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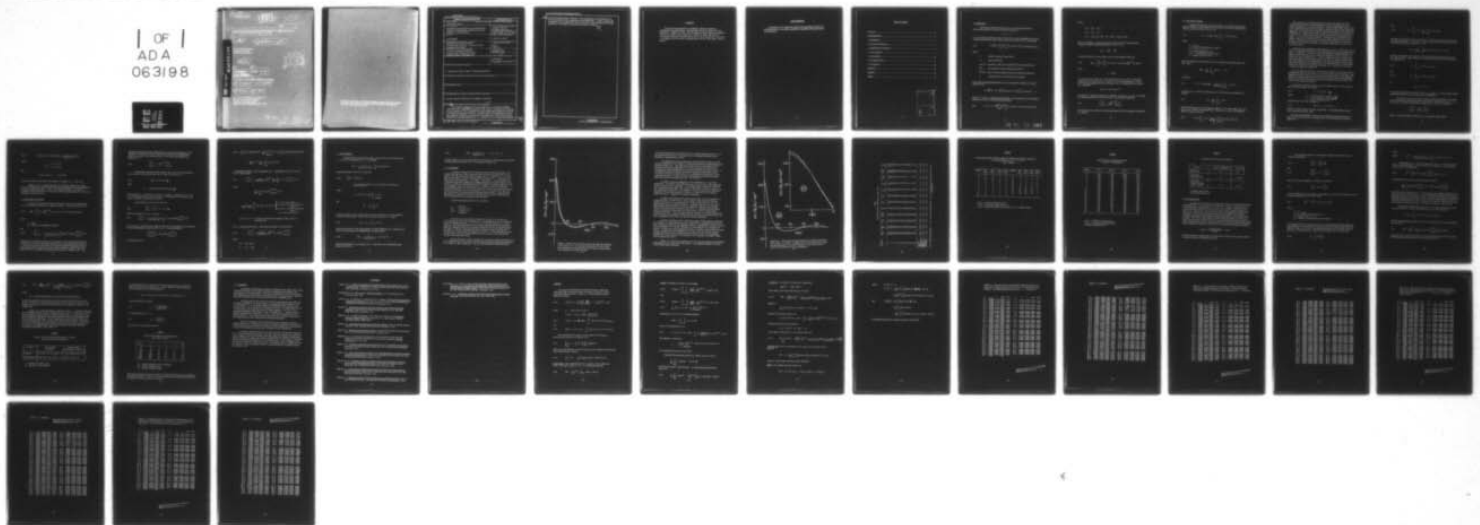
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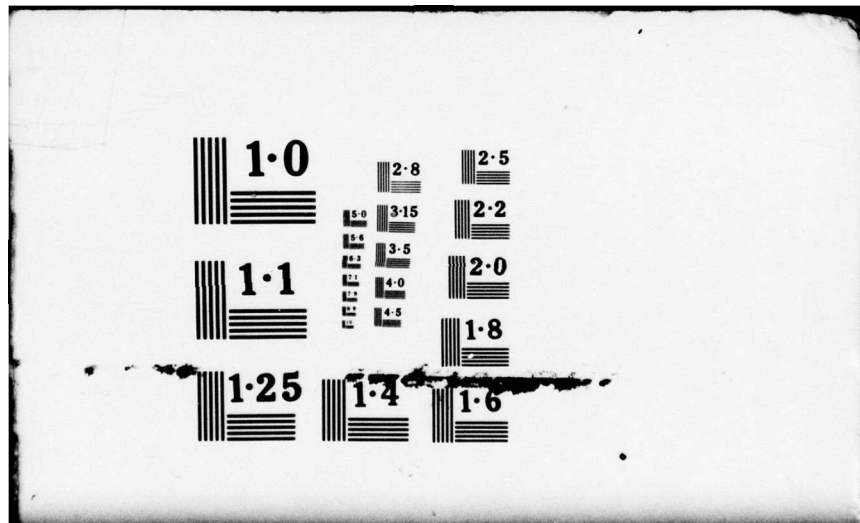
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The Ohio State University
Research Foundation
Columbus, Ohio 43212

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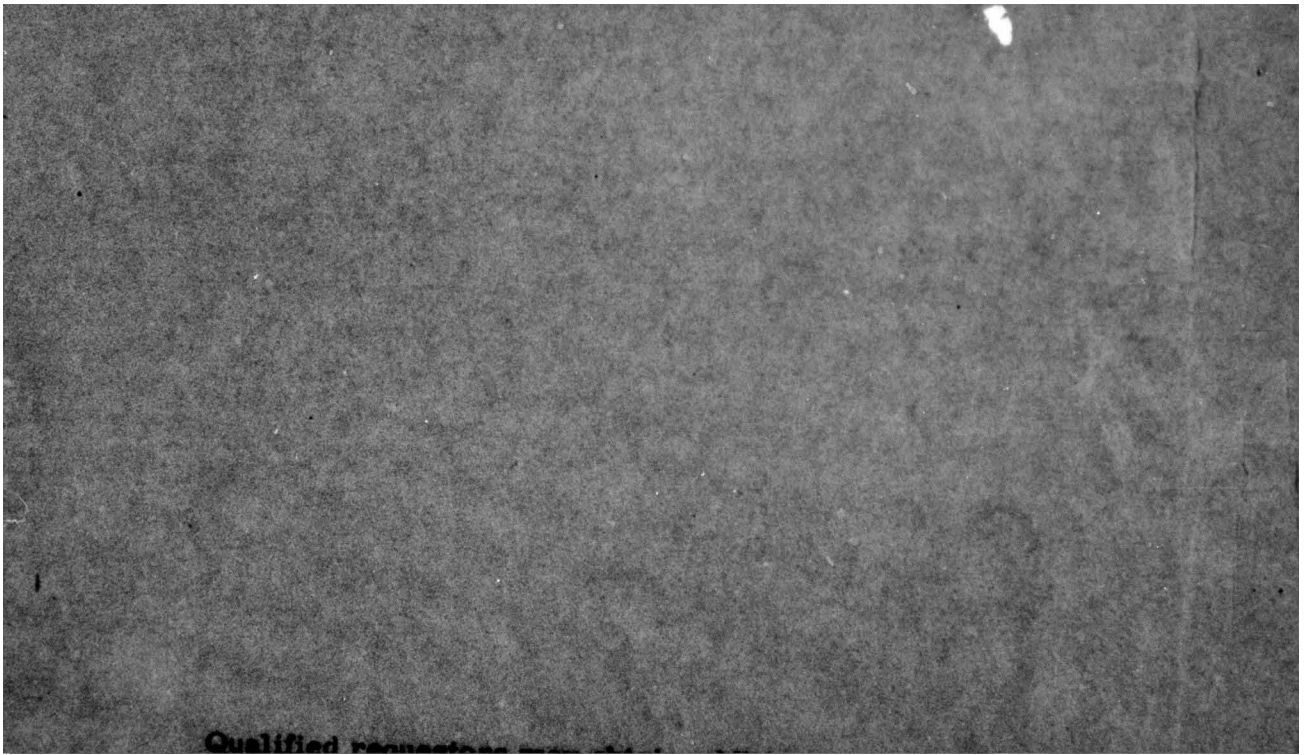
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cont (10%) were found among all solutions. The computations by collocation were at least 30 times more time consuming than the integrations. In order to reduce the computation time in applying the former method for larger systems (such as 5^{deg} anomalies), an alternative computation procedure is outlined.



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Foreword

This report was prepared by Lars Sjöberg, Research Associate, Department of Geodetic Science, The Ohio State University, under Air Force Contract No. F19628-76-C-0010, The Ohio State University Research Foundation Project No. 710335, Project supervisor, Richard H. Rapp. The contract covering this research is administered by the Air Force Geophysics Laboratory, L. G. Hanscom Air Force Base, Massachusetts, with Mr. Bela Szabo, Contract Monitor.

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1. Introduction

The gravity potential of the earth (W) can be decomposed into the gravitational potential (V) and the rotational potential (Φ):

$$W = V + \Phi$$

As V is harmonic outside the surface of the earth, it can be expanded into a series of spherical harmonics in this region (we assume that the series is convergent):

$$(1.1) \quad V = \frac{GM}{r} \left\{ 1 + \sum_{n=2}^{\infty} \left(\frac{a}{r} \right)^n \sum_{m=0}^n (\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\sin \varphi) \right\}$$

where

G Newton's constant of gravitation

M mass of the earth

(r, φ, λ) geocentric spherical coordinates of the computation point

$\bar{P}_{nm}()$ normalized associated Legendre polynomial

$\bar{C}_{nm}, \bar{S}_{nm}$ fully normalized spherical harmonic potential coefficients

a equatorial radius of the defined earth ellipsoid

In the study of the geopotential field it is most convenient to subtract a selected reference field:

$$U = \frac{GM}{r} \left\{ 1 + \hat{C}_{20} \left(\frac{a}{r} \right)^2 \bar{P}_{20}(\sin \varphi) + \hat{C}_{40} \left(\frac{a}{r} \right)^4 \bar{P}_{40}(\sin \varphi) \right\} + \Phi$$

where $\hat{C}_{20}$ and $\hat{C}_{40}$ are selected coefficients, from the potential W . The residual potential (T) is the so called disturbing potential:

$$(1.2) \quad T = W - U = \frac{GM}{a} \sum_{n=2}^{\infty} \left(\frac{a}{r} \right)^{n+1} \sum_{m=0}^n (\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\sin \varphi)$$

where

$$\bar{C}_{20} = \bar{C}_{20} - \hat{C}_{20}$$

$$\bar{C}_{40} = \bar{C}_{40} - \hat{C}_{40}$$

$$\bar{C}_{nm} = \bar{C}_{nm}, \bar{S}_{nm} = \bar{S}_{nm} \quad \text{for } (n, m) \neq (2, 0) \text{ or } (4, 0)$$

Now, by inserting (1.2) into the spherical approximation of "the fundamental boundary condition" (Heiskanen and Moritz, 1967, p. 88):

$$\Delta g = - \frac{\partial T}{\partial r} - \frac{2T}{r}$$

we obtain the gravity anomaly (Δg) in terms of the potential coefficients:

$$(1.3) \quad \Delta g = \gamma \sum_{n=2}^{\infty} (n-1) \sum_{m=0}^n (\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \left(\frac{a}{r}\right)^{n+2} \bar{P}_{nm}(\sin \varphi)$$

where

$$\gamma = GM/a^2$$

It is obvious from formula (1.1) that the coefficients $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are dependent on the choice of a , but that $\bar{C}_{nm} a^n$ and $\bar{S}_{nm} a^n$ are invariant quantities. On the other hand, for a fixed γ the gravity anomaly in (1.3) is independent of the choice of a , whenever:

$$\bar{C}_{nm} a^{n+2} \quad \text{and} \quad \bar{S}_{nm} a^{n+2}$$

are invariant. This means that for two different a -values (a_1, a_2) and $\gamma = \text{constant}$ the gravity anomalies in (1.3) are the same, if (with obvious notations):

$$(1.4) \quad \left\{ \begin{array}{c} \bar{C}_{nm} \\ \bar{S}_{nm} \end{array} \right\}_{a_1} = \left(\frac{a_2}{a_1} \right)^{n+2} \left\{ \begin{array}{c} \bar{C}_{nm} \\ \bar{S}_{nm} \end{array} \right\}_{a_2}$$

This relation will be useful in the following application of estimating the coefficients $\bar{C}_{nm}$ and $\bar{S}_{nm}$.

2. Covariance Functions

Suppose that the gravity field is harmonic, homogeneous and isotropic. Then the spatial covariance function of the point free-air gravity anomalies Δg_i and Δg_j is given by (Moritz, 1972, p. 89):

$$(2.1) \quad c_{ij} = \text{cov}(\Delta g_i, \Delta g_j) = \sum_{n=2}^{\infty} c_n s^{n+2} P_n(\cos \psi_{ij})$$

where

$$\begin{aligned} s &= r_s^2 / r_i r_j \\ r_s &= \text{radius of the Bjerhammar sphere} \\ r_k &= r_s + h_k, \quad k = i, j \\ r_s &= \text{radius of mean sea level} \\ h_i, h_j &= \text{elevations of points } P_i \text{ and } P_j \\ c_n &= \text{degree variances of } \Delta g \end{aligned}$$

The corresponding covariance function of the mean gravity anomalies $\overline{\Delta g}_i$ and $\overline{\Delta g}_j$, where:

$$\overline{\Delta g}_k = \frac{1}{\Delta \sigma_k} \iint_{\Delta \sigma_k} \Delta g \, d\sigma; \quad k = i, j$$

is given by

$$(2.2) \quad \overline{c}_{ij} = \text{cov}(\overline{\Delta g}_i, \overline{\Delta g}_j) = \frac{1}{\Delta \sigma_i \Delta \sigma_j} \iint_{\Delta \sigma_i} \iint_{\Delta \sigma_j} c_{ij} \, d\sigma \, d\sigma$$

In the same way we obtain the following cross-covariance function between Δg_i and $\overline{\Delta g}_j$:

$$(2.3) \quad \overline{c}_{ij} = \frac{1}{\Delta \sigma_j} \iint_{\Delta \sigma_j} c_{ij} \, d\sigma$$

In these formulae $\Delta \sigma$ is a part of the unit sphere σ . For small regions $\Delta \sigma_i$ and $\Delta \sigma_j$ we may, without loss of significance, assume that r_i and r_j are constants. Then (2.2) and (2.3) become:

$$(2.2') \quad \overline{c}_{ij} = \sum_{n=2}^{\infty} c_n s^{n+2} \frac{1}{\Delta \sigma_i \Delta \sigma_j} \iint_{\Delta \sigma_i} \iint_{\Delta \sigma_j} P_n(\cos \psi_{ij}) \, d\sigma \, d\sigma$$

The common way to determine the spherical harmonic coefficients of the earth's gravity field from terrestrial data is by means of integration of mean gravity anomalies over a mean earth sphere. See for example Rapp (1977a). Due to the orthogonal functions the coefficients are obtained directly in the integrations. Rigorously, it is required that the gravity anomalies are located on the sphere of integration, but in reality, due to the variation of the elevation of the terrain, this is not the case. This terrain deviation can be corrected for by adding the Molodensky G_1 term to the mean anomalies. Pellinen (1962) has indicated that the neglect of this term can cause errors in the low degree coefficients of 10 to 20 percent. Numerical results of Rapp (1977a) agree with this error estimate, but the results were based on a number of assumptions relating the G_1 term to the terrain correction.

In practice, the computation of the Molodensky correction term for the terrain may be very difficult and laborious, especially in areas with rapidly varying topography. It is therefore of interest to find a technique for the determination of the potential coefficients that does not include the computation of the G_1 terms, yet retains the rigor of that procedure. One such method was given by Rapp (1977b), where least-squares collocation was used for an upward continuation of the terrestrial mean anomalies to a bounding sphere. Once the anomalies are given at the sphere the integration can be applied strictly for the determination of the potential coefficients. Rapp (1977b) found that the neglect of G_1 caused errors less than 7.5% for harmonics up to degree 40.

