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MRC Technical Summary Report #1895

A CONSTRUCTIVE APPROACH TO KERGIN INTERPOLATION IN R^k

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November 1978

Received September 12, 1978

Approved for public release
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U. S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

In this paper we construct an interpolation map from $C^{(n)}(R^k)$ into $\pi_n(R^k)$ (polynomials of total degree $\leq n$) whose existence and uniqueness was proved by P. Kergin.

AMS(MOS) Subject Classification: 41A05, 41A63

Key Words: multidimensional interpolation
B-splines
divided differences

Work Unit #6 - Spline Functions and Approximation Theory

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

SIGNIFICANCE AND EXPLANATION

Polynomial interpolation to a univariate function is a thoroughly studied subject and when properly executed provides a numerically efficient means of approximation.

Very little seems to be known about polynomial interpolation of multivariate functions. However, Kergin recently established the existence and uniqueness of a natural extension of univariate interpolation to a multivariate setting. In this paper we provide a formula for Kergin interpolation. This formula is based on the Newton form for univariate polynomial interpolation. The error in approximating by Kergin interpolation is also obtained in a convenient form which allows us to assess the quality of this scheme. In particular, we establish that Kergin interpolation converges for analytic functions of several variables.

There seems to be a close connection between Kergin interpolation and the multidimensional B-spline introduced by de Boor. In univariate approximation B-splines play a central role in both the theory and numerical application of spline functions. An extremely useful numerically stable recurrence formula for univariate B-splines is known and widely used. We provide here a similar formula for multivariate B-splines.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A CONSTRUCTIVE APPROACH TO KERGIN INTERPOLATION IN R^k

Charles A. Micchelli

In his recent doctoral thesis P. Kergin [3] introduced a novel method of interpolating smooth functions of several variables. In this note, we give an explicit representation for Kergin's interpolation scheme.

We let $\pi_n(R^k)$ denote the space of polynomials of total degree $\leq n$ and $C^{(n)}(R^k)$ be all functions with n continuous derivatives on R^k.

Kergin proves the following

Theorem. Given any $x^0,\ldots,x^n \in R^k$, not necessarily distinct, there is a unique linear map $P : C^{(n)} \to \pi_n$ with the property that for every $f \in C^{(n)}$, every homogeneous differential operator q of order ℓ, $0 \leq \ell \leq n$, and every subset $J \subseteq \{0,1,\ldots,n\}$ with card $J = \ell+1$ there exists a point x in the convex hull $\{x^j | j \in J\}$ such that $qPf(x) = qf(x)$.

Pf necessarily interpolates f at $x^0,\ldots,x^n$ as may be seen by choosing $\ell = 0$. If a point is repeated with some multiplicity then all the derivatives of Pf agree with f at that point with the same multiplicity. In particular, when all the points coincide $x^i = x^0$ then Pf is the Taylor polynomial of f at x^0,

$$Pf(x) = f(x^0) + D_{x-x^0} f(x^0) + \ldots + \frac{1}{n!} D^n_{x-x^0} f(x^0).$$

In this formula, $D_{x-x^0} f$ is the directional derivative of f in the direction $x-x^0$.

For the general form of Kergin's map we use the notation

$$\int_{[x^0,\ldots,x^n]} f = \int_{S^n} f(v_0x^0 + v_1x^1 + \ldots + v_nx^n)dv_1\ldots dv_n$$

where

$$S^n = \{v = (v_0,\ldots,v_n) | v_j \geq 0, \sum_0^n v_j = 1\} .$$

We will show that Kergin's map is given by

$$(1) \qquad Pf(x) = f(x^0) + \int_{[x^0,x^1]} D_{x-x^0} f + \ldots + \int_{[x^0,x^1,\ldots,x^n]} D_{x-x^0} \cdots D_{x-x^{n-1}} f.$$

This formula implies that Kergin's map for the points $x^0,\ldots,x^n$ is obtained by adding to the map corresponding to the points $x^0,\ldots,x^{n-1}$ the expression

$$\int_{[x^0,\ldots,x^n]} D_{x-x^0} \cdots D_{x-x^{n-1}} f .$$

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This property is reminiscent of Newton divided difference expansion for polynomial interpolation in one variable. As we will see this similarity is not accidental as the proof of (1) is based on Newton form for polynomial interpolation.

