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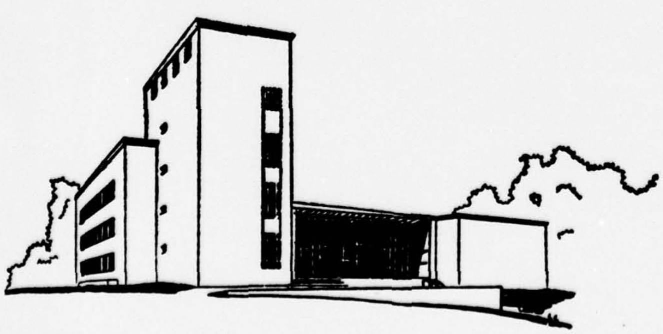

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(6) COMPUTING THE CORE OF A MARKET GAME

by

(10) Gerald L. Thompson

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COMPUTING THE CORE OF A MARKET GAME

by

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ABSTRACT

The assignment market game was defined by Shapley in 1955 and received a very full treatment by Shubik and Shapley in 1972. The present paper contains the following extensions: (a) the assignment game is generalized to a market game; (b) the two distinguished core points found for the assignment game are also shown to exist for the market game; (c) in the non dual degenerate case it is shown that the skeletons of the buyer and seller cores are isomorphic k-graphs; and (d) an algorithm is presented for computing skeletons of the buyer and seller cores of a market game.

The results are illustrated with examples. At the end some remarks are made on the limiting sizes of cores.

Key Words

Market game

Core of a market game

Primal degenerate transportation problems

Dual degenerate transportation problems

Mathematical economics

Computing the Core of a Market Game

by

Gerald L. Thompson

1. INTRODUCTION

The assignment market game was introduced by L. S. Shapley in 1955 and later received a very full treatment by Shapley and M. Shubik [7] in 1972. The present author became interested in these games and their generalizations while writing a paper on auctions [11].

The basic observation in [7] was that the extreme points of the core of an assignment game can be found by computing all basic solutions to the dual of an assignment problem. In the present paper it is shown that the core of a market game can likewise be found by computing all basic solutions to the dual of a transportation problem. In each case it was found that the size of the core was <u>increased</u> by primal degeneracy and <u>decreased</u> by dual degeneracy of the corresponding assignment or transportation problem. We are thus led back to the degeneracy questions in such linear programming models which occupied Charnes and Cooper [2], Dantzig [4] and Orden [6] in the early days of linear programming.

Shapley and Shubik [7] show that the core of an assignment game has two distinguished points. We extend that result to market games, and show in Section 4 that the degeneracy prevention technique of Orden [6] can be used to easily compute these two points. In Section 4 it is also shown that, if the transportation problem associated with the market game is not dual degenerate, and the basic solution of the primal transportation problem has $k + 1$ basic cells on which there is zero shipment, then every extreme point of the buyer or seller core has exactly k neighbors. If we let the skeleton of the core be the graph consisting of the extreme points and edges of the core then we can reformulate that result simply as: the skeleton of the buyer or seller core is a k-graph (i.e. every vertex has degree k).

In Section 5 an algorithm is given for computing the skeletons of the buyer and seller cores. As a by-product of the algorithm we also prove: the skeletons of the buyer and seller cores are isomorphic.

Some numerical examples of the application of the algorithm are given in Section 6 together with some remarks on the variation in the size of the core as the number of players increases. Although some preliminary observations are made it is clear that the full answer to the latter question requires another paper.

The existence of current fast codes [1, 5, 9] for solving transportation problems makes finding a single solution to a very large market game easy. However the possible existence of a huge number of extreme basic dual solutions makes it unlikely that anyone will compute all the extreme points of the core of a market game having many players except in the case in which the problem is highly dual degenerate. In Section 6 it is suggested that the computation of the two distinguished extreme points, together with a few "threads," i.e., paths on the skeleton, connecting these two extreme points, will probably suffice for large problems having "fat" cores.

2. NOTATION FOR MARKET GAMES

We denote the index set of the sellers by

$$I = \{1,2,\dots,m\} \tag{1}$$

and denote the index set of the buyers by

$$J = \{1,2,\dots,n\} \tag{2}$$

We assume that seller $i \epsilon I$ has

$$a_i > 0 \tag{3}$$

units of a good to sell, and that buyer $j \epsilon J$ wants to buy

$$b_j > 0 \tag{4}$$

units of the good. We let

$$c_{ij} \geq 0 \tag{5}$$

be the bid of buyer j for one unit of seller i's goods. The nonnegativity requirement in (5) means that seller i can dispose of his goods without charge in case no one bids a positive amount for it.

We make the same economic assumptions as do Shapley and Shubik [7] in their treatment of the assignment market game, namely:

(a) Utility is identified with money

(b) Side payments are permitted

(c) The objects of trade are indivisible

(d) Supply and demand functions are inflexible.

The remarks they make about these assumptions are pertinent here and will not be repeated.

As in the assignment game [7], the only profitable coalitions are those containing some buyers and some sellers. Also, because of assumption (5) and the side payment condition (b), the only important coalition is the all-player coalition $S = I \cup J$. We shall concentrate on evaluating $v(S)$ for this coalition only, since the same techniques can be used, if desired, for any other coalition.

Let x_{ij} be the number of units i sells to j. The value $v(I \cup J)$ is obtained by solving the linear program:

$$\text{Maximize} \sum_{i \in I} \sum_{j \in J} x_{ij} c_{ij}$$

Subject to (6)

$$\sum_{j \in J} x_{ij} \leq a_i$$

$$\sum_{i \in I} x_{ij} \leq b_j$$

$$x_{ij} \geq 0$$

The nonnegativity requirement on x_{ij} means that the exchange of property is from seller i to buyer j. The maximization objective in (6) means that we seek a set of transactions that maximizes the total gain of the coalition $I \cup J$ of all sellers and buyers (see Shapley and Shubik [7]).

If $b_j = 1$ for $j \in T$ the problem is called a <u>semi-assignment market game</u>. If, in addition, $a_i = 1$ for $i \in I$ the problem is called <u>an assignment market game</u>.

