Let b be the solution of $-b_t + \Delta b = 0$, $b|_p = r|_p$, and let $r^* = r - b$; define s^* in a similar manner. Then $\limsup r(x,t) = \limsup r^*(x,t)$ and r^* and s^* satisfy equations with the same right hand side as in (4.14). We apply Lemma 4.11 to the second equation in (4.14) to obtain

$$\limsup \|s(t)\|_\infty \le K\Lambda_2 \limsup \|r^*(t)\|_\infty \ ; \tag{4.23}$$

we next apply Lemma 4.12 and (4.23) to the equation satisfied by r^* to obtain

$$\limsup \|\nabla_{(x)} r^*(t)\|_\infty \le R_5 \limsup \|r^*(t)\|_\infty \ , \tag{4.24}$$

where

$$R_5 = JK\Lambda_1^{1+\delta}\Lambda_2 + J(1+\delta)\Lambda_1^\delta \Lambda_2 \ . \tag{4.25}$$

Now suppose that $\limsup \|r^*(t)\|_\infty = \lim_{n\to\infty} r^*(x_n,t_n)$ for some sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ and $\{x_n\} \subseteq \Omega$ with $\lim x_n = x_0$. Let $D_n = \{x \in \Omega : |x - x_n| < (2R_5)^{-1}\}$, so that $\lim_{n\to\infty} \mathrm{dist}(x_n, D_0) = 0$ for $x \in D_0$. By (4.24), we have that for $x \in D_0$ that

$$\tfrac{1}{2}\|r^*(t_n)\|_\infty \le r^*(x,t_n) + \varepsilon_1(t_n) \ , \tag{4.26}$$

where $\lim_{n\to\infty} \varepsilon_1(t_n) = 0$. If $D_0 \cap \partial\Omega \ne \phi$, we see that $\limsup \|r^*(t)\|_\infty = 0$; this fact, together with (4.23) completes the proof of Theorem 4.10 in this case.

Suppose now that $D_0 \cap \partial\Omega = \phi$. Let $\|e\|_\infty = 1$; multiply (4.26) by e and integrate over D_0 to obtain

$$1/2 \int_{D_0} e\, dx \cdot \|r^*(t_n)\|_\infty \le \int_{D_0} er^*(x,t_n) dx + \varepsilon_2(t_n) \ , \tag{4.27}$$

where $\lim_{n\to\infty} \varepsilon_2(t_n) = 0$. Let $D \subseteq D_0$ be a ball with center x_0 and radius $1/4R_5$ so that

$$\int_{D_0} e\, dx \ge \min_{x\in D} e(x) \cdot \mathrm{vol}(D) \ .$$

By the strong maximum principle for the Laplacian, $de/dn < 0$ on $\partial\Omega$. Using the fact that $\|e\|_\infty = 1$, we may next apply the method introduced in the proof of Lemma 4.9 to obtain explicit lower bounds for e of the form

$$e(x) \ge \begin{cases} L_1 & \text{if } \mathrm{dist}(x,\partial\Omega) \ge d_0 > 0 \\ L_2\, \mathrm{dist}(x,\partial\Omega) & \text{if } \mathrm{dist}(x,\partial\Omega) \le d_0 \ , \end{cases}$$

where d_0, L_1, L_2 depend only on Ω. (We shall not give the details of this estimate.) Hence, we see that

$$\int_{D_0} e\,dx \geq \min(L_1, L_2/4R_5)\ \mathrm{vol}(D) \ ,$$

and from (4.27) we see that

$$\|r^*(t_n)\|_\infty \leq R_3 \int_\Omega er(t_n)dx + \epsilon_3(t_n) \ ,$$

where $\lim_{n\to\infty} \epsilon_3(t_n) = 0$ and where we define

$$R_3 = 2\max(\max(L_1^{-1}, 4R_4/L_2)(4R_5)^m/B_m \ , \ 1) \ , \tag{4.28}$$

B_m is the volume of the m-dimensional unit ball. Hence with this choice of R_4, we see that (4.22) holds, since $\limsup\|er(t)\|_\infty \leq \limsup\|r(t)\|_\infty$.

