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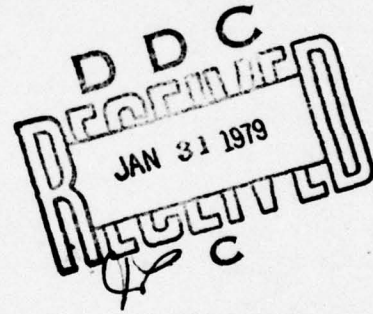3 -79
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REMARKS ON THE SCHRÖDINGER OPERATOR
WITH SINGULAR COMPLEX POTENTIALS

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ABSTRACT

22 p.

Let $A = -\Delta + V(x)$ be a Schrödinger operator on an (arbitrary) open set $\Omega \subset \mathbb{R}^m$, where $V \in L^1_{loc}(\Omega)$ is a complex valued function. We consider the "maximal" realization of A in $L^2(\Omega)$ under Dirichlet boundary condition, that is

$$D(A) = \{u \in H^1_0(\Omega); \; Vu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + Vu \in L^2(\Omega)\} .$$

When $\Omega = \mathbb{R}^m$ we also consider the operator

$$A_1 = -\Delta + V$$

with domain

$$D(A_1) = \{u \in L^2(\Omega); \; Vu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + Vu \in L^2(\Omega)\} .$$

A special case of our main results is:

Theorem: Let $m \geq 3$; assume that the function $\max\{-\mathrm{Re}\, V, 0\}$ belongs to $L^\infty(\Omega) + L^{m/2}(\Omega)$ and also to $L^{(m/2)+\varepsilon}_{loc}(\Omega)$ for some $\varepsilon > 0$. Then A (resp. A_1) is closable and $\bar{A} + \lambda$ (resp. $\bar{A}_1 + \lambda$) is m-accretive for some real constant λ.

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SIGNIFICANCE AND EXPLANATION

Schrödinger operators of the form $A = -\Delta + V(x)$, where Δ is the Laplacian and V is a scalar potential, arise in quantum mechanics and other areas. Delicate questions concerning what domain should be assigned to A must be settled in order to have a good theory. These questions are answered here for a very general class of potentials V which may even have complex values.

REMARKS ON THE SCHRÖDINGER OPERATOR WITH SINGULAR COMPLEX POTENTIALS

Haïm Brezis[1,3] and Tosio Kato[2,4]

1. Introduction

Let $A = -\Delta + V(x)$ be a Schrödinger operator on an (arbitrary) open set $\Omega \subset \mathbb{R}^m$, where $V \in L^1_{loc}(\Omega)$ is a complex valued function. We consider the "maximal" realization of A in $L^2(\Omega)$ under Dirichlet boundary condition, that is

$$D(A) = \{u \in H^1_0(\Omega); \ Vu \in L^1_{loc}(\Omega) \ \text{ and } \ -\Delta u + Vu \in L^2(\Omega)\} \ .$$

When $\Omega = \mathbb{R}^m$ we also consider the operator

$$A_1 = -\Delta + V$$

with domain

$$D(A_1) = \{u \in L^2(\Omega); \ Vu \in L^1_{loc}(\Omega) \ \text{ and } \ -\Delta u + Vu \in L^2(\Omega)\} \ .$$

We state now our main results (see Theorems 3.1 and 3.2) in a special case.

Theorem: Let $m \geq 3$; assume that the function $\max\{-\operatorname{Re} V, 0\}$ belongs to $L^\infty(\Omega) + L^{m/2}(\Omega)$ and also to $L^{(m/2)+\varepsilon}_{loc}(\Omega)$ for some $\varepsilon > 0$. Then A (resp. A_1) is closable and $\bar{A} + \lambda$ (resp. $\bar{A}_1 + \lambda$) is m-accretive for some real constant λ.

We emphasize the fact that $\max\{\operatorname{Re} V, 0\}$ and $\operatorname{Im} V$ could be arbitrary functions in $L^1_{loc}(\Omega)$.

Our methods rely on some measure theoretic arguments and standard techniques of DeGiorgi-Moser-Stampacchia type, related to the weak form of the maximum principle.

The distributional inequality

$$\Delta |u| \geq \operatorname{Re}[\Delta u \ \operatorname{sign} \bar{u}]$$

proved in [3] plays a crucial role. We also use a result from [1] concerning a property of Sobolev spaces.

In order to describe our method in a simple case we begin in Section 2 with real valued potentials. The main results in Section 2 are essentially known (see [3], [4], [8]) - except perhaps for Theorem 2.2 when $m \leq 4$.

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In Section 3 we turn to the case of complex potentials. Schrödinger operators with complex potentials have been studied by Nelson [6]. His results were extended in [5]. Here we allow more general singularities.

We thank Professors R. Jensen and B. Simon for useful suggestions and discussions (with the first author).

