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	STABILITY OF A COMPRESSED ROD	
	By: Karl Kreutzer	
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STABILITY OF A COMPRESSED ROD Karl Kreutzer

Introduction

The elasticity problem of a break [1] is dealt with here from the viewpoint which considers that a break process is inherently a problem of elasticity theory of finite deformations.

It is assumed that the tensions which appear are within the validity field of the expanded Hook law for finite deformations. Included here is the limitation of fixed mass and surface forces, that is independent from deformation.

The method used is an energetic one which takes into account the finiteness of deformations along all three coordinates, as developed by E.Trefftz [2] and reported at the international Congress for Technical Mechanics in Stockholm (1930).

The following nomenclature is used:

", ", ", " coordinates of points

 $X_1, X_2, X_3$  components of mass force per volume unit of the non-deformed body along the three directions

E, E, E, components of surface force per area unit of the non-deformed surface along the three directions

"", "", "" displacement components in the three directions

deter, butos, dutos

components of state changes from equilibrium (of disturbances)

-1-

internal energy of the whole body.

E

An elastic state is a stable equilibrium state when with each finite displacement, compatible with the geometric conditions the increase in internal energy is larger than the work available from external forces. Thus when

 $AE > \{\} \{ \sum_{i} X_{i} \delta u^{(i)} dx^{(i)} dx^{(i)} dx^{(i)} dx^{(i)} + \} \{ \sum_{i} Z_{i} \delta u^{(i)} do. \}$ Developing both sides according to exponents of  $\delta_{u}^{(i)}$  and their derivations gives:

 $\Delta E := \delta E + \delta^{*} E + \ldots > \iiint \sum X_{v} \delta u^{vr} dx^{vr} dx^{vr$ 

On the right appear no exponents of  $\partial u^{(n)}$  due to the limitation to fixed external forces  $X_r$  and  $\Xi_r$ .

Should now under any conditions compatible kinematically with J." the left term be larger than the right one, then the linear terms must disappear. Thus is should be

 $\delta E = \iiint \sum X_y \, \delta \, u^{(p)} \, d \, x^{(p)} \, d \, x^{(p)} \, d \, x^{(p)} + \iiint \sum E_y \, \delta \, u^{(p)} \, d \, 0.$ 

The content of this equation is described as the "principle of virtual displacements". It expresses that for each virtual displacement from equilibrium the change of internal energy equals the work of external forces.

Should the equilibrium state be stable, then the quadratic members of the left side of (1) should outbalance the members on the right side of (1); as a consequence of limiting ourselves to constant external forces, the work of these forces is spent by the linear members. The stability condition is thus reduced to  $A^* k > 9$ .

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The stability limit is reached when for at least one system of displacements  $\delta_{u}$ , the second variation of the internal energy disappears, that is  $\delta^{2}E_{m}$ .

These very general points will be, in what follows, applied to the stability problem of the elasticity theory; particularly to the case of obtaining break loads of a compressed rod.

I. Elasticity theory of finite deformations 1. The distorted state

To define the particles of an elastic body "substance coordinates" are used. These are denoted in a non-deformed state as rectangular normal coordinates and indicated by "" In a distorted state they become curved coordinates corresponding to each mass particle.

Any mass particle before distortion has the coordinates  $x^{m}, x^{m}, x^{m}$ . It includes a point P towards which, in fixed space intersection lead three vertical vectors **C.C.C.** and the location vector  $t = \sum_{i=1}^{n} x^{m} \cdot C_{i}$ . A neighboring particle with coordinates  $x^{m} + dx^{m}, x^{m} + dx^{m}$  includes a point Q. If  $d_{i}$  is a vector of P according to Q then  $d_{i} = \sum_{i=1}^{n} dx^{m} \cdot C_{i}$ , and the linear element is

For the coefficients  $G_{\cdot,\mu} = C_{\cdot,\nu} C_{\mu}$  of the linear element under normal coordinates the following scheme applies:

$$\|G_{\star\mu}\| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Through an elastic displacement  $\mathbf{b} = \sum \mathbf{x}^{in} \mathbf{C}$ , the body is distorted. The point  $P(\mathbf{z})$  takes a new position  $\overline{P}(\mathbf{r} = \mathbf{z} + \mathbf{v})$  and its neighboring point Q a new position  $\overline{Q}(\mathbf{r} + d\mathbf{x} = \mathbf{z} + d\mathbf{z} + \mathbf{v} + d\mathbf{v})$ . Besides, between the components  $\mathbf{z}^{in}$  of the location vector  $\mathbf{r}$ after deformation and the components  $\mathbf{z}^{in}$  of the location vector  $\mathbf{z}$ before deformation there is a relationship

 $\xi^{(\nu)} = x^{(\nu)} + u^{(\nu)}, \quad d\xi^{(\nu)} = dx^{(\nu)} + du^{(\nu)}.$ 

The linear element after distortion is

 $do^{\circ} = \sum_{\mu} \sum_{\mu} \sum_{\mu} \prod_{\mu} dx^{(\mu)} dx^{(\mu)},$ 

and where

 $\Gamma_{\star\mu} = \sum_{\mu} \frac{\partial}{\partial x^{(\mu)}} \frac{\delta}{\partial x^{(\mu)}} = G_{\star\mu} + \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} + \frac{\partial}{\partial x^{(\mu)}} + \sum_{\mu} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} + \sum_{\mu} \frac{\partial}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} + \sum_{\mu} \frac{\partial}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} + \sum_{\mu} \frac{\partial}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} + \sum_{\mu} \frac{\partial}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} + \sum_{\mu} \frac{\partial}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x^{(\mu)}} + \sum_{\mu} \frac{\partial}{\partial x^{(\mu)}} \frac{\partial u^{(\mu)}}{\partial x$ 

The linear element  $ds^2$  has thus been distorted into a linear element  $ds^2$ . Comparing both changes  $\gamma_{*,*} = \Gamma_{*,*} - G_{*,*}$  for the linear elements coefficients. we get:

$$y_{*,k} = \frac{\partial u^{(m)}}{\partial x^{(m)}} + \frac{\partial u^{(m)}}{\partial x^{(m)}} + \sum_{k} \frac{\partial u^{(m)}}{\partial x^{(m)}} \frac{\partial u^{(m)}}{\partial x^{(m)}}.$$

The r. are the "distortion magnitudes" describing the tension and angular changes suffered by each volume element. Due to substitution rules r. there are only six distinguishable distortion magnitudes. These form a tensor; it corresponds to each point of an elastic body and is its -4symmetrical distortion tensor.

The linear element do' is expressed in normal rectangu-

 $d \sigma^{2} = (1 + \gamma_{11}) dx^{(1)2} + (1 + \gamma_{22}) dx^{(1)2} + (1 + \gamma_{33}) dx^{(2)2}$  $+ 2\gamma_{12} dx^{(1)} dx^{(2)} + 2\gamma_{23} dx^{(2)} dx^{(2)} + 2\gamma_{31} dx^{(3)} dx^{(3)},$ 

where

$$\gamma_{11} = 2 \frac{\partial}{\partial} \frac{u^{(1)}}{x^{(1)}} + \left( \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(1)}} \right)^{2} + \left( \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(2)}} \right)^{2} + \left( \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(2)}} \right)^{2}$$

$$\gamma_{12} = \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(2)}} + \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(2)}} + \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(2)}} + \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(2)}} \frac{\partial}{x^{(2)}} + \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(2)}} \frac{\partial}{x^{(2)}} + \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(2)}} \frac{\partial}{x^{(2)}} + \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(2)}} \frac{\partial}{x^{(2)}} \frac{u^{(2)}}{x^{(2)}} \right)$$

$$(2)$$

etc, through cyclic substitution.

These non-linear equations transform into linear equations of the classical elasticity theory when the products and squares of definition may be neglected when compared to linear expressions. 2. Tension state and the equilibrium condition

In order to describe the tension state of a mass particle  $(x^{(m)}, x^{(m)}, x^{(m)})$  we regard the square angle paralelopiped which in the non-deformed condition is formed by the elements parallel to the respective axis element  $dx^{(m)}, dx^{(m)}, dx^{(m)}$ from the latter, after deformation a general paralelopiped is formed having edges  $\int_{a}^{b_{1}} dx^{(m)}, \int_{a}^{b_{1}} dx^{(m)}, \int_{a}^{b_{1}} dx^{(m)}$ . The vectors  $e_{\mu} = \int_{a}^{b_{1}} dx^{(m)}, \int_{a}^{b_{1}} dx^{(m)}, \int_{a}^{b_{1}} dx^{(m)}$ . The vectors  $e_{\mu} = \int_{a}^{b_{1}} dx^{(m)}, \int_{a}^{b_{1}} dx^{(m)}, \int_{a}^{b_{1}} dx^{(m)}$ .

When the force for data data acts in the direction of growing and boundary surface of the parallelopiped , then  $I^{(n)}$  is called the tension vector for surfaces  $x^{(1)} = \text{constant}$ . The same corresponds to the remaining surfaces.  $I^{(n)}$  is the tension vector for the surface element  $x^{(n)} = \text{constant}$ and signifies a force per unit of the non-deformed surface.

Each of the three tension vectors may be decomposed according to the grid vectors

. . . . (3),

by which nine tension components are created, which completely describe the tension state.

Im = Shuten .