If we now set $g = \limsup\|er(t)\|_\infty + \limsup \int_\Omega es\,dx$, we see from (4.21) and (4.22) that

$$g \leq 2R_3R_4(\gamma_1 + \gamma_2 + (1 + \mathrm{vol}(\Omega))\delta)g \ .$$

If we define $R = \max(2R_3R_4, R_2)$, ((4.16), (4.20), (4.28)), and if we define $\gamma = \gamma_1 + \gamma_2 + (1 + \mathrm{vol}(\Omega))\delta$, ((4.18), (4.19)), we see that $R\gamma < 1$ implies that $g = 0$, and thus, that $\limsup\|r(t)\|_\infty = 0$. From (4.23), we see that $\limsup\|s(t)\|_\infty = 0$ also. ■

__Remarks:__ Note that $R\gamma < 1$ implies assumption (4.16). Also, in our definition of H, it may be the case that H exceeds Λ_1, (this will be the case if Λ_1 and Λ_2 are both small). However, the conclusion of Theorem 4.10 follows trivially since in this case we have that $\|\phi_1\|_\infty \leq \Lambda_1 < H$. Finally, we note that a similar proof can be applied to systems of the form (1.6) with arbitrary positive r_i, d_i, and δ_i. The expressions R and γ, in addition to their other arguments, will now depend on r/d and $|\delta|$, where $r = \max(r_1, r_2)$, $d = \min(d_1, d_2)$ and $|\delta| = \max(\delta_i)$.

__Section 5.__

We shall now show that if condition (1.5) is violated, the steady state solution may bifurcate. In particular, we consider the system

$$\begin{aligned} \ddot{u} &= uv^2 \quad u(-1) = u(1) = M \\ \ddot{v} &= u^2v \quad v(-1) = v(1) = M \ . \end{aligned} \tag{5.1}$$

For all values of $M \geq 0$ there exists a symmetric solution $u = v = w$ where w is the solution of

$$\ddot{w} = w^3 \ , \ w(-1) = w(1) = M \ . \tag{5.2}$$

(Existence of solutions to (5.1) and (5.2) can be obtained from Theorem 3.4.) For small M, $u = v = w$ is the unique solution of (5.1), whereas for sufficiently large M, there exist at least three solutions of (5.1).

Numerical experiments indicate that the bifurcation diagram is as given in Figure 5.1. There is "exchange of stabilities" at M_0, whereupon the branch $u \neq v$ through M_0 becomes asymptotically stable for the associated parabolic system for all $M > M_0$, and the branch $u = v$, together with any other bifurcated branches become asymptotically unstable. Moreover, the branch $u \neq v$ emanating from M_0, appears to be non-degenerate, i.e., it does not admit secondary bifurcations.

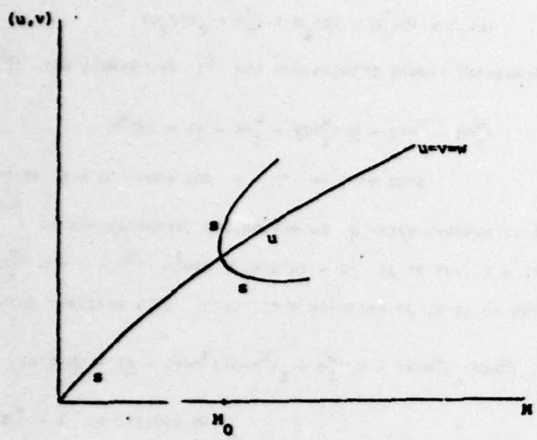

Figure 5.1

We begin with a description of the behavior of w as $M \to \infty$. Henceforth, we shall denote the solution of (5.2) by $w_M(x)$.