The dual linear programming problem to (6) is easily written as

$$\text{Minimize} \sum_{i \in I} a_i u_i + \sum_{j \in J} b_j v_j$$

Subject to (7)

$$u_i + v_j \geq c_{ij} \quad \text{for } i \in I, j \in J$$

$$u_i \geq 0 \quad \text{for } i \in I$$

$$v_j \geq 0 \quad \text{for } j \in J$$

where u_i and v_j are the dual variables associated with the first and second constraints in (6), respectively.

The <u>core</u> of the market game is the set of all solutions to the dual problem (7). This was argued in [7] for the assignment case, and the same result holds here. Because of the non-negativity conditions (3), (4), (5) and well-known linear programming results, the core is a bounded convex polyhedral set.

We can turn (6) into the classical transportation problem of linear programming by adding a dummy seller $\{m+1\}$ and dummy buyer $\{n+1\}$ giving extended seller and buyer index sets

$$I' = I \cup \{m+1\} \tag{8}$$

$$J' = J \cup \{n+1\} \tag{9}$$

We define the bids of these dummy players to be

$$c_{m+1,j} = 0 \quad \text{for} \quad j \epsilon J' \tag{10}$$

$$c_{i,n+1} = 0 \quad \text{for} \quad i \epsilon I' \tag{11}$$

and note that (10) can be interpreted as a "free gift" option for the buyers and (11) can be interpreted as a "free disposal" option for the sellers. To determine the amount sold by the dummy seller and the amount bought by the dummy buyer, we first define

$$S = \sum_{i \epsilon I} a_i \tag{12}$$

$$T = \sum_{j \epsilon J} b_j \tag{13}$$

and then define

$$a_{m+1} = \frac{1}{2}[|T - S| + (T - S)] \tag{14}$$

$$b_{n+1} = \frac{1}{2}[|S - T| + (S - T)] \tag{15}$$

as the amount sold by the dummy seller and the amount purchased by dummy purchaser, respectively. It is easy to see that at least one (and possibly both) of a_{m+1} and b_{n+1} is 0. In any case we retain both dummy players in the transportation problem for reasons that will become clear later.

We now use the above definitions to state a transportation problem from which the solution to (6) can be obtained.

$$\text{Maximize} \sum_{i\in I'} \sum_{j\in J'} x_{ij} c_{ij}$$

Subject to (16)

$$\sum_{j\in J'} x_{ij} = a_i \quad \text{for } i\in I'$$

$$\sum_{i\in I'} x_{ij} = b_j \quad \text{for } j\in J'$$

$$x_{ij} \geq 0$$

The dual problem to (16) is

$$\text{Minimize} \sum_{i\in I'} a_i u_i + \sum_{j\in J'} b_j v_j$$

Subject to (17)

$$u_i + v_j \geq c_{ij} \quad \text{for } i\in I',\ j\in J'$$

Clearly the only difference between (7) and (17) is the nonnegativity requirements on the dual variables which are present in (7) but missing in (17).

It is well known that the set of dual solutions to (17) is unbounded, since given any solution u_i^0, v_j^0, we can get infinitely many others from it by the transformation $u_i^0 + \delta$ for $i\in I'$, $v_j^0 - \delta$ for $j\in J'$, where δ is an arbitrary number. Also, if the c_{ij}'s are chosen arbitrarily then there may be no nonnegative solutions to (17). However, given assumption (5) that $c_{ij} \geq 0$ we will be able to show that the set of nonnegative solutions to (17) is non-empty and bounded, and we will give a constructive way of generating all extreme solutions. Hence we impose the nonnegativity constraints

$$u_i \geq 0 \text{ for } i\in I' \text{ and } v_j \geq 0 \text{ for } j\in J' \tag{18}$$

on the solutions to (17).

As indicated in [7] when the market game is an auction the interpretation of the u_i's are the selling prices received by the sellers for their goods and the v_j's are the buyer surplues attained by the sellers. For elaboration of these interpretations, see [7].

3. PRELIMINARY RESULTS

We concentrate on the solution to the transportation problem (16) and its dual (17), bringing in the nonnegativity constraints (18) when appropriate.

Given a non-empty mode set N and edge set E of pairs (i,j) of nodes $i,j \in N$ we let $G = (N,E)$ be the graph with nodes in N and edges in E. A tree is a connected graph with no cycles.

DEFINITION 1. A basis B consists of a subset of $m+n+1$ cells of $I' \times J'$ such that the graph $G = (I' \cup J', B)$ is a tree.

A pendant node of a graph is one that is incident to a single edge. It is well known that every tree having one or more nodes has at least one pendant node. Given any basis B, we can solve for a primal solution $X(B)$ satisfying the first two constraints in (16) by the following well-known procedure: find a pendant node i; solve for the x_{ij} corresponding to edge (i,j) incident to i; eliminate edge (i,j); repeat until $m+n+1$ values for x_{ij}'s are determined; all other x_{ij}'s are set equal to zero. (Note that this process may give some negative x_{ij} values.)

DEFINITION 2. Given a basis B let $X(B)$ be the corresponding solution for the x_{ij}'s satisfying the first two constraints in (16). If all the x_{ij} values in $X(B)$ are nonnegative then B is said to be primal feasible. If $X(B)$ solves the optimization problem (16) then B is said to be primal optimal.

Given a basis B, we can solve for a one parameter family of solutions $U(B)$ and $V(B)$ to the dual problem (17) by the following procedure: select any u_i or v_j arbitrarily and give it the value δ; say $u_1 = \delta$; then determine the values of the dual variables $v_{j_1}, v_{j_2}, \ldots, v_{j_k}$ corresponding to basis cells $(1,j_1), (1,j_2), \ldots, (1,j_k)$ in row 1; next find the values of the dual variables not yet determined in rows corresponding to basis cells in the columns $j_1, \ldots, j_k$ just used; etc.; repeat until all dual variables are determined.

DEFINITION 3. Given a basis B let $U_\delta(B)$ and $V_\delta(B)$ be the corresponding solutions for the u_i's and v_j's, depending on the parameter δ. If these u_i's and v_j's satisfy the constraints of (17) then B is <u>dual feasible</u>. If $U_\delta(B)$ and $V_\delta(B)$ solve the optimization problem in (17) then B is <u>dual optimal</u>. If B is dual optimal and there is a value δ^* of δ such that the dual solutions $U_{\delta^*}(B)$ and $V_{\delta^*}(B)$ are nonnegative, i.e., they satisfy (18), then B is said to be <u>non-negative dual optimal</u>.