The equilibrium against torsion in any chosen direction demands the disappearance of the momentums' sum of all the forces acting upon the paralelopiped considered at its center, thus:  $\Sigma M = \Sigma (c_r \times 1^{m}) dx^m dx^m = 0.$ 

For the equilibrium against a displacement in any direction it is necessary to consider the action of force  $i^{(1)} dx^{(0)} dx^{(0)}$  on the paralelopipeds' surface element  $x^{(1)}$ =constant on to the surface element  $x^{(1)}$ =constant; the acting force is then  $i^{(0)} dx^{(0)} dx^{(0)} dx^{(0)} dx^{(0)} dx^{(0)} dx^{(0)} dx^{(0)}$ .

-6-

$$d\mathfrak{R} = \frac{\nabla \partial f^{(m)}}{\partial x^{(m)}} dx^{(m)} dx^{(m)} dx^{(m)}.$$

Besides a mass force acts upon the paralelopiped

 $d\mathfrak{X} = \mathfrak{P}dx^{(1)}dx^{(2)}dx^{(3)} = \sum_{\mu} I^{(\mu)} \mathfrak{e}_{\mu} dx^{(1)}dx^{(2)}dx^{(3)},$ 

where **¥** is the mass force per volume unit of the non--deformed body and *P*<sup>m</sup> its component in the direction of the

th grid vector. The equilibrium against displacement requires

dik tak o,

from here after abbreviating through the product d.r. d.r.

North States a North Contraction

Figure 1

-7-

Using formula (3) and differentiating it follows:

 $\frac{\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \partial k^{n,n} c_{n} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} k^{n,n} \frac{\partial c_{n}}{\partial x^{n,n}} + \sum_{n=1}^{\infty} I^{n,n} c_{n} = 0,$   $\frac{\partial c_{n}}{\partial x^{n,n}} = \sum_{n=1}^{\infty} {\mu n \choose n} c_{n},$ 

thus it is

where  $\binom{\mu}{\lambda}$  are Christoffer's symbols of second class. At the same time:

$$\frac{\sum \sum_{\mu} \partial k^{(\mu)} e_{\mu} + \sum \sum_{\mu} \sum_{\mu} k^{\mu} \mu \begin{pmatrix} \mu & \nu \\ \mu \end{pmatrix} e_{\mu} + \sum P^{(\mu)} e_{\nu} = 0.$$

Should the base vector be everywhere called ., and index exchange has to be effected. If we decompose this vectorial equation into its components we get

This equation delivers for each of the three directions h=1,2,3 a partial differential equation. The middle member considers the bending of the coordinate curves, however the differentiation in (4) follows the substance coordinates which are curved in the deformed state.

3. Internal energy

The approach up to now gives each point a distortion and tension state. To represent the interconnection of both we have to consider the internal energy. Its existence follows for all reversible static phenomena from thermodynamic considerations. The following meanings are used:

e=[e, e, e]=1/[Inv] volume of each grid unit
dV=edx<sup>(n)</sup> d.<sup>(n)</sup> d.<sup>(n)</sup> d.<sup>(n)</sup> d.<sup>(n)</sup> d.<sup>(n)</sup> volume of an infinitesimal paralelopiped
e Internal energy por Volume unit - non deformed
internal energy of each grid unit booly
d\*E=edV internal energy of volume dV

In order to represent the internal energy e as a function of the distortion magnitudes  $p_{r_{i}^{m}}$ , we have to find the invariants of the distorted state. An absolute invariant for all values of parameter  $\lambda$  is

 $\frac{1}{|G_{*\mu}|}|\gamma_{*\mu}-\lambda G_{*\mu}|.$ 

By developing the determinants by exponents of  $\lambda$  a 3d degree function in  $\lambda$  results. Since this is an invariant for all  $\lambda$  the coefficients of the cubic form must also be invariants. Thus three invariants are formed:

$$\begin{split} I_{1} &= \frac{1}{\mid G_{r,\mu} \mid} \left\{ \gamma_{11} \left( G_{22} G_{33} - G_{23}^{2} \right) + \gamma_{22} \left( G_{33} G_{11} - G_{31}^{2} \right) + \gamma_{33} \left( G_{11} G_{22} - G_{12}^{2} \right) \\ &+ 2 \gamma_{12} \left( G_{51} G_{23} - G_{12} G_{33} \right) + 2 \gamma_{23} \left( G_{12} G_{13} - G_{23} C_{11} \right) + 2 \gamma_{31} \left( G_{23} G_{12} - G_{31} G_{22} \right) \right\}, \\ I_{2} &= \frac{1}{\mid G_{r,\mu} \mid} \left\{ G_{11} \left( \gamma_{22} \gamma_{33} - \gamma_{22}^{2} \right) + G_{22} \left( \gamma_{33} \gamma_{11} - \gamma_{31}^{2} \right) + G_{33} \left( \gamma_{11} \gamma_{22} - \gamma_{12}^{2} \right) \\ &+ 2 G_{12} \left( \gamma_{51} \gamma_{23} - \gamma_{12} \gamma_{33} \right) + 2 G_{23} \left( \gamma_{12} \gamma_{13} - \gamma_{23} \gamma_{11} \right) + 2 G_{31} \left( \gamma_{23} \gamma_{12} - \gamma_{31} \gamma_{22} \right) \right\}, \\ I_{3} &= \frac{\mid \gamma_{r,\mu} \mid}{\mid G_{r,\mu} \mid}. \end{split}$$

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For  $\mathcal{J}_{\mathcal{V}\mathcal{H}}$  I<sub>1</sub> is linear, I<sub>2</sub> is quadratic and I<sub>3</sub> cubical. further, I<sub>2</sub> follows from I<sub>1</sub> through a substitution of G with  $\gamma$ . For rectangular normal coordinates it is:

 $I_{1} = \gamma_{11} + \gamma_{22} + \gamma_{33},$   $I_{2} = \gamma_{11} \gamma_{22} + \gamma_{22} \gamma_{33} + \gamma_{33} \gamma_{11} - \gamma_{12}^{2} - \gamma_{23}^{2} - \gamma_{31}^{2},$   $I_{3} = |\gamma_{12}|,$ 

What is to be now attached for e? From numerous possibilities we take the simplest which corresponds to the classical theory. Thus  $\mathcal{E}\mathcal{E}$  is represented by the simplest quadratic invariant of the distorted state:  $e = \frac{V|G_{1,n}|}{V|I_{1,n}|} \left\{ \frac{a}{2} I_{1,n}^{2} - \beta I_{2} \right\}.$ 

The forward factor is as a volume ratio of the paralelopiped between the non-deformed and deformed states - an invariant. The constants  $\alpha$  and  $\beta$  are the two independent elasticity constants of the classical theory

 $\alpha = \frac{G m - 1}{2 m - 2}, \quad \beta = \frac{G}{2}.$ 

G means here the thrust modulus and m PoXisson's number of the cross-sectional contraction of the material.

The internal energy for each grid unit is thus

$$c = V[G_{1,p}] \left\{ \frac{a}{2} I_1^2 - \beta I_2 \right\}$$
. (5).

From the condition that e has to be positive it follows just as in the classical elasticity theory  $2 \leq m \leq \infty$ . For this span (5) is applicable, since (cc) is a positive, definite, quadratic homogeneous form in  $r_{rr}$ . For rectangular normal coordinates it is

$$e \varepsilon = \frac{\alpha}{2} (\gamma_{11} + \gamma_{22} + \gamma_{33})^2 - \beta (\gamma_{11} \gamma_{22} + \gamma_{22} \gamma_{33} + \gamma_{33} \gamma_{11} - \gamma_{12}^2 - \gamma_{23}^2 - \gamma_{31}^2).$$

The infinitely small deformations may easily be arranged into (5), when the linear expressions are introduced for the distortion magnitudes.

4. Tension-expansion equations and Hooke's law extended to finite deformations

The equilibrium conditions (4) are not sufficient alone to determine all tension and distortion magnitudes. For this we also need the relationships between the acting forces and the deformations caused by them.

 $\frac{\partial k^{(n)}}{\partial x^{(n)}} dx^{(n)} dx^{(2)} dx^{(3)}$ 

### Figure 2

As in the classical theory a possibility is available when observing the internal energy, to deduce tension-expansion equations, if we part from the internal energy of each grid unit.

We will take a real displacement  $\mathcal{V}$  and add to it an additional displacement  $\mathcal{SU}$  and the  $\mathcal{Y}_{\mathcal{V}\mathcal{H}}$  will change by  $\mathcal{S}_{\mathcal{V}\mathcal{V}\mathcal{H}}$ . The increase in internal energy is equal to the work exerted by the tension forces upon a volume element. Point A suffers an additional displacement  $\partial v + \frac{\partial \partial v}{\partial x^{(i)}} \frac{dx^{(i)}}{2}$ and point B  $\partial v - \frac{\partial \delta v}{\partial x^{(i)}} \frac{dx^{(i)}}{2}$ . With this additional displacement an internal energy change is obtained  $t^{(i)} \cdot \frac{\partial \delta v}{\partial x^{(i)}} dx^{(i)} dx^{(i)} dx^{(i)}$ Through cyclical substitutions internal energy changes are obtained for the remaining two directions. Their sum comprises per grid unit:

$$\delta(ce) = \sum_{i} f^{in} \cdot \frac{\partial \delta v}{\partial x^{in}} = \sum_{i} \sum_{\mu} \frac{v}{\mu} k^{in} e_{\mu} \cdot \frac{\partial \delta v}{\partial x^{in}}$$

now

and because  $\Gamma_{\mu} = e_{\mu} \cdot e_{\mu}$  it is  $\frac{\partial \Gamma_{\nu\mu} = e_{\mu} \cdot \partial e_{\mu} + e_{\mu} \cdot \partial e_{\mu}}{\partial e_{\mu} + e_{\mu} \cdot \partial e_{\mu}}$ 