2. Real valued potentials

Let Ω be an (arbitrary) open subset of $\mathbb{R}^m$ and let $H = L^2 = L^2(\Omega;\mathbb{C})$. Let $q \in L^1_{loc}(\Omega)$ be a real valued function. Set

$$q^+ = \max(q,0), \quad q^- = \max(-q,0) .$$

Assume

(1) $$q^- \in L^\infty(\Omega) + L^p(\Omega)$$

with

$$\begin{cases} p = \dfrac{m}{2} & \text{when} \quad m \geq 3 \\ p > 1 & \text{when} \quad m = 2 \\ p = 1 & \text{when} \quad m = 1 . \end{cases}$$

Consider the operator A defined in H by

$$A = -\Delta + q(x)$$

with

$$D(A) = \{u \in H^1_0(\Omega); qu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + qu \in L^2(\Omega)\} .$$

The main results are the following:

Theorem 2.1. A is self-adjoint and $A + \lambda_1$ is m-accretive for some real constant λ_1. Furthermore $u,v \in D(A)$ imply $q|u|^2 \in L^1(\Omega)$, $q|v|^2 \in L^1(\Omega)$ and

(2) $$(Au,v) = \int \text{grad} u \ \text{grad} \bar{v} + \int qu\bar{v} .$$

When $\Omega = \mathbb{R}^m$ we also consider the operator A_1 defined in H by

$$A_1 = -\Delta + q(x)$$

with

$$D(A_1) = \{u \in L^2(\Omega); qu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + qu \in L^2(\Omega)\} .$$

Only when $m = 3$ or $m = 4$ we will make the additional assumption:

(3) $\quad q^- \in L^{p+\epsilon}_{loc}(\Omega)$ with $p = \dfrac{3}{2}$ when $m = 3$ and $p = 2$ when $m = 4$, for some arbitrarily small $\epsilon > 0$.

More precisely we assume that for each $x_0 \in \mathbb{R}^m$ there exists a neighborhood U of x_0 and some $\epsilon > 0$ (depending on x_0) such that $q^- \in L^{p+\epsilon}(U)$.

Theorem 2.2: Under the assumptions (1) and (3), $A_1 = A$.

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Our first lemma is well known:

Lemma 2.1: Assume (1). Then for every $\varepsilon > 0$, there exists a constant λ_ε such that

$$\int q^- |u|^2 \leq \varepsilon \|\text{grad}\, u\|_{L^2}^2 + \lambda_\varepsilon \|u\|_{L^2}^2 \qquad \forall u \in H_0^1(\Omega) \ .$$

In particular

$$\int q^- |u|^2 \leq \|\text{grad}\, u\|_{L^2}^2 + \lambda_1 \|u\|_{L^2}^2 \qquad \forall u \in H_0^1(\Omega) \ .$$

Proof: Write $q^- = q_1 + q_2$ with $q_1 \in L^\infty(\Omega)$ and $q_2 \in L^p(\Omega)$. Then for each $k > 0$ we have

$$\int q^- |u|^2 \leq \|q_1\|_{L^\infty} \|u\|_{L^2}^2 + \int_{[|q_2|>k]} |q_2||u|^2 + k \int_{[|q_2|\leq k]} |u|^2$$

$$\leq (\|q_1\|_{L^\infty} + k) \|u\|_{L^2}^2 + \|q_2\|_{L^p([|q_2|>k])} \|u\|_{L^t}^2$$

with

$$\frac{1}{p} + \frac{2}{t} = 1 \ .$$

In case $m \geq 3$ we find $t = 2^*$ where 2^* is the Sobolev exponent, that is $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{m}$. By the Sobolev imbedding theorem we have

$$\|u\|_{L^t} \leq C \|\text{grad}\, u\|_{L^2} \qquad \forall u \in H_0^1(\Omega) \ .$$

When $m = 2$ we find $2 < t < \infty$ and it is known that

$$\|u\|_{L^t} \leq C(\|\text{grad}\, u\|_{L^2} + \|u\|_{L^2}) \qquad \forall u \in H_0^1(\Omega) \ .$$

When $m = 1$ we find $t = \infty$ and it is known that

$$\|u\|_{L^\infty} \leq C(\|\text{grad}\, u\|_{L^2} + \|u\|_{L^2}) \qquad \forall u \in H_0^1(\Omega) \ .$$

We reach the conclusion of Lemma 2.1 in all the cases by choosing k large enough so that

$$c^2 \|q_2\|_{L^p([|q_2|>k])} < \varepsilon \ .$$

Remark 2.1: Assumption (1) is used in all the results of this paper only through Lemma 2.1 and it may in fact be weakened to a "locally uniform L^p-condition":

(1') $$\|q^-\|_{L^p(\Omega \cap B_r(y))} \to 0 \quad \text{as } r \to 0 \text{ uniformly in } y \in \Omega \ ,$$

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where

$$B_r(y) = \{x \in \mathbb{R}^m; \ |x - y| \leq r\} \ .$$

Indeed let $\varphi \in \mathcal{D}_+(\mathbb{R}^m)$ with $\text{supp } \varphi \subset B_r(0)$ and $\|\varphi\|_{L^2} = 1$. Then, writing $\varphi_y(x) = \varphi(x - y)$,

$$\int q^- |u|^2 = \int dy \int q^- |u\varphi_y|^2 \leq \int \|q^-\|_{L^p(B_r(y))} \|u\varphi_y\|_{L^t}^2 \, dy \ .$$

Here $\|q^-\|_{L^p(B_r(y))} \leq \delta$ for any small δ by (1') if r is chosen small. So

$$\int q^- |u|^2 \leq \delta \int \|u\varphi_y\|_{L^t}^2 \, dy \leq C\delta \int \|\text{grad}(u\varphi_y)\|_{L^2}^2 \, dy$$

$$\leq 2C\delta \int (\|\varphi_y \, \text{grad} u\|_{L^2}^2 + \|u \, \text{grad} \varphi_y\|_{L^2}^2) \, dy$$

$$= 2C\delta (\|\text{grad} u\|_{L^2}^2 + C_r \|u\|_{L^2}^2) \ .$$

Choosing δ so that $2C\delta = \varepsilon$, one gets the conclusion of Lemma 2.1. Such a locally uniform L^p-condition was used by Simader [7].

We recall a result of [1] which will be used in the proof of Theorem 2.1[1].