In the present study the idea is to estimate the potential coefficients directly by applying least-squares collocation to the terrestrial mean gravity anomalies. The integration is then taken care of in the cross-covariance matrix. The advantage of using such a method is that the terrain correction is easily included and that the various accuracies of the mean anomalies can be taken into account, which is not obvious in the integration approach.

In collocation a physical quantity V may be predicted from a vector of (mean) gravity anomalies Δg by the relation:

$$(1.5) \quad V = c_v^T (G + D)^{-1} \Delta g$$

where

c_v^T = cross-covariance matrix ($V, \Delta g$)

C = auto-covariance matrix ($\Delta g, \Delta g$)

D = error covariance matrix

The prediction errors are estimated by

$$(1.6) \quad m_v^2 = C_0 - c_v^T (C + D)^{-1} c_v$$

Where C_0 is the variance of V prior to prediction. For further details on these basic formulae see Moritz (1972).

The collocation technique requires that the relevant covariance functions are known. In the next section we are going to study these functions for the present application.

and

$$(2.3') \quad \bar{c}_{ij} = \sum_{n=2}^{\infty} c_n s^{n+2} \frac{1}{\Delta\sigma_j} \iint_{\Delta\sigma_j} P_n(\cos \psi_{ij}) d\sigma$$

Formulae (2.2') and (2.3') are very laborious to compute in practice due to the many numerical integrations. Approximate mean covariance functions may be obtained by using the β_n function of Meissl (1971, p. 23):

$$\beta_n = \frac{1}{1 - \cos \psi_0} \frac{1}{2n+1} [P_{n-1}(\cos \psi_0) - P_{n+1}(\cos \psi_0)]$$

where ψ_0 is the radius of a circular cap of area equal to the relevant block size of the mean anomalies. Then, the above covariance functions become (approximately) (cf. Figure 2 a-b):

$$(2.2'') \quad \bar{c}_{ij} = \sum_{n=2}^{\infty} c_n s^{n+2} \beta_n^2 P_n(\cos \psi_{ij})$$

and

$$(2.3'') \quad \bar{c}_{ij} = \sum_{n=2}^{\infty} c_n s^{n+2} \beta_n P_n(\cos \psi_{ij})$$

As

$$\beta_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

it is usually sufficient to truncate the series (2.2'') and (2.3'') at a few hundred degrees (dependent upon the block size) without loss of significance.

We also give the autocovariance relations between the potential coefficients. In Moritz (1970) the relations are given for the anomaly coefficients ($\bar{a}_{nm}$, $\bar{b}_{nm}$). As $\bar{a}_{nm}$ and $\bar{b}_{nm}$ are related to the potential coefficients $\bar{C}_{nm}$ and $\bar{S}_{nm}$ according to:

$$\begin{Bmatrix} \bar{C}_{nm} \\ \bar{S}_{nm} \end{Bmatrix} = \frac{1}{\gamma(n-1)} \begin{Bmatrix} \bar{a}_{nm} \\ \bar{b}_{nm} \end{Bmatrix}$$

where γ is the mean gravity at sea-level, we obtain from Moritz (ibid.):

$$(2.4) \quad \text{cov} (\bar{C}_{nm}, \bar{C}_{pq}) = \text{cov} (\bar{S}_{nm}, \bar{S}_{pq}) = \frac{c_n}{(2n+1)(n-1)^2 \gamma^2} \delta_{np} \delta_{mq}$$

where

$$\delta_{np} = \begin{cases} 1 & \text{if } n = p \\ 0 & \text{if } n \neq p \end{cases}$$

and

$$\text{cov} (\bar{C}_{nm}, \bar{S}_{pq}) = 0 \quad \text{in any case.}$$

The above covariance relations will be useful in our application of collocation.

Finally, we like to mention that the mean covariance functions (2.2') and (2.3') can in addition, be approximated by the corresponding point covariance functions simply by increasing the radii r_1 and r_2 by a feasible constant. This type of smoothed covariance functions was discussed in Tscherning and Rapp (1974, Section 10) and Schwarz (1976, Section 7).

3. Application of Collocation

We assume that the external gravity field of the earth may be expanded into a series of spherical harmonics at a sphere of radius R . Then we have [cf. (1.3)]:

$$(3.1a) \quad \Delta g_R = \gamma \sum_{n=2}^{\infty} (n-1) \left(\frac{r_B}{R} \right)^{n+2} \sum_{m=0}^n (\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\sin \varphi)$$

where

$$\gamma = GM/a^2$$

r_B = radius of the Bjerhammar sphere

and

$$(3.1b) \quad \begin{Bmatrix} \bar{C}_{nm} \\ \bar{S}_{nm} \end{Bmatrix}_{r_B} = \frac{1}{4\pi\gamma(n-1)(r_B/R)^{n+2}} \iint_{\sigma} \Delta g_R \bar{P}_{nm}(\sin \varphi) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} d\sigma$$

Formula (3.1b) is the basic equation we are going to use for estimating the potential coefficients. The coefficients determined by (3.1b) are independent of the choice of R . Thus by choosing R as the radius of the Brioullin sphere (bounding all the mass of the earth) we have a theoretically most attractive situation, because the series expansion (3.1a) is rigorously convergent at this sphere (cf. Sjöberg, 1977). The

standard representation of the coefficients $\bar{C}_{nm}$ and $\bar{S}_{nm}$ in the literature is with reference to the equatorial radius of the earth (a) (and not to the Bjerhammar sphere as in (3.1b)). The conversion from (3.1b) to this representation was given in (1.4):

$$(3.2) \quad \begin{Bmatrix} \bar{C}_{nm} \\ \bar{S}_{nm} \end{Bmatrix}_a = \left(\frac{r_B}{a}\right)^{n+2} \begin{Bmatrix} \bar{C}_{nm} \\ \bar{S}_{nm} \end{Bmatrix}_{r_B}$$

Theoretically, the point gravity anomaly Δg_R in (3.1a) can be estimated from a vector of mean gravity anomalies by means of formula (1.5):

$$(3.3) \quad \Delta g_R = c_R^T (C + D)^{-1} \underline{\Delta g}$$

where

$$c_R^T = \text{cross-covariance matrix } (\Delta g, \underline{\Delta g})$$

The element $(c_R^T)_{1j}$ is given by (2.3') with $r_1 = R$ and C_{1j} is given by (2.2'). The error covariance matrix D can be estimated by the diagonal matrix formed by the a priori estimated mean anomaly variances.

By inserting (3.3) into (3.1b) we arrive at:

$$(3.4) \quad \begin{Bmatrix} \bar{C}_{nm} \\ \bar{S}_{nm} \end{Bmatrix} = \begin{Bmatrix} c_c^T \\ c_s^T \end{Bmatrix} (C + D)^{-1} \underline{\Delta g}$$

where the elements of c_c and c_s become:

$$(3.5) \quad \begin{Bmatrix} (c_c)_{1j} \\ (c_s)_{1j} \end{Bmatrix} = \frac{1}{4\pi\gamma(n-1)(r_B/R)^{n+2}} \iint_{\sigma} \bar{c}_{1j} \bar{P}_{nm}(\sin \varphi) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} d\sigma$$

We are going to simplify (3.5) by taking into account the orthogonality of the spherical harmonics (Heiskanen and Moritz, 1967, p. 29). Using the notations:

$$\begin{Bmatrix} \bar{R}_{nm}(\varphi, \lambda) \\ \bar{S}_{nm}(\varphi, \lambda) \end{Bmatrix} = \bar{P}_{nm}(\sin \varphi) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix}$$

we obtain from (2.3'):

$$\begin{aligned}
(3.6) \quad \frac{1}{4\pi} \iint \bar{c}_{1j} \bar{R}_{nn}(\varphi, \lambda) d\sigma &= \frac{1}{\Delta\sigma_j} \iint \sum_{n=2}^{\infty} c_n s^{n+2} \frac{1}{4\pi} \iint P_n(\cos \psi_{ij}) \bar{R}_{nn}(\varphi, \lambda) d\sigma d\sigma \\
&= \frac{c_n}{2n+1} s^{n+2} \frac{1}{\Delta\sigma_j} \iint \bar{R}_{nn}(\varphi, \lambda) d\sigma
\end{aligned}$$

A completely analogous result is obtained for $\bar{S}_{nn}$. Inserting the result of (3.6) into (3.5) we finally obtain:

$$(3.7) \quad \begin{Bmatrix} (c_c)_{1j} \\ (c_s)_{1j} \end{Bmatrix} = \frac{c_n}{\gamma(2n+1)(n-1)} \left(\frac{r_B}{r_j}\right)^{n+2} \frac{1}{\Delta\sigma_j} \iint \bar{P}_{nn}(\sin \varphi) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} d\sigma$$

where

$$\begin{aligned}
&\frac{1}{\Delta\sigma_j} \iint \bar{P}_{nn}(\sin \varphi) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} d\sigma = \\
&= \frac{1}{\sin \varphi_N - \sin \varphi_S} \int_{\varphi_S}^{\varphi_N} \bar{P}_{nn}(\sin \varphi) \cos \varphi d\varphi \times \begin{cases} 1 & \text{if } \cos m\lambda \text{ with } m=0 \\ (\sin m\lambda_E - \sin m\lambda_W)/m & \text{if } \cos m\lambda \\ & \text{with } m \neq 0 \\ (\cos m\lambda_W - \cos m\lambda_E)/m & \text{if } \sin m\lambda \\ & \text{with } m \neq 0 \\ 0 & \text{if } \sin m\lambda \text{ with } m=0 \end{cases}
\end{aligned}$$

$\varphi_S, \varphi_N, \lambda_E, \lambda_W$ = geocentric latitudes and longitudes of the corners of the block $\Delta\sigma_j$

If $\bar{c}_{1j}$ is approximated by (2.3'') the formula analogous to (3.7) becomes:

$$(3.7') \quad \begin{Bmatrix} (c_c)_{1j} \\ (c_s)_{1j} \end{Bmatrix} = \frac{c_n \beta_n}{\gamma(2n+1)(n-1)} \left(\frac{r_B}{r_j}\right)^{n+2} \bar{P}_{nn}(\sin \bar{\varphi}) \begin{Bmatrix} \cos m\bar{\lambda} \\ \sin m\bar{\lambda} \end{Bmatrix}$$

where

$$\bar{\varphi} = (\varphi_N + \varphi_S)/2$$

$$\bar{\lambda} = (\lambda_E + \lambda_W)/2$$

4. Error Analysis

Suppose that the gravity anomaly Δg_1 used in formula (3.1b) has the error ϵ_1 . The error propagation to $\bar{C}_{nm}$ is accordingly:

$$d\bar{C}_{nm} = \frac{1}{4\pi \gamma(n-1)(r_B/R)^{n+2}} \iint_{\sigma} \epsilon_1 \bar{R}_{nm}(\varphi, \lambda) d\sigma$$

and the mean square error of $\bar{C}_{nm}$ becomes:

$$(4.1) \quad \overline{m}_{C_{nm}}^2 = M \{d\bar{C}_{nm}\}^2 = \left\{ \frac{1}{\gamma(n-1)(r_B/R)^{n+2}} \right\}^2 M_1 \{ M_J \{ \sigma_{1j} \bar{R}_{nm}(\varphi_1, \lambda_1) \bar{R}_{nm}(\varphi_j, \lambda_j) \} \}$$

where

$$\sigma_{1j} = M \{ \epsilon_1 \epsilon_j \} = \frac{1}{4\pi} \iint \epsilon_1 \epsilon_j d\sigma$$

$\psi_{1j} = \text{const.}$

and

$$M_1 = \frac{1}{4\pi} \iint_{\sigma} d\sigma_1$$

As Δg of formula (3.1b) is estimated by means of collocation, σ_{1j} is the prediction covariance of that method. Moritz (1972), formula (3-39) with $A = 0$ gives

$$(4.2) \quad \sigma_{1j} = C_{1j} - \bar{c}_1^T (C + D)^{-1} \bar{c}_j$$

where the covariances of the right member are those defined in (3.3). Inserting (4.2) into (4.1) and carrying out the integrations, we finally obtain:

$$(4.3a) \quad \overline{m}_{C_{nm}}^2 = \frac{c_n}{(2n+1)(n-1)^2 \gamma^2} - c_c^T (C + D)^{-1} c_c$$

where the elements of c_c are given by (3.7). In the same way we obtain the mean square error for $\bar{S}_{nm}$:

$$(4.3b) \quad \overline{m}_{sna}^2 = \frac{c_n}{(2n+1)(n-1)^2 \gamma^2} - c_s^T (C + D)^{-1} c_s$$

We notice that (4.3a-b) give exactly the error estimates we would expect in collocation with the prediction formula (3.4). Cf. formulae (1.5) and (1.6).

5. Computations

A global coverage of $416 \cdot 10^6$ equal area free air gravity anomalies were available as input data. (These anomalies had been derived using the data of Rapp (1977a) and the methods described by Hajela (1975).) In all computed covariance functions the degree variances implied by the subroutine COVA of Tschebning and Rapp (1974) were used with $c_2 = 7.5 \text{ mgal}^2$. The cross-covariance functions were computed by numerical integration according to formula (3.7) [derived from (2.3')]. However, for the auto-covariance function we felt that it was unreasonable to use the corresponding, very laborious formula (2.2'). Instead, we tried two different approximate formulae. In order to save computer time the auto-covariance function for the mean anomalies for zero-elevations ($h_i = h_j = 0$) was stored in a table and the current values were interpolated among the tabulated values. When elevation information was included in the process the table was used only to determine the auto-covariances between ocean-block mean anomalies.

The following reference data were anticipated:

$$\begin{aligned} a &= 6378140 \text{ m} \\ \overline{C}_{20} &= -484.198 \times 10^{-6} \\ \overline{C}_{40} &= 0.790333 \times 10^{-6} \\ f &= 1/298.247 \end{aligned}$$

In a first test of the prediction formulae (3.4) and (4.3 a-b), the auto-covariance function was approximated by the corresponding point covariance function at the best fitting elevation ($h_i = h_j = 142.3 \text{ km}$, cf. the end of section 2). Figure 1 indicates a fair agreement between this covariance function and the one implied by a numerical integration of the point covariance function over $10^\circ \times 10^\circ$ blocks (around the equator). However, the prediction result was poor and especially the error estimates were useless (because of negative variances!). It was apparent from the test that choice of mean covariance function was very critical, especially for the outcome of the prediction errors.