The following representations are obtained for the internal energy change:

 $v_{\mu} = (v_{\mu} + \frac{\partial v}{\partial v_{\mu}})$  with  $\frac{\partial \partial v}{\partial v_{\mu}}$   $\Gamma_{\nu\mu} = G_{\nu\mu} + \gamma_{\nu\mu}$ , with  $\delta \Gamma_{\nu\mu} = \delta \gamma_{\nu\mu}$ ,

On the other hand based on formula (5)

 $\delta(e\varepsilon) = \frac{\delta(e\varepsilon)}{\delta\gamma_{11}} \delta\gamma_{11} + \frac{\delta(e\varepsilon)}{\delta\gamma_{22}} \delta\gamma_{22} + \frac{\delta(e\varepsilon)}{\delta\gamma_{23}} \delta\gamma_{33} + \frac{\delta(e\varepsilon)}{\delta\gamma_{12}} \delta\gamma_{12} + \frac{\delta(e\varepsilon)}{\delta\gamma_{23}} \delta\gamma_{23} + \frac{\delta(e\varepsilon)}{\delta\gamma_{23}} \delta\gamma_{33} + \frac{\delta(e\varepsilon)}{\delta\gamma_{33}} \delta\gamma$ 

Comparing both expressions we get the six equations

 $k^{11} = \frac{2 \partial (c \epsilon)}{\partial \gamma_{11}}, \qquad k^{12} = \frac{\partial (c \epsilon)}{\partial \gamma_{12}},$   $k^{22} = \frac{2 \partial (c \epsilon)}{\partial \gamma_{22}}, \qquad k^{23} = \frac{\partial (c \epsilon)}{\partial \gamma_{23}},$   $k^{33} = \frac{2 \partial (c \epsilon)}{\partial \gamma_{23}}, \qquad k^{31} = \frac{\partial (c \epsilon)}{\partial \gamma_{31}}$ (7).

These are the tension-expansion equations of the finite defor-

mations theory. They express that the tension magnitudes  $k^{\nu\mu}$ may be gotten through a partial differentiation of specific internal energy (ee) according to the distortion magnitudes  $\lambda_{\nu\mu}$ . They are valid in form (7) even when not starting from normal coordinates.

According to (5) the internal energy (ee) is a homogeneous quadratic form of the distortion magnitudes  $V_{UH}$ . The  $k^{UH}$ are according to (7) linear homogeneous functions of the distortion magnitudes and are:

$$k^{11} = \frac{2}{\gamma |G_{r\mu}|} [a I_1 (G_{22} G_{33} - G_{23}^2) - \beta (G_{22} \gamma_{33} + G_{33} \gamma_{11} - 2 G_{23} \gamma_{23})],$$
  

$$k^{12} = \frac{2}{\gamma |G_{r\mu}|} [a I_1 (G_{13} G_{23} - G_{12} G_{33}) - \beta (G_{23} \gamma_{13} + G_{13} \gamma_{23} - G_{33} \gamma_{12} - G_{12} \gamma_{23})]$$
(8)

etc, through cyclical substitutions.

These equations are Hook's law extended to finite deformations. For rectangular normal coordinates it is

$k^{11} = G\left(\gamma_{11} + \frac{\Phi}{m-2}\right).$	$k^{12} = G \gamma_{12},$	
$k^{22} = G\left(\gamma_{22} + \frac{\Phi}{m-2}\right),$	$k^{23}=G\gamma_{23},$	(9).
$k^{33} = G\left(\gamma_{33} + \frac{\Phi}{m-2}\right),$	$k^{a_1} = G \dot{\gamma}_{a_1}$	

where

\$= 711 + 7 11 + 7 11 ist.

Equations (9) convert into the classical tension-expansion equations when infinitely small deformations are allowed. -13Then with

$$\begin{split} \frac{\partial}{\partial} \frac{u^{(1)}}{x^{(1)}} &= u_x, \qquad \frac{\partial}{\partial} \frac{u^{(2)}}{x^{(2)}} = v_y, \qquad \frac{\partial}{\partial} \frac{u^{(3)}}{x^{(3)}} = w_z, \\ \Phi &= 2\left(u_x + v_y + w_z\right) = 2\Theta, \\ k^{11} &= 2G\left(u_x + \frac{\Theta}{m-2}\right), \qquad k^{12} = G\left(u_y + v_x\right), \\ k^{22} &= 2G\left(v_y + \frac{\Theta}{m-2}\right), \qquad k^{23} = G\left(v_z + w_y\right), \\ k^{32} &= 2G\left(w_z + \frac{\Theta}{m-2}\right), \qquad k^{31} = G\left(w_x + u_z\right). \end{split}$$

Equations (9) supply the connection between the tension components and the distortion magnitudes. If the tension component's  $k^{\nu\mu}$ 

values from (9) are put into the equilibrium conditions (4) we obtain for the determination of the equilibrium the necessary differential equations for the displacement components  $\mathcal{U}_{i}^{(u)}, \mathcal{U}_{i}^{(2)}$ ,  $\mathcal{U}_{i}^{(3)}$ . For these differential equations (exactly as in the classical elasticity theory) correspond limiting conditions which ma refer to displacements or to surface forces. 5. Stability of equilibrium

Above we presented equations which serve to ascertain he equilibrium. For the study of stability it is anticipated that these equations be integrated so that we may know the tensions and displacements of the equilibrium studied for stability. To apply the theory we shall satisfy ourselves with a relatively simple particular case - a compressed rod- whose tension and displacement states may simply be overlooked. To deal with the stability we have to compare the internal energy of the body in equilibrium state with the internal energy in a neighboring state. Thus, besides considering the equilibrium displacements  $2i_{1}^{(2)}2i_{2}^{(2)}$  we have to consider the neighboring displacements  $n^{(2)} n^{(2)} n^{$ 

E = fff (c e) d x" d x" d x"

in the periphery of the equilibrium state by exponents of dur and eventually developed by their derivation.

According to the statements in the introduction the equilibrium will be stable when for each allowable (thus compatible with the geometric conditions) system of "disturbances"  $\delta_{2}$  ( $\omega$ the so called second variation  $\delta^{2}\mathcal{E}$  (which when developed contains the quadratic members) is always positive.

If we describe the differentiations by the following indices, for exampel  $u_i^{(m)} = \frac{\partial u^{(m)}}{\partial x^{(0)}}$ . then the second variation will acquire the following form 4)

$$\begin{split} \delta^{2} E &= \int \int \int \left\{ \frac{1}{2} \sum_{h} \left[ 4 \, a \, (1 + u_{h}^{(h)})^{2} + 2 \, \beta \, (u_{h+1}^{(h)})^{2} + 2 \, \beta \, (u_{h+2}^{(h)})^{2} + k^{h\,h} \right] \, \delta \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \frac{1}{2} \sum_{h,k} \left[ 2 \, \beta \, (1 + u_{h}^{(h)})^{2} + 2 \, \beta \, (u_{2}^{(h)} + k)^{2} + 4 \, a \, (u_{h}^{(h)})^{2} + k^{h\,h} \right] \, \delta \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \sum_{h,k} \left[ 2 \, (\alpha - \beta) \, (1 + u_{h}^{(h)}) \, u_{h}^{(h)} + k^{h\,h} \right] \, \delta \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \sum_{h,k} \left[ 4 \, a \, (1 + u_{h}^{(h)}) \, u_{h}^{(h)} + 2 \, \beta \, (1 + u_{h}^{(h)}) \, u_{h}^{(h)} + 2 \, \beta \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \sum_{h,k} \left[ 4 \, a \, (1 + u_{h}^{(h)}) \, u_{h}^{(h)} + 2 \, \beta \, (1 + u_{h}^{(h)}) \, u_{h}^{(h)} + 2 \, \beta \, u_{h}^{(h)} \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \sum_{h,k} \left[ 4 \, (\alpha - \beta) \, (1 + u_{h}^{(h)}) \, (1 + u_{h}^{(k)}) + 2 \, \beta \, u_{h}^{(h)} \, u_{m}^{(h)} \right] \, \delta \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \sum_{h,k,m} \left[ 2 \, (2 \, a - \beta) \, u_{m}^{(h)} \, u_{m}^{(h)} + 2 \, \beta \, u_{h}^{(h)} \, u_{m}^{(h)} \right] \, \delta \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \sum_{h,k,m} \left[ 2 \, \beta \, (1 + u_{h}^{(h)}) \, (1 + u_{h}^{(h)}) + 4 \, (\alpha - \beta) \, u_{h}^{(h)} \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \sum_{h,k,m} \left[ 2 \, \beta \, (1 + u_{h}^{(h)}) \, u_{h}^{(k)} + 2 \, \beta \, (1 + u_{h}^{(h)}) \, u_{m}^{(k)} \right] \, \delta \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \sum_{h,k,m} \left[ 2 \, \beta \, (1 + u_{h}^{(h)}) \, u_{h}^{(k)} + 2 \, \beta \, (1 + u_{h}^{(h)}) \, u_{m}^{(k)} \right] \, \delta \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \sum_{h,k,m} \left[ 2 \, \beta \, (1 + u_{h}^{(h)}) \, u_{h}^{(k)} + 2 \, \beta \, (1 + u_{h}^{(h)}) \, u_{m}^{(k)} \right] \, \delta \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \\ &+ \sum_{h,k,m} \left[ 2 \, \beta \, (1 + u_{h}^{(h)}) \, u_{h}^{(k)} + 4 \, \alpha \, u_{h}^{(m)} \, u_{h}^{(k)} + 2 \, \beta \, (1 + u_{h}^{(k)}) \, \delta \, u_{h}^{(h)} \, \delta \, u_{h}^{(h)} \, \delta \, u_{h}^{(m)} \right] \, d \, x^{(1)} \, d \, x^{(2)} \, d \, x^{(3)} \\ \end{array} \right\}$$

The indices go through values from one to three and two differently numbered indices may never acquire the same value. Values above three of the indices are to be reduced by modulus three.

