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A NOTE ON THE APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES B--ETC(U)

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OF FUNCTIONS OF ONE VARIABLE

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A NOTE ON THE APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES

BY SUMS OF FUNCTIONS OF ONE VARIABLE

C. T. Kelley

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ABSTRACT

For a class of functions of several variables, which contains the continuous functions, we show that there exists a sum of functions of one variable that minimizes the distance from the given function to the space of such sums. For functions of two variables we show that such a minimizing sum may be constructed by an iterative scheme.

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SIGNIFICANCE AND EXPLANATION

Often it is desirable to approximate a given function as closely as possible by a member of a class of functions that are, in some sense, simpler than the original function.

In this paper we consider the approximation of a function of several variables by sums of functions of one variable relative to the supremum norm. It is not obvious that a best such approximation exists. We prove that such an approximation does, in fact, exist if the domain of the function to be approximated is a rectangle in a generalized sense, and if the function is in a certain class which includes the continuous functions. Also, if we consider functions of two variables, we show that a best such approximation may be found by an iterative method.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A NOTE ON THE APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES

BY SUMS OF FUNCTIONS OF ONE VARIABLE

C. T. Kelley

I. Introduction

Let $\{\Omega_i\}_{i=1}^m$ be compact subsets of the reals. Let $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$. Let $L_\infty(\Omega)$ be the Banach space of essentially bounded real-valued functions on Ω with the supremum norm. Let $S(\Omega)$ denote the closed subspace of $L_\infty(\Omega)$ consisting of sums of the form $\sum_{i=1}^m f_i$, with $f_i \in L_\infty(\Omega_i)$.

Let $K(\Omega)$ be the closure of all finite sums of the form $\sum_{k=1}^M \alpha_k \sum_{j=1}^m \varphi_{kj}$, where $\varphi_{kj} \in L_\infty(\Omega_j)$ for $1 \leq j \leq m$. $K(\Omega)$ is a closed subalgebra of $L_\infty(\Omega)$; $K(\Omega)$ contains the continuous functions on Ω ; $S(\Omega) \subset K(\Omega)$. In fact, $K(\Omega)$ is the smallest subalgebra of $L_\infty(\Omega)$ containing $S(\Omega)$. For $k \in L_\infty(\Omega)$ we define a functional $\mu(k)$ by

$$(1.1) \quad \mu(k) = \inf_{f \in S(\Omega)} \|k - f\|.$$

In [1], Diliberto and Straus considered the problem of finding a sequence $\{f_n\} \subset S(\Omega)$ so that $\lim_{n \rightarrow \infty} \|k - f_n\| = \mu(k)$. They were able to do this and for continuous k , their sequence possessed a convergent subsequence. Hence, for continuous k , the infimum in (1.1) is attained. The purpose of this note is to show that the infimum in (1.1) is attained for all $k \in K(\Omega)$, and that for $m = 2$, the iteration scheme of Diliberto and Straus converges for all $k \in K(\Omega)$. These results partially answer questions raised in [1].

II.

For $k \in L_\infty(\Omega)$ and $1 \leq j \leq m$ define $H_j(k) \in L_\infty(\Omega_j)$ by,

$$(2.1) \quad H_j(k)(x_j) = \frac{1}{2} \left(\operatorname{ess\,sup}_{\substack{x_i \in \Omega_i \\ i \neq j}} k(x) + \operatorname{ess\,inf}_{\substack{x_i \in \Omega_i \\ i \neq j}} k(x) \right), \quad \text{a.e. } x_j.$$

The sequence $\{f_n\}$ of Diliberto and Straus is defined as follows. Let $\{k_n\}$ be given by

$$(2.1) \quad \begin{aligned} k_0 &= k, \\ k_1 &= k - H_1(k), \\ k_2 &= k - H_1(k) - H_2(k - H_1(k)), \\ &\vdots \\ k_{mp+r} &= k_{mp+r-1} - H_r(k_{mp+r-1}) \quad \text{for } 1 \leq r \leq m. \end{aligned}$$

We define f_n by:

$$(2.2) \quad f_n = k - k_n.$$

The following theorem is due to Diliberto and Straus.