LEMMA 1. Given the nonnegativity assumption (5) each primal optimal basis B is also nonnegative dual optimal.

PROOF. Given a primal optimal basis B let X(B) be the optimal primal solution and $U_\delta(B)$, $V_\delta(B)$ the optimal dual solutions. Then we have

$$u_i + v_j \geq c_{ij} \geq 0 \quad \text{for} \quad i \in I', \; j \in J' \tag{18}$$

Suppose some u_i is negative; choose the most negative one, say it is u_1. Then

$$u_1 + v_j \geq 0 \quad \text{for} \quad j \in J' \quad \text{implies} \quad v_j \geq - u_1 \geq 0 \quad \text{for} \quad j \in J'$$

Hence if we set $\delta^* = -u_1$ we have

$$u_i + \delta \geq 0 \quad \text{all} \quad i \in I' \quad \text{and} \quad v_j - \delta \geq 0 \quad \text{all} \quad j \in J'$$

so that $U_{\delta*}(B)$ and $V_{\delta*}(B)$ are optimal, nonnegative dual solutions. A similar argument holds if some v_j is negative.

REMARK 1. Because of Lemma 1 we can drop the adjectives "primal" and "nonnegative dual" before the words "optimal basis." We shall just speak of an optimal basis.

REMARK 2. Because of assumption (5) and (18) we know that there is no "more for less" transportation paradox, since the existence of such a paradox requires $u_i + v_j$ to sometimes be positive and sometimes negative; see Charnes and Klingman [3], Szwarc [12], and Srinivasan and Thompson [8].

THEOREM 1. Given assumption (5) the dual problems defined by (16), (17), and (18) have solutions $X(B)$, $V(B)$ and $V(B)$ with $u_{m+1} = 0$ and $v_{n+1} = 0$. Restricting these solutions to the index sets I and J give (not necessarily basic) solutions to the dual problems defined by (6) and (7).

PROOF. Solve (16) and (17) by one of the standard transportation methods such as the MODI method. Let B be the optimal basis so obtained. Use the method of Lemma 1 to get nonnegative, optimal dual solutions $U(B)$ and $V(B)$ for some δ (which we do not designate). Since $c_{i,n+1} = 0$ for all $i \epsilon I'$ we have $u_i + v_{n+1} \geq c_{i,n+1} = 0$ for all $i \epsilon I'$ and, because there is at least one cell $(i^*, n+1) \epsilon B$, it follows that $u_i + v_{n+1} = 0$. Since $u_i \geq 0$ and $v_{n+1} \geq 0$ we have $u_i = v_{n+1} = 0$. A similar argument shows $u_{m+1} = 0$.

It is clear that restricting $X(B)$, $U(B)$, and $V(B)$ to I and J give optimal solutions to (6) and (7) since (16) and (17) were derived from them by adding slack variables. These optimal solutions will be basic for (6) and (7) only if B restricted to $I \times J$ is also a basis.

In the example worked in Section 6 some restricted solutions are basic and others are not basic.

DEFINITION 4. (a) Problem (16) (and (17)) is primal degenerate if there are two bases $B^1 \neq B^2$ such that $X(B^1) = X(B^2)$.

(b) Problem (17) (and (16)) is dual degenerate if there are two bases $B^1 \neq B^2$ such that both $U_{\delta_1}(B^1) = U_{\delta_2}(B^2)$ and $V_{\delta_1}(B^1) = V_{\delta_2}(B^2)$.

The next two lemmas give alternate characterizations of primal and dual degeneracy. These results are well-known, hence the proofs are not given.

LEMMA 2. (a) Problem (16) is primal degenerate if and only if there is a feasible basis B with solution $X(B)$ such that $x_{ij} = 0$ for some $(i,j) \in B$.

(b) Problem (16) is primal degenerate if and only if there are subsets $I_1 \subseteq I$ and $J_1 \subseteq J$, with at least one of I_1, J_1 a proper subset, such that

$$\sum_{i \in I_1} a_i = \sum_{j \in J_1} b_j .$$

LEMMA 3. (a) Problem (17) is dual degenerate if and only if there is a feasible basis B, with dual solutions $U(B)$ and $V(B)$, such that

$$c_{ij} - u_i - v_j = 0 \text{ for some } (i,j) \notin B.$$

(b) Problem (17) is dual degenerate if and only if there is a cycle Ω such that

$$\sum_{(i,j) \in \Omega} c_{ij} = 0.$$

(Note: A cycle Ω is a set of cells (arcs) $(i,j) \in I \times J$ such that each row and each column of the matrix c contains either no cells or exactly two cells of Ω.)

REMARK 3. If (16) is not primal degenerate then there is a one to one correspondence between optimal bases and optimal primal solutions $X(B)$. Similarly, if (17) is not dual degenerate then there is a one to one correspondence between optimal bases B and optimal dual solutions $U(B)$, $V(B)$.

REMARK 4. Some problems are both primal and dual degenerate. For instance, suppose $m = n = 2$, $c = I$, $a_1 = a_2 = b_1 = b_2 = 1$. Then the corresponding problem given by (16) and/or (17) has the following tableau:

$$\begin{array}{|ccc|c}
\hline
1^{1} & 0 & 0 & 1 \\
0 & 1^{1} & 0 & 1 \\
0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 &
\end{array}$$

An optimal basis B consists of the two circled cells together with any three of the four cells (2,1), (2,3), (3,1) and (3,3). For each of these four bases the optimal primal and dual solutions are:

(a) $x_{11} = x_{22} = 1$, all other $x_{ij} = 0$

(b) $u_1 = v_2 = 1$ all other u_i's and v_j's $= 0$.

In the remainder of this paper we will work with problems (such as assignment problems) which are highly primal degenerate, but assume (for expositional purposes) they are not dual degenerate in order to make the description of the algorithm for finding all dual solutions easy. As is well known a small perturbation of the c_{ij}'s is sufficient to insure dual nondegeneracy.

4. CHARACTERIZATION OF THE CORE

In [7] Shapley and Shubik characterized the core of an assignment market game as the set of nonnegative dual solutions to an assignment problem, and showed that there were two distinguished points, one that maximizes seller surplus and another that maximizes buyer surplus. Here we extend their results to general market games, and provide theorems that provide computational techniques for the algorithm of the next section.