__Lemma 2.2:__ Let $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ and let $u \in H^1_0(\Omega)$ be such that a.e. on Ω

$$\text{Re} T \cdot \bar{u} \geq f$$

for some real valued function $f \in L^1(\Omega)$. Then $\text{Re } T \cdot \bar{u} \in L^1(\Omega)$ and

$$\text{Re}\langle T, u \rangle = \int \text{Re} T \cdot \bar{u}$$

where $\langle T, u \rangle$ denotes the Hermitian scalar product in the duality between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

The proof of Theorem 2.1 is divided into 4 steps.

__Step 1:__ $A + \lambda$ is onto for $\lambda > \lambda_1$. Set $q_n^+ = \min(q^+, n)$; by a Theorem of Lax-Milgram there exists a unique function $u_n \in H^1_0(\Omega)$ which satisfies

(4)
$$-\Delta u_n + (q_n^+ - q^-)u_n + \lambda u_n = f \ .$$

---

[1]The use of this sort of lemma in this context was suggested by M. Crandall.

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(Note that by Lemma 2.1 the sesquilinear form $\int q^- u \bar{v}$ is continuous on $H_0^1(\Omega)$).
Multiplying (4) by $\bar{u}_n$ we find a constant C independent of n such that

(5)
$$\|u_n\|_{H^1} \leq C ,$$

(6)
$$\int q_n^+ |u_n|^2 \leq C .$$

Choose a subsequence denoted again by u_n such that $u_n \rightarrow u$ weakly in $H_0^1(\Omega)$ and $u_n \rightarrow u$ a.e. on Ω. It follows from Faton's Lemma and (6) that $q^+|u|^2 \epsilon L^1(\Omega)$. We deduce that $qu \epsilon L_{loc}^1(\Omega)$; indeed

$$q^+ |u| \leq \frac{1}{2} q^+ (|u|^2 + 1) \epsilon L_{loc}^1(\Omega) ,$$

$$q^- |u| \leq \frac{1}{2} q^- (|u|^2 + 1) \epsilon L_{loc}^1(\Omega) .$$

We pass now to the limit in (4) and prove that $-\Delta u + qu + \lambda u = f$ in $D'(\Omega)$. It suffices to show that

$$(q_n^+ - q^-)u_n \rightarrow qu \quad \text{in} \quad L_{loc}^1(\Omega) .$$

For this purpose we adapt a device due to W. Strauss [9] and extensively used in the study of strongly nonlinear equations. In view of Vitali's convergence theorem, it suffices to verify that given $\omega \subset\subset \Omega$, then $\forall \epsilon > 0, \exists \delta > 0$ such that $E \subset \omega$ and $|E| < \delta$ imply $\int_E |q_n^+ - q^-||u_n| < \epsilon$ for all n. But for every $R > 0$ we have

$$q_n^+ |u_n| \leq \frac{1}{2} q_n^+ (R + \frac{1}{R} |u_n|^2)$$

and thus, by (6),

$$\int_E q_n^+ |u_n| \leq \frac{1}{2} R \int_E q^+ + \frac{1}{2R} C .$$

We fix R large enough so that $\frac{C}{R} < \epsilon$ and then $\delta > 0$ so small that $R \int_E q^+ < \epsilon$. We proceed similarly with $q^- |u_n|$.

<u>Step 2</u>: $A + \lambda_1$ is accretive. Let $u \epsilon D(A)$ and set $T = qu$. Since $T \epsilon H^{-1}(\Omega) \cap L_{loc}^1(\Omega)$ and

$$\text{Re } T\bar{u} = q|u|^2 \geq -q^- |u|^2 \epsilon L^1(\Omega)$$

it follows from Lemma 2.2 that $q|u|^2 \epsilon L^1$ and

$$\text{Re}(T,u) = \int q|u|^2 .$$

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But $qu = Au + \Delta u$ and so

$$\text{Re}\langle Au, u\rangle - \int |\text{grad} u|^2 = \int q|u|^2 .$$

Since $Au \in L^2(\Omega)$ we have in fact

$$\text{Re}(Au, u) = \int |\text{grad} u|^2 + \int q|u|^2 \geq -\lambda_1 \int |u|^2$$

by Lemma 2.1.

<u>Step 3</u>: $u \in D(A)$ implies $q|u|^2 \in L^1(\Omega)$ and (2) holds. We have just seen in Step 2 that $u \in D(A)$ implies $q|u|^2 \in L^1(\Omega)$. Now let $u, v \in D(A)$ and set $T = qu$. We have $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ and

$$\text{Re } T \cdot \bar{v} = \text{Re } qu\bar{v} \geq -\frac{1}{2}|q||u|^2 - \frac{1}{2}|q||v|^2 \in L^1(\Omega)$$

and therefore

$$\text{Re}\langle T, v\rangle = \int \text{Re } qu\bar{v} .$$

Thus

$$\text{Re}(Au, v) - \text{Re} \int \text{grad} u \text{ grad} \bar{v} = \text{Re} \int qu\bar{v} .$$

Changing u into iu we find

$$(Au, v) = \int \text{grad} u \text{ grad} \bar{v} + \int qu\bar{v} .$$

<u>Step 4</u>: A is self-adjoint. Indeed $A + \lambda_1$ is m-accretive and symmetric. Therefore $A + \lambda_1$ is self-adjoint and so is A.