Second, the auto-covariance function was computed according to formula (2.3''). It was found that the series could be accepted when truncated at degree 200 (except for $\psi = 0$, where $n = 200$ is sufficient). The very good agreement between the series

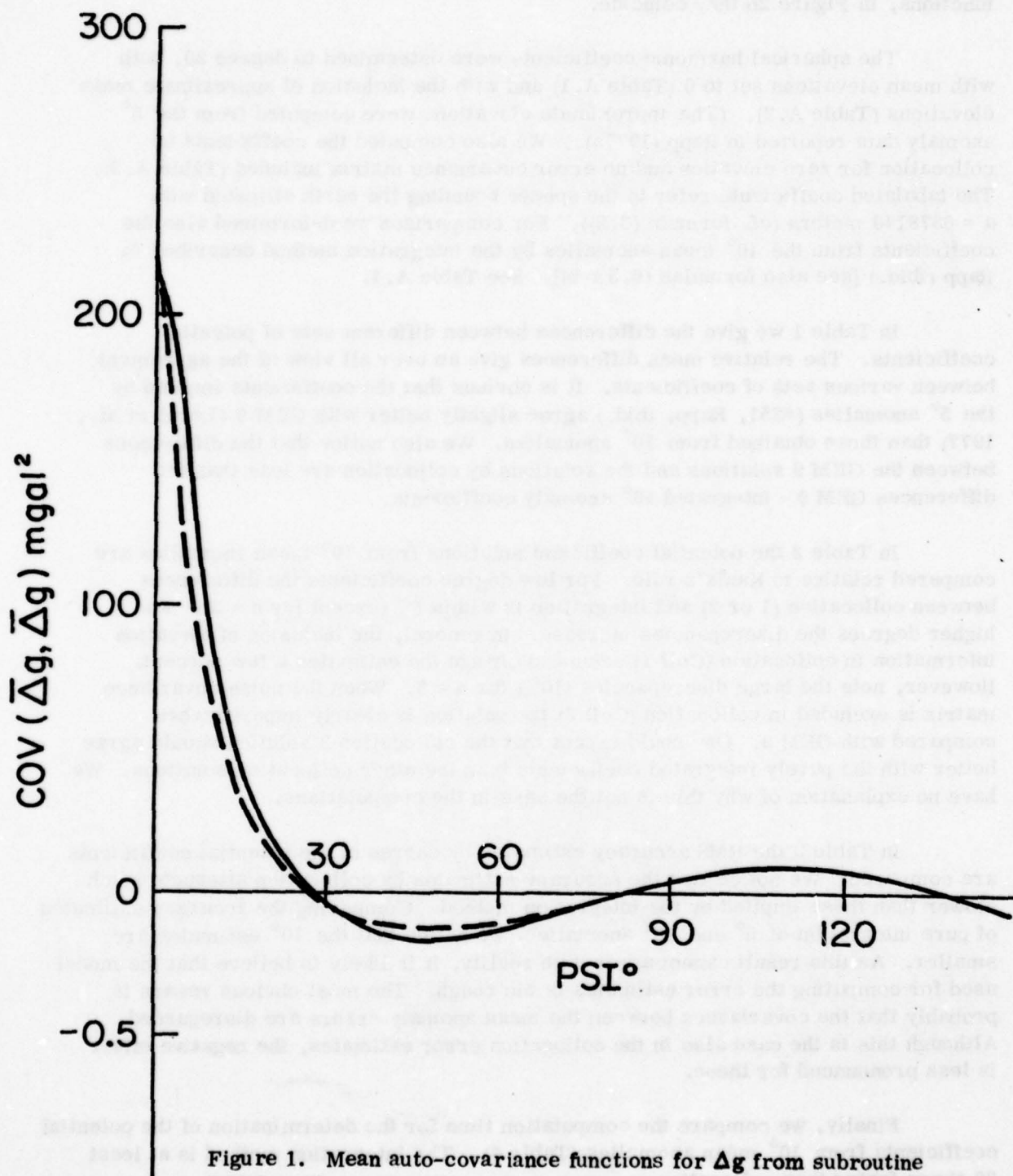


Figure 1. Mean auto-covariance functions for Δg from subroutine COVA (with $c_2 = 0$). The selected elevation ($h_A = h_B = 142.3$ km) gave the point covariance function that best agreed with the numerically integrated covariance function.

(2.2'') and the numerically integrated $10^\circ \times 10^\circ$ covariance function [formula (2.2')] is illustrated in Figure 2a-b. In Figure 2a we can distinguish between the two functions, in Figure 2b they coincide.

The spherical harmonic coefficients were determined to degree 20, both with mean elevations set to 0 (Table A.1) and with the inclusion of approximate mean elevations (Table A.2). (The approximate elevations were computed from the 5° anomaly data reported in Rapp (1977a).) We also computed the coefficients by collocation for zero elevation and no error covariance matrix included (Table A.3). The tabulated coefficients refer to the sphere bounding the earth ellipsoid with $a = 6378140$ meters (cf. formula (3.2)). For comparison we determined also the coefficients from the 10° mean anomalies by the integration method described in Rapp (ibid.) [see also formulae (6.5a-b)]. See Table A.4.

In Table 1 we give the differences between different sets of potential coefficients. The relative mean differences give an over all view of the agreement between various sets of coefficients. It is obvious that the coefficients implied by the 5° anomalies (#251, Rapp, ibid.) agree slightly better with GEM 9 (Lerch et al., 1977) than those obtained from 10° anomalies. We also notice that the differences between the GEM 9 solutions and the solutions by collocation are less than the differences GEM 9 - Integrated 10° anomaly coefficients.

In Table 2 the potential coefficient solutions from 10° mean anomalies are compared relative to Kaula's rule. For low degree coefficients the differences between collocation (1 or 2) and integration is within 6% (except for $n = 3$). For higher degrees the discrepancies increase. In general, the inclusion of elevation information in collocation (Coll 1) seems to change the estimates a few percent. However, note the large discrepancies (10%) for $n = 3$. When the noise covariance matrix is excluded in collocation (Coll 3) the solution is clearly impaired when compared with GEM 9. One could expect that the collocation 3 solution should agree better with the purely integrated coefficients than the other collocation solutions. We have no explanation of why this is not the case in the computations.

In Table 3 the RMS accuracy estimates by degree of the potential coefficients are compared. We notice that the accuracy estimates by collocation attenuate much slower than those implied by the integration method. Comparing the accuracy estimates of pure integration of 5° and 10° anomalies, we notice that the 10° estimates are smaller. As this result cannot agree with reality, it is likely to believe that the model used for computing the error estimates is too rough. The most obvious reason is probably that the covariances between the mean anomaly errors are disregarded. Although this is the case also in the collocation error estimates, the negative effect is less pronounced for these.

Finally, we compare the computation time for the determination of the potential coefficients from 10° mean anomalies (Table 4). The integration method is at least 30 times faster than collocation.

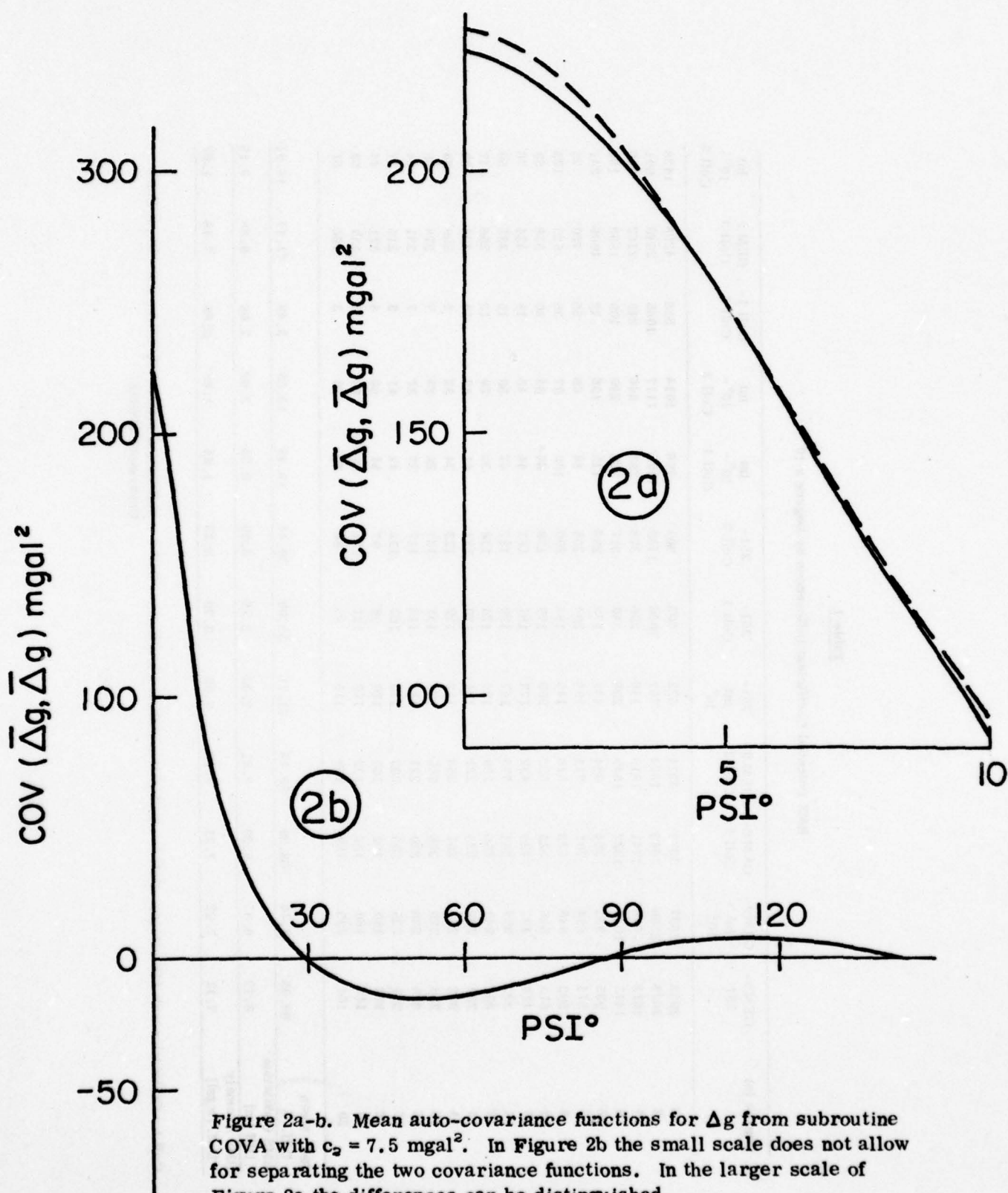


Table 1

RMS Potential Coefficient Differences by Degree x 10¹⁰

Degree (n)	GEM 9- 251	GEM 9- Int 10°	GEM 9- Coll 1	GEM 9- Coll 2	251- Int 10°	251- Coll 1	251- Coll 2	Int 10° - Coll 1	Int 10° - Coll 2	Coll 1 - Coll 2	GEM 9- Coll 3	Int 10° - Coll 3
2	2838	2897	3373	3514	522	603	801	706	1014	515	4268	1438
3	2625	2599	2646	1834	267	1056	1139	1101	1117	1085	2010	991
4	1612	1645	1741	1819	146	296	378	295	346	240	2162	746
5	1473	1549	1527	1549	208	340	317	223	240	109	1563	182
6	935	954	984	984	141	173	206	115	136	72	1006	241
7	574	694	701	711	194	204	206	84	65	55	720	73
8	592	546	523	516	145	194	200	100	97	35	529	123
9	612	526	525	522	200	179	179	64	54	20	538	63
10	444	427	425	432	173	181	187	44	45	18	424	76
11	332	377	381	378	147	129	127	47	50	15	383	50
12	254	249	256	259	124	124	126	32	29	13	258	47
13	222	232	223	221	124	120	121	41	42	10	234	32
14	252	249	251	253	112	122	123	34	32	8	259	38
15	228	270	266	265	138	137	137	29	29	7	278	26
16	205	236	220	221	118	116	117	37	37	7	231	24
17	217	235	203	203	110	102	102	47	47	5	215	37
18	172	190	160	160	120	98	97	54	56	4	171	44
19	145	160	132	132	143	121	121	59	60	3	137	52
20	164	206	159	159	115	85	84	70	70	3	166	61
Rel. Mean												
Diff. %	68.03	73.40	68.29	68.15	31.71	30.09	30.44	13.43	13.50	3.49	71.17	14.91
RMS Undulation												
Diff [m]	8.72	8.81	9.20	8.81	2.03	2.76	3.00	2.35	2.62	2.05	9.88	3.15
RMS Anomaly												
Diff. [mgal]	7.17	7.62	7.21	7.18	3.55	3.30	3.33	1.64	1.67	0.69	7.48	1.67

Coll 1 = Collocation with Elevations, Coll 2 = Collocation without Elevations, Coll 3 = Collocation without Elevations and without Noise
Covariance Matrix

Table 2

Ratios between RMS Potential Coefficient Differences by Degree Implied by
 10° Anomalies and $(C_{nm}, S_{nm}) = 10^{-5}/n^2$.
 Units: Percent

Degree	Int- Coll 1	Int- Coll 2	Int- Coll 3	Coll 1- Coll 2	Degree	Int- Coll 1	Int- Coll 2	Int- Coll 3	Coll 1- Coll 2
2	3	4	6	2	12	5	4	7	2
3	10	10	9	10	13	7	7	5	2
4	5	6	12	4	14	7	6	7	2
5	6	6	5	3	15	7	7	6	2
6	4	5	9	3	16	9	9	6	2
7	4	3	4	3	17	14	14	11	1
8	6	6	8	2	18	17	18	14	1
9	5	4	5	2	19	21	22	19	1
10	4	4	8	2	20	28	28	24	1
11	6	6	6	2					

Coll 1 = Collocation with Elevations

Coll 2 = Collocation without Elevations

Coll 3 = Collocation without Elevations or Noise Covariance Matrix

Table 3

RMS Accuracy Estimates by Degree
for Potential Coefficients

Degree	# 251	Int 10°	Coll 1	Coll 2
2	1911	1722	2054	1779
3	956	861	1674	2247
4	637	574	992	1207
5	478	430	691	807
6	382	344	468	438
7	318	286	443	486
8	273	245	370	378
9	239	214	341	360
10	212	190	301	300
11	191	171	285	289
12	173	155	272	277
13	159	142	266	266
14	147	130	250	248
15	136	121	243	243
16	127	113	237	237
17	119	105	231	231
18	112	100	224	224
19	106	94	217	217
20	100	89	209	209

Coll 1 = Collocation with Elevations
Coll 2 = Collocation without Elevations
All values multiplied by 10^{10}

Table 4

Computation Times for Various Methods

Method	Degree of Expansion	CPU Time
Integration	25	19 ^s
Collocation with Elevations	20	12 ^m 13 ^s
Collocation without Elevations (COVA in Table)	20	9 ^m 21 ^s

Computer: IBM 370/168
No. of Observations: 416

6. An Extended View

An extension of the previous computations would be to determine the potential coefficients from 5° mean anomalies by collocation. However, as the number of observations then increases from 416 to 1654, it is no longer a standard procedure to invert the auto-covariance matrix of the system. Most computers cannot even store such a large matrix. We are going to estimate the necessary computer time as follows. In Table 5 the total number of necessary multiplications $f(M)$ for computing the potential coefficients by collocation to degree n_0 for M observations are given. The direct method implies that the method of Cholesky is used for the inversion of the auto-covariance matrix. Let us assume that the total computer time $T(M)$ is proportional to $f(M)$. (We disregard the time needed for addition operations.) Then we obtain from Tables 4 and 5 for $n_0 = 20$ (no elevation information included):

$$T(1654) = \frac{T(416) f(1654)}{f(416)} = 3^h 50^m$$

Thus the necessary computer time is so large that we should really consider whether it could be reduced by modifying the method.