Theorem (2.1). For $k \in L_\infty(\Omega)$, let k_n be given by (2.1). Then

$$\lim_{n \rightarrow \infty} \|k_n\| = \lim_{n \rightarrow \infty} \|k - f_n\| = \mu(k).$$

Moreover, for $n \geq 1$, $\|k_n\| \leq \|k_{n-1}\|$, and hence

$$\|f_n\| \leq 2\|k\|.$$

We list some obvious properties of the functions $H_i(k)$ in the following lemma.

Lemma (2.1). Let $k \in L_\infty(\Omega)$, let $\varphi_i \in L_\infty(\Omega_i)$, and let $\{E_r^i\}_{r=1}^R$ be finitely many disjoint measurable sets in Ω_i such that $\Omega_i = \bigcup_{r=1}^R E_r^i$. Let $\chi_{E_r^i}$ denote the characteristic function of E_r^i . Let $\{\alpha_{rs}\}_{s,r=1}^R$ be real, and let $\{p_s\}_{s=0}^R \subset L_\infty(\Omega)$ be independent of x_i . Then,

- (a) $H_i(k + \varphi_i) = H_i(k) + \varphi_i$,
- (b) $\|k - H_i(k)\| \leq \|k\|$,
- (c) $H_i(\varphi_i, k) = \varphi_i H_i(k)$,
- (d) $H_i\left(\sum_{r=1}^R \sum_{s=1}^R \alpha_{rs} X_{E_r}^i P_s - P_0\right) = \sum_{r=1}^R X_{E_r}^i H_i\left(\sum_{s=1}^R \alpha_{rs} P_s - P_0\right)$.

For $f \in S(\Omega)$, we may write $f = \sum_{i=1}^m \varphi_i$, with $\varphi_i \in L_\infty(\Omega_i)$. This representation of f is unique in the sense that if $f = \sum_{i=1}^m \varphi_i = \sum_{i=1}^m \psi_i$, then there are constants δ_i , so that $\sum_{i=1}^m \delta_i = 0$, and $\psi_i + \delta_i = \varphi_i$. Now let $f \in S(\Omega)$ and let $f = \sum_{i=1}^m \varphi_i$; $\varphi_i \in L_\infty(\Omega_i)$. For $k \in L_\infty(\Omega)$ define $Q_k(f) = \sum_{i=1}^m \psi_i$, where $\psi_i \in L_\infty(\Omega_i)$ is given by

$$(2.3) \quad \psi_i = H_i\left(k - \sum_{j=1}^{i-1} \psi_j - \sum_{j=i+1}^m \varphi_j\right).$$

Note that each individual ψ_i depends on the representation $f = \sum_{i=1}^m \varphi_i$. However, $Q_k(f)$ does not depend on the representation of f . Indeed if $f = \sum_{i=1}^m \varphi_i + \delta_i$, where the δ_i 's are constants such that $\sum \delta_i = 0$, let $\hat{\psi}_i$ be defined by (2.3) with φ_i replaced by $\varphi_i + \delta_i$. We have $\hat{\psi}_1 = H_1\left(k - \sum_{i=2}^m (\varphi_i + \delta_i)\right) = H_1\left(k - \sum_{i=2}^m \varphi_i\right) = \sum_{i=2}^m \delta_i = \psi_1 + \delta_1$. Hence $\hat{\psi}_2 = \psi_2 - \delta_1 - \sum_{i=3}^m \delta_i = \psi_2 + \delta_2$. Continuing in this way we obtain $\hat{\psi}_i = \psi_i + \delta_i$, for all i , and hence $\sum_{i=1}^m \hat{\psi}_i = \sum_{i=1}^m \psi_i$. Note that Q is continuous as a map on $S(\Omega)$.