Now the matter is to decide in special cases if such allowable disturbances  $\partial u'$  may be found for which the second variations becomes negative or if it remains positive. Now then the displacements and tensions of the equilibrium state are to be regarded as given and we should now look for the "most cangerous" disturbances, such  $\partial u''$  which would possibly allow the second variation to be negative.

6. Determination of the stability limit

To determine the stability limit of an elastic equilibrium state we need to consider the second variation of the internal energy. It may be obtained from expression (10) if we introduce into it the values corresponding to this special case for displacement and tension components. Since the quadratic form in  $\int u^{(u)}$  under the integral and its derivations may always be written as a difference of two positively defined forms, we should write:

 $\delta^* E = Q_1 - Q_2$ 

Where  $Q_1$  and  $Q_2$  are integrals over positively definite quadratic forms in the derivations of  $\delta q^{(\nu)}$ .

To find if the difference could become negative, we write:  $N^* E = Q_1(\lambda - 1), \quad \lambda = \frac{Q_1}{Q_1}.$ 

The smaller  $\lambda$  becomes the higher the danger for  $\delta_{z} \mathcal{E}$  to become

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negative. The "most dangerous" displacement from equilibrium state is such for which  $\lambda$  becomes a minimum.

. . . . . 111

If  $\lambda$  becomes a minimum then  $\delta \lambda = \frac{Q_s \delta Q_1 - Q_1 \delta Q_2}{Q_2^2}$ must disappear, thus it should be

for all allowable variations  $dou^{(u)}$  .

3Q, =10Q, . .

Equation (11) has a solution  $\mathcal{A}=\mathcal{I}$  which may be larger, equal or smaller than one. The case  $\mathcal{J}=/$  gives the stability limit.

Since we are asking only for a stability limit, the problem may be simplified further. Equation (11) says: for the "most dangerous" displacement  $\delta x'' \delta x''$ ,  $\delta x''$  from equilibrium state is for all changes  $\delta (\delta x'')$ 

# 89, = 289,

Now, instead of finding the value  $\lambda$  for all the given forces and then ask for which forces  $\lambda = 1$ , we may directly put  $\lambda = 1$ .

It should be noted that the stability limit may also be determined from the isoperimetric problem: among all the allowable displacement variations from the equilibrium state the "most dangerous" are those for which the integral Q<sub>1</sub> becomes a minimum for a neighboring condition  $Q_2=1$ . With  $\mathcal{A}$  as the Lagrange factor for the neighboring condition it becomes identical to equation  $\delta Q_1 = \lambda \delta Q_2$ 

which agrees with (11).

The stability limit criterion expressed here in connection with the variation calculation may also be converted into a homogeneous system of differential equations and homogeneous limiting conditions for the "most dangerous" variation. Such a homogeneous problem has several solutions different from zero when the parameters characterizing the equilibrium state (loads) acquire definite critical values (break values).

This method will be used below with the example of a compressed rod.

II. Stability of a Compressed Rod7. Equilibrium state

To carry out the Trefftz method in order to find the stability limit of a compressed rod the definitions should be fitted to this particular problem.

The coordinates are designated x,y,z. The coordinate system is layed out so that the z-axis coincides with the rod axis and the remaining axis fall within the cross-section of the rod.

The equilibrium displacements are indicated with capital letters U,V,W. Variations of U,V,W will be indicated by small  $u,v,\overset{\omega}{\lambda}$  and the variations of these latter will be indicated by  $\delta u$ ,  $\delta v$ ,  $\delta w$ .

The rod considered should be "fixed". This means that all points of the lower and all points of the uppper cross-sections

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have suffered the same vertical displacement; thus W should be predetermined for the cross-sections of the extremes. From all the disturbances of the equilib ium state only those are acceptable whose w=0 at the frontal surfaces of the rod, and thus also

$$\frac{\delta w}{\delta x} = 0$$
 and  $\frac{\delta w}{\delta y} = 0$ .

The rod is held by its lower end (z=0) and compressed by a portion  $\Delta L$  by a force P=p•f (f is the cross-section). then the displacements are

 $U = a_x x; \quad V = a_y y; \quad W = a_z z,$ 

and these may be neglected.

The connection between the expansions  $a_x, a_y, a_z$  and pressure p is obtained from the tension-expansion equations (9) which now appar as  $\kappa^{xx} = G\left(r_{xx} + \frac{\Phi}{m-2}\right) = 0$ ,



it is

$$\Phi = -\frac{m-2}{m+1}\frac{p}{G},$$

 $\gamma_{xx} = \gamma_{yy} = \frac{p}{(m+1)} G'$ 

Yzz

and

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$\gamma_{xx} = 2 a_x + a_x^2,$	$a_x = \sqrt{1 + \gamma_{xx}} - 1,$
$\gamma_{yy} = 2 a_y + a_y^2,$	$a_y = \frac{1}{1 + \gamma_{yy}} - 1,$
$\gamma_{zz}=2a_z^2+a_z^2,$	$a_z = \frac{1}{1 + \gamma_{zz}} - 1.$

The equilibrium conditions (4) are (as in any tension state) self-fulfilling.

8. Second variation of the internal energy

The homogeneous equilibrium loses above a "critical"magnitude of pressure p its stability.

To find the stability limit the second internal energy variation has to be considered. It results in :

$$\begin{split} \delta^{\beta} E &= \iiint \{2 a \left[ (1 + a_{x})^{2} u_{x}^{2} + (1 + a_{y})^{2} v_{y}^{2} + (1 + a_{z})^{2} w_{z}^{2} \right] \\ &+ 4 \left( a - \beta \right) \left[ (1 + a_{x}) \left( 1 + a_{y} \right) u_{x} v_{y} + (1 + a_{y}) \left( 1 + a_{z} \right) v_{y} w_{z} + (1 + a_{z}) \left( 1 + a_{x} \right) w_{z} u_{x} \\ &+ \beta \left\{ (1 + a_{x})^{2} \left( u_{y}^{2} + u_{z}^{2} \right) + (1 + a_{y})^{2} \left( v_{x}^{2} + v_{z}^{2} \right) + (1 + a_{z})^{2} \left( w_{x}^{2} + w_{y}^{2} \right) \\ &+ 2 \left[ (1 + a_{x}) \left( 1 + a_{y} \right) v_{x} v_{y} + 2 \left( 1 + a_{y} \right) \left( 1 + c_{z} \right) w_{y} v_{z} + 2 \left( 1 + a_{z} \right) \left( 1 + a_{x} \right) u_{z} w_{x} \right] \right\} dx dy dz \\ &= \frac{p}{2} \iiint \left( u_{z}^{2} + v_{z}^{2} + w_{z}^{2} \right) dx dx dz \,. \end{split}$$

 $\overline{w} = (1 + u_z) w$ 

introducing instead of disturbances u, v, w

 $\hat{u} = (1 + a_y) u$ ,  $\tilde{v} = (1 + a_y) \tilde{v}$ ,

and with  $a = \frac{Gm}{2m} \frac{1}{2}, \beta = \frac{G}{2}$  it is

and with

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 $\delta^{*} E = \iiint \left\{ \frac{\hat{u}_{x}}{2} \frac{1}{m} - \frac{1}{2} (u_{x} + \bar{v}_{y} + \bar{w}_{z})^{2} - 2 (\hat{u} | 1 \, u_{x} \, \bar{v}_{y} + 1 \, \bar{v}_{y} \, \bar{w}_{z} \right. \\ \left. + 1 \, \bar{w}_{z} \, u_{x} - (\hat{u}_{y} + \bar{v}_{x})^{2} - (\bar{v}_{z} + \bar{w}_{y})^{2} - (\bar{w}_{x} + \bar{u}_{z})^{2} \right\} dx \, dy \, dz \\ \left. \frac{p}{2} \iiint \left[ \frac{\hat{u}_{z}^{2}}{(1 + a_{x})^{2}} + \frac{\bar{v}_{z}^{2}}{(1 + a_{y})^{2}} + \frac{\bar{w}_{z}^{2}}{(1 + a_{z})^{2}} \right] dx \, dy \, dz \, .$ 

The first integral is equal exactly to the form change work of the classical elasticity theory. It shows the form modification work involved in small displacements  $\widetilde{\mathcal{U}}, \widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}$  when these alone are present. The second integral contains the acting pressure p. And so the second variation of the internal energy of a compressed rod is written as a difference of two integrals over positively defined quadratic forms in derivations of  $\widetilde{\mathcal{U}}, \widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}$ ,

thus:

 $\delta^2 E = Q_1 - Q_2$ 

9. Jacobi's equations

After developments in 6, we reach the stability limit for an elastic equilibrium state; if there is a "most dangerous" variation u,v,w for which with every acceptable variation  $\mathcal{S}_{24}, \mathcal{O}_{V}, \mathcal{O}_{W}$  it is  $\mathcal{O}_{1} = \mathcal{O}_{2}$  and for the case of a compressed rod  $Q_1$  and  $Q_2$  have the above values.