One possibility might be to determine the coefficients and their accuracy estimates according to:

$$(6.1) \quad \begin{Bmatrix} \bar{C}_{na} \\ \bar{S}_{na} \end{Bmatrix} = \begin{Bmatrix} h_c^T \\ h_s^T \end{Bmatrix} \Delta \bar{g}$$

and

$$(6.2) \quad \begin{Bmatrix} \bar{m}_c^2 \\ \bar{m}_s^2 \end{Bmatrix} = C_0 - \begin{Bmatrix} c_c^T h_c \\ c_s^T h_s \end{Bmatrix}$$

where the vectors of unknowns (h_c and h_s) are given by:

$$(6.3) \quad \begin{Bmatrix} c_c \\ c_s \end{Bmatrix} = (C + D) \begin{Bmatrix} h_c \\ h_s \end{Bmatrix}$$

For each coefficient to be determined, the weights (h_c or h_s) can be computed iteratively by the following formula (cf. Miller, 1974):

$$(6.4) \quad \underset{M \ 1}{h^{(\nu+1)}} = \underset{M \ 1}{h^{(\nu)}} + \underset{M \ 1}{\beta} \{ \underset{M \ 1}{c} - \underset{M \ M}{(C + D)} \underset{M \ 1}{h^{(\nu)}} \}$$

where

$$0 < \beta < 2/\lambda_{max}$$

λ_{max} = maximum eigen value of $C + D$

ν = iterative step: 0, 1, 2, ...

M = number of observations (mean anomalies)

The starting value $h^{(0)}$ for the iteration is most conveniently given in the spherical approximation. By assuming that all mean anomalies are located on the mean earth sphere of radius r_e we arrive at the following formula from (3.1b) after replacing Δg and $\bar{C}_{na}$ by $\Delta \bar{g}$ and $\beta_n \bar{C}_{na}$, respectively [cf. Rapp, 1977a, formula (30)]:

$$(6.5a) \quad \bar{C}_{na} = \sum_{k=1}^M h_k^{(0)} \Delta \bar{g}_k$$

where

$$(6.5b) \quad h_k^{(0)} = \frac{1}{4\pi\gamma\beta_n(r_B/r_a)^{n+2}(n-1)} \iint_{\Delta\sigma_k} \bar{P}_{nm}(\sin\varphi) \cos m\lambda \, d\sigma$$

Substituting $\cos m\lambda$ under the integral by $\sin m\lambda$ we obtain the weights $(h_k^{(0)})$ for and coefficients $\bar{S}_{nm}$ in (6.5b) and (6.5a), respectively. Even simpler approximations are obtained for:

$$(6.6) \quad h_k^{(0)} = \left(\frac{r_a}{r_B}\right)^{n+2} \bar{P}_{nm}(\sin\bar{\varphi}) \left\{ \frac{\cos m\bar{\lambda}}{\sin m\bar{\lambda}} \right\} \Delta\sigma_k / [4\pi\gamma(n-1)]$$

which formula is given from (6.5b) by the approximation:

$$\frac{1}{\Delta\sigma_k} \iint_{\Delta\sigma_k} \bar{P}_{nm}(\sin\varphi) \left\{ \frac{\cos m\lambda}{\sin m\lambda} \right\} d\sigma \approx \beta_n \bar{P}_{nm}(\sin\bar{\varphi}) \left\{ \frac{\cos m\bar{\lambda}}{\sin m\bar{\lambda}} \right\}$$

where $\bar{\varphi}$ and $\bar{\lambda}$ are given in (3.7'). As the elevation of the highest mountain is less than 0.2% of the mean earth radius, we can expect that the iteration error in (6.4) is insignificant after a few iterations. Again, it should be emphasized that the spherical harmonic coefficients so determined refer to the Bjerhammar sphere (of radius r_B) and should be multiplied by $(r_B/a)^{n+2}$ in order to be consistent with other coefficient determinations, which usually refer to the sphere of radius $a = 6378.140$ km.

In the approximate formula (6.5b) and (6.6) we have disregarded the noise covariance function D. When considering the noise covariance function:

$$d(P, Q) = \sum_{n=2}^{\infty} d_n \beta_n^2 (r_B^2/r_P r_Q)^{n+2} P_n(\cos\psi_{PQ})$$

the following weight function can be derived for the spherical case [see the Appendix, formula (A.9)]:

$$(6.7) \quad h_k^{(0)} = \left(\frac{r_a}{r_B}\right)^{n+2} \frac{c_n}{c_n + d_n} \bar{P}_{nm}(\sin\varphi) \left\{ \frac{\cos m\bar{\lambda}}{\sin m\bar{\lambda}} \right\} \Delta\sigma_k / 4\pi\gamma(n-1)$$

In the same way, if we assume that the errors between the blocks are uncorrelated, the following weight function can be derived [formula (A.11)]:

$$(6.8) \quad h_k^{(0)} = \frac{\Delta \sigma_k}{\gamma(n-1)4\pi} \left(\frac{r_B}{r_n} \right)^{n+2} \frac{c_n \beta_n^2}{c_n \beta_n^2 \left(\frac{r_B}{r_n} \right)^{2(n+2)} + (2n+1) \epsilon_k^2} \bar{P}_{nm}(\sin \bar{\varphi}_k) \begin{Bmatrix} \cos m \bar{\lambda} \\ \sin m \bar{\lambda} \end{Bmatrix}$$

where

$\bar{\epsilon}_k^2$ = estimated mean square error of the observation in block k.

By using the iterative formula (6.4) we avoid the inversion of the auto-covariance matrix. Formula (6.5) is theoretically attractive in $h^{(0)}$, because it implies that the iterative collocation is carried out with the solution of the integration method as the original step.

Finally, we compare the number of necessary matrix operations for computing the coefficients and their accuracy estimates by direct collocation [formulae (3.4) and (4.3a-b)] and the proposed iterative method [formulae (6.1), (6.2) and (6.3)]. In the direct method the computations of the matrix inverse and the accuracy estimates are the most laborious operations. For the comparison we assume that $h^{(0)}$ of (6.4) is a priori given and that ν_0 steps are necessary in the iterative method. The numbers of necessary operations for a determination of the accuracy estimates to degree n_0 [i.e. for $(n_0+1)^2$ coefficients] are summarized in Table 5 (the direct method in accordance with Westlake, 1968, Table 7.1).

Table 5

Number of Necessary Matrix Operations to Compute
Accuracy Estimates to Degree n_0

Operation	Direct Method (Cholesky)	Iterative Method [formulae (6.2) and (6.3)]
Addition	$M^3 - 2M^2 + M + (M^2 + M)(n_0 + 1)^2$	$\{(M + 1)M\nu_0 + M\}(n_0 + 1)^2$
Multiplication	$\frac{1}{2} M^3 + \frac{3}{2} M^2 - M + 2M^2(n_0 + 1)^2$	$(M^2\nu_0 + M)(n_0 + 1)^2$

ν_0 = number of iterative steps

M = number of mean anomalies

As the multiplications are the most time-consuming operations, we limit the following comparison to those figures. Then we obtain from the table that the iterative method is more efficient whenever:

$$(M \nu_0 + 1) M (n_0 + 1)^2 < \frac{1}{2} M^3 + \frac{3}{2} M^2 - M + 2 M^2 (n_0 + 1)^2$$

From this inequality we obtain

$$(6.8) \quad n_0 < \sqrt{\frac{M^2 + 3M - 2}{2M(\nu_0 - 2) + 2}} - 1$$

or, approximately, for $\nu_0 > 2$

$$(6.8') \quad n_0 < \sqrt{\frac{M + 3}{2(\nu_0 - 2)}} - 1$$

Formula (6.8) is illustrated in Table 6.

Table 6

The Maximum Integer (n_0) Satisfying (6.8)
for Various M and ν_0

$M \nu_0$	2	3	4	5	10
100	70	6	4	3	1
416	294	13	9	7	4
1000	707	21	14	11	6
1654	1169	27	19	15	9
5000	3535	49	34	27	16
10000	7071	69	49	39	24

n_0 = maximum degree of series expansion

M = number of observations

ν_0 = number of iterative steps

The table shows that the iterative method is favorable merely for up to 2 or possibly 3 necessary iterations. However, as earlier discussed, the available approximations $h^{(0)}$ could very well meet such a requirement.

7. Conclusions

Geopotential coefficients determined by collocation were found to agree somewhat better with the GEM 9 coefficients than the coefficients determined by pure integration of 10° mean anomalies. This result is probably due to the incorporation of a weighting of the observations with respect to their a priori accuracies. However, this gain is achieved at the cost of several times more computation time.

By the inclusion of the elevation information in collocation, the coefficients to degree 20 changed by 3% on the average. A surprisingly large difference of 10% was obtained for $n = 3$. The RMS changes of the undulation and the anomaly were 2 meters and 0.7 mgal, respectively. In general, however, we might expect that the 10° blocks give a too rough approximation to the topography to reveal any more significant magnitudes of the changes of the coefficients when including a correction for the topography (the Molodensky G_1 term). A possible explanation of the 10% differences for $n = 3$ in various methods might be the non-symmetric distribution of the continents between the northern and the southern hemisphere.

From the comparison of the coefficient accuracy estimates for various methods (Table 3) we conclude that the error covariances between the mean anomalies should be taken into account in the computations by direct integration. In collocation it seems less important to include these covariances in the computations.

A natural continuation of the above study would be to compute the coefficients for 5° mean anomalies by collocation. However, due to the difficult task to invert an auto-covariance matrix for more than 1600 observations, the original method should first be modified according to the iterative method described in section 6. The method includes the solution by integration as a preliminary step. As this technique avoids the inversion of the auto-covariance matrix, a considerable gain in computer time can be expected. Another possibility would be to determine the auto-covariance matrix in an iterative way.

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Appendix

It is desired to determine the spherical harmonic coefficients (referring to the Bjerhammar sphere of radius r_b) implied by a mean anomaly field $\overline{\Delta g}$ on a sphere of radius r_n . The following covariance functions are given:

$$(A.1) \quad \left\{ \begin{matrix} c_c \\ c_s \end{matrix} \right\} (P) = \left\{ \begin{matrix} \text{cov} (\overline{C}_{n' n'}, \overline{\Delta g_p}) \\ \text{cov} (\overline{S}_{n' n'}, \overline{\Delta g_p}) \end{matrix} \right\} = b_{n'} \left(\frac{r_b}{r_p} \right)^{n'+2} Y_{n' n'}(P)$$

where $b_{n'} = c_{n'} \beta_{n'} / \gamma (2n' + 1) (n' - 1)$

$$Y_{n' n'}(P) = P_{n'} |m'| (\sin \phi_p) \begin{cases} \cos m' \lambda P \\ \sin |m'| \lambda P \end{cases}$$

$$(A.2) \quad c(Q, P) = \text{cov} (\overline{\Delta g_Q}, \overline{\Delta g_P}) = \sum_{n=-\infty}^{\infty} c_n \beta_n^2 (r_b^2 / r_Q r_P)^{n+2} P_n(\cos \phi_{QP})$$

and

$$(A.3) \quad d(Q, P) = \text{cov} (\epsilon_Q, \epsilon_P) = \sum_{n=-\infty}^{\infty} d_n \beta_n^2 (r_b^2 / r_Q r_P)^{n+2} P_n(\cos \phi_{QP})$$

The solutions for $\overline{C}_{n' n'}$ and $\overline{S}_{n' n'}$ by the method of least squares collocation are given by (see Sjöberg, 1978)

$$(A.4) \quad \left\{ \begin{matrix} \hat{C}_{n' n'} \\ \hat{S}_{n' n'} \end{matrix} \right\} = \frac{1}{4\pi} \iint \left\{ \begin{matrix} h_c(Q) \\ h_s(Q) \end{matrix} \right\} \overline{\Delta g}(Q) d\alpha_Q$$

where the weight functions $h_c(Q)$ and $h_s(Q)$ are given by $h(Q)$ of the following Wiener-Hopf integral equations

$$(A.5) \quad \left\{ \begin{matrix} c_c \\ c_s \end{matrix} \right\} (P) = \frac{1}{4\pi} \iint h(Q) \left\{ c(Q, P) + d(Q, P) \right\} d\alpha_Q$$

Proposition 1: The weight functions for $\hat{C}_{nn}$ and $\hat{S}_{nn}$ in (A.4) implied by (A.1) - (A.3) and (A.5) for $r_p = r_Q = r_n = \text{constant}$ are given by:

$$(A.6) \quad h(Q) = \left(\frac{r_n}{r_b} \right)^{n+2} \frac{c_n}{c_n + d_n} Y_{nn}(Q) / \gamma \beta_n (n-1)$$

Proof: We expand (A.2) and (A.3) accordingly:

$$(A.2') \quad c(Q, P) = \sum_{n=2}^{\infty} \sum_{m=-n}^n \frac{c_n \beta_n^2}{2n+1} \left(\frac{r_P}{r_B} \right)^{2(n+2)} Y_{nm}(Q) Y_{nm}(P)$$

and

$$(A.3') \quad d(Q, P) = \sum_{n=2}^{\infty} \sum_{m=-n}^n \frac{d_n \beta_n^2}{2n+1} \left(\frac{r_P}{r_B} \right)^{2(n+2)} Y_{nm}(Q) Y_{nm}(P)$$

$$(A.7) \quad \frac{1}{4\pi} \iint Y_{nm} Y_{pq} d\sigma = \begin{cases} 1 & \text{if } n=p \text{ and } m=q \\ 0 & \text{otherwise} \end{cases}$$

Inserting (A.1), (A.2'), (A.3') and the expansion

$$h(Q) = \sum_{n=2}^{\infty} \sum_{m=-n}^n h_{nm} Y_{nm}(Q)$$

into (A.5) we obtain from (A.7):

$$(A.8) \quad b_{n'} (r_B / r_B)^{n+2} Y_{n'm'}(P) = \sum_{n,m} h_{nm} \frac{c_n + d_n}{(2n+1)} \beta_n^2 (r_B / r_B)^{2(n+2)} Y_{nm}(P)$$

This identity is satisfied by

$$h_{nm} = \begin{cases} \frac{c_n}{c_n + d_n} \left(\frac{r_B}{r_B} \right)^{n+2} / \gamma \beta_n (n-1) & \text{if } n=n' \text{ and } m=m' \\ 0 & \text{otherwise} \end{cases}$$

The proposition follows from this result.