<u>Proof of Theorem 2.2</u>: Clearly $A \subset A_1$. Let $u \in D(A_1)$ and set $f = A_1 u + \lambda u$ with some $\lambda > \lambda_1$. Let $u^* \in D(A)$ be the unique solution of

$$Au^* + \lambda u^* = f .$$

We have

$$A_1(u - u^*) + \lambda(u - u^*) = 0 .$$

Since $(u - u^*) \in L^1_{loc}(\mathbb{R}^m)$ and $\Delta(u - u^*) \in L^1_{loc}(\mathbb{R}^m)$ we may apply Lemma A in [3] to conclude that

$$\Delta|u - u^*| \geq \text{Re}[\Delta(u - u^*) \text{ sign}(\bar{u} - \bar{u}^*)] \text{ in } D'(\mathbb{R}^m) ,$$

and thus in $D'(\mathbb{R}^m)$ we find,

$$\Delta|u - u^*| \geq \text{Re}[(q + \lambda)|u - u^*|] \geq (-q^- + \lambda)|u - u^*| .$$

Using the next lemma we conclude that $u = u^*$ (and hence $D(A_1) = D(A)$).

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<u>Lemma 2.3</u>: Assume (1) and (3). Let $v \in L^2(\mathbb{R}^m)$ be a real valued function with $q^- v \in L^1_{loc}(\mathbb{R}^m)$ satisfying

$$-\Delta v - q^- v + \lambda v \le 0 \quad \text{in} \quad D'(\mathbb{R}^m)$$

with some $\lambda > \lambda_1$. Then $v \le 0$ a.e. on $\mathbb{R}^m$.

The proof of Lemma 2.3 relies on the following crucial result. Since we shall need it in Section 3 for a general domain $\Omega \subset \mathbb{R}^m$ we work now again in Ω.

<u>Theorem 2.3</u>: Assume (1). Let $g \in L^2(\Omega) \cap L^\infty(\Omega)$ and let $\psi \in H^1_0(\Omega)$ be the unique solution of

(7) $$-\Delta\psi - q^- \psi + \lambda\psi = g \quad \text{in} \quad \Omega \quad (\lambda > \lambda_1) .$$

Then

a) $g \ge 0$ a.e. on Ω implies $\psi \ge 0$ a.e. on Ω;

b) $\psi \in \bigcap_{2 \le p < \infty} L^p(\Omega)$.

<u>Proof of Theorem 2.3</u>: a) Multiplying (7) by $-\psi^-$ we find

$$\int |\text{grad } \psi^-|^2 - \int q^- |\psi^-|^2 + \lambda \int |\psi^-|^2 \le 0$$

and thus $\psi^- = 0$.

b) We have to consider only the case $m \ge 3$ (when $m \le 2$, $\psi \in H^1_0(\Omega)$ implies $\psi \in \bigcap_{2 \le p < \infty} L^p(\Omega)$).

We can always assume that $g \ge 0$ a.e. on Ω so that $\psi \ge 0$ a.e. on Ω. We truncate q^- by $q_k^- = \min(q^-, k)$ and define ψ_k to be the unique solution of

$$\begin{cases} \psi_k \in H^1_0(\Omega) \\ -\Delta\psi_k - q_k^- \psi_k + \lambda\psi_k = g \quad \text{in} \quad \Omega . \end{cases}$$

It is clear that $\psi_k \to \psi$ weakly in $H^1_0(\Omega)$ as $k \to \infty$. We shall prove that for every $p \in [2, \infty)$, $\psi_k \in L^p(\Omega)$ and

(8) $$\|\psi_k\|_{L^p} \le C_p (\|g\|_{L^2} + \|g\|_{L^\infty}) ,$$

where C_p is independent of k, but it depends on q^- through the use of Lemma 2.1. For simplicity we drop now the subscript k on ψ_k and write

(9) $$-\Delta\psi - q_k^- \psi + \lambda\psi = g .$$

Set $\psi_n = \min(\psi, n)$ and let $2 \leq p < \infty$; since $(\psi_n)^{p-1} \in H_0^1(\Omega)$ we can multiply (9) by $(\psi_n)^{p-1}$ and we get

$$(p - 1) \int (\psi_n)^{p-2} |\text{grad } \psi_n|^2 \leq \int g(\psi_n)^{p-1} + \int q_k^-(\psi_n)^p + \int_{[\psi > n]} kn^{p-1} \psi ,$$

that is

$$\frac{4(p - 1)}{p^2} \int |\text{grad } \psi_n^{p/2}|^2 \leq \|g\|_{L^p} \|\psi_n\|_{L^p}^{p-1} + \int q^-(\psi_n)^p + kn^{p-1} \int_{[\psi > n]} \psi$$

$$\leq \|g\|_{L^p} \|\psi_n\|_{L^p}^{p-1} + \varepsilon \|\text{grad} \psi_n^{p/2}\|_{L^2}^2 + \lambda_\varepsilon \|\psi_n\|_{L^p}^p + k \int_{[\psi > n]} \psi^p$$

by Lemma 2.1 (here $\int_{[\psi > n]} \psi^p$ is possibly infinite). Choosing $\varepsilon > 0$ small enough (for example $\varepsilon = \frac{2(p - 1)}{p^2}$) we see that

$$\int |\text{grad } \psi_n^{p/2}|^2 \leq C_p [\|g\|_{L^p}^p + \|\psi\|_{L^p}^p + k \int_{[\psi > n]} \psi^p]$$

where C_p is independent of k and n. Using Sobolev's inequality we find

$$(10) \qquad \|\psi\|_{L^{p2^*/2}}^p \leq C_p \left[\|g\|_{L^p}^p + \|\psi\|_{L^p}^p + k \int_{[\psi > n]} \psi^p \right] .$$

<u>Assuming</u> now that $\psi \in L^p(\Omega)$ and passing to the limit in (10) as $n \to \infty$ we obtain that $\psi \in L^{p2^*/2}(\Omega)$ and

$$\|\psi\|_{L^{p2^*/2}} \leq C_p [\|g\|_{L^p} + \|\psi\|_{L^p}] .$$

Iterating this process from $p = 2$ we obtain finally for every $p \in [2, \infty)$

$$\|\psi\|_{L^p} \leq C_p [\|g\|_{L^2} + \|g\|_{L^\infty}] .$$

More precisely we have proved (8). The conclusion of Theorem 2.3 follows since $\psi_k \to \psi$ weakly in $H_0^1(\Omega)$ as $k \to \infty$.