DEFINITION 5. The <u>core</u> of the market game (6) is the set of all nonnegative solutions to its dual problem (7); i.e., the core is the set of all solutions to (17) and (18). We denote the core by $C = (C(U), C(V))$ where $C(U)$ is the set of row dual solutions U which we call the <u>seller core</u>, and $C(V)$ is the set of column dual solutions V which we call the <u>buyer core</u>.

REMARK 5. From Theorem 1 it follows immediately that the core is nonempty. From standard linear programming theory we know the core is a bounded convex polyhedral set having a finite number of extreme points.

DEFINITION 6. Given a market game the <u>maximum seller surplus</u> u_i^* for seller i is given by

$$u_i^* = \underset{U \in C(U)}{\text{Maximum}} \; u_i \tag{19}$$

The <u>minimum seller surplus</u> u_{*i} is defined by replacing the word Maximum by the word Minimum in (19). The vectors u^* and u_* with components u_i^* and u_{*i} are the maximum and minimum seller surplus vectors.

DEFINITION 7. Given a market game the <u>maximum buyer suplus</u> v_j^* for buyer j is given by

$$v_j^* = \underset{V \in C(V)}{\text{Maximum}} \; v_j \tag{20}$$

The <u>minimum buyer surplus</u> v_{*j} is defined by replacing the Maximum by the word Minimum in (20). The vectors v^* and v_* with components v_j^* and v_{*j} are the maximum and minimum buyer surplus vectors.

THEOREM 2. Given a market game (6), the vector pairs (u^*, v_*) and (u_*, v^*)

(a) are in the core;

(b) are the furthest distance apart of any two vectors in the core;

(c) individually and collectively maximize, or minimize, buyer or seller surpluses.

PROOF. (a) Let B be any optimal basis for (16) and (17) and let $u_i(B)$ and $v_j(B)$ be the components of the optimal dual solution. Then

$$u_i(B) + v_j(B) \geq c_{ij} \quad \text{for } i \in I' \text{ and } j \in J' \tag{21}$$

Since (21) holds for all optimal bases we have from (19)

$$u_i^* + v_j(B) \geq c_{ij} \quad \text{for } i \in I' \text{ and } j \in J' \tag{22}$$

Since u_i^* *is a constant, we have from the definition of* v_{*j} that (22) is true for each optimal basis B. Hence

$$u_i^* + v_{*j} \geq c_{ij} \quad \text{for } i \in I' \text{ and } j \in J' \tag{23}$$

and it follows that (u^*, v_*) is in the core. A similar argument holds for (u_*, v^*).

(b) If (u, v) is any vector in the core then $u_{*i} \leq u_i \leq u_i^*$ for $i \in I'$ and $v_{*j} \leq v_j \leq v_j^*$ for $j \in J'$ so that the euclidean distance between any two vectors in C must be less than the distance between (u^*, v_*) and (u_*, v^*). As Shapley and Shubik point out, the same minimum distance result also holds for any other distance function between two points whose definition depends only on the absolute differences of vector components.

(c) Definition (19) shows that in the core, u_i^* maximizes the surplus of buyer i surplus as an individual; if we add together the components of u^*, that is, we compute $\$ = \Sigma u_i^*$, it follows that the collective surplus is also maximized at u^*. Similar remarks hold for u_*, v^* and v_*.

Theorem 2 characterizes the two "end points" of the core. The next theorem shows how these two distinguished core solutions can be calculated by making use of well-known perturbation techniques for the transportation problem.

We define two kinds of perturbation (P1) and (P2), by the following transformations, where the arrow "→" means "is replaced by":

$$\text{(P1)}\begin{cases} a_i \to a_i \text{ for } i\epsilon I, \quad a_{m+1} \to a_{m+1} + n\epsilon \\ b_j \to b_j + \epsilon \text{ for } j\epsilon J, \quad b_{n+1} \to b_{n+1} \\ \text{where } \; 0 < \epsilon < \dfrac{1}{2(n+1)} \end{cases}$$

$$\text{(P2)}\begin{cases} a_i \to a_i + \epsilon \text{ for } i\epsilon I, \quad a_{m+1} \to a_{m+1} \\ b_j \to b_j \text{ for } j\epsilon J, \quad b_{n+1} \to b_{n+1} + m\epsilon \\ \text{where } \; 0 < \epsilon < \dfrac{1}{2(m+1)} \end{cases}$$

As shown in Orden [6] or Dantzig [4] either of these perturbations, when applied to a transportation problem, gives a primal non-degenerate problem. Srinivasan and Thompson [10] showed that, given integer rim data, the primal solution $X(\epsilon)$ to the perturbed problem, when scientifically rounded, yields an optimal integer primal solution $T(X(\epsilon))$ to the original problem. We make use of this fact in the proof of the next theorem.

THEOREM 3. Let (6) be a market game with integer rim data and let (16) and (17) be the corresponding transportation problem; we assume (for convenience) the latter is dual non-degenerate.

(A) The dual solutions to (16), (17) after applying perturbation (P1) give the core vector pair (u^*, v_*).

(B) the dual solutions to (16), (17) after applying perturbation (P2) give the core vector pair (u_*, v^*).

PROOF. (A) Perform (P1) to the data for the dual problems (16) and (17) and solve; call the solutions $X(\epsilon)$, $U(\epsilon)$, $V(\epsilon)$ and let $B(\epsilon)$ be the optimal basis. By Theorem 1, $u_{m+1} = 0$ and $v_{n+1} = 0$ at the optimum, so that the dual problem (17) can be written as

$$\left.\begin{array}{ll} \text{Minimize} & \sum_{i\epsilon I} a_i u_i + \sum_{j\epsilon J} b_y v_j + \epsilon(\sum_{j\epsilon J} v_j) \\ \text{Subject to} & u_i + v_j \geq c_{ij} \quad \text{for } i\epsilon I',\ j\epsilon J' \end{array}\right\} \qquad (24)$$