For $k \in L_\infty(\Omega)$ let f_n be given by (2.2), then for $p \geq 1$

$$(2.4) \quad f_{np} = Q_k^p(0).$$

Also by Theorem (2.1) for any $k \in L_\infty(\Omega)$ and $f \in S(\Omega)$ we have $\|k - Q_k(f)\| \leq \|k - f\|$. Hence for each k , Q_k maps bounded sets in $S(\Omega)$ to bounded sets in $S(\Omega)$.

Theorem (2.2). Let $k \in K(\Omega)$. Q_k is a compact map on $S(\Omega)$. Hence $\{Q_k^p(0)\}_{p=1}^\infty$ has a convergent subsequence and the infimum in (1.1) is attained.

Proof. We give the proof for $m = 2$. The proof for arbitrary m is similar.

Note first that for $j = 1, 2$ and $k_1, k_2 \in L_\infty(\Omega)$, we have

$$(2.5) \quad \max_{x_j} |H_j(k_1)(x_j) - H_j(k_2)(x_j)| \leq \|k_1 - k_2\|.$$

If $f(x_1, x_2) = \varphi_1(x_1) + \varphi_2(x_2)$, then

$$(2.6) \quad Q_k(f) = H_1(k - \varphi_2) + H_2(k - H_1(k - \varphi_2)).$$

Hence, for any $f \in S(\Omega)$, $k_1, k_2 \in L_\infty(\Omega)$,

$$(2.7) \quad \|Q_{k_1}(f) - Q_{k_2}(f)\| \leq 3\|k_1 - k_2\|.$$

Let $\varepsilon > 0$. As $k \in K(\Omega)$, we may find finitely many disjoint measurable sets $\{E_r^i\}_{r=1}^R$ in Ω_i such that $\bigcup_{r=1}^R E_r^i = \Omega_i$, and real numbers $\{a_{rs}\}_{r,s=1}^R$, so that

$$(2.8) \quad \|k - \sum_{r,s=1}^R a_{rs} \chi_{E_r^1} \chi_{E_s^2}\| < \varepsilon/3.$$

Now let $\hat{k} = \sum_{r,s=1}^R a_{rs} \chi_{E_r^1} \chi_{E_s^2}$. For $f = \varphi_1 + \varphi_2 \in S(\Omega)$, we have, by Lemma (2.1), that

$$Q_{\hat{k}}(f) = \sum_{r=1}^R \chi_{E_r^1} H_1 \left(\sum_{s=1}^R a_{rs} \chi_{E_s^2} - \varphi_2 \right) + \sum_{s=1}^R \chi_{E_s^2} H_2 \left(\sum_{r=1}^R a_{rs} \chi_{E_r^1} - H_1(\hat{k} - \varphi_2) \right).$$

As $\chi_{E_s^2}$ and φ_2 are independent of x_1 , $H_1 \left(\sum_{s=1}^R a_{rs} \chi_{E_s^2} - \varphi_2 \right)$ is constant for each r .

Similarly $H_2 \left(\sum_{r=1}^R a_{rs} \chi_{E_r^1} - H_1(\hat{k} - \varphi_2) \right)$ is constant. Hence $Q_{\hat{k}}$ has finite dimensional range.

We apply (2.7) twice to obtain

$$(2.9) \quad \|Q_k(f) - Q_{\hat{k}}(f)\| \leq 3\|k - \hat{k}\| < \varepsilon.$$

Hence Q_k is the uniform limit of maps on $S(\Omega)$ which have finite dimensional range. This completes the proof.

We note that Theorem (2.2) is in a sense a converse of a theorem in [2]. Golomb showed, in a general Banach space setting, that if one assumes that the minimum in (1.1) is attained, then Theorem (2.1) holds.

The reader should note that if k is continuous on Ω , so is $Q^p(0)$ for each $p \geq 1$. Hence if k is continuous the infimum in (1.1) is attained at a continuous $f \in S(\Omega)$.

In order to prove the final result we require the following theorem of Diliberto and Straus.