<u>Proof of Lemma 2.3</u>: By assumption $q^- v \in L_{loc}^1(\mathbb{R}^m)$ and

$$\int v(-\Delta \varphi - q^- \varphi + \lambda \varphi) \leq 0 \quad \forall \varphi \in D_+(\mathbb{R}^m) .$$

An easy density argument (smoothing by convolution) shows that

$$(11) \quad \int v(-\Delta \varphi - q^- \varphi + \lambda \varphi) \leq 0 \quad \forall \varphi \in H^2(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m), \text{ supp } \varphi \text{ compact}, \varphi \geq 0 \text{ a.e. } .$$

Fix $g \in D_+(\mathbb{R}^m)$ and let $\psi_k \in H^1(\mathbb{R}^m)$ be the unique solution of

(12)
$$-\Delta\psi_k - q_k^-\psi_k + \lambda\psi_k = g \quad \text{in} \quad \mathbb{R}^m .$$

We know by Theorem 2.3 that $\psi_k \geq 0$ a.e.

$$\psi_k \in \bigcap_{2 \leq p < \infty} L^p(\mathbb{R}^m) \quad \text{with} \quad \|\psi_k\|_{L^p} \leq C_p ,$$

and also $\|\text{grad } \psi_k\|_{L^2} \leq C$. In addition we derive from (12) that

$$\psi_k \in H^2(\mathbb{R}^m) \cap L_{loc}^\infty(\mathbb{R}^m) .$$

Fix $\zeta \in D_+(\mathbb{R}^m)$ satisfying $\zeta(x) = 1$ for $|x| \leq 1$ and set $\zeta_n(x) = \zeta(\frac{x}{n})$. In (11) we choose $\varphi = \psi_k \zeta_n$. Note that by (12)

$$-\Delta\varphi - q^-\varphi + \lambda\varphi = \zeta_n g - (\Delta\zeta_n)\psi_k - 2\text{grad}\zeta_n \text{ grad}\psi_k - \zeta_n\psi_k(q^- - q_k^-) ,$$

and therefore

$$\int v\zeta_n g \leq \frac{C}{n^2} + \frac{C}{n} + \int v\zeta_n\psi_k(q^- - q_k^-) .$$

First we $\underline{\text{fix } n}$ and let $\underline{k \to \infty}$. We distinguish two cases:

a) $m \geq 5$,

b) $m < 5$.

a) When $m \geq 5$ we have $q^- - q_k^- \to 0$ in $L_{loc}^{m/2}(\mathbb{R}^m)$. Let $p \in [2,\infty)$ be such that $\frac{1}{2} + \frac{2}{m} + \frac{1}{p} = 1$; we have

$$\left|\int v\zeta_n\psi_k(q^- - q_k^-)\right| \leq \|v\|_{L^2}\|\psi_k\|_{L^p}\|\zeta_n(q^- - q_k^-)\|_{L^{m/2}} \to 0 .$$

Consequently

$$\int v\zeta_n g \leq \frac{C}{n^2} + \frac{C}{n} .$$

b) When $m < 5$ we use the assumption (3) (or (1)): $q^- \in L_{loc}^{m/2+\epsilon}(\mathbb{R}^m)$ with some $\epsilon > 0$. It follows from (12) that ψ_k remains bounded in $W_{loc}^{2,q}(\mathbb{R}^m)$ for some $q > \frac{m}{2}$ (when $m \geq 2$) as $k \to \infty$. We conclude that ψ_k remains bounded in $L_{loc}^\infty(\mathbb{R}^m)$ as $k \to \infty$ (in case $m = 1$, ψ_k is bounded in $L^\infty(\mathbb{R})$ since it is bounded in $H^1(\mathbb{R})$). Therefore

$$\int v\zeta_n\psi_k(q^- - q_k^-) \to 0 \quad \text{as} \quad k \to \infty$$

since $\|\zeta_n v(q^- - q_k^-)\|_{L^1} \to 0$ by the dominated convergence theorem (recall that

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$q^- v \in L^1_{loc}(\mathbf{R}^m))$. In both cases we find

$$\int v\zeta_n g \leq \frac{C}{n^2} + \frac{C}{n} \quad \forall n .$$

As $n \to \infty$ we see that

$$\int vg \leq 0 \quad \forall g \in D_+(\mathbf{R}^m)$$

and therefore $v \leq 0$ a.e. on $\mathbf{R}^m$.

Remark 2.2: The conclusion of Lemma 2.3 fails in $\mathbf{R}^3$ and in $\mathbf{R}^4$ if we do not assume (3). Ancona (personal communication) has constructed in $\mathbf{R}^3$ and in $\mathbf{R}^4$ functions $q^- \in L^{m/2}(\mathbf{R}^m)$ and $u \in L^{m/m-2}(\mathbf{R}^m) \cap L^2(\mathbf{R}^m)$ such that

$$-\Delta u - q^- u + u = 0 \quad \text{in } D'$$

with $\|q^-\|_{L^{m/2}}$ as small as we please and $u \neq 0$.