Using the following approximation (cf. Meissl, 1971, pp. 22-23)

$$\frac{1}{\Delta\sigma} \iint_{\Delta\sigma} Y_{nm}(Q) d\sigma_Q = \beta_n Y_{nm}(\bar{Q})$$

where $\bar{Q}$ is the center of the block $\Delta\sigma$, we obtain the following relation from (A.6):

$$(A.9) \quad \frac{1}{4\pi} \iint_{\Delta\sigma_k} h(Q) d\sigma = \left(\frac{r_B}{r_B} \right)^{n+2} \frac{c_n}{c_n + d_n} Y_{nm}(\bar{Q}_k) \Delta\sigma_k / 4\pi \gamma(n-1)$$

Corollary 1: For $d(Q, P)$ of Proposition 1 replaced by

$$d(Q, P) = \epsilon^2(Q) \delta(\psi_{QP})$$

where $\delta(\psi_{QP})$ is the Dirac delta function, we obtain

$$(A.10) \quad h(Q) = \frac{c_n \beta_n}{\gamma(n-1)} \left(\frac{r_B}{r_n} \right)^{n+2} \frac{1}{c_n \beta_n^2 (r_B/r_n)^{2(n+2)} + (2n+1) \epsilon^2(Q)} Y_{nm}(Q)$$

Proof: As

$$\frac{1}{4\pi} \iint \epsilon^2(Q) \delta(\psi_{QP}) Y_{nm}(Q) d\sigma_Q = \epsilon^2(P) Y_{nm}(P)$$

formula (A.8) becomes in this case

$$b_{n'} (r_B/r_n)^{n'+2} Y_{n'm'}(P) \equiv \sum_{n,n'} h_{nm} \left(\frac{c_n \beta_n}{2n+1} \left(\frac{r_B}{r_n} \right)^{2(n+2)} + \epsilon^2(P) \right) Y_{nm}(P)$$

and the proof follows after noting that

$$h_{nm} = 0 \text{ for } n \neq n' \text{ and } m \neq m'.$$

In accordance with formula (A.9) we obtain in this case:

$$(A.11) \quad \frac{1}{4\pi} \iint_{\Delta\sigma_k} h(Q) d\sigma \approx \frac{c_n \beta_n^2}{\gamma(n-1)} \left(\frac{r_B}{r_n} \right)^{n+2} \frac{1}{c_n \beta_n^2 (r_B/r_n)^{2(n+2)} + (2n+1) \epsilon^2(Q_k)} Y_{nm}(\bar{Q}_k) \frac{\Delta\sigma_k}{4\pi}$$

Proposition 2: The error estimates for $\hat{C}_{nm}$ and $\hat{S}_{nm}$ of (A.4) and (A.5) are given by

$$\overline{m}^2 = C_0 - \frac{1}{16\pi^2} \iiint \iiint h(Q) h(Q') \{c(Q, Q') + d(Q, Q')\} d\sigma_Q d\sigma_{Q'}$$

where C_0 is the a priori variance of the coefficients.

Proof: We consider only the estimate $\hat{C}_{nm}$.

$$\overline{m}_c^2 = E \{ (\hat{C}_{nm} - \bar{C}_{nm})^2 \} = E \{ \bar{C}_{nm}^2 \} + E \{ \hat{C}_{nm}^2 \} - 2E \{ \bar{C}_{nm} \hat{C}_{nm} \}$$

where

$$E \{ \bar{C}_{nn}^2 \} = C_0$$

$$\begin{aligned} E \{ \hat{C}_{nn}^2 \} &= \frac{1}{16\pi^2} \iiint h(Q) h(Q') E \{ \Delta g_Q \Delta g_{Q'} \} d\alpha_Q d\alpha_{Q'} \\ &= \frac{1}{16\pi^2} \iiint h(Q) h(Q') \{ c(Q, Q') + d(Q, Q') \} d\alpha_Q d\alpha_{Q'} \end{aligned}$$

and

$$\begin{aligned} E \{ \bar{C}_{nn} \hat{C}_{nn} \} &= \frac{1}{4\pi} \int h(Q) E \{ \bar{C}_{nn} \Delta g_Q \} d\alpha_Q \\ &= \frac{1}{4\pi} \int h(Q) c_c(Q) d\alpha_Q \\ &= \frac{1}{16\pi^2} \iiint h(Q) h(Q') \{ c(Q, Q') + d(Q, Q') \} d\alpha_Q d\alpha_{Q'} \end{aligned}$$

From these deductions the proposition follows immediately.

Table A.1: Fully Normalized Geopotential Coefficients Determined from 416 10° Mean Anomalies by Collocation. No Mean Elevations Included. All Coefficients multiplied by 10⁶. a = 6378140 m.

N	M	C	S	SIGMA	SIGMA	N	M	C	S	SIGMA	SIGMA
2		-483.4460		0.2016		2	2	2.4400	-1.3037	0.1678	0.1097
2	1	0.3000	-0.0906	0.1799	0.1708	3	2	1.1793	-0.4719	0.2263	0.2243
3		0.7133		0.2304		4	2	0.3170	0.3939	0.1223	0.1227
3	1	1.0349	0.1234	0.2263	0.2261	4	4	-0.2265	0.4.66	0.1151	0.1141
3	3	0.7581	1.5310	0.2193	0.2203	5	2	0.4107	-0.2016	0.0820	0.0824
4		0.0957		0.1324		5	4	-0.0921	-0.0276	0.0771	0.0769
4	1	-0.4564	-0.4057	0.1219	0.1217	6	2	0.2816	-0.3643	0.0464	0.0459
4	3	0.7955	-0.3966	0.1160	0.1109	6	4	-0.1000	-0.4041	0.0426	0.0423
5		0.2236		0.0896		6	6	0.0097	-0.2402	0.0371	0.0375
5	1	-0.1576	-0.2482	0.0821	0.0817	7	2	0.2858	0.1194	0.0497	0.0499
5	3	-0.2440	-0.1369	0.0811	0.0811	7	4	-0.1572	-0.1660	0.0490	0.0409
5	5	0.1064	-0.4974	0.0740	0.0766	7	6	-0.3078	0.2008	0.0448	0.0449
6		-0.1376		0.0540		8	2	0.1333	0.0993	0.0399	0.0404
6	1	0.1128	0.0633	0.0466	0.0457	8	4	-0.2167	0.0313	0.0392	0.0307
6	3	-0.0173	-0.0623	0.0459	0.0433	8	6	-0.0006	0.1900	0.0358	0.0354
6	5	-0.3769	-0.5370	0.0363	0.0396	8	8	-0.1417	0.0911	0.0343	0.0338
7		0.2002		0.0547		9	2	0.1171	-0.0043	0.0300	0.0304
7	1	0.2010	0.0466	0.0513	0.0510	9	4	-0.0390	0.0490	0.0360	0.0363
7	3	0.1615	-0.1005	0.0501	0.0501	9	6	0.0014	0.1023	0.0347	0.0344
7	5	0.0896	-0.0060	0.0466	0.0472	9	8	0.2030	0.0063	0.0327	0.0324
7	7	-0.0323	-0.0921	0.0450	0.0449	10	2	-0.0657	-0.0457	0.0324	0.0326
8		0.0422		0.0441		10	4	-0.0674	-0.0963	0.0309	0.0303
8	1	-0.0423	0.0575	0.0492	0.0480	10	6	-0.0475	-0.0479	0.0299	0.0296
8	3	0.0704	-0.0234	0.0308	0.0303	10	8	0.0413	-0.0533	0.0273	0.0270
8	5	-0.0176	0.0345	0.0366	0.0373	10	10	0.1159	-0.0309	0.0277	0.0274
8	7	0.0303	0.0713	0.0338	0.0336	11	2	-0.0219	-0.0922	0.0317	0.0310
9		0.1336		0.0409		11	4	-0.1044	-0.0707	0.0302	0.0299
9	1	0.1723	-0.0228	0.0302	0.0301	11	6	-0.0112	0.0100	0.0209	0.0207
9	3	-0.1706	-0.0303	0.0373	0.0372	11	8	0.0092	0.0613	0.0266	0.0263
9	5	-0.0572	-0.0396	0.0362	0.0367	11	10	-0.0269	0.0139	0.0263	0.0233
9	7	-0.0023	-0.0215	0.0336	0.0336	12	2	0.0038	-0.0309	0.0303	0.0306
9	9	-0.0159	0.0237	0.0333	0.0333	12	4	-0.0653	-0.0343	0.0280	0.0207
10		0.0303		0.0331		12	6	0.0233	0.0496	0.0273	0.0272
10	1	0.1146	-0.0147	0.0326	0.0320	12	8	0.0100	0.0264	0.0263	0.0261
10	3	-0.0416	-0.0954	0.0310	0.0319	12	10	-0.0094	-0.0265	0.0254	0.0253
10	5	-0.0261	-0.0113	0.0301	0.0306	12	12	0.0134	-0.0094	0.0263	0.0264
10	7	0.0024	-0.0107	0.0270	0.0279	13	2	0.0003	-0.0414	0.0289	0.0293
10	9	0.1013	-0.0222	0.0263	0.0263	13	4	0.0019	-0.0060	0.0277	0.0273
11		-0.0933		0.0306		13	6	-0.0246	-0.0006	0.0263	0.0263
11	1	-0.0214	0.0107	0.0310	0.0311	13	8	-0.0237	0.0200	0.0258	0.0253
11	3	-0.0747	-0.1126	0.0306	0.0304	13	10	0.0359	-0.0149	0.0243	0.0240
11	5	0.0123	0.0074	0.0200	0.0292	13	12	-0.0016	0.0943	0.0242	0.0243
11	7	0.0373	-0.1030	0.0279	0.0200						
11	9	-0.0456	0.0017	0.0263	0.0264						
11	11	0.0769	-0.0164	0.0271	0.0271						
12		0.0330		0.0321							
12	1	-0.0213	-0.0334	0.0297	0.0298						
12	3	0.0097	0.0307	0.0295	0.0294						
12	5	0.0376	-0.0023	0.0201	0.0203						
12	7	-0.0240	0.0267	0.0272	0.0271						
12	9	0.0106	0.0169	0.0232	0.0231						
12	11	0.0009	0.0064	0.0250	0.0231						
13		0.0473		0.0303							
13	1	-0.0016	0.0106	0.0200	0.0292						
13	3	-0.0144	0.0421	0.0207	0.0206						
13	5	0.0506	0.0397	0.0271	0.0273						
13	7	-0.0171	0.0137	0.0239	0.0239						
13	9	-0.0140	0.0334	0.0240	0.0247						
13	11	0.0043	0.0201	0.0243	0.0246						
13	13	-0.0349	0.0637	0.0237	0.0260						
14		0.0000		0.0200							

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Table A.1 (continued)

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14 1	0.0204	0.0084	0.0276	0.0209	14 2	-0.0307	-0.0173	0.0272	0.0272
14 3	0.0096	0.0002	0.0272	0.0270	14 4	-0.0016	-0.0200	0.0263	0.0260
14 5	0.0147	-0.0070	0.0254	0.0254	14 6	-0.0085	0.0010	0.0251	0.0250
14 7	0.0342	-0.0140	0.0242	0.0241	14 8	-0.0276	-0.0234	0.0243	0.0237
14 9	0.0154	0.0416	0.0234	0.0234	14 10	0.0160	-0.0076	0.0223	0.0225
14 11	0.0336	-0.0752	0.0224	0.0223	14 12	0.0011	-0.0205	0.0220	0.0231
14 13	0.0365	0.0207	0.0220	0.0220	14 14	-0.0171	0.0117	0.0244	0.0245
15 1	-0.0027	0.0259	0.0259		15 2	0.0045	0.0133	0.0266	0.0267
15 3	0.0337	0.0269	0.0270	0.0274	15 4	0.0042	-0.0123	0.0260	0.0260
15 5	0.0441	0.0452	0.0262	0.0262	15 6	-0.0070	-0.0254	0.0243	0.0246
15 6	0.0092	0.0135	0.0247	0.0248	15 8	-0.0437	0.0190	0.0234	0.0231
15 7	0.0563	0.0110	0.0240	0.0239	15 10	-0.0242	0.0002	0.0226	0.0226
15 9	0.0024	-0.0033	0.0230	0.0231	15 12	-0.0100	0.0046	0.0220	0.0221
15 11	-0.0043	0.0114	0.0222	0.0220	15 14	0.0119	-0.0212	0.0227	0.0227
15 13	-0.0103	0.0177	0.0220	0.0226					
15 15	-0.0263	0.0083	0.0244	0.0243					
16 1	0.0214	0.0254			16 2	-0.0106	0.0019	0.0258	0.0258
16 3	-0.0003	0.0170	0.0260	0.0270	16 4	0.0314	0.0019	0.0252	0.0252
16 5	-0.0176	0.0246	0.0256	0.0256	16 6	-0.0010	-0.0315	0.0237	0.0239
16 7	-0.0123	0.0211	0.0245	0.0243	16 8	-0.0502	0.0340	0.0230	0.0223
16 9	-0.0118	-0.0064	0.0236	0.0236	16 10	-0.0059	-0.0083	0.0224	0.0223
16 11	-0.0049	-0.0589	0.0222	0.0226	16 12	0.0161	0.0012	0.0217	0.0216
16 13	0.0064	-0.0104	0.0221	0.0217	16 14	-0.0010	-0.0239	0.0225	0.0224
16 15	0.0011	0.0153	0.0221	0.0215	16 16	-0.0178	-0.0172	0.0242	0.0243
17 1	-0.0090	-0.0335	0.0231	0.0219	17 2	-0.0307	0.0173	0.0217	0.0217
17 3	0.0019	0.0197			17 4	-0.0194	0.0290	0.0231	0.0230
17 5	0.0034	0.0228	0.0296	0.0297	17 6	-0.0207	-0.0351	0.0220	0.0230
17 7	0.0030	-0.0140	0.0270	0.0270	17 8	0.0212	-0.0092	0.0226	0.0224
17 9	-0.0165	0.0114	0.0247	0.0240	17 10	-0.0110	0.0120	0.0216	0.0217
17 11	0.0192	-0.0231	0.0231	0.0232	17 12	-0.0166	0.0002	0.0213	0.0212
17 13	-0.0101	-0.0336	0.0210	0.0222	17 14	-0.0120	0.0190	0.0215	0.0215
17 15	-0.0013	0.0034	0.0219	0.0214	17 16	-0.0112	0.0141	0.0223	0.0222
17 17	0.0174	0.0102	0.0222	0.0204					
17 19	0.0146	0.0172	0.0235	0.0200	18 2	-0.0054	0.0039	0.0309	0.0309
18 1	-0.0312	0.0061	0.0230	0.0237	18 4	0.0064	0.0053	0.0292	0.0292
18 3	0.0069	0.0302			18 6	0.0164	-0.0022	0.0256	0.0256
18 5	-0.0199	-0.0390	0.0165	0.0167	18 8	0.0203	-0.0076	0.0229	0.0230
18 7	-0.0030	-0.0107	0.0153	0.0154	18 10	0.0104	-0.0049	0.0214	0.0213
18 9	0.0044	0.0130	0.0102	0.0103	18 12	0.0063	-0.0171	0.0211	0.0210
18 11	-0.0016	-0.0010	0.0206	0.0206	18 14	0.0022	-0.0193	0.0209	0.0210
18 13	0.0050	0.0087	0.0210	0.0213	18 16	0.0100	0.0129	0.0219	0.0217
18 15	-0.0211	-0.0076	0.0213	0.0209	18 18	-0.0041	-0.0116	0.0292	0.0151
18 17	-0.0052	-0.0320	0.0225	0.0100	19 2	0.0200	-0.0094	0.0247	0.0247
18 19	-0.0410	-0.0271	0.0247	0.0102	19 4	0.0164	-0.0116	0.0197	0.0197
19 1	0.0091	-0.0097	0.0220	0.0217	19 6	0.0065	0.0166	0.0160	0.0160
19 3	0.0001	0.0253			19 8	0.0220	0.0013	0.0179	0.0179
19 5	0.0003	0.0176	0.0216	0.0217	19 10	-0.0133	-0.0092	0.0190	0.0190
19 7	0.0007	-0.0136	0.0237	0.0237	19 12	-0.0021	-0.0030	0.0204	0.0202
19 9	-0.0089	0.0049	0.0269	0.0262	19 14	0.0159	-0.0066	0.0202	0.0203
19 11	-0.0026	0.0045	0.0259	0.0259	19 16	-0.0247	0.0009	0.0200	0.0200
19 13	0.0049	0.0039	0.0230	0.0231	19 18	0.0336	-0.0092	0.0230	0.0197
19 15	0.0003	0.0060	0.0210	0.0200					
19 17	0.0077	-0.0205	0.0230	0.0173	20 2	-0.0036	0.0035	0.0227	0.0227
19 19	0.0009	-0.0153	0.0241	0.0163	20 4	-0.0050	-0.0193	0.0190	0.0190
20 1	0.0100	-0.0064	0.0214	0.0213	20 6	0.0115	0.0036	0.0221	0.0221
20 3	0.0190	0.0095	0.0226	0.0226	20 8	0.0069	-0.0022	0.0244	0.0245
20 5	-0.0036	0.0234			20 10	-0.0000	0.0142	0.0220	0.0220
20 7	-0.0119	-0.0031	0.0216	0.0216	20 12	-0.0191	0.0070	0.0206	0.0206
20 9	-0.0076	0.0007	0.0233	0.0233	20 14	0.0123	-0.0019	0.0198	0.0199
20 11	-0.0040	0.0011	0.0230	0.0230	20 16	-0.0163	-0.0090	0.0200	0.0201
20 13	-0.0179	-0.0065	0.0178	0.0179	20 18	-0.0094	-0.0109	0.0210	0.0196
20 15	0.0202	-0.0070	0.0161	0.0163	20 20	0.0152	0.0043	0.0217	0.0210
20 17	0.0221	0.0037	0.0102	0.0102					
20 19	0.0039	-0.0004	0.0223	0.0162					
20 21	0.0005	-0.0074	0.0220	0.0159					
20 23	-0.0014	0.0067	0.0202	0.0202					
20 25	-0.0023	0.0011	0.0200	0.0200					