Along with formula (1) we also have available a convenient expression for the error in approximating by Kergin's map, namely,

$$(2) \qquad Pf(x) - f(x) = \int_{[x,x^0,\ldots,x^n]} D_{x-x^0} \cdots D_{x-x^n} f \, .$$

This useful formula makes it possible to obtain error estimates for approximating by Kergin's map. For instance, since the volume of S^n is $1/n!$ we have

$$\|Pf-f\|_{L^\infty(K)} \le \frac{1}{(n+1)!}(d_q(K))^{n+1}\Big(\sum_{|\alpha|=n+1} \|D^\alpha f\|^p_{L^\infty(K)}\Big)^{1/p}$$

where $\frac{1}{p}+\frac{1}{q}=1$, K = convex hull $\{x^j \mid 0 \le j \le n\}$ and $d_q(K)$ = diameter of K in ℓ^q. Thus if F is a bounded linear functional on $L^\infty(K)$ which annihilates $\pi_n(R^k)$ then

$$|Ff| \le \|F\|_{L^\infty(K)} \frac{(d_q(K))^{n+1}}{(n+1)!}\Big(\sum_{|\alpha|=n+1} \|D^\alpha f\|^p_{L^\infty(K)}\Big)^{1/p}$$

- a quick ("constructive") proof of the Bramble-Hilbert Lemma.

In one dimension it is known that polynomial interpolation to a function which is analytic in a sufficiently large region containing the interpolation points converges geometrically fast (as the number of points increase) to the function. Formula (2) provides us with a means to extend this result to higher dimensions. We will show that if $Q = \{x = (x_1,\ldots,x_k) \mid |x_j| \le 1\}$ (the unit cube in R^k) and f is a function on $C^n(Q)$ which has an analytic extension to the polydisk $D(\rho) = \{z = (z_1,\ldots,z_n) \mid |z_i| \le 2\rho+1\}$ $\rho > 1$, then for $K \subseteq Q$

$$(3) \qquad \|Pf - f\|_{L^\infty(Q)} \le 3\,\frac{(n+k)^{k-1}}{\rho^n}\,\|f\|_{D(\rho)}$$

(it seems to matter here that Q is a cube rather than a parallelepiped).

Clearly, formula (1) shows that P is a continuous map on $C^{(n)}(K)$. Kergin proved that when $k < n$ and the points $x^0,\ldots,x^n$ are in general position, i.e. every subset of $k+1$ points of $\{x^0,\ldots,x^n\}$ form a simplex of dimension k then P is a continuous map on $C^{k-1}(R^k)$. This fact is not obvious from (1). However, we will show by elementary means that when $x^0,\ldots,x^n$ are in general position and q is any differential operator of order $\leq n$, the linear functional

(4) $$Pf = \int_{[x^0,\ldots,x^n]} qf\ , \quad f \in C^{(n)}(K)$$

has an extension as a bounded linear functional on $C^{(k-1)}(R^k)$. In the case $k=1$ such an extension is given by the Hermite-Genocchi formula for the divided difference of a function,

(5) $$\int_{[x_0,\ldots,x_n]} f^{(n)} = [x_0,\ldots,x_n]f\ , \qquad x_j \in R\ ,$$

see Nörlund [4, p. 16] (subscripts will be used for scalars, real or complex). Our remarks concerning this aspect of Kergin's map leads us directly to some simple properties of multi-dimensional B-splines introduced by Carl de Boor in [2]. Needless to say, there are many intriguing questions still to be settled concerning these B-splines.