Lemma 5.1. There exists a function $w_*(x)$ defined on $|x| < 1$ which satisfies the equation $\ddot{w}_* = w_*^3$ and has the property that

$$w_*(x) = \lim_{M\to\infty} w_M(x) \quad . \tag{5.3}$$

Proof: We begin by noting that there is a unique solution to (5.2); indeed, if w_1 and w_2 are two solutions, then

$$\frac{d^2}{dx^2}(w_1-w_2) = (w_1^2 + w_1w_2 + w_2^2)(w_1-w_2), \ (w_1-w_2)\Big|_{\pm 1} = 0 \ ;$$

since the coefficient $w_1^2 + w_1w_2 + w_2^2$ is non-negative, we have that $w_1-w_2 = 0$. Note also that

$$\frac{d^2}{dx^2} w_M(-x) = w_M^3(-x) \quad , \quad w_M(-x)\Big|_{x=\pm 1} = M \ ;$$

hence $w(-x)$ is also a solution of (5.2), so that $w(x) = w(-x)$. Since $\ddot{w} = w^3 \geq 0$ if $M \geq 0$, w satisfies a maximum principle on each subinterval of $[-1,1]$; this fact combined with the symmetry of w about zero imply that if $m_M = \min_{-1\leq x\leq 1} w_M(x)$, then $m_M = w_M(0)$ and that $\dot{w}_M(0) = 0$. Hence, w_M coincides with the solution of the system

$$\begin{aligned} \dot{w} &= z \quad , \quad w(0) = m \\ \dot{z} &= w^3 \quad , \quad z(0) = 0 \quad , \end{aligned} \tag{5.4}$$

when we choose $m = m_M$.

We now compute the maximum interval of existence $|x| < x_{max}$ of the solution of (5.4) in terms of m. If $m > 0$, then $w \neq 0$, so that $dw/dz = z/w^3$; hence

$$\dot{w} = z = \left[\frac{1}{2}(w^4-m^4)\right]^{1/2} \quad ; \tag{5.5}$$

by integrating (5.5) we obtain

$$x = \int_m^{w(x)} 2^{1/2}(w^4-m^4)^{-1/2}\, dw \quad , \tag{5.6}$$

for each x for which $w(x)$ is defined. Let

$$x_{max}(m) = \int_m^{\infty} 2^{1/2}(w^4-m^4)^{-1/2}\, dw \ ;$$

if we put $h = w/m$, we see that

$$x_{max}(m) = 2^{1/2}/m \int_1^{\infty} (h^4-1)^{-1/2}\, dh \quad . \tag{5.7}$$

Now note that $h^4-1 = 4(h-1) + 6(h-1)^2 + r(h-1)^3 + (h-1)^4$, so that if we set $\xi = h-1$,

$$\int_1^{\infty} (h^4-1)^{-1/2}\, dh = \int_0^{\infty} (4\xi + 6\xi^2 + 4\xi^3 + \xi^4)^{-1/2}\, d\xi \leq 3/2 \quad .$$

Thus $x_{max}(m) \leq 3/(2^{1/2}m)$. From (5.6) it is clear that w cannot be defined for $x \geq x_{max}$, so that (w,z) must leave every compact set in the w-z plane as $|x| \to x_1$ for some $x_1 \le x_{max}$. We see that $\lim_{|x|\to x_1} w(x) = \infty$, since if this was not the case, w would be a solution of (5.2), taking $|x| \le x_1$ and $M = w(x_1)$. But in this case, $\lim_{|x|\to x_1} z(x) = \pm\infty$; the $|\cdot|_{1+\delta}$ Schauder estimates applied to (5.2) imply that $|w|_{1+\delta} \le \text{const}\cdot w(x_1) < \infty$, contradicting the unboundedness of $z = \dot w$ on $|x| < x_1$. Thus $x_1 = x_{max}$ and $\lim_{|x|\to x_{max}} w(x) = \infty$. From (5.7) we see that $x_{max}(m)$ is a strictly decreasing function of m with $x_{max}(\infty) = 0$, $x_{max}(0) = \infty$, so that for some unique choice of $m = m_*$, we have that $x_{max}(m_*) = 1$. Let $w_*(x)$ be the solution of (5.4) with $m = m_*$. Equation (5.3) then follows for $|x| < 1$ from the continuous dependence of the solution of (5.4) on its initial data. ■