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3. Complex potentials

Let Ω be an (arbitrary) open subset of $\mathbb{R}^m$. Assume $q(x)$ and $q'(x)$ are real valued functions such that $q, q' \in L^1_{loc}(\Omega)$ and set

$$V(x) = q(x) + iq'(x).$$

We assume

(13) $\qquad \underline{\text{either}}\ q' \in L^{1+\varepsilon}_{loc}(\Omega)\ \underline{\text{or}}\ q^- \in L^{(m/2)+\varepsilon}_{loc}(\Omega) \qquad$ when $m \geq 2$,

for some arbitrarily small $\varepsilon > 0$. Define

$$A = -\Delta + V(x)$$

with

$$D(A) = \{u \in H^1_0(\Omega);\ Vu \in L^1_{loc}(\Omega)\ \text{and}\ -\Delta u + Vu \in L^2(\Omega)\}\ .$$

The main results are the following

<u>Theorem 3.1</u>: Assume (1) and (13). Then A is closable in $L^2(\Omega)$ and $\bar{A} + \lambda_1$ is m-accretive. In addition $u \in D(\bar{A})$ implies that $u \in H^1_0(\Omega)$, $q|u|^2 \in L^1(\Omega)$ and

(14) $\qquad\qquad\qquad \text{Re}(\bar{A}u, u) = \int |\text{grad} u|^2 + \int q|u|^2\ .$

<u>Remark 3.1</u>: In case we assume

(15) $\qquad\qquad\qquad |q'(x)| \leq Mq^+(x) + h(x)\ $ for a.e. $x \in \Omega$

with $h \in L^{2m/(m+2)}_{loc}(\Omega)$ and $m \geq 3$ then A is closed in $L^2(\Omega)$. (Note that (15) corresponds essentially with the assumption made in [5]). Indeed let $u_n \in D(A)$ be such that $u_n \to u$ in $L^2(\Omega)$ and $Au_n \to f$ in $L^2(\Omega)$. It follows from Lemma 2.1 and (14) that $u_n \to u$ in $H^1_0(\Omega)$ and $\sqrt{q^+}u_n \to \sqrt{q^+}u$ in $L^2(\Omega)$. From (15) we deduce easily that $Vu \in L^1_{loc}(\Omega)$ and that $-\Delta u + Vu = f$ in $D'(\Omega)$. Therefore $u \in D(A)$ and $Au = f$.

When $\Omega = \mathbb{R}^m$ we consider also the operator A_1 defined in $L^2(\mathbb{R}^m)$ by

$$A_1 = -\Delta + V(x)$$

with

$$D(A_1) = \{u \in L^2(\mathbb{R}^m);\ Vu \in L^1_{loc}(\mathbb{R}^m)\ \text{and}\ -\Delta u + Vu \in L^2(\mathbb{R}^m)\}\ .$$

<u>Theorem 3.2</u>: Assume (1), (3) and (13). Then A_1 is closable and $\overline{A_1} = \bar{A}$.

In the proof of Theorem 3.1 we shall use the following

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Lemma 3.1: Let $v \in H_0^1(\Omega)$ be a real valued function. Assume (1) and

$$-\Delta v - q^- v + \lambda v \leq 0 \quad \text{in} \quad D'(\Omega)$$

with $\lambda > \lambda_1$. Then $v \leq 0$ a.e. on Ω.

Proof of Lemma 3.1: We have, for every $\varphi \in D_+(\Omega)$

$$\int \text{grad} v \, \text{grad} \varphi - \int q^- v \varphi + \lambda \int v \varphi \leq 0 .$$

Now we use the fact (pointed out by G. Stampacchia) that $D_+(\Omega)$ is dense in $\{u \in H_0^1(\Omega); \ u \geq 0 \ \text{a.e. on } \Omega\}$ for the H^1 norm [1] to derive that

$$\int \text{grad} v \, \text{grad} \varphi - \int q^- v \varphi + \lambda \int v \varphi \leq 0 \quad \forall \varphi \in H_0^1(\Omega), \ \varphi \geq 0 .$$

Choosing $\varphi = v^+$ we obtain

$$\int |\text{grad} v^+|^2 - \int q^- |v^+|^2 + \lambda \int |v^+|^2 \leq 0$$

and therefore $v^+ = 0$.

The proof of Theorem 3.1 is divided into five steps.

Step 1: $R(A + \lambda) \supset L^2(\Omega) \cap L^\infty(\Omega)$ for $\lambda > \lambda_1$.

Indeed let $f \in L^2(\Omega) \cap L^\infty(\Omega)$ and let $u_n \in H_0^1(\Omega)$ be the unique solution of

$$(16) \qquad -\Delta u_n + V_n u_n + \lambda u_n = f$$

where $V_n = q_n^+ - q^- + i q_n'$ and

$$q_n' = \begin{cases} n & \text{if} \quad q' > n \\ q' & \text{if} \quad |q'| \leq n \\ -n & \text{if} \quad q' \leq -n . \end{cases}$$

The existence of u_n follows from a theorem of Lax-Milgram. Multiplying (16) by $\overline{u_n}$ we find

$$(17) \qquad \|u_n\|_{H^1} \leq C$$

$$(18) \qquad \int q_n^+ |u_n|^2 \leq C .$$

---

[1] Indeed let $u \in H_0^1(\Omega)$ with $u \geq 0$ a.e. on Ω; let $u_n \in D(\Omega)$ be such that $u_n \to u$ in $H^1(\Omega)$. We claim that $|u_n| \to |u| = u$ in $H^1(\Omega)$ because $\|\,|u_n|\,\|_{H^1} = \|u_n\|_{H^1}$ and $|u_n| \to |u|$ weakly in $H^1(\Omega)$. On the other hand $|u_n|$ can be smoothed by convolution and for fixed n, $\rho_\varepsilon * |u_n| \to |u_n|$ in $H^1(\Omega)$ as $\varepsilon \to 0$.