Let $T(X(\epsilon))$ be the vector of scientifically rounded values of $X(\epsilon)$; by Theorem 2 in [10] $T(X(\epsilon))$ is an optimal integer solution to the unperturbed problem (16) with the same basis $B(\epsilon)$ as $X(\epsilon)$. Since the dual solution to (24) depends only on the basis $B(\epsilon)$ we see that U and V solve both the perturbed and unperturbed versions of (17). Hence $\sum_{i\epsilon I} a_i u_i + \sum_{j\epsilon J} b_j v_j = K$, where K is the value of mutual objective values of unperturbed problems (16) and (17). We see that (24) becomes

$$\left.\begin{array}{ll} \text{Minimize} & \epsilon(\sum_{j\epsilon J} v_j) + K \\ \text{Subject to} & u_i + v_j \geq c_{ij} \quad \text{for } i\epsilon I',\ j\epsilon J' \end{array}\right\} \qquad (25)$$

and, since K can be ignored and $\epsilon > 0$, the solutions to this problem must minimize the sum $\sum_{j \in J} v_j$. By Theorem 2 the vector pair of the core that solves (25) is (u^*, v_*).

The proof of part (B) is similar.

The computational importance of Theorem 3 is immediately obvious. For by solving just two transportation problems it is possible to find the two distinguished extreme points of the core (u^*, v_*) and (u_*, v^*). By using one of the current transportation codes [1, 5, 9] this computation can be made in a few seconds or minutes, even for problems having hundreds of buyers and sellers. Since the core tends to be long and thin with the other points in the core usually lying quite close to the line segment between these two extreme points, finding them already gives a very good idea of what the core is like. The examples in Section 6 will illustrate this point.

The next theorem to be proved shows how to move from a given extreme point of the core to its neighboring extreme points. However, before we can state that result we must recall some notation and concepts from other papers. We put these as a series of remarks.

REMARK 6. Let B be a basis and $G = (I' \cup J', B)$ the corresponding basis graph. Let (p,q) be any cell in B, which is also an arc of G. If (p,q) is removed from B giving an arc set $B' = B - \{(p,q)\}$ then $G' = (I' \cup J', B')$ has two connected components G'_R and G'_C where p belongs to the node set of G'_R and q belongs to the node set of G'_C. Let I'_R and J'_R be the sets of indices of the rows and columns in G'_R; also let I'_C and J'_C be the sets of indices of the rows and columns in G'_C. Then $I'_R \cup I'_C = I'$ and $I'_R \cap I'_C = \emptyset$, that is I'_R and I'_C partition I'. Similarly J'_R and J'_C partition J'. Because of the rim positivity assumptions (3) and (4) it is easy to show that no set in any of these partitions is empty. The "scanning" routine on p. 218 of [8] can be used to find these partitions.

REMARK 7. If (p^1, q^1) and (p^2, q^2) are two different cells of B (neither of which is $(m+1, n+1)$) then $(I_R')^1 \neq (I_R')^2$ and $(J_C')^1 \neq (J_C')^2$. For suppose $(I_R')^1 = (I_R')^2$; then $(I_C')^1 = (I_C')^2$. But suppose we remove both (p^1, q^1) and (p^2, q^2) from B. The resulting graph has three components with p^1 and p^2 belonging to the same component and q^1 and q^2 to each of the others. From this it follows that $(I_C')^1 \cap (I_C')^2 = \emptyset$, which together with $(I_C')^1 = (I_C')^2$ implies that both sets are empty, contradicting the result in Remark 6.

REMARK 8. Define

$$\mu = - \underset{(i,j)\in I_C' \times J_R'}{\text{Maximum}} (c_{ij} - u_i - v_j) \tag{26}$$

As shown in [8, p. 231] the set $I_C' \times J_R'$ has no basis cells in B. Because of this and the fact that we assume (17) to be non dual degenerate, it follows that $\mu > 0$. Let (r,s) be the cell in $I_C' \times J_R'$ at which the maximum in (26) is taken on.

REMARK 9. As shown in [8, p. 241] the set $B^* = B - \{(p,q)\} + \{(r,s)\}$ is a basis.

REMARK 10. By "zero shifting" we mean a change in the basic solution of the following kind: Let $(p,q) \in B$ be a cell such that $x_{pq} = 0$; let I_R', I_C', J_R', J_C', μ, B^*, and (r,s) be as defined in Remarks 6-9; then carry out the following transformations (see [8], p. 232):

$$B \to B^* = B - \{(p,q)\} + \{(r,s)\}$$
$$X \to X^* = X$$

$$u_i \to u_i + \mu \quad \text{for } i \in I'_R, \; u_i \to u_i \quad \text{for } i \in I'_C$$

$$v_j \to v_j - \mu \quad \text{for } j \in J'_R, \; v_j \to v_j \quad \text{for } j \in J'_C$$

Let U^* and V^* be the transformed dual solutions. Then X^*, U^* and V^* are alternate optimal solutions to (16) and (17). The proof is contained in the above cited reference.

THEOREM 4. Assume (17) is not dual degenerate and let X be a basic primal solution to (16) with basis B. Let $k+1$ be the number of cells (i,j) in B such that $x_{ij} = 0$. Then every extreme point of $C(U)$ and every extreme point of $C(V)$ has exactly k distinct neighbors; each of these can be found by the "zero shifting" process of Remark 10.

PROOF. For each $x_{ij} = 0$ and $(i,j) \in B$ (except for cell $(m+1, n+1)$) calculate μ_{ij} as in Remark 8 and carry out the zero shift as in Remark 10 creating new dual solutions U_{ij} and V_{ij}. By the results in Remarks 6 and 7 the partitions induced in both I' and J' are different. Since $\mu_{ij} > 0$, see Remark 8, the new dual solutions are distinct, completing the proof.

DEFINITION 8. The <u>skeleton</u> of the core is the graph (P, E) where P is the set of extreme points and E the set of edges connecting <u>adjacent extreme points</u> determined as in Theorem 4.

DEFINITION 9. A <u>k-graph</u> is a graph in which every vertex is connected by k edges to adjacent vertices.

Using these two definitions we can reformulate the results of Theorem 4 in the following succinct fashion.

THEOREM 5. Consider a market game for which the associated dual problem (17) is not dual degenerate; then the skeletons of the row and column cores, $C(U)$ and $C(V)$, are k-graphs.