The satisfaction of equation (12) for any variations of u,v,w leads to a system of three partial differential equations. It represents Jacobi's equations adapted to an equilibrium problem in a form of a variation's problem (6).

To simplify the derivation the crosslines above  $\mathcal{U}$ ,  $\mathcal{V}$ ,  $\mathcal{W}$  are again omitted. The partial derivatives of the integrants of  $Q_1$  along the break magnitudes  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$  etc are in formal accordance with the classical theory designated by the tensions  $\mathcal{O}_{\mathcal{X}}$ ,  $\mathcal{T}_{\mathcal{X}\mathcal{Y}}$  etc.

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$$\sigma_x = 2 \ G \left\{ \frac{\partial u}{\partial x} + \frac{\Theta}{m-2} \right\} \ \tau_{xy} = G \left\{ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\}$$

etc. where  $\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ 

Since Q1 has the shape of Hooks form change work, the first variation of  $Q_1$  results as when  $\theta = u_x + v_y + w_z$ 

$$\delta Q_{1} = 2 \iiint \delta u \ G \left( \Delta u + \frac{m}{m-2} \frac{\delta \Theta}{\delta x} \right) dx \ dy \ dz \\ + 2 \iiint \delta v \ G \left( \Delta v + \frac{m}{m-2} \frac{\delta \Theta}{\delta y} \right) dx \ dy \ dz \\ + 2 \iiint \delta w \ G \left( \Delta w + \frac{m}{m-2} \frac{\delta \Theta}{\delta z} \right) dx \ dy \ dz \\ + 2 \iiint \delta w \ G \left( \Delta w + \frac{m}{m-2} \frac{\delta \Theta}{\delta z} \right) dx \ dy \ dz \\ = 5 \lim_{Mantel} \left[ \delta u \left[ \sigma_{x} \cos \left( n, x \right) + \tau_{xy} \cos \left( n, y \right) \right] + \delta v \left[ \tau_{xy} \cos \left( n, x \right) + \sigma_{y} \cos \left( n, y \right) \right] \right] \\ + \delta w \left[ \tau_{zx} \cos \left( n, x \right) + \tau_{zy} \cos \left( n, y \right) \right] do \\ - \left\{ \left[ \left[ \delta u \tau_{zx} + \delta v \tau_{zy} + \delta w \sigma_{z} \right] do + \left[ \left[ \left[ \delta u \tau_{zx} + \delta v \tau_{zy} + \delta w \sigma_{z} \right] do \right] \right] \right] \\ = 0$$

further

 $\partial Q_z = \frac{2p}{(1+a_x)^2} \iiint \partial u \frac{\partial^2 u}{\partial z^2} dx dy dz + \frac{2p}{(1+a_y)^2} \iiint \partial v \frac{\partial^2 v}{\partial z^2} dx dy dz$  $+ \frac{2p}{(1+a_{z})^{2}} \int \int \partial w \frac{\partial^{2} w}{\partial z^{2}} dx dy dz$  $\begin{array}{c}
\left| \left( \begin{array}{c} p & \partial u \\ (1+a_x)^2 & \partial z \end{array} \right)^n + \left( \begin{array}{c} p & \partial v \\ (1+a_y)^2 & \partial z \end{array} \right)^n + \left( \begin{array}{c} 1+a_y \end{array} \right)^2 & \partial z \end{array} \right)^n + \left( \begin{array}{c} p & \partial u \\ (1+a_z)^2 & \partial z \end{array} \right)^n \\
+ \left| \left( \begin{array}{c} p & \partial u \\ (1+a_x)^2 & \partial z \end{array} \right)^n + \left( \begin{array}{c} 1+a_y \end{array} \right)^2 & \partial z \end{array} \right)^n \\
= \left| \begin{array}{c} nut & \text{stirud} \\ \text{fower Front surface} \end{array} \right|^n \\
\quad \text{Lower Front surface} \\
\end{array}$ 

Equation SQ = QQ may be possible for all acceptable variations only when the integrants in the space integrals are mutually equal. It follows:

 $G\left(Au + \frac{m}{m} \frac{\partial}{2} \frac{\partial}{\partial x}\right) - \frac{p}{(1+a_x)^2} \frac{\partial^2 u}{\partial z^2},$  $G\left(Av + \frac{m}{m-2} \frac{\partial}{\partial y}\right) = \frac{p}{(1+a_y)^2} \frac{\partial^2 v}{\partial z^2},$ (13).  $G\left(Aw + \frac{m}{m-2}\frac{\partial}{\partial z}\Theta\right) = \frac{p}{(1+a_z)^2}\frac{\partial^2 w}{\partial z^2}$ 

These are Jacobi's equations.

Limiting conditions belong to the above and follow from the surface integrals.

a) Integrals touched only by SQ over the shell plane have to dissappear. It follows from this that on the shell surface following equations apply:

 $\sigma_x \cos{(n, x)} + \tau_{xy} \cos{(n, y)} = 0$  $\tau_{xy}\cos\left(n,x\right)+\sigma_y\cos\left(n,y\right)=0$  $\tau_{xz}\cos\left(n,x\right)+\tau_{yz}\cos\left(n,y\right)=0$ 

 $(\cos n, z=0)$ 

They indicate that a force-free shell plane corresponds to the displacements u,v,w.

b) The integrals of  $\mathcal{Q}_{1}$  and  $\mathcal{Q}_{2}$  taken over the front surfaces must be equal. Since according to 7, for all allowable disturbances and their variations at both front surfaces are w=0 and  $\mathcal{Q}_{W}=0$  for all points, also w<sub>x</sub>=0 and w=0, it follows that for the front surfaces  $\mathcal{Q}_{2X} = \mathcal{G}\mathcal{U}_{2}$  and  $\mathcal{Q}_{XY} = \mathcal{G}\mathcal{V}_{2}$ . So for the upper

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front surface expression only

$$\iint \delta u \ G \ \frac{\partial u}{\partial z} \ dx \ dy = \iint \frac{\mu}{(1+u_x)^2} \ \frac{\partial u}{\partial z} \ \delta u \ dx \ dy$$

remains and for any chosen  $\delta u$ ,

$$\frac{\partial u}{\partial z} = 0.$$

The same applies to the lower front surface

$$\frac{\partial v}{\partial z} = 0$$
.

With the condition w = 0 we have for the limiting conditions at the front surfaces

 $w = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0 \quad \text{oder} \quad w = 0, \quad \tau_{zx} = 0, \quad \tau_{zy} = 0.$ 

Finally we have the "most dangerous displacement" u,v,w is sufficient for the homogeneous differential equations

$$G\left(Au + \frac{m}{m-2}\frac{\partial}{\partial x}\theta\right) = \frac{p}{(1+a_x)^2}\frac{\partial^2 u}{\partial z^2},$$
  

$$G\left(Av + \frac{m}{m-2}\frac{\partial}{\partial y}\theta\right) = \frac{p}{(1+a_y)^2}\frac{\partial^2 v}{\partial z^2},$$
  

$$G\left(Au + \frac{m}{m-2}\frac{\partial}{\partial z}\theta\right) = \frac{p}{(1+a_y)^2}\frac{\partial^2 w}{\partial z^2},$$

with homogeneous limiting or boundary conditions:

a) for the shell

. front surfaces

 $\begin{aligned} \tau_{x,y}\cos{(n,y)} + \sigma_x\cos{(n,x)} &= 0, \\ \sigma_y\cos{(n,y)} + \tau_{x,y}\cos{(n,x)} &= 0, \\ \tau_{y,z}\cos{(n,y)} + \tau_{y,z}\cos{(n,x)} &= 0, \end{aligned}$ 

b) for the front surfaces

w 0, n. 0, r. ..0,

and the tensions according to the classical theory

$$\sigma_x = 2 G \left( \frac{\partial u}{\partial x} + \frac{\Theta}{m-2} \right), \qquad \tau_{xy} = G \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right).$$

Equations (13) give a simple meaning which gives the connection of the extended theory with the elementary approximation

theory. When in the equilibrium the equations of the classical  $G\left(Au + \frac{m}{m-2}\frac{\partial}{\partial x}\theta\right) + X = 0,$   $G\left(Av + \frac{m}{m-2}\frac{\partial}{\partial y}\theta\right) + Y = 0,$  $G\left(Av + \frac{m}{m-2}\frac{\partial}{\partial z}\theta\right) + Z = 0$ 

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and

 $X = -\frac{p}{(1+a_x)^2} u_{xx}, \quad Y = -\frac{p}{(1+a_y)^2} v_{zz}, \quad Z = -\frac{p}{(1+a_z)^2} w_{zz}$ are put as the volume forces we obtain equations (13). This reveals that the integration of these equations is identical to the task dealing with finding the equilibrium profile of the rod when, at a proper p-value, the loads per each volume unit are proportional to the magnitudes  $\mathcal{U}_{22}$ ,  $\mathcal{V}_{22}$ ,  $\mathcal{V}_{22}$ .