Theorem (2.3). Let $m = 2$, $k \in K(\Omega)$. Let k_n be given by (2.1), and let k_* be any limit point of the sequence $\{k_n\}_{n=1}^\infty$. We have

$$H_1(k)(x_1) = 0, \quad H_2(k)(x_2) = 0 \quad \text{for a.e. } (x_1, x_2).$$

Theorem (2.4). The sequence $\{k_n\}$, and hence the sequence $\{f_n\}$, converges in $L_\infty(\Omega)$.

Proof. Let k_* be any limit point of the sequence $\{k_{2n}\}_{n=1}^\infty$. Write $k = k_* + \varphi_1 + \varphi_2$ with $\varphi_i \in L_\infty(\Omega_i)$. Then for $n \geq 2$ we have $k_{2n} = k_* + \varphi_1^{(n)} + \varphi_2^{(n)}$, and

$$(2.10) \quad \begin{aligned} \varphi_1^{(n)} &= -H_1(k + \varphi_2^{(n-1)}), \\ \varphi_2^{(n)} &= -H_2(k + \varphi_1^{(n)}). \end{aligned}$$

As $H_1(k) = H_2(k) = 0$ a.e. we have, for $n \geq 2$

$$(2.11) \quad \begin{aligned} -\text{ess inf}_{x_2 \in \Omega_2} \varphi_2^{(n)}(x_2) &\leq \text{ess sup}_{x_1 \in \Omega_1} \varphi_1^{(n)}(x_1) \leq -\text{ess inf}_{x_2 \in \Omega_2} \varphi_2^{(n-1)}(x_2), \\ \text{ess sup}_{x_1 \in \Omega_1} \varphi_2^{(n)}(x_2) &\leq -\text{ess inf}_{x_1 \in \Omega_1} \varphi_1^{(n)}(x_1) \leq \text{ess sup}_{x_2 \in \Omega_2} \varphi_2^{(n-1)}(x_2). \end{aligned}$$

Now there is, by assumption, a subsequence $\{k_{2n_j}\}_{j=1}^\infty$ of $\{k_{2n}\}_{n=1}^\infty$ which converges uniformly to k_* . This means $\lim_{j \rightarrow \infty} \|\varphi_1^{(n_j)} + \varphi_2^{(n_j)}\| = 0$. This in turn implies that there is a real number c , so that

$$(2.12) \quad \lim_{j \rightarrow \infty} \varphi_1^{(n_j)} = c = -\lim_{j \rightarrow \infty} \varphi_2^{(n_j)}.$$

Now choose $\epsilon > 0$. There is j_0 so that $j \geq j_0$ implies that

$$(2.13) \quad \|\varphi_1^{(n_j)} - c\| < \epsilon/2, \quad \|\varphi_2^{(n_j)} + c\| < \epsilon/2.$$

But (2.11) implies that, for all $k \geq 0$, and a.e. (x_1, x_2) ,

$$(2.14) \quad c - \epsilon/2 \leq \varphi_1^{(n_{j_0}+k)}(x_1) \leq c + \epsilon/2,$$

$$-c - \epsilon/2 \leq \varphi_2^{(n_{j_0}+k)}(x_2) \leq -c + \epsilon/2.$$

Hence for all $n \geq n_{j_0}$, we have

$$(2.15) \quad \|\varphi_1^{(n)} + \varphi_2^{(n)}\| \leq \epsilon.$$

Hence $\lim_{n \rightarrow \infty} k_{2n} = k_*$. As $\lim_{n \rightarrow \infty} \|k_{2n} - k_{2n+1}\| = \lim_{n \rightarrow \infty} \|H_1(k_{2n})\| = 0$, $\lim_{n \rightarrow \infty} k_{2n+1} = k_*$. This completes the proof.

We note that these results generalize directly to the case where each Ω_i is a compact Hausdorff space endowed with a positive, regular Borel measure μ_i , and Ω is given the measure $\mu_1 \times \mu_2 \times \dots \times \mu_n$. The functions k , $H_i(k)$, and f may be allowed to have values in $\mathbb{R}^n$ if supremums and infimums are understood to be taken componentwise.

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- 2) M. Golomb, Approximation by functions of fewer variables. "On Numerical Approximation", University of Wisconsin Press, Madison, WI, 1959, 275-327.

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