Returning to formula (1), our first task is to provide its proof. As we said earlier the Newton form for the polynomial of degree n which interpolates f at $x_0,\ldots,x_n$

(1') $$pf(x) = f(x_0) + (x-x_0)[x_0,x_1]f + \ldots + (x-x_0)\ldots(x-x_{n-1})[x_0,\ldots,x_n]f$$

is the basis for the proof of (1). It is also well-known that the remainder formula for polynomial interpolation is given by

(2') $$pf(x) - f(x) = [x,x_0,\ldots,x_n]f\ .$$

To verify that Pf satisfies the property of Kergin's theorem we first observe that for the function $f(x) = e^{i(\lambda\cdot x)} = g(\lambda\cdot x)$, into (1) gives $Pf(x) = p_\lambda g(\lambda\cdot x)$ where p_λ is the polynomial of degree $\leq n$ which interpolates g at $\lambda\cdot x^0,\ldots,\lambda\cdot x^n$. Now, let $J = \{x^{i_0},\ldots,x^{i_\ell}\}$ be a subset of $\{x^0,\ldots,x^n\}$ and q a homogeneous differential operator, of order ℓ then

$$\begin{aligned}\int_{[x^{i_0},\ldots,x^{i_\ell}]} qf &= i^{-\ell}q(i\lambda) \int_{[\lambda\cdot x^{i_0},\ldots,\lambda\cdot x^{i_\ell}]} g^{(\ell)} \\ &= i^{-\ell}q(i\lambda)[\lambda\cdot x^{i_0},\ldots,\lambda\cdot x^{i_\ell}]g \\ &= i^{-\ell}q(i\lambda)[\lambda\cdot x^{i_0},\ldots,\lambda\cdot x^{i_\ell}]p_\lambda g \\ &= \int_{[x^{i_0},\ldots,x^{i_\ell}]} qPf\ .\end{aligned}$$

Since $\{e^{i\lambda\cdot x} | \lambda \in R^k\}$ spans a dense subset of $C^{(n)}(R^k)$ we have shown that

$$\int_{[x^{i_0},\dots,x^{i_\ell}]} q(f-Pf) = 0\,, \qquad f \in C^{(n)}(R^k)$$

and thus P is the Kergin map.

The proof of (2) follows from (2') in a similar way.

For the proof of (3) we require Cauchy integral formula in C^k, see Bockner and Martin [1, p. 33],

$$D^{\alpha}f(x) = \frac{\alpha!}{(2\pi i)^k}\int_{c_k}\dots\int_{c_k}\frac{f(\zeta)d\zeta}{(x_1-\zeta_1)^{\alpha_1+1}\dots(x_k-\zeta_k)^{\alpha_k+1}}$$

$$x = (x_1,\dots,x_k),\quad \zeta = (\zeta_1,\dots,\zeta_k),\quad \alpha! = \alpha_1!\dots\alpha_k!\,,$$

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_k^{\alpha_k}}\,,\quad |\alpha| = \alpha_1+\dots+\alpha_k$$

and c_j is the contour $\{z \mid |z| = 2\rho'+1\}$ where $1 < \rho' < \rho$. Thus we have

$$|D^{\alpha}f(x)| \le \left(\frac{3}{2}\right)^k \frac{\alpha!}{(2\rho)^{|\alpha|}}\|f\|_{D(\rho)}\,,\qquad x \in Q\,,$$

and since

$$\|D_{x-x^0}\dots D_{x-x^n}f\|_{L^{\infty}(Q)}$$

$$\le \prod_{\ell=0}^{n}\|x-x_\ell\|_{\ell^{\infty}}\sum_{|\alpha|=n+1}\|D^{\alpha}f\|_{L^{\infty}(Q)}$$

$$\le \left(\frac{3}{2}\right)^k \rho^{-n-1}\sum_{|\alpha|=n+1}\alpha!\,\|f\|_{D(\rho)}\,,$$