We shall now prove that the solution $u = v = w$ bifurcates for large M. We employ the standard technique of linearizing an appropriate Fredholm mapping of Hilbert spaces about the trivial solution; the reader is referred to [9] for a detailed exposition of this method.

Let $\Omega = (-1,1)$, and let $X = [H^2(\Omega) \cap H_0^1(\Omega)]^2$, $Y = L^2(\Omega)^2$. Let $F : X \times \mathbb{R}_+ \to Y$ be defined by

$$F(a,b,M) = (\ddot a - (a+w_M)(b+w_M)^2 + w_M^3,\ \ddot b - (a+w_M)^2(b+w_M) + w_M^3) .$$

It is easily verified that (u,v) is a solution of (5.1) if and only if $u = a + w_M$, $v = b + w_M$, where $F(a,b,M) = 0$. If we let $A = (a,b)$ and $L_M = F_A|_{A=0}$ be the (Fréchet) derivative of F with respect to the variables A evaluated at $A = 0$, then for $A \in X$, we have that

$$L_M(A) = (\ddot a - aw_M^2 - 2bw_M^2,\ \ddot b - 2aw_M^2 - bw_M^2) .$$

<u>Lemma 5.2</u>. The linear map L_M has non-trivial kernel precisely when the map

(5.8) $$(d^2/dx^2 + w_M^2) : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$$

has non-trivial kernel. If $\sigma_0 \not\equiv 0$ is such that $\ddot\sigma_0 + w_M^2\sigma_0 = 0$, then $\ker L_M$ is spanned by the vector $x_0 = (\sigma_0, -\sigma_0)$.

<u>Proof</u>: If $\ddot\sigma_0 + w_M^2\sigma_0 = 0$, then it is clear that $(\sigma_0, -\sigma_0) \in \ker L_M$. Conversely, if we have that

$$\ddot a = w_M^2 a + 2w_M^2 b \ , \ \ddot b = 2w_M{}^2 a + w_M^2 b \ ,$$

for some $(a,b) \in X$, then $(a+b)'' = 3w_M^2(a+b)$. Since $a + b \in H_0^1(\Omega)$ and $3w_M^2 \ge 0$, we have that $a = -b$ and that $\ddot a + w_M^2 a = 0$. Since $d^2/dx^2 + w_M^2$ is a second order self-adjoint ordinary differential operator, it can have only simple eigenvalues. Hence this must also be true for L_M. ■

<u>Lemma 5.3</u>. There exists an $M_0 > 0$ such that the operator (5.8) has non-trivial kernel when $M = M_0$.

<u>Proof</u>: Let $\lambda_1(M)$ be the largest eigenvalue of $d^2/dx^2 + w_M^2$. Then

(5.9) $$\lambda_1(M) = \max\{\int_{-1}^1 (-\dot\varphi^2 + w_M^2\varphi^2)dx : \|\varphi\|_{L^2} = 1,\ \varphi \in H_0^1(\Omega)\} .$$