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On the other hand we have

$$\Delta |u_n| \geq \mathrm{Re}[\Delta u_n \ \mathrm{sign} \ \overline{u}_n] \quad \text{in} \quad D'(\Omega)$$

which leads to

$$-\Delta |u_n| - q^- |u_n| + \lambda |u_n| \leq |f| \quad \text{in} \quad D'(\Omega) \ .$$

Let $\psi \in H_0^1(\Omega)$ be the solution of

(19)
$$-\Delta \psi - q^- \psi + \lambda \psi = |f| \ .$$

It follows from Lemma 3.1 that

(20)
$$|u_n| \leq \psi \quad \text{a.e. on} \quad \Omega \ .$$

By Theorem 2.3 we know that $\psi \in L^p(\Omega)$ for every $p \in [2,\infty)$. We extract a subsequence, denoted again by u_n such that $u_n \to u$ weakly in $H_0^1(\Omega)$, $u_n \to u$ a.e. on Ω. We see as in the proof of Theorem 2.1 (Step 1) that $(q_n^+ - q^-)u_n \to qu$ in $L_{loc}^1(\Omega)$. Therefore we have only to verify that $q_n' u_n \to q'u$ in $L_{loc}^1(\Omega)$. We distinguish two cases:

a) $q' \in L_{loc}^{1+\varepsilon}(\Omega)$,

b) $q^- \in L_{loc}^{(m/2)+\varepsilon}(\Omega)$.

<u>Case a)</u> From (20) we deduce that $u_n \to u$ in every L^p space, $2 \leq p < \infty$ and so $q_n' u_n \to q'u$ in $L_{loc}^1(\Omega)$.

<u>Case b)</u> Since $q^- \psi \in L_{loc}^q(\Omega)$ for some $q > \frac{m}{2}$, it follows from (19) that $\psi \in L_{loc}^\infty(\Omega)$. We deduce from the dominated convergence theorem that $q_n' u_n \to q'u$ in $L_{loc}^1(\Omega)$.

<u>Step 2</u>: $A + \lambda_1$ is accretive. Let $u \in D(A)$ and set $T = Vu$. We have $T \in H^{-1}(\Omega) \cap L_{loc}^1(\Omega)$ and

$$\mathrm{Re} \ T \cdot \overline{u} = q|u|^2 \geq -q^- |u|^2 \in L^1(\Omega) \ .$$

It follows from Lemma 2.2 that $q|u|^2 \in L^1(\Omega)$ and

$$\int q|u|^2 = \mathrm{Re} \langle T,u \rangle = \mathrm{Re} \langle Au + \Delta u, u \rangle \ .$$

Therefore

(21)
$$\mathrm{Re}(Au,u) = \int |\mathrm{grad} u|^2 + \int q|u|^2 \geq -\lambda_1 \int |u|^2 \ .$$

<u>Step 3</u>: $D(A)$ is dense in $L^2(\Omega)$. Given $f \in L^2(\Omega) \cap L^\infty(\Omega)$ we solve for large n the equation

(22)
$$u_n + \frac{1}{n} Au_n = f.$$

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We shall prove that $u_n \to f$ in $L^2(\Omega)$ as $n \to \infty$ — and as a consequence $D(A)$ is dense in $L^2(\Omega)$. By (21) we have

$$\int |u_n|^2 + \frac{1}{n} \int |\mathrm{grad}\, u_n|^2 + \frac{1}{n} \int q |u_n|^2 = \mathrm{Re}(f, u_n) \ .$$

In particular we deduce that

$$(23) \qquad \limsup_{n \to \infty} \|u_n\|_{L^2} \leq \|f\|_{L^2}$$

$$(24) \qquad \frac{1}{n} \int q^+ |u_n|^2 \leq C$$

$$(25) \qquad \frac{1}{n} \int |\mathrm{grad}\, u_n|^2 \leq C \ .$$

Next we have (as in the proof of Step 1)

$$|u_n| - \frac{1}{n} \Delta |u_n| - \frac{1}{n} q^- |u_n| \leq |f| \quad \text{in } D'(\Omega) \ .$$

On the other hand let $\psi \in H_0^1(\Omega)$ be the solution of

$$-\Delta \psi - q^- \psi + \lambda \psi = |f|$$

for some fixed $\lambda > \lambda_1$. Since $|u_n| \geq \lambda \left| \frac{u_n}{n} \right|$ for $n \geq \lambda$, we deduce from Lemma 3.1 that $\left| \frac{u_n}{n} \right| \leq \psi$ a.e. Choose a subsequence, denoted again by u_n such that $u_n \to u$ weakly in $L^2(\Omega)$, $\frac{1}{n} u_n \to 0$ a.e. (this is possible since $\frac{1}{n} u_n \to 0$ in $L^2(\Omega)$). For every $\varphi \in D(\Omega)$ we have

$$(26) \qquad \int u_n \bar{\varphi} - \frac{1}{n} \int u_n \Delta \bar{\varphi} + \frac{1}{n} \int V u_n \bar{\varphi} = \int f \bar{\varphi} \ .$$