(3) follows from the observation that

$$\sum_{|\alpha|=n+1}\alpha! \le \sum_{|\alpha|=n+1}|\alpha|! = (n+1)!\sum_{|\alpha|=n+1}1 \le (n+1)!\,\frac{(n+k)^{k-1}}{(k-1)!}$$

and that the vol $S^{n+1} = 1/(n+1)!$

In one dimension, polynomial interpolation at any (triangular) sequence of points is bound to diverge for some continuous function while for interpolation at the zeros of the Chebyshev polynomial convergence holds for lip 1 functions. It would be interesting to extend these results in R^k for Kergin interpolation.

We now turn to the matter of extending the linear functional defined in (4). We begin by observing that

$$Lf = \int_{[x^0,\dots,x^n]} D_y f , \qquad f \in C^{(1)}(K)$$

has an extension as a bounded linear functional on $C(K)$ if and only if y is in the vector space generated by $x^1-x^0,\dots,x^n-x^0$. The necessity of this condition is a consequence of the following: Suppose $\lambda \in R^k$ and $\lambda \perp x^j-x^0$, $j=1,\dots,n$. Given any $\delta \in R$, let $f(x) = e^{i\delta\lambda\cdot x} = g(\delta\lambda\cdot x)$ then $|Lf| = |\delta||y\cdot\lambda|$ while $\|f\|_{L^\infty(K)} = 1$. Thus $\lambda \perp y$ and we conclude that y is in the span of $x^1-x^0,\dots,x^n-x^0$. If on the other hand, $y = \sum_{j=1}^{n} a_j(x^j-x^0)$ then for $\delta = 1$

$$\begin{aligned}
Lf &= i^{-k+1}\, y\cdot\lambda[\lambda\cdot x^0,\dots,\lambda\cdot x^n]g \\
&= i^{-k+1}\sum_{j=1}^{n} a_j\lambda\cdot(x^j-x^0)[\lambda\cdot x^0,\dots,\lambda\cdot x^n]g \\
&= i^{-k+1}\sum_{j=1}^{n} a_j\{[\lambda\cdot x^0,\dots,\lambda\cdot x^{j-1},\lambda\cdot x^{j+1},\dots,\lambda\cdot x^n]g - [\lambda\cdot x^1,\dots,\lambda\cdot x^n]g
\end{aligned}$$

and we get

$$(6)\qquad \int_{[x^0,\dots,x^n]} D_y f = \sum_{j=1}^{n} a_j\Big\{\int_{[x^0,\dots,x^{j-1},x^{j+1},\dots,x^n]} f - \int_{[x^1,\dots,x^n]} f\Big\} .$$

Again, since $\{e^{i\lambda\cdot x}|\lambda \in R^k\}$ spans a dense subset of $C^{(1)}(R^k)$ this formula provides the extension of L to $C(K)$. In particular, if for $n \geq k$ $\{x^1-x^0,\dots,x^n-x^0\}$ spans R^k then L is a bounded linear functional on $C(K)$ for <u>all</u> y.

If the points $x^0,\dots,x^n$ are in general position then the above observation extends by induction to show that for any $y^1,\dots,y^\ell \in R^k$ $\ell \leq n$, $n \geq k$ and $q_\ell(x) = \prod_{i=1}^{\ell} y^i\cdot x$ the linear functional

$$\int_{[x^0,\dots,x^n]} q_\ell f$$

has a bounded extension on $C^{(k-1)}(R^k)$. Since any differential operator of order $\leq n$ is a sum of q_ℓ the linear functional given by (5) has a bounded extension on $C^{(k-1)}(R^k)$.