In Section 6 we present two examples: a 2×2 assignment market game in which these skeletons are 2-graphs, and a 3×3 assignment market game in which these skeletons are 3-graphs (cubic graphs).

5. ALGORITHM FOR COMPUTING THE CORE

The results of the preceeding section permit us to state an algorithm for computing all the extreme points and edges, that is, the skeleton of the buyer and seller cores. The algorithm starts at the point (u_*, v^*) and works its way "upward" through the buyer core. To measure the "upward" direction we compute $s = \sum_{i\in I} u_i$ for each solution and choose new solutions in such a way that s never decreases. Three lists are maintained; L, the list of extreme points computed but not all of whose neighbors have been computed; P, the list of all extreme points whose neighbors are fully computed; and E, the list of all edges in the skeletons. The precise statement of the algorithm is now given.

<u>Algorithm for finding the skeletons of the buyer and seller cores</u>. Let X be the fixed primal solution. For each optimal basis B let $S = \{B, u, v, s\}$ where u and v are the dual solutions and $s = \sum_{i\in I} u_i$.

(0) Use perturbation (P2) to calculate $S_1 = \{B_1, u_*, v^*, s_1\}$. Set $L = \{S\}$, $P \neq 0$, and $E = \emptyset$.

(1) Find $S_i \in L$ with smallest sum s_i.

(2) Suppose there are t unmarked cells (p,q) in B_i such that $x_{pq} = 0$. For each $j = 1,\ldots,t$ calculate S_j by shifting the zero at (p,q) as in Remark 10. Put edge (S_i, S_j) in E.

(a) Is S_j in L? If yes go to (b). If no go to (c).

(b) Mark the zero cell just shifted in S_j. If all cells in S_j are marked, take S_j out of L and put it in P.

(c) Put S_j in L after marking the zero-cell just shifted.

(3) Take S_i out of L and put it in P. If $L = \emptyset$ go to (4). Else go to (1).

(4) List P contains all the extreme points and list E contains all the extreme edges of the buyer and seller skeletons. Stop.

A by-product of this algorithm is the fact that we compute extreme points and edges of both cores simultaneously and there is a 1-1 correspondence between them explicitly exhibited by the steps of the algorithm. We summarize this by the following theorem.

THEOREM 6. The skeleton graphs of the buyer and seller cores are isomorphic.

It is possible to apply this algorithm even when the market game is dual degenerate (as is the 3×3 example on p. 122 of [7].) All that changes is that there are fewer extreme points to compute, and the degrees of each point are not necessarily the same.

6. EXAMPLES

We present two examples to illustrate the ease with which the algorithm can be applied.

The first is the 2×2 assignment market game whose tableau is given by

3^1	2	1
1	3^1	1
1	1	

The optimal primal solution is marked. The complete set of tableaus for the various solutions constructed by the algorithm of the preceeding section is shown in Figure 1. The skeletons of the buyer and seller cores are shown in Figure 2. The isomorphism (in this case even the congruence) of these two figures is evident by making a 180° rotation of either figure.

The core was also computed for the following 3×3 assignment market game.

$$\begin{array}{ccc|c} 7^{1} & 6 & 5 & 1 \\ 3 & 7^{1} & 4 & 1 \\ 1 & 2 & 7^{1} & 1 \\ \hline 1 & 1 & 1 & \end{array}$$

The skeleton of the seller core for this example appears in Figure 3. Note that it has 20 vertices, 30 edges and 12 faces; these numbers are the same as for the dodecahedron. However this figure is clearly not a dodecahedron because it has 3 faces with 4 sides, 3 faces with 6 sides, and 6 faces with 5 sides, whereas a dodecahedron has 12 five sided faces. The actual extreme points for both buyer and seller cores are listed in Figure 4. The extreme edges can be found in Figure 3.

The author has also used the algorithm to solve the 3×3 assignment game solved by Shapley and Shubik on p. 122 of [7]. Because of dual degeneracy the cores of that game have 6 extreme points each instead of the twenty of the example in Figure 3.

Although the computations of the algorithm are elementary and can be done very quickly, the number of extreme points of the core increases very rapidly with the number of players in a non-dual degenerate case. Therefore it is doubtful that anyone will ever completely compute the core in that case for games having more than 10 or 20 players.

The algorithm of Section 5 can easily be modified to compute a "thread" on the surface of the skeleton of the core, by simply computing for each extreme point only a single neighbor in the "upward" direction. For most purposes, the knowledge of the two distinguished extreme points and a few threads connecting them will provide a sufficiently accurate idea of the core.

Although there are a number of important economic models for which it can be proved that the core size shrinks as the number of participants tends to infinity, nothing like that can be proved here. In [7] Shapley and Shubik discuss this problem to a certain extent. We make here only the following two remarks, leaving a fuller discussion of the problem for another paper. Let the core size be the number of extreme points of the core.

(A) In a market game the core size varies directly with the degree of primal degeneracy and inversely with the degree of dual degeneracy (when such degrees are suitably defined.)

(B) It is possible to construct market games having arbitrarily many players with either (i) single element cores or (ii) maximal size cores.