We claim that $\frac{1}{n} \int V u_n \bar{\varphi} \to 0$ as $n \to \infty$. Indeed by (24) and (25) we have

$$\frac{1}{n} \left| \int q^+ u_n \bar{\varphi} \right| \leq \frac{C}{\sqrt{n}} \quad \text{and} \quad \frac{1}{n} \left| \int q^- u_n \bar{\varphi} \right| \leq \frac{C}{\sqrt{n}} \ .$$

Thus we have only to verify that $\frac{1}{n} \int q' u_n \bar{\varphi} \to 0$. We distinguish two cases:

a) if $q^- \in L_{loc}^{(m/2)+\epsilon}(\Omega)$, we have $\psi \in L_{loc}^\infty(\Omega)$ and we deduce from the dominated convergence theorem that $\frac{1}{n} \int q' u_n \bar{\varphi} \to 0$;

b) if $q' \in L_{loc}^{1+\epsilon}(\Omega)$ we use the fact that $\left| \frac{u_n}{n} \right| \leq \psi \in L^p(\Omega)$ for every $2 \leq p < \infty$ to deduce that $\frac{u_n}{n} \to 0$ in $L^p(\Omega)$ and so $\frac{1}{n} \int q' u_n \bar{\varphi} \to 0$.

In all the cases, we derive from (26) that

$$\int u\bar{\varphi} = \int f\bar{\varphi} \qquad \forall \varphi \in D$$

and consequently $u = f$. We conclude using (23) that $u_n \to f$ in $L^2(\Omega)$.

<u>Step 4</u>: A is closable and $\bar{A} + \lambda_1$ is m-accretive. This is a standard fact, see e.g. Theorem 3.4 in [2].

<u>Step 5</u>: $u \in D(\bar{A})$ implies that $u \in H_0^1(\Omega)$, $q|u|^2 \in L^1(\Omega)$ and (14).

We already know (Step 2) that $v \in D(A)$ implies $q|v|^2 \in L^1(\Omega)$ and

$$(27) \qquad \qquad \operatorname{Re}(Av,v) = \int |\operatorname{grad}v|^2 + \int q|v|^2 .$$

Now let $u \in D(\bar{A})$ and let $u_n \in D(A)$ be such that $u_n \to u$, $Au_n \to \bar{A}u$. It follows from (27) applied to $v = u_n - u_m$ that $u_n \to u$ in $H_0^1(\Omega)$ and $\int q^+ |u_n - u|^2 \to 0$ (since u_n is a Cauchy sequence in $H_0^1(\Omega)$ and in $L^2(\Omega)$ with weight q^+). In particular $q|u|^2 \in L^1(\Omega)$ and (14) holds.

<u>Proof of Theorem 3.2</u>: Clearly $A \subset A_1$. Now let $u \in D(A_1)$ and let $\lambda > \lambda_1$. Set $f = A_1 u + \lambda u$, and let u^* be the unique solution of

$$\bar{A}u^* + \lambda u^* = f .$$

Thus, there exists a sequence $u_n^* \to u^*$ in $L^2(\mathbb{R}^m)$ with $u_n^* \in D(A)$ and

$$Au_n^* + \lambda u_n^* = f_n \to f \quad \text{in} \quad L^2(\mathbb{R}^m) .$$

In particular we have

$$A_1(u_n^* - u) + \lambda(u_n^* - u) = f_n - f$$

and therefore

$$-\Delta|u_n^* - u| - q^-|u_n^* - u| + \lambda|u_n - u| \leq |f_n - f| \quad \text{in} \quad D'(\mathbb{R}^m) .$$

We deduce from Lemma 2.3 that $|u_n^* - u| \leq \psi_n$ a.e. on $\mathbb{R}^m$ where $\psi_n \in H^1(\mathbb{R}^m)$ is the solution of

$$-\Delta\psi_n - q^-\psi_n + \lambda\psi_n = |f_n - f| .$$

Hence $\|\psi_n\|_{H^1} \to 0$ and in particular $u_n^* - u \to 0$ in $L^2(\mathbb{R}^m)$. It follows that $u^* = u$, that is $A_1 \subset \bar{A}$. We have $A \subset A_1 \subset \bar{A}$ and therefore A_1 is closable with $\overline{A_1} = \bar{A}$.

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Let $A = -\Delta + V(x)$ be a Schrödinger operator on an (arbitrary) open set $\Omega \subset \mathbb{R}^m$, where $V \in L^1_{loc}(\Omega)$ is a complex valued function. We consider the "maximal" realization of A in $L^2(\Omega)$ under Dirichlet boundary condition, that is

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20. ABSTRACT - cont'd.

$$D(A) = \{u \in H_0^1(\Omega); \; Vu \in L_{loc}^1(\Omega) \quad \text{and} \quad -\Delta u + Vu \in L^2(\Omega)\} \; .$$

When $\Omega = \mathbb{R}^m$ we also consider the operator

$$A_1 = -\Delta + V$$

with domain

$$D(A_1) = \{u \in L^2(\Omega); \; Vu \in L_{loc}^1(\Omega) \quad \text{and} \quad -\Delta u + Vu \in L^2(\Omega)\} \; .$$

A special case of our main results is:

<u>Theorem</u>: Let $m \geq 3$; assume that the function $\max\{-\text{Re } V, 0\}$ belongs to $L^\infty(\Omega) + L^{m/2}(\Omega)$ and also to $L_{loc}^{(m/2)+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. Then A (resp. A_1) is closable and $\bar{A} + \lambda$ (resp. $\bar{A}_1 + \lambda$) is m-accretive for some real constant λ.