One may easily verify that, in general, if $q = \sum_{j=0}^{m} q_j$, q_j a homogeneous polynomial of degree j, $\int_{[x^0,\dots,x^n]} qf$ has a bounded extension to $C^{(\ell)}(K)$ then whenever $\lambda \perp x^j-x^0$, $j=1,\dots,n$ it follows that $q_{\ell+1}(\lambda) = \dots = q_m(\lambda) = 0$. Whether or not this condition is also sufficient for the boundedness of $\int_{[x^0,\dots,x^n]} qf$ has not been settled.

Finally, we conclude with some remarks on the multidimensional B-splines introduced in [2]. It is not hard to see that if $x^1-x^0,\dots,x^n-x^0$ span R^k then there exists a function $M(x|x^0,\dots,x^n)$ such that

$$(7)\qquad \int_{[x^0,\dots,x^n]} f \quad = \quad \int_K f(x)M(x|x^0,\dots,x^n)dx \ .$$

Note that the expression on the left is an n-fold integral over the simplex S^n while the right hand expression is a k-fold ($k \le n$) integral over the convex hull of $x^0,x^1,\dots,x^n$. Repeated application of (6) implies that if $x^0,x^1,\dots,x^n$ are in general position then

$$\int_{[x^0,\dots,x^n]} D^\alpha f = \int_K D^\alpha f(x)M(x|x^0,\dots,x^n)dx \ , \qquad |\alpha| = n-k+1$$

is a bound linear functional on $C(K)$. Moreover, it is a sum of integrals of f over hyperplanes formed by subsets of k points of $x^0,\dots,x^n$. Thus M is a polynomial of total degree $\le n-k$ in every region bounded by such hyperplanes and globally has $n-k-1$ continuous derivatives on K, facts already mentioned in [2].

A formula for $M(x)$, which reveals its geometric interpretation, may be obtained by "lifting" the points $x^0,\dots,x^n$ into R^n. We let $\hat{x}^0,\dots,\hat{x}^n$ be any vectors in R^n such that $\hat{x}^j|_{R^k} = x^j$ and that the simplex σ determined by these points is of dimension n. Then

$$(8)\qquad M(x\ x^0,\dots,x^n) = \frac{1}{n!}\ \frac{\mathrm{vol}_{n-k}\{u \in \sigma|u|_{R^k} = x\}}{\mathrm{vol}_n \sigma}$$

(For $k=n$, $M(x|x^0,\dots,x^n) = \frac{1}{k!}\chi_\sigma(x)\ /\mathrm{vol}_n\sigma$). Thus M is (with a different normalization) de Boor's multidimensional B-spline [2].

In spite of this formula's attractive geometric content it seems more convenient to derive properties of M through the defining equation (7). In particular, according to formula (6)

$$(9)\qquad D_y M(\cdot|x^0,\dots,x^n) = \sum_{j=1}^n a_j\{M(\cdot|x^0,\dots,x^{j-1},x^{j+1},\dots,x^n)-M(\cdot|x^1,\dots,x^n)\}$$

$$y = \sum_{j=1}^n a_j(x^j-x^0) \ .$$

When $k=1$, this formula yields the equation

$$(10) \qquad M'(x|x_0,\ldots,x_n) = \frac{M(x|x_0,\ldots,x_{n-1}) - M(x|x_1,\ldots,x_n)}{x_n - x_0}$$

(see [2]) and with the initial condition

$$M(x|x_0,x_1) = \frac{1}{x_1-x_0}\begin{cases} 1, & x_0 \le x \le x_1 \\ 0, & x \notin (x_0,x_1) \end{cases}$$

all the (univariate) B-splines can be generated from (10). A similar remark applies for (9).