Let $\sigma(x) = w_*(x)^{-1}$ and $\varphi = \|\sigma\|_{L^2}^{-1}\sigma$. Note that from (5.5) we have that

$$\dot\sigma^2 = \frac{1}{2}(w_*^4 - m_*^4)w_*^{-4} = \frac{1}{2}(1 - (m_*\sigma)^4) \ ;$$

hence

$$\lambda_1(M) \ge \int_{-1}^1 (-\dot\varphi^2 + w_M^2\varphi^2)dx$$

$$\ge \|\sigma\|_{L^2}^{-2} \int_{-1}^1 \left[\left(\frac{w_M^2}{w_*^2} - \frac{1}{2}\right) + \frac{1}{2}(m_*\sigma)^4\right]dx .$$

Since w_M^2/w_*^2 increases monotonically to unity, pointwise for $|x| < 1$, we have that $\int_{-1}^1 (w_M^2/w_*^2 - 1/2)dx > 0$ for sufficiently large M. Hence $\lambda_1(M) > 0$ for large M. Since $\lambda_1(0) < 0$ and λ_1 depends continuously on M, there must exist $M_0 > 0$ such that $\lambda_1(M_0) = 0$. ■

<u>Theorem 5.4</u>. The solution $u = v = w_M$ of (5.1) bifurcates at $M = M_0$. There exist neighborhoods V of 0 in X and $\mathcal{M}$ of M_0 in $\mathbb{R}$ such that

$$\{(a,b) \in V \mid F(a,b,M) = 0, \text{ and } M \in \mathcal{M}\}$$

consists of two smooth curves in V which intersect transversally at $M = M_0$.

<u>Proof:</u> It suffices to show that the mapping F satisfies the hypotheses of Theorem 3.2.2 in [9]. We first show that F is a smooth mapping from $X \times \mathbb{R} \to Y$. Note that since $\dim \Omega = 1$, X is continuously embedded in $BC^0(\Omega)^2$, so that F is in fact a well defined continuous mapping of the indicated spaces. The smoothness of F is then easily verified.

We must show that

(i) $F_M(0,0,M_0) = 0$;

(ii) $\ker L_{M_0}$ is one dimensional and is spanned by $x_0 = (\sigma_0, -\sigma_0)$.

(iii) $Y_1 = \operatorname{range}(L_{M_0})$ has codimension 1

(iv) $F_{MM}\Big|_{\substack{A=0\\M=M_0}} \in Y_1$ and $F_{MA}\Big|_{\substack{A=0\\M=M_0}}(x_0) \notin Y_1$.

Note that (i) follows since $F(0,0,M) \equiv 0$, and (ii) follows from Lemmas 5.2 and 5.3. Property (iii) follows if we note that $L_{M_0} : Y \to Y$ defines a (densely defined) self-adjoint differential operator with closed range and domain X, so that range $(L_{M_0}) = \operatorname{cl}(\operatorname{range}(L_{M_0})) = (\ker L_{M_0})^{\perp}$.

We now verify (iv). The first statement is obvious. Let $z = \partial w_M/\partial M$. Then z is a solution of

$$\ddot{z} = 3w_M^2 z \quad , \quad z(-1) = z(1) = 1 \ .$$

The solution of the equation is non-negative by the maximum principle. If we define $P = F_{MA}\Big|_{\substack{A=0\\M=M_0}}$, then

$$P(A) = (-2w_M za - 4w_M zb, \ -4w_M za - 2w_M zb) \ .$$

If $x_0 = (\sigma_0, -\sigma_0)$ where $\ddot{\sigma}_0 + w_{M_0}^2 \sigma_0 = 0$, then by the Fredholm alternative, $P(x_0) \notin R(L_{M_0})$ if and only if $(P(x_0), x_0)_{L^2} \neq 0$. Since

$$(P(x_0), x_0)_{L^2} = \int_{-1}^{1} 4w_M z\sigma_0^2 dx > 0 \ ,$$

we see that property (iv) holds. ■

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This paper give conditions which imply the existence of a unique, globally attracting steady state solution of a certain class of reaction-diffusion systems with inhomogeneous Dirichlet conditions. In addition, an example is given which shows that when the above conditions are not satisfied the steady state may bifurcate.

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