When the volume load over the cross-section of the rod is integrated we obtain loads per length unit of here rod  $\int X dF = -\frac{P}{(1+a_x)^2} \frac{d^2[u]}{dz^2}, \quad \int Y dF = -\frac{P}{(1+a_y)^2} \frac{d^2[v]}{dz^2}, \quad \int Z dF = -\frac{P}{(1+a_z)^2} \frac{d^2[w]}{dz^2},$ where  $\frac{d^2[w]}{dz^2}$  erc. are the average values. If these average values are replaced by the differential ratios  $\frac{d^2[u]}{dz^2}, \frac{d^2[u]}{dz^2}$ taken for the rod's centreline, and the small magnitude  $\mathbf{Q}$  in the denominator is neglected, the task arises to find the equilibrium profile of the rod which is loaded vertically with forces  $P^{d^2[u]}$   $d^2[u]$   $d^2[u]$ 

 $P\frac{d}{dz^2} - P\frac{d^2v}{dz^2}$  *EIw*<sup>IV</sup> = - *Pw*<sup>II</sup> which follows from the elementary theory.

III. Numerical Results

The Jacobi's equations together with the boundary conditions of page 361 (23) tell us that there exists a neighboring equilibrium state next to the initial conditions. The displacements that transfer the initial conditions into the neighboring conditions are the similar factors  $(I + \alpha_x)^2$  etc. solutions u,v,w. Since the Jacobi's equations and the boundary conditions are homogeneous we have a proper value problem. The proper value is the "critical" load p.

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In this section we solve the proper value problem for circular and rectangular cross-sections. In the first case - by integrating by development in series; in the second by the Ritz method.

10 Circular cross-section

Equation (13)

)	$G\left(\Delta u + \frac{m}{m-2}\frac{\partial}{\partial x}\theta\right) = \mu \frac{\partial^2 u}{\partial z^2},$	
	$G\left(1v + \frac{m}{m-2}\frac{\partial}{\partial}\frac{\partial}{y}\right) = p\frac{\partial^2 v}{\partial z^2},$	$u_x + v_y + w_z$
	$G\left(Aw + \frac{m}{m-2}\frac{\partial}{\partial}\frac{\partial}{z}\right) = p\frac{\partial^2 w}{\partial z^2}$	

is transformed for cylindrical coordinates. This means we seek the equilibrium profile of a rod loaded per volume unit by forces:

 $-p\frac{\partial^2 w}{\partial z^2} = X, \quad -p\frac{\partial^2 v}{\partial z^2} = Y, \quad -p\frac{\partial^2 w}{\partial z^2} = Z$ 

where the very small magnitudes  $\alpha_{\chi}, \alpha_{\gamma}, \alpha_{z}$  are negligible when compared to 1.

When the cylindrical coordinates are introduced  $r, \vartheta, z$  and the displacements are: radial  $\varrho$ , tangential  $\iota$  and  $\mathfrak{N}$ axial  $\mathfrak{M}$ , the problem is to find the equilibrium profile for a rod loaded in the three directions by volume forces  $-r_{\vartheta z}^{\vartheta e}, -r_{\vartheta z}^{\vartheta m}, -r_{\vartheta z}^{\vartheta m}$ 

when 6, 19, 02, 1, 0, 192, 1, 2

are the cylindrical coordinates of the tension forces.

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The equilibrium conditions [5] are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_r}{\partial \theta} + \frac{\partial \tau_r}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = p \frac{\partial^2 \theta}{\partial z^2},$$

$$\frac{\partial \tau_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_\theta}{\partial z} + \frac{2 \tau_r}{r} = p \frac{\partial^2 \tau}{\partial z^2},$$

$$\frac{\partial \tau_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_\theta}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_r}{r} = p \frac{\partial^2 w}{\partial z^2},$$

and the tension-eapansion equations are

$$\sigma_{r} = 2 G \left( \frac{\partial \varrho}{\partial r} + \frac{\Theta}{m-2} \right), \qquad \tau_{\vartheta z} = G \left( \frac{1}{r} \frac{\partial w}{\partial \vartheta} + \frac{\partial \tau}{\partial z} \right),$$
  

$$\sigma_{\vartheta} = 2 G \left( \frac{1}{r} \frac{\partial \tau}{\partial \vartheta} + \frac{\varrho}{r} + \frac{\Theta}{m-2} \right), \qquad \tau_{z r} = G \left( \frac{\partial \varrho}{\partial z} + \frac{\partial w}{\partial r} \right),$$
  

$$\sigma_{z} = 2 G \left( \frac{\partial w}{\partial z} + \frac{\Theta}{m-2} \right), \qquad \tau_{\vartheta r} = G \left( \frac{\partial \tau}{\partial r} - \frac{\tau}{r} + \frac{1}{r} \frac{\partial \varrho}{\partial z} \right),$$
  

$$\bullet \qquad \Theta = \frac{\partial \varrho}{\partial r} + \frac{1}{r} \frac{\partial \tau}{\partial \vartheta} + \frac{\varrho}{r} + \frac{\partial w}{\partial z}.$$

By eliminating tensions in the equilibrium conditions through a tension expansion equation we have Jacobi's equations with cylindrical coordinates

$$\begin{aligned}
\Delta &= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \\
G \left[ \Delta \varrho + \frac{m}{m-2} \frac{\partial \Theta}{\partial r} - \frac{2}{r^2} \frac{\partial \tau}{\partial \theta} - \frac{\varrho}{r^2} \right] &= p \frac{\partial^2 \varrho}{\partial z^2}, \\
G \left[ \Delta \tau + \frac{m}{m-2} \frac{1}{r} \frac{\partial \Theta}{\partial r} + \frac{2}{r^2} \frac{\partial \varrho}{\partial \theta} - \frac{\tau}{r^2} \right] &= p \frac{\partial^2 \tau}{\partial z^2}, \\
G \left[ \Delta r + \frac{m}{m-2} \frac{\partial \Theta}{\partial r} + \frac{2}{r^2} \frac{\partial \varrho}{\partial \theta} - \frac{\tau}{r^2} \right] &= p \frac{\partial^2 \tau}{\partial z^2}, \\
G \left[ \Delta w + \frac{m}{m-2} \frac{\partial \Theta}{\partial z} \right] &= p \frac{\partial^2 w}{\partial z^2}
\end{aligned}$$
(14).

Their boundary conditions

and b) m = 0,  $n_2 = 0$ ,  $n_3 = 0$  for the cylindrical shell b) m = 0,  $n_2 = 0$ ,  $n_3 = 0$  for the front planes (z=0, z=L). To integrate (14) we chose

 $\begin{array}{c}
\varrho = P(\mathbf{r})\cos n\,\vartheta\cos r\,z, \\
z = Q(\mathbf{r})\sin n\,\vartheta\cos r\,z, \\
w = R(\mathbf{r})\cos n\,\vartheta\sin r\,z
\end{array}$ (15)

It is sufficient for boundary conditions at the front planes when we take

For functions P(H) Q(H), R(H) three common differential equations follow from (14)

17.

Where the vertical lines signify derivatives accombing to and q equals p/G. -29-

Through the inotroduction of cylindrical coordinates it was succeeded to reduce the partial system (14) to a system with three common differential equations of second order.

Equations (17) supply the case of a break, that is a linear bending of a solid cylinder and a hollow wall column for n=1. The equation s are, when the cross-contraction value is 10/3:

> $r^{2}P'' + rP' + \frac{2}{7}r^{2}(q-1)r^{2}P - \frac{9}{7}P - \frac{9}{7}Q + \frac{5}{7}rQ' + \frac{5\nu}{7}r^{2}R' = 0,$   $r^{2}Q'' + rQ' + r^{2}(q-1)r^{2}Q - \frac{9}{2}Q - \frac{9}{2}P - \frac{5}{2}rP' - \frac{5\nu}{2}rR = 0,$  $r^{2}R'' + rR' + r^{2}\left(q - \frac{7}{2}\right)R - R - \frac{5\nu}{2}rP - \frac{5\nu}{2}r^{2}P' - \frac{5\nu}{2}rQ = 0$ (18).

The boundary conditions for the shell are simplified by using (15) in equations

 $\left.\begin{array}{c}
7r_{a}P'+3(P+Q)+3\nu r_{a}R=0,\\
P+Q-r_{a}Q'=0,\\
R'-\nu P=0
\end{array}\right\} \ldots \ldots \ldots (19).$ 

and in case of a hollow cylinder there are three analogous equations with  $\gamma_i$  instead of  $\gamma_6$  .

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behave in a regular manner. This regularity requirement supplies three more data for the determination of constants.