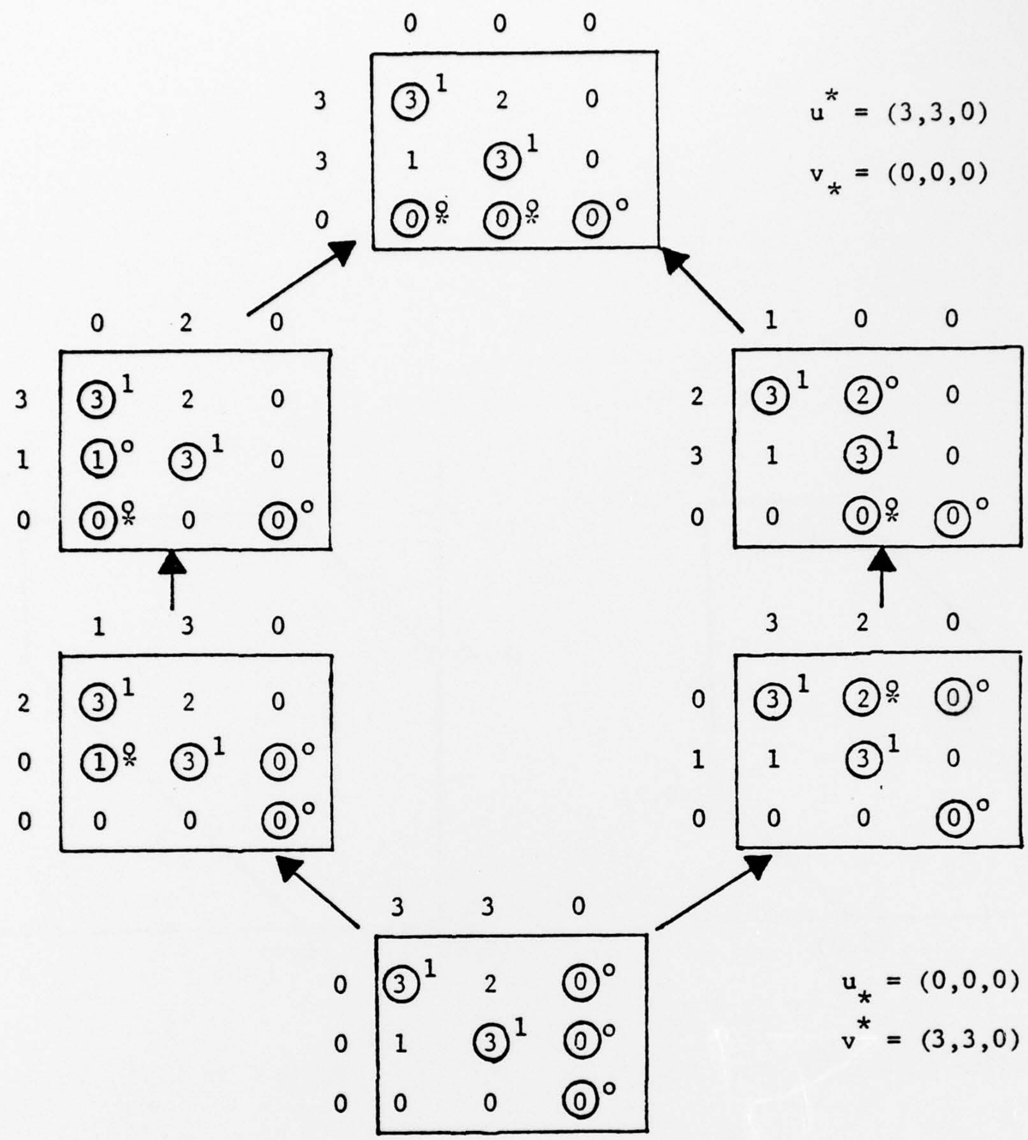

FIGURE 1. Algorithm computations for 2x2 example

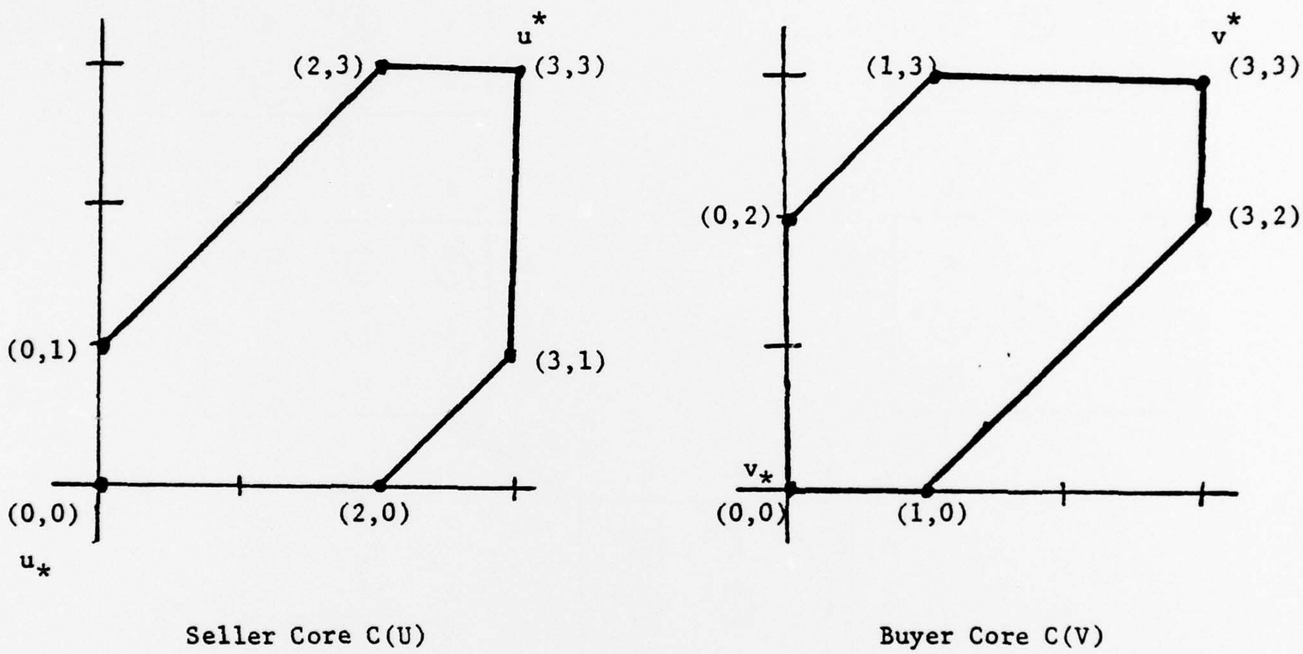

FIGURE 2. Skeletons of the Buyer and Seller Cores for 2x2 Example

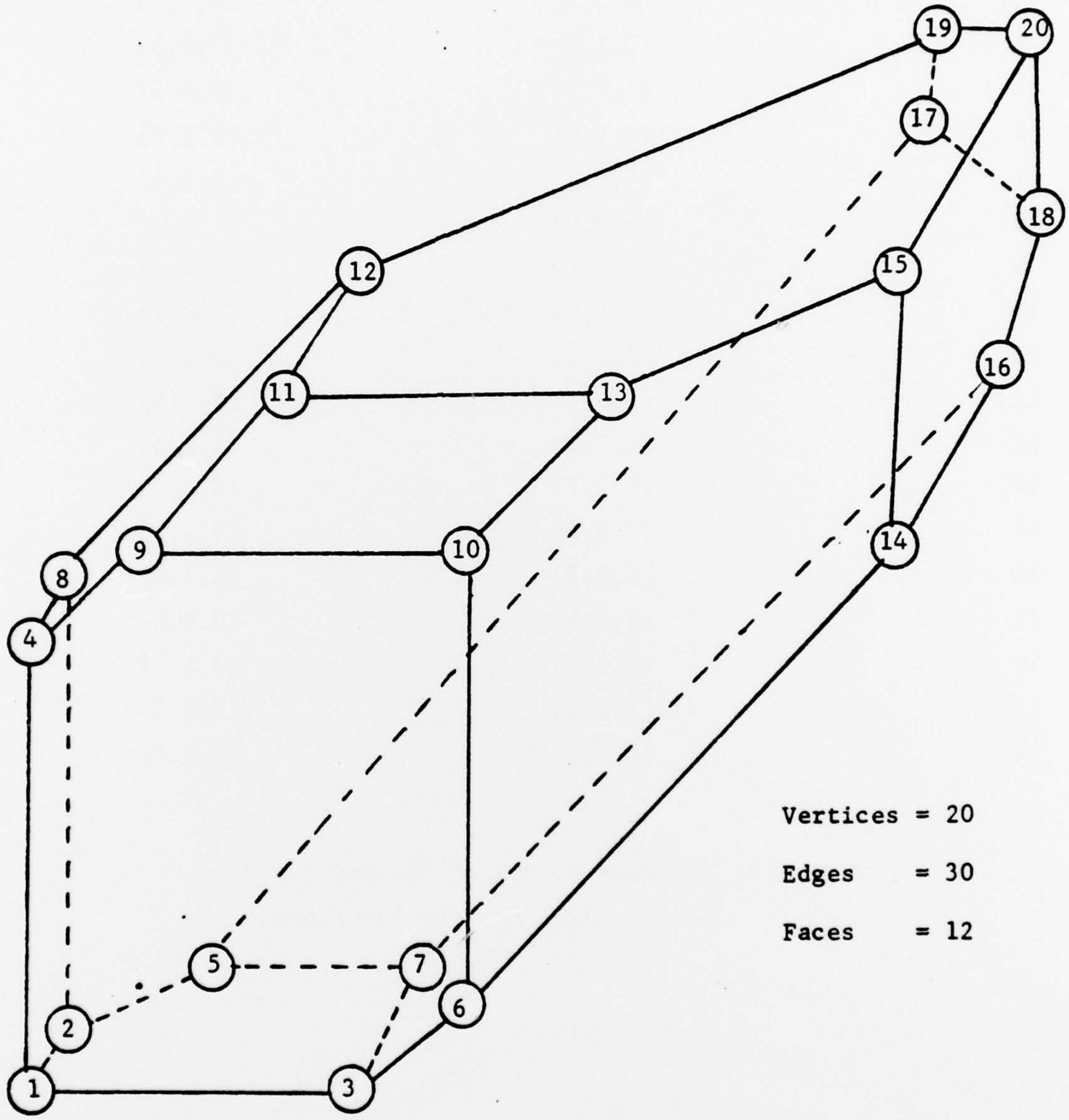

FIGURE 3. Skeleton of the Seller Core for the 3x3 Example

Extreme Points

Number	Seller Core	Buyer Core
1	(0,0,0)	(7,7,7)
2	(0,1,0)	(7,6,7)
3	(0,0,2)	(7,7,5)
4	(0,1,2)	(7,6,5)
5	(1,0,3)	(6,7,4)
6	(4,0,0)	(3,7,7)
7	(4,0,3)	(3,7,4)
8	(6,2,0)	(1,5,7)
9	(4,5,0)	(3,2,7)
10	(6,5,0)	(1,2,7)
11	(7,3,1)	(0,4,6)
12	(7,6,1)	(0,1,6)
13	(6,7,2)	(1,0,5)
14	(5,4,7)	(2,3,0)
15	(7,3,6)	(0,4,1)
16	(5,6,7)	(2,1,0)
17	(7,7,2)	(0,0,5)
18	(7,4,7)	(0,3,0)
19	(6,7,7)	(1,0,0)
20	(7,7,7)	(0,0,0)

FIGURE 4. Extreme Points of the Buyer and Seller Cores for the 3x3 Example.

Bibliography

[1] Bradley, G. H., G. G. Brown, and G. W. Graves, "Design and Implementation of Large Scale Primal Transshipment Algorithms," Management Science 24 (1977), 1-34.

[2] Charnes, A. and W. W. Cooper, Management Models and Industrial Applications of Linear Programming, Vols. I and II, John Wiley and Sons, New York, 1961.

[3] Charnes, A. and D. Klingman, "The More for Less Paradox in the Distribution Model," Cahiérs de Centre des Études Operationelle, 13 (1971), 11-22.