Table A.2: Fully Normalized Geopotential Coefficients Determined from 416 10° Mean Anomalies by Collocation. Approximate Mean Elevations Included. All coefficients multiplied by 10⁶. a = 6378140 m.

N	M	C	S	SIGMA	SIGMA	N	M	C	S	SIGMA	SIGMA
2		-483.4651		0.2212		2	2	2.3810	-1.3976	0.1927	0.1806
2	1	0.2721	-0.0049	0.2098	0.2196	3					
3		0.6295		0.1772		3	2	1.3741	-0.5472	0.1674	0.1690
3	1	1.0080	-0.0201	0.1730	0.1604	4					
3	3	0.6623	1.5512	0.1580	0.1662	4	2	-0.3700	0.4063	0.1036	0.1019
4		0.0659		0.1096		4	4	-0.2159	0.3966	0.0968	0.0983
4	1	-0.4433	-0.3007	0.1001	0.0967	5					
4	3	0.7904	-0.3962	0.0947	0.0976	5	2	0.4138	-0.1909	0.0721	0.0705
5		0.2030		0.0792		5	4	-0.0911	-0.0493	0.0662	0.0645
5	1	-0.1429	-0.2469	0.0699	0.0673	6					
5	3	-0.2435	-0.1402	0.0694	0.0709	6	2	0.2776	-0.3591	0.0490	0.0409
5	5	0.1198	-0.4990	0.0617	0.0667	6	4	-0.1730	-0.4046	0.0455	0.0459
6		-0.1464		0.0583		6	6	0.0041	-0.2293	0.0402	0.0407
6	1	0.1175	0.0490	0.0480	0.0406	7					
6	3	-0.0168	-0.0632	0.0487	0.0405	7	2	0.2941	0.1235	0.0450	0.0450
6	5	-0.3037	-0.5336	0.0399	0.0424	7	4	-0.1582	-0.1714	0.0450	0.0448
7		0.2042		0.0504		7	6	-0.3094	0.2030	0.0410	0.0401
7	1	0.2090	0.0453	0.0473	0.0460	8					
7	3	0.1597	-0.1926	0.0452	0.0453	8	2	0.1375	0.1005	0.0389	0.0392
7	5	0.0835	-0.0001	0.0421	0.0434	8	4	-0.2114	0.0454	0.0386	0.0378
7	7	-0.0367	-0.0903	0.0411	0.0414	8	6	-0.0163	0.2066	0.0347	0.0345
8		0.0410		0.0430		8	8	-0.1430	0.0959	0.0344	0.0331
8	1	-0.0450	0.0544	0.0393	0.0391	9					
8	3	0.0795	-0.0220	0.0370	0.0375	9	2	0.1169	-0.0836	0.0360	0.0365
8	5	-0.0193	0.0311	0.0359	0.0369	9	4	-0.0619	0.0510	0.0340	0.0342
8	7	0.0445	0.0735	0.0336	0.0328	9	6	0.0007	0.1023	0.0330	0.0325
9		0.1377		0.0391		9	8	0.2041	0.0072	0.0310	0.0304
9	1	0.1770	-0.0223	0.0364	0.0360	10					
9	3	-0.1794	-0.0320	0.0351	0.0350	10	2	-0.0667	-0.0425	0.0323	0.0326
9	5	-0.0594	-0.0361	0.0343	0.0349	10	4	-0.0704	-0.0950	0.0309	0.0305
9	7	-0.0027	-0.0223	0.0346	0.0317	10	6	-0.0492	-0.0493	0.0299	0.0297
9	9	-0.0179	0.0253	0.0314	0.0317	10	8	0.0410	-0.0524	0.0277	0.0272
10		0.0264		0.0353		10	10	0.1172	-0.0333	0.0270	0.0273
10	1	0.1151	-0.0175	0.0320	0.0320	11					
10	3	-0.0410	-0.0970	0.0319	0.0319	11	2	-0.0200	-0.0225	0.0312	0.0314
10	5	-0.0270	-0.0110	0.0302	0.0307	11	4	-0.1059	-0.0790	0.0297	0.0294
10	7	0.0020	-0.0201	0.0279	0.0279	11	6	-0.0107	0.0090	0.0285	0.0283
10	9	0.1031	-0.0220	0.0265	0.0267	11	8	0.0099	0.0609	0.0263	0.0250
11		-0.0966		0.0332		11	10	-0.0250	0.0136	0.0260	0.0250
11	1	-0.0221	0.0074	0.0307	0.0307	12					
11	3	-0.0741	-0.1140	0.0302	0.0300	12	2	0.0053	-0.0303	0.0290	0.0302
11	5	0.0114	0.0070	0.0284	0.0280	12	4	-0.0650	-0.0350	0.0283	0.0282
11	7	0.0372	-0.1035	0.0276	0.0276	12	6	0.0272	0.0490	0.0270	0.0267
11	9	-0.0494	0.0021	0.0259	0.0261	12	8	0.0101	0.0269	0.0259	0.0256
11	11	0.0760	-0.0169	0.0267	0.0269	12	10	-0.0092	-0.0274	0.0250	0.0248
12		0.0312		0.0317		12	12	0.0140	-0.0106	0.0261	0.0260
12	1	-0.0245	-0.0337	0.0294	0.0293	13					
12	3	0.0114	0.0300	0.0290	0.0289	13	2	0.0095	-0.0421	0.0284	0.0289
12	5	0.0565	-0.0022	0.0275	0.0270	13	4	0.0024	-0.0069	0.0273	0.0270
12	7	-0.0249	0.0277	0.0267	0.0267	13	6	-0.0243	-0.0092	0.0261	0.0259
12	9	0.0100	0.0190	0.0248	0.0246	13	8	-0.0234	0.0199	0.0254	0.0250
12	11	0.0103	0.0060	0.0246	0.0247	13	10	0.0351	-0.0129	0.0239	0.0235
13		0.0460		0.0299		13	12	-0.0004	0.0966	0.0230	0.0241
13	1	-0.0020	0.0183	0.0284	0.0287						
13	3	-0.0135	0.0413	0.0283	0.0282						
13	5	0.0593	0.0406	0.0266	0.0269						
13	7	-0.0172	0.0163	0.0255	0.0254						
13	9	-0.0137	0.0347	0.0244	0.0243						
13	11	0.0047	0.0204	0.0241	0.0242						
13	13	-0.0345	0.0650	0.0235	0.0236						
14		-0.0007		0.0201							

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Table A.2 (continued)

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14 1	0.0192	0.0074	0.0277	0.0200	14 2	-0.0379	-0.0154	0.0273	0.0274
14 3	0.0103	0.0093	0.0272	0.0271	14 4	-0.0014	-0.0273	0.0263	0.0261
14 5	0.0157	-0.0069	0.0253	0.0255	14 6	-0.0092	0.0013	0.0251	0.0251
14 7	0.0334	-0.0143	0.0243	0.0242	14 8	-0.0273	-0.0243	0.0241	0.0230
14 9	0.0157	0.0411	0.0236	0.0233	14 10	0.0157	-0.0040	0.0230	0.0226
14 11	0.0327	-0.0742	0.0223	0.0223	14 12	0.0003	-0.0204	0.0220	0.0202
14 13	0.0370	0.0306	0.0230	0.0230	14 14	-0.0104	0.0203	0.0243	0.0247
15	-0.0036		0.0269		15 2	0.0047	0.0137	0.0204	0.0208
15 1	0.0340	0.0267	0.0270	0.0273	15 4	0.0031	-0.0122	0.0239	0.0250
15 3	0.0446	0.0449	0.0262	0.0263	15 6	-0.0075	0.0253	0.0245	0.0246
15 5	0.0104	0.0135	0.0247	0.0240	15 8	-0.0430	0.0194	0.0234	0.0231
15 7	0.0359	0.0124	0.0239	0.0239	15 10	-0.0051	-0.0066	0.0226	0.0226
15 9	0.0027	-0.0029	0.0230	0.0232	15 12	-0.0107	0.0049	0.0221	0.0221
15 11	-0.0064	0.0167	0.0223	0.0220	15 14	0.0119	-0.0200	0.0227	0.0227
15 13	-0.0179	0.0179	0.0229	0.0226					
15 15	-0.0262	0.0071	0.0243	0.0243					
16	0.0203		0.0254		16 2	-0.0103	0.0033	0.0250	0.0250
16 1	-0.0089	0.0137	0.0260	0.0270	16 4	0.0317	0.0619	0.0252	0.0251
16 3	-0.0174	0.0233	0.0256	0.0256	16 6	-0.0026	-0.0322	0.0237	0.0239
16 5	-0.0111	0.0204	0.0243	0.0243	16 8	-0.0503	0.0330	0.0229	0.0228
16 7	-0.0123	-0.0033	0.0243	0.0236	16 10	-0.0062	-0.0093	0.0224	0.0223
16 9	-0.0032	-0.0306	0.0223	0.0226	16 12	0.0156	-0.0003	0.0210	0.0216
16 11	0.0060	-0.0113	0.0221	0.0217	16 14	-0.0019	-0.0238	0.0223	0.0223
16 13	0.0010	0.0153	0.0221	0.0213	16 16	-0.0173	-0.0172	0.0242	0.0240
16 15	-0.0107	-0.0338	0.0231	0.0219					
17	0.0020		0.0197		17 2	-0.0303	0.0176	0.0217	0.0217
17 1	0.0053	0.0226	0.0203	0.0204	17 4	-0.0106	0.0300	0.0231	0.0230
17 3	0.0040	-0.0144	0.0270	0.0270	17 6	-0.0213	-0.0357	0.0220	0.0220
17 5	-0.0161	0.0111	0.0247	0.0248	17 8	0.0211	-0.0093	0.0223	0.0224
17 7	0.0100	-0.0234	0.0231	0.0231	17 10	-0.0110	0.0127	0.0217	0.0217
17 9	-0.0173	-0.0353	0.0217	0.0222	17 12	-0.0167	0.0002	0.0213	0.0212
17 11	-0.0011	0.0033	0.0219	0.0214	17 14	-0.0126	0.0103	0.0215	0.0215
17 13	0.0100	0.0103	0.0223	0.0204	17 16	-0.0112	0.0134	0.0222	0.0222
17 15	0.0145	0.0166	0.0230	0.0200					
17 17	-0.0314	0.0060	0.0230	0.0217	18 2	-0.0054	0.0042	0.0309	0.0309
18	0.0066		0.0302		18 4	0.0060	0.0037	0.0292	0.0292
18 1	-0.0200	-0.0390	0.0163	0.0166	18 6	0.0163	-0.0027	0.0256	0.0256
18 3	-0.0060	-0.0106	0.0153	0.0153	18 8	0.0201	-0.0077	0.0229	0.0228
18 5	0.0043	0.0142	0.0181	0.0182	18 10	0.0104	-0.0042	0.0214	0.0213
18 7	-0.0015	-0.0023	0.0203	0.0206	18 12	0.0056	-0.0172	0.0211	0.0209
18 9	0.0063	0.0094	0.0210	0.0214	18 14	0.0022	-0.0194	0.0209	0.0210
18 11	-0.0210	-0.0076	0.0212	0.0200	18 16	0.0103	0.0126	0.0219	0.0217
18 13	-0.0055	-0.0525	0.0223	0.0108	18 18	-0.0044	-0.0124	0.0292	0.0150
18 15	-0.0413	-0.0201	0.0237	0.0102	19 2	0.0209	-0.0090	0.0247	0.0247
18 17	0.0093	-0.0093	0.0220	0.0217	19 4	0.0163	-0.0116	0.0197	0.0196
19	-0.0003		0.0253		19 6	0.0060	0.0166	0.0159	0.0160
19 1	0.0001	0.0174	0.0216	0.0217	19 8	0.0222	0.0017	0.0179	0.0178
19 3	0.0001	-0.0130	0.0237	0.0237	19 10	-0.0134	-0.0091	0.0190	0.0197
19 5	-0.0000	0.0049	0.0269	0.0269	19 12	-0.0010	-0.0031	0.0204	0.0202
19 7	-0.0025	0.0042	0.0239	0.0239	19 14	0.0161	-0.0065	0.0201	0.0202
19 9	0.0049	0.0043	0.0227	0.0230	19 16	-0.0243	0.0091	0.0200	0.0206
19 11	0.0007	0.0050	0.0210	0.0200	19 18	0.0336	-0.0092	0.0230	0.0196
19 13	0.0002	-0.0294	0.0230	0.0174					
19 15	0.0010	-0.0160	0.0241	0.0163	20 2	-0.0036	0.0029	0.0227	0.0227
19 17	0.0194	-0.0063	0.0214	0.0212	20 4	-0.0047	-0.0107	0.0198	0.0198
19 19	0.0199	0.0091	0.0226	0.0226	20 6	0.0116	0.0033	0.0220	0.0220
20	-0.0056		0.0234		20 8	0.0069	-0.0024	0.0244	0.0245
20 1	-0.0129	-0.0036	0.0216	0.0216	20 10	-0.0013	0.0143	0.0228	0.0228
20 3	-0.0076	0.0001	0.0233	0.0233	20 12	-0.0191	0.0008	0.0206	0.0204
20 5	-0.0040	0.0011	0.0230	0.0230	20 14	0.0123	-0.0015	0.0198	0.0198
20 7	-0.0179	-0.0069	0.0170	0.0179	20 16	-0.0167	-0.0068	0.0200	0.0201
20 9	0.0204	-0.0077	0.0160	0.0162	20 18	-0.0096	-0.0107	0.0217	0.0196
20 11	0.0230	0.0037	0.0182	0.0182	20 20	0.0153	0.0044	0.0217	0.0218
20 13	0.0040	-0.0006	0.0223	0.0162					
20 15	0.0003	-0.0060	0.0220	0.0159					
20 17	-0.0016	0.0071	0.0202	0.0202					
20 19	-0.0022	0.0006	0.0200	0.0200					