Equations (18) are integrated normally by developing exponential series for the solid cylinder. The succesive calculations of coefficients delivers, designating the coefficients  $c_1, c_2, c_3$ , the series:

$$P(\mathbf{r}) = \frac{c_1}{\nu} \left[ 1 + \frac{147 - 22}{7 \cdot 192} q(q-1) \mathbf{r}^s \mathbf{r}^s + \frac{434 - 343}{7 \cdot 96 \cdot 96} q^2 (q-1) \mathbf{r}^s \mathbf{r}^s + \dots \right] + \mathbf{r} \mathbf{c}_2 \left[ \mathbf{r}^2 + \frac{154 - 29}{7 \cdot 24} \mathbf{r}^2 \mathbf{r}^s + \frac{441 - 357}{7 \cdot 12 \cdot 96} q + \frac{41}{7 \cdot 12 \cdot 96} q^2 \mathbf{r}^s \mathbf{r}^s + \dots \right]$$

$$+ \mathbf{c}_3 \left[ \frac{140 - 15}{7 \cdot 96} \mathbf{r}^s \mathbf{r}^s + \frac{420 - 315}{7 \cdot 48 \cdot 96} q + \frac{20}{7 \cdot 48 \cdot 96} \mathbf{r}^s \mathbf{r}^s + \dots \right],$$

$$= Q(\mathbf{r}) = \frac{c_1}{\nu} \left[ 1 + 2(q-1) \mathbf{r}^2 \mathbf{r}^2 + \frac{231 - 206}{7 \cdot 192} q(q-1) \mathbf{r}^s \mathbf{r}^s + \frac{2450 - 4375}{49 \cdot 32 \cdot 144} q + \frac{2050}{7 \cdot 96} q^2 (q-1) \mathbf{r}^s \mathbf{r}^s + \dots \right]$$

$$+ \mathbf{r} \mathbf{c}_2 \left[ 19 \mathbf{r}^2 + \frac{266 - 241}{7 \cdot 24} \mathbf{r}^2 \mathbf{r}^s + \frac{2793 - 5061}{49 \cdot 32 \cdot 144} q + \frac{2399}{2} \mathbf{r}^s \mathbf{r}^s + \dots \right]$$

$$\left[ \frac{5rr^{2} + \frac{280 - 255q}{7 \cdot 96}r^{3}r^{4} + \frac{2940 - 5355q + 2540q^{2}}{49 \cdot 32 \cdot 144}r^{5}r^{4} + \ldots \right],$$

$$\left[ \frac{5}{7}(n-1)r^{2}r^{4} + \frac{70 - 45q}{7}r^{6} + \frac{735 - 945q + 335q^{2}}{7}r^{6} + \frac{735 - 945q + 335q + 335q^{2}}{7}r^{6} + \frac{735 - 945q + 335q + 335q^{2}}{7}r^{6} + \frac{735 - 945q + 335q + 335q$$

(20).

+ 0

$$\begin{aligned} & \left[ r^{2}(r) = \frac{1}{r} \left[ \frac{1}{8} (q-1) r^{3} r^{3} + \frac{13}{7 \cdot 192} (q-1) r^{5} r^{8} + \frac{133 - 945 q + 335 q^{2}}{49 \cdot 96 \cdot 96} (q-1) r^{7} r^{7} + \dots \right] \\ & + rc_{2} \left[ \frac{5 r r^{3} + \frac{70 - 45 q}{7 \cdot 24} r^{3} r^{8} + \frac{735 - 945 q + 335 q^{2}}{49 \cdot 12 \cdot 96} r^{5} r^{7} + \dots \right] \\ & + c_{3} \left[ -r + \frac{9 + q}{8} r^{2} r^{3} + \frac{133 - 76 q - 7 q^{2}}{7 \cdot 192} r^{4} r^{8} + \frac{1421 - 1743 q + 523 q^{2} + 49 q^{3}}{49 \cdot 96 \cdot 96} r^{6} r^{7} + \dots \right] \end{aligned}$$

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Apaet from these solutions the system (18) also contains logarhytmically singular solutions which fall out because of the regularity requirement.

Multiplying the series by the trigonometric functions of and z, accompding to (15), when q is found, we obtain the values for the "most dangerous" displacements from equilibrium for the cylindrical rod. We still have the three arbitrary integration constants  $c_1, c_2, c_3$  which have to be determined from the boundary conditions (19). They are:

> $7r_{a} l' + 3(l' + Q) + 3rr_{a} k = 0,$   $P + Q - r_{a} Q' = 0,$ R' - r P = 0

and, say, that the shell becomes tension free.

If we introduce P,Q,R values for  $r=r_a$  in (20) then the boundary conditions form a system of three homogeneous linear cequations for the constants  $c_{1}, c_{2}, c_{3}$ . The magnitude  $\mathcal{V} = \frac{2\pi}{L}$  and the radius  $\mathcal{T}_{L}$  appear with same exponents so that for their product  $k = \frac{2\pi r_{e}}{L}$  this system may be solved for values different from zero only if the determinant of the system disappears. Since the determinant contains the unknown load magnitude q, it can be set in such a waybthat the determinant becomes zero.

For the following calculations the series for P,Q,R we used the values given in (20). An ample calculation of the determinant

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leads to the equation

$$\left. \begin{array}{c} \frac{40}{3} q \, k^{2} - \frac{546 - 1099 \, q + 503 \, q^{2}}{63} \, k^{3} \\ - \frac{12495 - 36013 \, q + 31781 \, q^{2} - 3358 \, q^{3}}{49 \cdot 114} \, k^{6} - \ldots = 0 \end{array} \right\}$$

Further calculation is possible only numerically. Because of the complex structure of (21) with respect to q, load values are used to solve the series. Since q=p/G and the push module G is very high, q must be small. With G=800,000 kg/cm<sup>2</sup> we obtain for the loads p, which are within Hook's law, p=800kg/cm<sup>2</sup> and p=1,600kg/cm<sup>2</sup> the values q=1/1,000 or q=1/500.

The results of equation (21) were compared with the elementary theory where the break lenghth is given by  $I_{k}^{*} = i a^{\frac{EI}{P_{k}}}$ 

The result is that there occurs only a small deviation from Euler's formula. It is

 $q = \frac{1}{1000}, \qquad L^2 = 4\pi^2 \frac{E}{P_{\pi}} (1 - 0.0012),$  $q = \frac{1}{500}, \qquad L^2 = 4\pi^2 \frac{E}{P_{\pi}} (1 - 0.0027).$ 

. (21).

Thus within the calculation accuracy Euler's break formula is confirmed.

11. Rectangular cross-section

The rod has a lenghth L. The rectangle's dimensions are 2a and 2b(a > b). The position and the tension relationships are the same as before.

The stability limit is dominated by equation  $\mathcal{Q}_{p} = \mathcal{O}_{Q}$  where  $Q_{1}$  and  $Q_{2}$  have the meaning of page 360 (23).

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The Ritz method is used to solve the equation.

For displacements u, v, w we chose

 $u = A(xy)\cos vz$ ,  $v = B(xy)\cos vz$ ,  $w = C(xy)\sin vz$ ,  $v = \frac{2\pi}{L}$  .... (22) they reduce the spatial problem to a plane problem. It satisfies the boundary conditions for the front planes, where for z=0 and z=L it has to be w=0,  $u_z=0$ , 2z = 0. The integration of  $Q_1$  and  $Q_2$ over the rod lenghth L gives

$$Q_{1} = \frac{G}{2} \frac{L}{a-b} \int_{-a-b}^{+a} \left\{ \frac{m-1}{m-2} (A_{x} + B_{y} + \nu C)^{2} - 2A_{x} B_{y} - 2\nu B_{y} C - 2\nu A_{x} C + \frac{1}{2} (A_{y} + B_{x})^{2} + \frac{1}{2} (C_{y} - \nu B)^{2} + \frac{1}{2} (C_{x} - \nu A)^{2} \right\} dx dy, \qquad (23).$$

$$Q_{2} = \frac{\nu^{2}}{2} \frac{L}{b} \int_{-a-b}^{+a} \left[ A^{2} + B^{2} + C^{2} \right] dx dy$$

For A, B, C we chose

$$A = \frac{a_{0}}{\nu} + a_{1}x + a_{2}y + a_{11}\nu x^{2} + 2a_{12}\nu x y + a_{22}\nu y^{2},$$

$$B = \frac{\beta_{0}}{\nu} + \beta_{1}x + \beta_{2}y + \beta_{11}\nu x^{2} + 2\beta_{12}\nu x y + \beta_{22}\nu y^{2},$$

$$C = \frac{\gamma_{0}}{\nu} + \gamma_{1}x + \gamma_{2}y + \gamma_{11}\nu x^{2} + 2\gamma_{12}\nu x y + \gamma_{22}\nu y^{2}$$
(24).

•••

Introduced in (23) we have

$$\begin{split} h_{1} &= 2 \ a \ b \ L \ G \left\{ \left[ \frac{7}{4} \left( a_{1}^{2} + \beta_{2}^{2} + \gamma_{0}^{2} \right) + \frac{1}{2} \left( a_{0}^{2} + \beta_{0}^{2} + a_{2}^{2} + \beta_{1}^{2} + \gamma_{1}^{2} + \gamma_{2}^{2} \right) \right. \\ &+ \frac{3}{2} \left( a_{1} \beta_{2} + a_{1} \gamma_{0} + \beta_{2} \gamma_{0} \right) + a_{2} \beta_{1} - \beta_{0} \gamma_{2} - a_{0} \gamma_{1} \right] \\ &+ \left[ \frac{a_{1}^{2} + \beta_{1}^{2}}{2} + 7 \left( a_{11}^{2} + \beta_{12}^{2} \right) + 2 \left( \beta_{11}^{2} + \gamma_{11}^{2} + a_{12}^{2} + \gamma_{12}^{2} \right) + \frac{7}{4} \gamma_{1}^{2} \right. \\ &+ \left. a_{0} a_{11} + \beta_{0} \beta_{21} + \frac{7}{2} \gamma_{0} \gamma_{1} \right] + \left[ \frac{a_{1}}{2} a_{1} \gamma_{11} + \frac{3}{2} \beta_{2} \gamma_{11} + 2 a_{11} \gamma_{1} + 6 a_{11} \beta_{12} + 3 \gamma_{1} \beta_{12} + 1 a_{12} \beta_{11} - \beta_{11} \gamma_{2} - 2 \beta_{1} \gamma_{12} \right] \frac{y^{2}}{3} \\ &+ \left[ \frac{a_{2}^{2} + \beta_{2}^{2}}{2} + 7 \left( a_{12}^{2} + \beta_{22}^{2} \right) + 2 \left( a_{22}^{2} + \gamma_{22}^{2} + \beta_{12}^{2} + \gamma_{12}^{2} \right) + \frac{7}{1} \gamma_{1} \right. \\ &+ a_{0} a_{22} + \beta_{0} \beta_{22} + \frac{7}{2} \gamma_{0} \gamma_{22} - \frac{1}{2} \beta_{2} \gamma_{22} + \frac{3}{2} a_{1} \gamma_{22} + 2 \beta_{22} \gamma_{2} \end{split}$$