In the univariate case, an efficient and numerically stable means of generating B-splines is available, [2]. The univariate B-spline over its interval of support is a nonnegative combination of two lower order B-splines, specifically

$$(11) \quad (n-1)\, M(x|x_0,\ldots,x_n) = \frac{x_n - x}{x_n - x_0} M(x|x_1,\ldots,x_n) + \frac{x - x_0}{x_n - x_0} M(x|x_1,\ldots,x_{n-1})$$

Using the identity

$$(12) \quad [x_0,\ldots,x_n] = \frac{x_n - z}{x_n - x_0}[z,x_1,\ldots,x_{n-1}] + \frac{z - x_0}{x_n - x_0}[z,x_0,\ldots,x_{n-1}]$$

for divided differences it follows by the line of reasoning used to prove (9) that

$$(13) \qquad M(x|x^0,x^1,\ldots,x^n) = t\, M(x|x^0,\ldots,x^{n-1},z) + (1-t)M(x|x^1,\ldots,x^{n-1},z)$$

$$z = t\, x^0 + (1-t)x^n .$$

In the univariate case, it may be easily proved that

$$(14) \qquad M(x|x_0,\ldots,x_n,x) = \frac{1}{n} M(x|x_0,\ldots,x_n)$$

and thus (14) with (13) proves (11) by choosing $x = t\,x_0 + (1-t)x_n$. In the multivariate case it is highly plausible that a formula like (14) also holds and this should lead to a multidimensional version of (11).

The following theorem provides the details for this intuition.

<u>Theorem 2.</u> <u>If</u> $x = \sum_{j=0}^{n} \lambda_j x^j$, $\sum_{j=0}^{n} \lambda_j = 1$

<u>then</u> <u>for</u> $n \ge k+1$

$$M(x|x^0,\ldots,x^n) = \frac{1}{n-k}\sum_{j=0}^{n} \lambda_j M(x|x^0,\ldots,x^{j-1},x^{j+1},\ldots,x^n)$$

<u>Proof</u>. We divide the proof into several ancillary results. First we require an observation about divided differences which generalizes formula (12).

Let $z = \sum_{j=0}^{n} \lambda_j x_j$, $\sum_{j=0}^{n} \lambda_j = 1$

then

$$(15) \qquad [x_0,\dots,x_n]f = \sum_{j=0}^{n} \lambda_j [z,x_0,\dots,x_{j-1},x_{j+1},\dots,x_n]f \, .$$

Note that in (15) the order of the divided difference on each side of the equal sign is the same.

Proof. The right hand side of (15) is a linear functional which agrees with the divided difference appearing on the left hand side on polynomials of degree n. The support of the linear functional on the right hand side is contained in $\{x_0,\dots,x_n,z\}$. However, since the coefficient of $f(z)$ is

$$\sum_{j=0}^{n} \lambda_j \frac{1}{(z-x_0)\dots(z-x_{j-1})(z-x_{j+1})\dots(z-x_n)}$$

$$= \frac{1}{(z-x_0)\dots(z-x_n)} \sum_{j=0}^{n} \lambda_j (z-x_j) = 0$$

the support is $\{x_0,\dots,x_n\}$ and thus the identity is proved.

We have already demonstrated in the proof of (9) how identities for divided differences may be extended to higher dimensions. This same approach applied to (15) yields the identity

$$\int_{[x^0,\dots,x^n]} f = \sum_{j=0}^{n} \lambda_j \int_{[z,x^0,\dots,x^{j-1},x^{j+1},\dots,x^n]} f$$

or equivalently

$$M(x|x^0,\dots,x^n) = \sum_{j=0}^{n} \lambda_j M(x|z,x^0,\dots,x^{j-1},x^{j+1},\dots,x^n)$$

(16)

$$z = \sum_{j=0}^{n} \lambda_j x^j \, .$$

The next identity we require is

$$(17) \qquad M(x|x,x^0,\dots,x^{n-1}) = \frac{1}{n-k} M(x|x^0,\dots,x^{n-1}).$$

We base the proof of this formula on the following general consideration.