Table A.3: Fully Normalized Geopotential Coefficients Determined from 416 10° Mean Anomalies by Collocation. No Elevations Included. Noise Covariance Matrix D = 0. a = 6378140 m. Coefficients multiplied by 10⁶.

N	M	C	S	SIGMA	SIGMA	N	M	C	S	SIGMA	SIGMA
2		-483.2466		0.1293		2	2	2.4441	-1.4526	0.0878	0.0860
2	1	0.2343	0.0751	0.0903	0.0808	3					
3		0.7037		0.1927		3	2	1.1602	-0.4638	0.2102	0.2093
3	1	1.7968	0.0687	0.2015	0.2093	4					
3	3	0.7672	1.5693	0.2093	0.2100	4	2	0.3340	0.4062	0.1070	0.1065
4		1.0257		0.1117		4	4	-0.2243	0.4157	0.1059	0.1061
4	1	-0.4403	-0.3936	0.1078	0.1076	5					
4	3	0.7718	-0.4191	0.1053	0.1057	5	2	0.3902	-0.1930	0.0695	0.0694
5		0.2040		0.0712		5	4	-0.0062	-0.0101	0.0674	0.0677
5	1	-0.1577	-0.2423	0.0699	0.0699	6					
5	3	-0.2319	-0.1332	0.0643	0.0681	6	2	0.2066	-0.3644	0.0288	0.0289
5	5	0.1089	-0.5070	0.0672	0.0691	6	4	-0.1792	-0.4131	0.0239	0.0240
6		-0.0914		0.0494		6	6	0.0017	-0.2346	0.0252	0.0257
6	1	0.1246	0.0478	0.0293	0.0292	7					
6	3	-0.0162	-0.0610	0.0263	0.0239	7	2	0.2920	0.1261	0.0411	0.0410
6	5	-0.3680	-0.5424	0.0237	0.0240	7	4	-0.1626	-0.1636	0.0303	0.0309
7		0.2051		0.0442		7	6	-0.3073	0.2070	0.0308	0.0304
7	1	0.2026	0.0449	0.0412	0.0412	8					
7	3	0.1539	-0.1040	0.0442	0.0399	8	2	0.1292	0.0908	0.0308	0.0307
7	5	0.0901	-0.0048	0.0384	0.0382	8	4	-0.2115	0.0304	0.0281	0.0283
7	7	-0.0293	-0.0949	0.0371	0.0391	8	6	-0.0877	0.1944	0.0269	0.0269
8		0.0619		0.0364		8	8	-0.1447	0.0913	0.0202	0.0202
8	1	-0.0443	0.0461	0.0307	0.0307	9					
8	3	0.0812	-0.0213	0.0290	0.0294	9	2	0.1212	-0.0779	0.0310	0.0310
8	5	-0.0170	0.0339	0.0269	0.0272	9	4	-0.0606	0.0554	0.0291	0.0291
8	7	0.0404	0.0712	0.0269	0.0270	9	6	0.0922	0.1084	0.0276	0.0271
9		0.1370		0.0315		9	8	0.2044	0.0008	0.0276	0.0276
9	1	0.1813	-0.0271	0.0310	0.0310	10					
9	3	-0.1873	-0.0303	0.0303	0.0302	10	2	-0.0715	-0.0428	0.0256	0.0256
9	5	-0.0502	-0.0317	0.0211	0.0201	10	4	-0.0673	-0.1026	0.0236	0.0236
9	7	-0.0836	-0.0199	0.0214	0.0276	10	6	-0.0516	-0.0487	0.0216	0.0211
9	9	-0.0129	0.0262	0.0206	0.0208	10	8	0.0441	-0.0371	0.0210	0.0211
10		0.0483		0.0217		10	10	0.1077	-0.0314	0.0233	0.0224
10	1	0.1216	-0.0106	0.0254	0.0253	11					
10	3	-0.0306	-0.0991	0.0246	0.0247	11	2	-0.0241	-0.0912	0.0250	0.0250
10	5	-0.0270	-0.0139	0.0221	0.0223	11	4	-0.1030	-0.0733	0.0240	0.0241
10	7	0.0064	-0.0170	0.0205	0.0209	11	6	-0.0007	0.0093	0.0220	0.0217
10	9	0.1061	-0.0191	0.0210	0.0216	11	8	0.0073	0.0666	0.0209	0.0210
11		-0.0929		0.0243		11	10	-0.0295	0.0158	0.0223	0.0212
11	1	-0.0222	0.0041	0.0258	0.0250	12					
11	3	-0.0806	-0.1206	0.0251	0.0251	12	2	0.0067	-0.0279	0.0237	0.0237
11	5	0.0152	0.0077	0.0229	0.0230	12	4	-0.0673	-0.0379	0.0241	0.0241
11	7	0.0337	-0.1130	0.0212	0.0215	12	6	0.0292	0.0513	0.0222	0.0219
11	9	-0.0474	0.0033	0.0213	0.0217	12	8	0.0092	0.0266	0.0208	0.0210
11	11	0.0798	-0.0171	0.0214	0.0234	12	10	-0.0117	-0.0266	0.0215	0.0214
12		0.0409		0.0200		12	12	0.0141	-0.0100	0.0234	0.0233
12	1	-0.0183	-0.0814	0.0216	0.0233	13					
12	3	0.0129	0.0426	0.0210	0.0240	13	2	0.0077	-0.0381	0.0254	0.0254
12	5	0.0569	-0.0016	0.0210	0.0230	13	4	0.0040	-0.0039	0.0239	0.0239
12	7	-0.0231	0.0020	0.0211	0.0214	13	6	-0.0242	-0.0111	0.0220	0.0220
12	9	0.0139	0.0167	0.0208	0.0207	13	8	-0.0230	0.0222	0.0206	0.0207
13		0.0090	0.0039	0.0217	0.0217	13	10	-0.0330	-0.0114	0.0207	0.0204
13	1	0.0497		0.0209		13	12	-0.0042	0.0994	0.0215	0.0217
13	3	-0.0842	0.0142	0.0214	0.0234						
13	5	-0.0195	0.0376	0.0216	0.0246						
13	7	0.0302	0.0403	0.0210	0.0231						
13	9	-0.0177	0.0132	0.0213	0.0214						
13	11	-0.0176	0.0323	0.0207	0.0201						
13	13	0.0065	0.0200	0.0213	0.0213						
13	15	-0.0379	0.0633	0.0213	0.0214						
14		0.0125		0.0214							

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Table A.3 (continued)

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14 1	0.0222	0.0077	0.0244	0.0244	14 2	-0.0426	-0.0187	0.0243	0.0243
14 3	0.0009	0.0050	0.0234	0.0234	14 4	-0.0006	-0.0232	0.0226	0.0226
14 5	0.0141	-0.0007	0.0219	0.0219	14 6	-0.0031	0.0039	0.0210	0.0210
14 7	0.0316	-0.0149	0.0199	0.0199	14 8	-0.0301	-0.0200	0.0194	0.0194
14 9	0.0155	0.0437	0.0189	0.0189	14 10	0.0234	-0.0040	0.0191	0.0191
14 11	0.0327	-0.0035	0.0191	0.0190	14 12	0.0016	-0.0238	0.0199	0.0202
14 13	0.0302	0.0305	0.0203	0.0203	14 14	-0.0176	0.0132	0.0223	0.0223
15 1	-0.0020		0.0215		15 2	0.0047	0.0103	0.0241	0.0242
15 3	0.0331	0.0300	0.0210	0.0240	15 4	0.0032	-0.0172	0.0229	0.0229
15 5	0.0439	0.0465	0.0215	0.0236	15 6	-0.0071	0.0269	0.0213	0.0213
15 7	0.0095	0.0159	0.0231	0.0221	15 8	-0.0427	0.0223	0.0197	0.0196
15 9	0.0611	0.0147	0.0205	0.0205	15 10	-0.0502	0.0000	0.0190	0.0189
15 11	-0.0026	-0.0030	0.0193	0.0194	15 12	-0.0155	0.0697	0.0194	0.0193
15 13	-0.0061	0.0160	0.0193	0.0192	15 14	0.0136	-0.0215	0.0207	0.0206
15 15	-0.0200	0.0173	0.0205	0.0203					
16 1	-0.0205	0.0096	0.0220	0.0223					
16 3	0.0294		0.0210						
16 5	-0.0009	0.0144	0.0214	0.0254	16 2	-0.0090	0.0000	0.0236	0.0236
16 7	-0.0236	0.0212	0.0217	0.0230	16 4	0.0350	0.0001	0.0220	0.0220
16 9	-0.0113	0.0219	0.0222	0.0222	16 6	0.0003	-0.0307	0.0215	0.0215
16 11	-0.0160	-0.0070	0.0207	0.0200	16 8	-0.0517	0.0340	0.0201	0.0200
16 13	-0.0037	-0.0393	0.0194	0.0193	16 10	-0.0025	-0.0115	0.0194	0.0193
16 15	0.0074	-0.0126	0.0192	0.0180	16 12	0.0190	-0.0011	0.0195	0.0193
17 1	0.0000	0.0160	0.0201	0.0193	16 14	-0.0046	-0.0240	0.0207	0.0206
17 3	-0.0111	-0.0350	0.0213	0.0203	16 16	-0.0179	-0.0109	0.0220	0.0220
17 5	0.0029		0.0173						
17 7	0.0064	0.0201	0.0207	0.0207	17 2	-0.0422	0.0199	0.0200	0.0200
17 9	0.0001	-0.0106	0.0214	0.0235	17 4	-0.0195	0.0202	0.0214	0.0214
17 11	-0.0104	0.0109	0.0220	0.0220	17 6	-0.0230	-0.0373	0.0212	0.0212
17 13	0.0223	-0.0230	0.0211	0.0211	17 8	0.0219	-0.0113	0.0201	0.0201
17 15	-0.0202	-0.0377	0.0116	0.0199	17 10	-0.0162	0.0155	0.0193	0.0193
18 1	0.0003	0.0029	0.0195	0.0190	17 12	-0.0160	0.0015	0.0191	0.0190
18 3	0.0191	0.0111	0.0210	0.0184	17 14	-0.0120	0.0201	0.0190	0.0199
18 5	0.0167	0.0101	0.0223	0.0192	17 16	-0.0110	0.0139	0.0210	0.0210
18 7	-0.0321	0.0069	0.0220	0.0227					
18 9	0.0102		0.0300						
18 11	-0.0251	-0.0419	0.0139	0.0138	18 2	-0.0057	0.0030	0.0300	0.0300
18 13	-0.0093	-0.0004	0.0119	0.0120	18 4	0.0000	0.0069	0.0256	0.0206
18 15	0.0032	0.0156	0.0160	0.0160	18 6	0.0199	-0.0010	0.0244	0.0244
19 1	-0.0044	-0.0025	0.0160	0.0191	18 8	0.0290	-0.0050	0.0211	0.0211
19 3	0.0071	0.0099	0.0142	0.0194	18 10	0.0246	-0.0052	0.0195	0.0195
19 5	-0.0234	-0.0090	0.0193	0.0189	18 12	0.0001	-0.0179	0.0193	0.0191
19 7	-0.0066	-0.0354	0.0214	0.0165	18 14	0.0003	-0.0205	0.0197	0.0197
19 9	-0.0442	-0.0291	0.0220	0.0163	18 16	0.0113	0.0123	0.0200	0.0207
19 11	0.0092	-0.0110	0.0210	0.0200	18 18	-0.0046	-0.0122	0.0290	0.0131
19 13	-0.0002		0.0250						
19 15	0.0013	0.0221	0.0205	0.0203	19 2	0.0229	-0.0107	0.0244	0.0244
19 17	0.0023	-0.0160	0.0219	0.0229	19 4	0.0174	-0.0095	0.0182	0.0182
20 1	-0.0093	0.0034	0.0216	0.0266	19 6	0.0041	0.0157	0.0134	0.0134
20 3	-0.0015	0.0064	0.0234	0.0254	19 8	0.0230	-0.0004	0.0162	0.0162
20 5	0.0029	0.0056	0.0213	0.0217	19 10	-0.0166	-0.0116	0.0163	0.0162
20 7	0.0003	0.0092	0.0195	0.0194	19 12	-0.0011	-0.0030	0.0189	0.0187
20 9	0.0097	-0.0260	0.0211	0.0131	19 14	0.0164	-0.0063	0.0190	0.0190
20 11	0.0030	-0.0150	0.0216	0.0146	19 16	-0.0254	0.0094	0.0190	0.0190
20 13	0.0193	-0.0076	0.0216	0.0208	19 18	0.0363	-0.0091	0.0223	0.0180
20 15	0.0190	0.0100	0.0211	0.0221					
20 17	-0.0056		0.0242						
20 19	-0.0100	-0.0052	0.0210	0.0210	20 2	-0.0054	0.0041	0.0223	0.0223
21 1	0.0100	0.0076	0.0210	0.0230	20 4	-0.0035	-0.0190	0.0186	0.0186
21 3	-0.0015	0.0009	0.0226	0.0226	20 6	0.0136	0.0041	0.0214	0.0213
21 5	-0.0102	-0.0009	0.0153	0.0163	20 8	0.0063	-0.0010	0.0242	0.0242
21 7	0.0293	-0.0094	0.0142	0.0143	20 10	0.0013	0.0170	0.0219	0.0219
21 9	0.0240	0.0010	0.0170	0.0170	20 12	-0.0199	0.0090	0.0196	0.0194
21 11	0.0035	-0.0023	0.0219	0.0139	20 14	0.0135	-0.0016	0.0180	0.0189
21 13	0.0090	-0.0093	0.0211	0.0144	20 16	-0.0104	-0.0092	0.0193	0.0194
21 15	-0.0014	0.0074	0.0195	0.0195	20 18	-0.0110	-0.0204	0.0214	0.0190
21 17	-0.0026	0.0002	0.0203	0.0203	20 20	0.0175	0.0045	0.0214	0.0213

Table A.4: Fully Normalized Geopotential Coefficients Determined from 416 10° Mean Anomalies by Integration. All coefficients multiplied by 10⁶. (Error Estimates not corrected by β_n^{-1} .)