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$$+ 6 a_{12} \beta_{22} + 3 \gamma_{2} a_{12} + 4 a_{22} \beta_{12} - a_{22} \gamma_{1} - 2 a_{2} \gamma_{12} \left| \frac{r^{2} h^{2}}{3} + \left[ 2 (a_{12}^{2} + \beta_{12}^{2}) + a_{11} a_{22} + \beta_{11} \beta_{22} + \frac{7}{2} (\gamma_{11} \gamma_{22} + 2 \gamma_{12}^{2}) \right] \left| \frac{r^{4} a^{2} b^{2}}{9} + \left[ \frac{a_{11}^{2} + \beta_{11}^{2}}{2} + \frac{7}{4} \gamma_{12}^{2} \right] \frac{r^{4} a^{4}}{5} + \left[ \frac{a_{22}^{2} + \beta_{22}^{2}}{2} + \frac{7}{4} \gamma_{22}^{2} \right] \frac{r^{4} b^{4}}{5} \right];$$

$$Q_{2} = a b L p \left\{ \left| a_{a}^{2} + \beta_{a}^{2} + \gamma_{a}^{2} \right| 2 a_{a} a_{11} + 2 \beta_{a} \beta_{11} + 2 \gamma_{a} \gamma_{11} \right] \frac{r^{2} a^{2}}{3} + \left[ a_{22}^{2} + \beta_{2}^{2} + \gamma_{2} \gamma_{2} \right] \frac{r^{2} b^{2}}{3} + \left[ a_{22}^{2} + \beta_{2}^{2} + \gamma_{2}^{2} + 2 \alpha_{2} a_{2} a_{2} + 2 \beta_{a} \beta_{22} + 2 \gamma_{a} \gamma_{22} \right] \frac{r^{2} b^{2}}{3} + \left[ a_{2}^{2} + \beta_{2}^{2} + \gamma_{2}^{2} + 2 a_{2} a_{2} a_{2} + 2 \beta_{a} \beta_{22} + 2 \gamma_{a} \gamma_{22} \right] \frac{r^{2} b^{2}}{3} + \left[ a_{12}^{2} + \beta_{12}^{2} + 4 \beta_{12}^{2} + 4 \gamma_{12}^{2} + 2 a_{11} a_{22} + 2 \beta_{11} \beta_{22} + 2 \gamma_{11} \gamma_{22} \right] \frac{r^{4} a^{2} b^{2}}{9} + \left[ a_{11}^{2} + \beta_{11}^{2} + \gamma_{11}^{2} \right] \frac{r^{4} a^{4}}{5} + \left[ a_{22}^{2} + \beta_{22}^{2} + \gamma_{22}^{2} \right] \frac{r^{4} b^{4}}{5} \right\}.$$

Integrals  $Q_1$  and  $Q_2$  became the quadratic functions of the 18 **different** coefficients. The equations for the stability limit  $\delta Q_1 = \delta Q_2$ , yields 18 linear equations for the coefficients according to  $\frac{\partial Q_1}{\partial a_{1x}} = \frac{\partial Q_2}{\partial a_{1x}}$ . The 18 equations system is distributed into 4 group's which contain the coefficients 1.  $a_2$ ;  $\beta_1$ ;  $\gamma_{12}$ 

2.  $a_1; \beta_2; \gamma_0; \gamma_{11}; \gamma_{22}$ 3.  $a_0; a_{,1}; a_{22}; \beta_{12}; \gamma_1$ 4.  $\beta_0; \beta_{11}; \beta_{22}; a_{12}; \gamma_2$ .

The individual groups are systems of homogeneous linear equations where coefficients have a non-zero solution only when the determinant disappears. If we take out one group and equate its deerminant to zero, this determines the break load, we will obtain the coefficients of this group up to a common factor.

The coefficients of the other group become zero because the determinants of the other group for the break load value found do not disappear. We are interested here in the determinant equation

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which supplies the break load, thus the question arises: which of the four groups should we take? Naturally such for which the lowest break load results; which one it is - is not difficult to recognize. If the rod breaks in the x direction then all the cross-section points have about the same displacement in the x direction; u cannot be thus an uneven function of x. Now the only group which has no uneven function for u is the third group. Putting the determinant of this group as equal to zero we should obtain the break load value which causes a break in the x-direction. The fourth group, which follows from the third group by substituting x and y, yields the break load for a y break. Thendeterminant equation is the same as for the third group when a and b are substituted. Groups 1 and 2, which with the substitution become one, give a kind of squeeze boundary, but this value has no physical meaning because these phenomena take place outside the limits of the expanded Hook's law.

We write the third group of equations as follows

 $\begin{aligned} & +18\,a_{i1}+12\,n^2\,a_{22}+\left\{42+12\,n^2+4\,n^2\,k^2\,(1-q)\right\}\beta_{12}+9\,\gamma_1=0\\ & -3\,(1-q)\,a_n+(1-q)\,k^2\,a_{11}+(1-q)\,n^2\,k^2\,a_{22}-3\,\gamma_1=0\\ & +5\,(1-q)\,a_n+5\,(1-q)\,k^2\,a_{11}+\left[9\,(1-q)\,n^2\,k^2+60\right]a_{22}+40\,\beta_{12}-15\,\gamma_1=0\\ & +5\,(1-q)\,a_n+\left[9\,(1-q)\,k^2+210\right]a_{11}+5\,(1-q)\,n^2\,k^2\,a_{22}+90\,\beta_{12}+30\,\gamma_1=0\\ & +6\,a_n+4\,k^3\,a_{11}-2\,n^2\,k^2\,a_{22}+6\,k^2\,\beta_{12}+\left[6+(7-2q)\,k^2\right]\gamma_1=0 \end{aligned}$ 

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And to simplify we take 
$$q = \frac{p}{Q}, \quad n = \frac{b}{a}, \quad a_F = \frac{2\pi a}{L} = k$$

The determinant for this system, when multiplied, is

$$q^{5} = \frac{15 n^{2} k^{2} + 9342 k^{2} + 51}{2 n^{2} k^{2}} q^{3} + \frac{26 n^{4} k^{4} + 291 n^{5} k^{2} + 432 n^{2} k^{2} + 93 n^{2} + 630 q^{3}}{4 n^{4} k^{4}} = \frac{100 n^{4} k^{5} + 2535 n^{4} k^{3} + 1296 n^{2} k^{4} + 4392 n^{4} k^{2} + 22797 n^{2} k^{2} + 3060 k^{2}}{4 n^{4} k^{6}} = \frac{1890 n^{4} + 14850 n^{2} + 288500}{4 n^{4} k^{6}} q^{2} + \frac{60 n^{4} k^{8} + 1518 n^{2} k^{4} + 1734 n^{2} k^{3}}{4 n^{4} k^{8}} = \frac{4896 n^{4} k^{4} + 38421 n^{2} k^{4} + 4230 k^{4} + 1890 n^{4} k^{2} + 11850 n^{2} k^{2} + 99090 k^{2} + 81090}{4 n^{4} k^{8}} q^{4} = \frac{156 n^{4} k^{6} + 2568 n^{4} k^{4} + 6200 n^{4} k^{2} + 1320 n^{2} k^{3} + 39735 n^{2} k^{2} + 5609 k^{2} + 280800}{16 n^{4} k^{6}} q^{4} = 0$$

5).

Since the coefficients of the exponents of q change with its sign, the equation may only have positive roots.

For a numerical calculation we note that q as well as k are very small. If we consider the limit case of very thin rods (we let k tend to zero) then after multiplying by  $k^8$  we are left with the last members in the linear and absolute members which yield the equation for q

$$13k^2 - 15q = 0$$

From this - the break tension

Introducing

$$G = \frac{m}{2(m+1)}E, \quad m = \frac{10}{3}, \quad k = \frac{2\pi a}{L}, \quad \frac{I}{F} = \frac{a^2}{3}, \quad P_k = p_k \cdot F,$$

we obtain exactly Euler's formula for the tensioned rod:

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 $P_k = \frac{4 \pi^2 E I}{l^2}.$ 

This formula is thus confirmed for the limit case of very thin rods. The lateral relationship n does not appear explicitly. Strictly speaking there is a need for a n-depending correction but in practically important cases this correction does not reach any noticeable effect.

#### Summary

As an example of the stability theory for elastic equilibrium developed by E.Trefftz the present work calculates the **rm** break limit for circular and rectangular rods. The first part outlines the elasticity theory for finite deformations. In particular, it is shown that the expression f r the elastic potential (internal energy of a volume unit) may be taken from the elasticity theory of small deformations when, instead of linearized distortion magnitudes of the classical theory the real distortion magnitudes for finite deformations are taken. (because of this the coefficients of the linear elements change)

From the obtained integraal for the overall inner elastic energy this work develops an expression for the second variation of internal energy, whose sign decides the stability. Then following common methods for variations' calculation, the stability limit is determined, that is the load limit above which the second variation is capable to accept negative values. The third section contains results of numerical calculations from which it follows that within the limits of calculation accuracy (up to fractions of one percent) Euler's formula is

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confirmed, which formula is obtained according to elementary beam-bending theory.

I wish to thank here Proffessor Dr. Trefftz for his encouragement in my work.

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