Let K be a convex set R^m and $y \in R^m$. We form the cone in R^{m+1} with base $K \times \{0\}$ and vertex $y \times \{1\}$, that is,

$$C = \mathrm{cohull}(K \times \{0\} \cup y \times \{1\}).$$

Then

$$(18) \qquad \mathrm{vol}_{m+1} C = \frac{1}{m+1} \mathrm{vol}_m K \, .$$

We prove this formula by first establishing it for a simplex. If K is the simplex with vertices $y^0,\dots,y^m$ then C is a simplex with vertices $y^0 \times \{0\},\dots,y^m \times \{0\}, y \times \{1\}$. The formula for the volume of the simplex K is

$$\mathrm{vol}_m K = \frac{1}{m!} \begin{vmatrix} 1 & \cdots & 1 \\ y_1^0 & \cdots & y_1^m \\ \vdots & & \vdots \\ y_m^0 & \cdots & y_m^m \end{vmatrix}$$

and applying this formula to C gives

$$\mathrm{vol}_{m+1} C = \frac{1}{(m+1)!} \begin{vmatrix} 1 & \cdots & 1 & 1 \\ y_1^0 & \cdots & y_1^m & y_1 \\ \vdots & & & \\ y_m^0 & \cdots & y_m^m & y_m \\ 0 & \cdots & 0 & 1 \end{vmatrix}$$

which readily simplifies to (18).

Now, if K_1 and K_2 are disjoint convex sets then their corresponding cones with the same vertex intersect only at the vertex. Hence (18) extends to any countable union of disjoint simplices and in particular for any convex set K.

To prove (17) we first lift the points $x^0,\dots,x^{n-1}$ into R^{n-1} by means of $\hat{x}^0,\dots,\hat{x}^{n-1}$ and apply the relation (8),

$$M(x|x^0,\dots,x^{n-1}) = \frac{1}{(n-1)!} \frac{\mathrm{vol}_{n-k-1}\{u \in \sigma | u|_{R^k} = x\}}{\mathrm{vol}_{n-1}\sigma}$$

where σ is the simplex with vertices $\hat{x}^0,\dots,\hat{x}^{n-1}$. The vector x is now also lifted to R^{n-1} in any fashion to $\hat{x} \in R^{n-1}$ and then we form the cone with vertex $\hat{x} \times \{1\} = \bar{x}$ and base $\sigma \times \{0\}$ and denote it by $\bar{\sigma}$. It is readily seen that the cone with base $\{u \in \sigma | u|_{R^k} = x\} \times \{0\}$ and vertex $\bar{x}$ is $\{u \in \bar{\sigma} | u|_{R^k} = x\}$ and thus two applications of formula (18) gives us

$$\begin{aligned} M(x|x^0,\dots,x^{n-1}) &= \frac{1}{n!} \frac{\mathrm{vol}_{n-k-1}\{u \in \sigma | u|_{R^k} = x\}}{\mathrm{vol}_n \bar{\sigma}} \\ &= \frac{(n-k)}{n!} \frac{\mathrm{vol}_{n-k}\{u \in \bar{\sigma} | u|_{R^k} = x\}}{\mathrm{vol}_n \bar{\sigma}} \\ &= (n-k) M(x|x,x^0,\dots,x^{n-1}) . \end{aligned}$$

If we now put together formula (16) with (17) when $\lambda_j = 0,\ j = 2,\dots,n$ and $x = z = \lambda_0 x^0 + \lambda_1 x^1$ we get

$$(19)\qquad (n-k)M(x|x^0,\dots,x^n) = \lambda_0 M(x|x^1,\dots,x^n) + \lambda_1 M(x|x^0,x^2,\dots,x^n),$$
$$x = \lambda_0 x^0 + \lambda_1 x^1 .$$

With these facts at hand we may now complete the proof of Theorem 2.