[4] Dantzig, G. B., Linear Programming and Extensions, Princeton University Press, Princeton, N. J., 1963.

[5] Glover, F., D. Karney, D. Klingman, and A. Napier, "A Computational Study on Start Procedures, Basis Change Criteria and Solution Algorithms for Transportation Problems," Management Science 20 (1974), 793-813.

[6] Orden, A., "The Transshipment Problem," Management Science 2 (1956), 276-285.

[7] Shapley, L. S. and M. Shubik, "The Assignment Game I: The Core," International Journal of Game Theory 1 (1972), 111-130.

[8] Srinivasan, V. and G. L. Thompson, "An Operator Theory of Parametric Programming for the Transportation Problem, I and II," Naval Research Logistics Quarterly, 19 (1972), 205-252.

[9] Srinivasan, V. and G. L. Thompson, "Benefit-Cost Analysis of Coding Techniques for the Primal Transportation Algorithm," Journal of the Association for Computing Machinery, 20 (1973), 194-213.

[10] Srinivasan, V. and G. L. Thompson, "Cost Operator Algorithms for the Transportation Problem," Mathematical Programming 12 (1977), 372-391.

[11] Thompson, G. L., "Pareto Optimal, Multiple Deterministic Models for Bid and Offer Auctions," in preparation.

[12] Szwarc, W., "The Transportation Paradox," Naval Research Logistics Quarterly, 18 (1971), 185-202.