N	M	C	S	SIGMA	SIGMA	N	M	C	S	SIGMA	SIGMA
2		-403.5679		0.1835		2	2	2.4380	-1.4632	0.1450	0.1461
2	1	0.2204	0.0726	0.1841	0.1898	3	2	1.1970	-0.4740	0.0876	0.0850
3		0.5031		0.0862		4	2	0.3324	0.4237	0.0616	0.0634
3	1	1.9103	-0.0421	0.0940	0.0965	4	4	-0.2205	0.4239	0.0464	0.0438
3	3	0.7925	1.6197	0.0696	0.0701	5	2	0.4009	-0.1949	0.0461	0.0453
4		0.0030		0.0600		5	4	-0.0861	-0.0195	0.0379	0.0372
4	1	-0.4561	-0.3934	0.0594	0.0593	6	2	0.2723	-0.3512	0.0350	0.0342
4	3	0.7911	-0.4269	0.0521	0.0547	6	4	-0.1774	-0.4802	0.0330	0.0333
5		0.1359		0.0437		6	6	0.0066	-0.2309	0.0256	0.0259
5	1	-0.1507	-0.2748	0.0436	0.0423	7	2	0.2006	0.1244	0.0271	0.0276
5	3	-0.2310	-0.1319	0.0446	0.0445	7	4	-0.1629	-0.1626	0.0293	0.0209
5	5	0.1026	-0.3222	0.0323	0.0334	7	6	-0.3065	0.2026	0.0228	0.0224
6		-0.1711		0.0354		8	2	0.1170	0.0995	0.0237	0.0246
6	1	0.1204	0.0476	0.0350	0.0343	8	4	-0.2054	0.0526	0.0257	0.0247
6	3	-0.0105	-0.0597	0.0364	0.0361	8	6	-0.0826	0.1837	0.0212	0.0213
6	5	-0.3573	-0.5234	0.0270	0.0290	8	8	-0.1370	0.0909	0.0162	0.0168
7		0.1833		0.0274		9	2	0.1191	-0.0779	0.0204	0.0210
7	1	0.1994	0.0313	0.0297	0.0298	9	4	-0.0500	0.0524	0.0209	0.0200
7	3	0.1613	-0.1813	0.0293	0.0295	9	6	0.0071	0.1790	0.0194	0.0194
7	5	0.0073	-0.0039	0.0254	0.0271	9	8	0.1900	0.0102	0.0139	0.0152
7	7	-0.0312	-0.0904	0.0210	0.0209	10	2	-0.0713	-0.0372	0.0176	0.0170
8		0.0152		0.0236		10	4	-0.0593	-0.0936	0.0177	0.0171
8	1	-0.0433	0.0461	0.0244	0.0241	10	6	-0.0821	-0.0476	0.0103	0.0102
8	3	0.0830	-0.0111	0.0233	0.0231	10	8	0.0445	-0.0325	0.0153	0.0145
8	5	-0.0156	0.0338	0.0230	0.0242	10	10	0.1135	-0.0315	0.0126	0.0132
8	7	0.0396	0.0639	0.0100	0.0103	11	2	-0.0233	-0.0003	0.0161	0.0163
9		0.1230		0.0202		11	4	-0.0903	-0.0603	0.0159	0.0154
9	1	0.1729	-0.0300	0.0207	0.0206	11	6	-0.0115	0.0052	0.0162	0.0160
9	3	-0.1760	-0.0299	0.0201	0.0201	11	8	0.0079	0.0614	0.0139	0.0133
9	5	-0.0560	-0.0279	0.0209	0.0221	11	10	-0.0216	0.0163	0.0116	0.0114
9	7	-0.0013	-0.0113	0.0177	0.0175	12	2	0.0015	-0.0231	0.0130	0.0145
9	9	-0.0175	0.0203	0.0151	0.0140	12	4	-0.0639	-0.0334	0.0133	0.0132
10		0.0210		0.0171		12	6	0.0237	0.0461	0.0137	0.0135
10	1	0.1146	-0.0170	0.0160	0.0183	12	8	0.0107	0.0244	0.0135	0.0120
10	3	-0.0330	-0.0946	0.0177	0.0178	12	10	-0.0072	-0.0251	0.0111	0.0111
10	5	-0.0241	-0.0129	0.0170	0.0106	12	12	0.0137	-0.0001	0.0101	0.0090
10	7	0.0707	0.0149	0.0161	0.0161	13	2	0.0078	-0.0309	0.0116	0.0124
10	9	0.0909	-0.0205	0.0136	0.0130	13	4	0.0042	-0.0044	0.0117	0.0114
11		-0.0963		0.0150		13	6	-0.0274	-0.0107	0.0123	0.0123
11	1	-0.0276	-0.0030	0.0149	0.0150	13	8	-0.0222	0.0215	0.0129	0.0122
11	3	-0.0729	-0.1145	0.0151	0.0149	13	10	0.0315	-0.0070	0.0102	0.0099
11	5	0.0135	0.0071	0.0151	0.0153	13	12	-0.0041	0.0925	0.0087	0.0089
11	7	0.0350	-0.1069	0.0156	0.0154						
11	9	-0.0478	0.0771	0.0131	0.0123						
11	11	0.0705	-0.0127	0.0113	0.0113						
12		0.0305		0.0142							
12	1	-0.0103	-0.0296	0.0129	0.0131						
12	3	0.0149	0.0395	0.0136	0.0136						
12	5	0.0544	-0.0016	0.0136	0.0142						
12	7	-0.0230	0.0310	0.0144	0.0141						
12	9	0.0139	0.0162	0.0110	0.0115						
12	11	0.0085	0.0047	0.0099	0.0101						
13		0.0461		0.0122							
13	1	-0.0090	0.0033	0.0115	0.0121						
13	3	-0.0145	0.0359	0.0126	0.0124						
13	5	0.0370	0.0400	0.0119	0.0123						
13	7	-0.0173	0.0141	0.0122	0.0121						
13	9	-0.0144	0.0302	0.0113	0.0111						
13	11	0.0029	0.0270	0.0096	0.0090						
13	13	-0.0374	0.0626	0.0056	0.0090						
14		0.0027		0.0104							

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Table A.4 (continued)

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14 1	0.0223	0.0101	0.0106	0.0113	14 2	-0.0470	-0.0171	0.0162	0.0162
14 3	0.0123	0.0023	0.0114	0.0111	14 4	-0.0019	-0.0223	0.0107	0.0103
14 5	0.0159	-0.0101	0.0103	0.0103	14 6	-0.0004	-0.0033	0.0109	0.0110
14 7	0.0274	-0.0132	0.0109	0.0108	14 8	-0.0234	-0.0250	0.0111	0.0110
14 9	0.0162	0.0394	0.0111	0.0108	14 10	0.0131	-0.0009	0.0097	0.0094
14 11	0.0209	-0.0766	0.0009	0.0008	14 12	0.0034	-0.0227	0.0004	0.0007
14 13	0.0374	0.0240	0.0076	0.0078	14 14	-0.0108	0.0103	0.0073	0.0080
15 1	-0.0047	0.0047	0.0049	0.0044	15 2	0.0033	0.0100	0.0096	0.0099
15 3	0.0407	0.0477	0.0093	0.0094	15 4	0.0037	-0.0173	0.0102	0.0099
15 5	0.0096	0.0144	0.0087	0.0090	15 6	-0.0091	-0.0251	0.0065	0.0068
15 7	0.0575	0.0120	0.0098	0.0097	15 8	-0.0394	0.0190	0.0079	0.0094
15 9	0.0053	-0.0063	0.0097	0.0097	15 10	-0.0267	0.0009	0.0095	0.0093
15 11	-0.0032	0.0146	0.0003	0.0001	15 12	-0.0170	0.0664	0.0078	0.0078
15 13	-0.0215	0.0004	0.0074	0.0077	15 14	0.0102	-0.0178	0.0068	0.0069
15 15	-0.0242	0.0102	0.0067	0.0071					
16 1	0.0201	0.0099	0.0099	0.0099	16 2	-0.0138	0.0014	0.0094	0.0095
16 3	-0.0213	0.0213	0.0079	0.0077	16 4	0.0369	0.0731	0.0091	0.0090
16 5	-0.0097	0.0204	0.0083	0.0084	16 6	-0.0040	-0.0314	0.0079	0.0082
16 7	-0.0107	-0.0072	0.0089	0.0089	16 8	-0.0337	0.0328	0.0088	0.0083
16 9	-0.0017	-0.0500	0.0083	0.0088	16 10	-0.0013	-0.0090	0.0087	0.0087
16 11	0.0084	-0.0150	0.0083	0.0082	16 12	0.0172	-0.0044	0.0073	0.0069
16 13	-0.0001	0.0140	0.0066	0.0072	16 14	-0.0034	-0.0253	0.0066	0.0067
16 15	-0.0116	-0.0367	0.0059	0.0063	16 16	-0.0223	-0.0170	0.0065	0.0059
17 1	0.0031	0.0090	0.0090	0.0090	17 2	-0.0534	0.0232	0.0081	0.0081
17 3	0.0007	0.0262	0.0059	0.0062	17 4	-0.0243	0.0340	0.0073	0.0072
17 5	0.0160	-0.0226	0.0077	0.0073	17 6	-0.0261	-0.0411	0.0067	0.0071
17 7	-0.0204	0.0126	0.0080	0.0083	17 8	0.0216	-0.0147	0.0082	0.0078
17 9	0.0233	-0.0245	0.0076	0.0077	17 10	-0.0153	0.0127	0.0077	0.0078
17 11	-0.0191	-0.0363	0.0074	0.0070	17 12	-0.0220	-0.0007	0.0072	0.0070
17 13	-0.0010	0.0063	0.0077	0.0076	17 14	-0.0009	0.0202	0.0061	0.0062
17 15	0.0171	0.0117	0.0034	0.0069	17 16	-0.0097	0.0151	0.0054	0.0054
17 17	0.0191	0.0182	0.0034	0.0064					
18 1	-0.0360	0.0078	0.0036	0.0034	18 2	-0.0112	0.0036	0.0024	0.0023
18 3	0.0122	0.0030	0.0030	0.0030	18 4	0.0083	0.0084	0.0042	0.0043
18 5	-0.0366	-0.0300	0.0030	0.0030	18 6	0.0083	-0.0023	0.0062	0.0064
18 7	-0.0091	-0.0129	0.0092	0.0091	18 8	0.0212	-0.0070	0.0073	0.0070
18 9	0.0000	0.0177	0.0070	0.0082	18 10	0.0236	-0.0047	0.0068	0.0068
18 11	-0.0040	-0.0022	0.0063	0.0063	18 12	0.0220	-0.0047	0.0068	0.0067
18 13	0.0099	0.0114	0.0060	0.0073	18 14	0.0043	-0.0221	0.0057	0.0055
18 15	-0.0192	-0.0127	0.0071	0.0069	18 16	0.0022	-0.0223	0.0054	0.0049
18 17	-0.0093	-0.0360	0.0040	0.0073	18 18	0.0114	0.0098	0.0056	0.0056
19 1	-0.0470	-0.0301	0.0043	0.0063	19 2	-0.0039	-0.0138	0.0024	0.0025
19 3	0.0073	-0.0103	0.0036	0.0043	19 4	0.0372	-0.0176	0.0044	0.0044
19 5	0.0001	0.0041	0.0041	0.0041	19 6	0.0234	-0.0172	0.0070	0.0069
19 7	-0.0062	0.0267	0.0073	0.0073	19 8	0.0060	0.0239	0.0077	0.0078
19 9	0.0030	-0.0273	0.0064	0.0063	19 10	0.0261	-0.0011	0.0067	0.0065
19 11	-0.0114	0.0066	0.0037	0.0039	19 12	-0.0207	-0.0132	0.0062	0.0063
19 13	0.0001	0.0031	0.0041	0.0043	19 14	-0.0030	-0.0074	0.0063	0.0062
19 15	0.0047	0.0063	0.0061	0.0063	19 16	0.0206	-0.0018	0.0053	0.0054
19 17	0.0028	0.0098	0.0061	0.0061	19 18	-0.0208	0.0113	0.0053	0.0043
19 19	0.0104	-0.0292	0.0039	0.0074		0.0427	-0.0127	0.0039	0.0046
20 1	0.0000	-0.0170	0.0034	0.0058	20 2	-0.0133	0.0030	0.0053	0.0053
20 3	0.0240	-0.0093	0.0049	0.0043	20 4	-0.0127	-0.0309	0.0069	0.0068
20 5	0.0323	0.0142	0.0043	0.0043	20 6	0.0207	0.0052	0.0054	0.0055
20 7	-0.0117	0.0040	0.0040	0.0040	20 8	0.0032	-0.0039	0.0032	0.0033
20 9	-0.0275	-0.0072	0.0057	0.0058	20 10	-0.0006	0.0283	0.0052	0.0053
20 11	-0.0166	0.0132	0.0043	0.0042	20 12	-0.0220	0.0070	0.0054	0.0055
20 13	-0.0053	0.0008	0.0039	0.0041	20 14	0.0142	-0.0057	0.0052	0.0050
20 15	-0.0274	-0.0137	0.0063	0.0064	20 16	-0.0192	-0.0174	0.0042	0.0043
20 17	0.0393	-0.0134	0.0063	0.0066	20 18	-0.0143	-0.0223	0.0038	0.0043
20 19	0.0342	-0.0007	0.0036	0.0037	20 20	0.0262	0.0032	0.0042	0.0036
21 1	0.0033	-0.0060	0.0033	0.0071					
21 3	0.0149	-0.0177	0.0033	0.0036					
21 5	0.0013	0.0098	0.0036	0.0041					
21 7	-0.0033	0.0003	0.0036	0.0039					