Suppose we write $x = \sum_{j=0}^{n} \lambda_j x^j$

in the form

$$x = (1-\lambda_n)z + \lambda_n x^n ,$$

$$z = \sum_{j=0}^{n-1} \mu_j x^j , \quad (1-\lambda_n)\mu_j = \lambda_j \qquad j = 0,1,\dots,n-1 .$$

Then for every integer ℓ, with $0 \le \ell \le n-1$ we have

$$(n-k)M(x|z,x^0,\dots,x^{\ell-1},x^{\ell+1},\dots,x^n)$$
$$= (1-\lambda_n)M(x|x^0,\dots,x^{\ell-1},x^{\ell+1},\dots,x^n) + \lambda_n M(x|z,x^0,\dots,x^{\ell-1},x^{\ell+1},\dots,x^{n-1}).$$

But according to (16)

$$\sum_{\ell=0}^{n-1} \mu_\ell M(x|z,x^0,\dots,x^{\ell-1},x^{\ell+1},\dots,x^n) = M(x|x^0,\dots,x^n)$$

and hence

$$(n-k)M(x|x^0,\dots,x^n) = \sum_{\ell=0}^{n-1} \lambda_\ell M(x|x^0,\dots,x^{\ell-1},x^{\ell+1},\dots,x^n)$$
$$+ \lambda_n \sum_{\ell=0}^{n-1} \mu_\ell M(x|z,x^0,\dots,x^{\ell-1},x^{\ell+1},\dots,x^{n-1}) .$$

Another application of (16) implies that the second sum above is $M(x|x^0,\dots,x^{n-1})$ and thus the theorem is proved.

Besides the obvious integral relation

$$M(x|x^0,\dots,x^n) = \frac{1}{(2\pi i)^k}\int_{R^k} e^{-i\lambda\cdot x}[i\lambda\cdot x^0,\dots,i\lambda\cdot x^n]g\,d\lambda, \quad g(t) = e^t ,$$

$$= \frac{1}{(2\pi i)^k}\int_{R^k} [i\lambda\cdot(x^0-x),\dots,i(\lambda\cdot(x^n-x))]g\,d\lambda$$

for M obtainable from (7) by setting $f(x) = e^{i\lambda\cdot x}$ and using the Fourier inversion formula there is also a connection between $M(x|x^0,\dots,x^n)$, the Randon transform and the univariate B-spline.

Recall that the Radon transform is a map of functions $f \in L^1(R^k)$ defined for $\theta \in \omega_{k-1}$ (the unit sphere in R^k) and $t \in R$ by

$$R_\theta f(t) = \int_{\theta\cdot x=t} f(x)\,d\omega_{k-1}(x) .$$

The Radon transform also has the characteristic property

$$\int_{-\infty}^{\infty} (R_\theta f)(t)g(t)dt = \int_{R^k} f(x)g(\theta\cdot x)dx, \quad g \in L^\infty(R).$$

Thus

$$(R_\theta M(\cdot|x^0,\ldots,x^n))(t) = M(t|\theta\cdot x^0,\ldots,\theta\cdot x^n)$$

a striking relationship between univariate and multivariate B-splines.

Acknowledgements.

We gratefully acknowledge our appreciation to Carl de Boor for setting the stage that led to (1) as well as providing answers to many questions concerning Kergin interpolation. Allan Hoffman assisted us with the proof of (18).

We would also like to add that we have been informed by P. Milman that he has obtained independently a formula quite similar to (1).

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

MRC-TSR REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1895	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER 9 Technical
4. TITLE (and Subtitle) 6 A CONSTRUCTIVE APPROACH TO KERGIN INTERPOLATION IN $R^{(k)}$,		5. TYPE OF REPORT & PERIOD COVERED Summary Report, - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 Charles A. Micchelli		8. CONTRACT OR GRANT NUMBER(s) 15 DAAG29-75-C-0024
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS #6-Spline Functions and Approximation Theory
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE 11 November 1978
		13. NUMBER OF PAGES 11
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) 12 15p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Multidimensional interpolation,
B-splines,
Divided differences

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we construct an interpolation map from $C^{(n)}(R^k)$ into $\pi_n(R^k)$ (polynomials of total degree $\leq n$) whose existence and uniqueness was proved by P. Kergin.

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