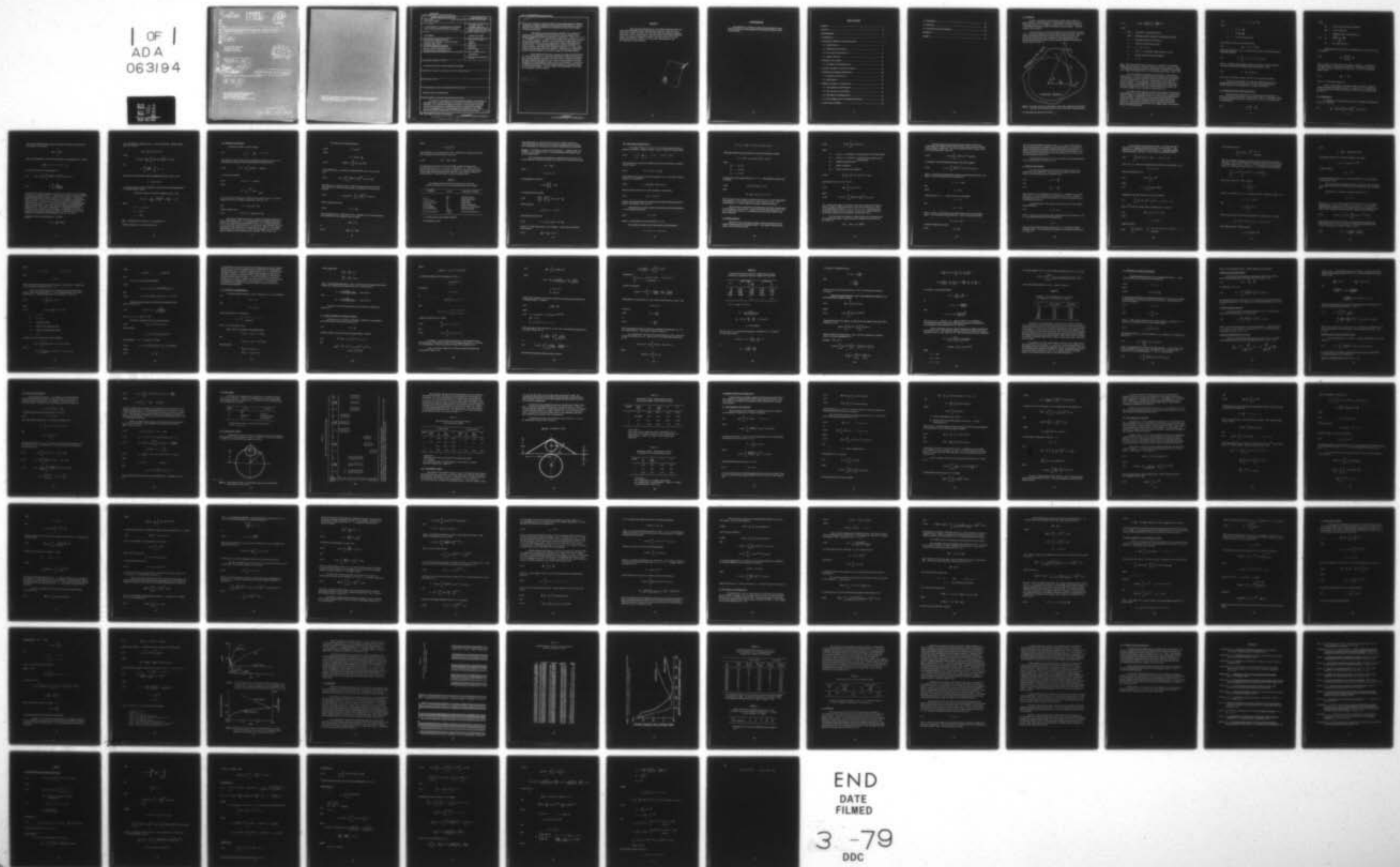
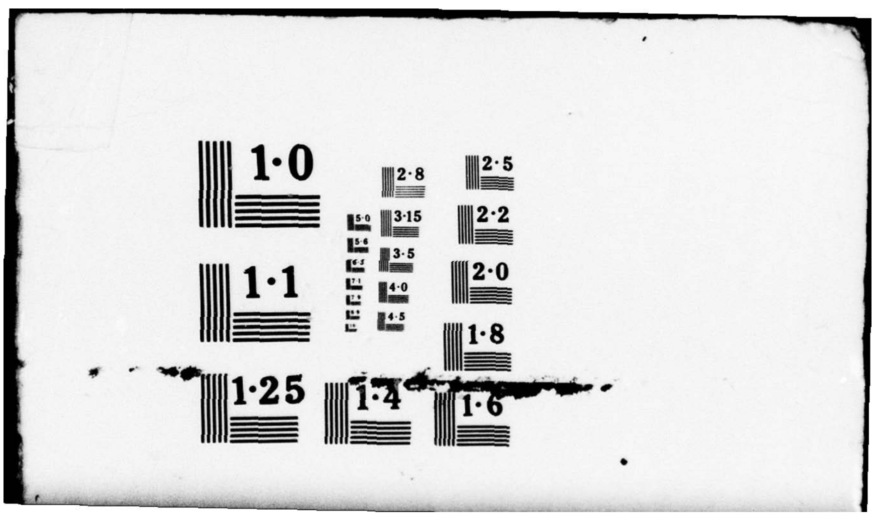
AD-A063 194 OHIO STATE UNIV COLUMBUS DEPT OF GEODETIC SCIENCE F/G 8/5
A COMPARISON OF BJERHAMMAR'S METHODS AND COLLOCATION IN PHYSICA--ETC(U)
JUL 78 L SJOEBERG F19628-76-C-0010
UNCLASSIFIED DGS-273 AFGL-TR-78-0203 NL

1 OF 1
AD A
063194



END
DATE
FILMED
3 -79
DDC


1·0
2·8
2·5
5·0
5·6
3·15
2·2
3·5
1·1
2·0
4·0
4·5
1·8
1·25
1·4
1·6

(18) (19)

AFGL-TR-78-0203

LEVEL IX

(12)

AD A063194

(6)

A COMPARISON OF BJERHAMMAR'S METHODS AND COLLOCATION IN PHYSICAL GEODESY,

(10)

Lars Sjöberg

The Ohio State University
Research Foundation
Columbus, Ohio 43212

(9) Interim rept.,

(11) Jul[redacted]78

(12) 89 p.

(14) DGS-273,
Scientific [redacted] 17

(15) F19628-76-C-0010

Approved for public release; distribution unlimited

(16) 7600 (17) 03

AIR FORCE GEOPHYSICS LABORATORY
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
HANSCOM AFB, MASSACHUSETTS 01731

400 254

79 01 12 003

unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFGL-TR-78-0203	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A COMPARISON OF BJERHAMMAR'S METHODS AND COLLOCATION IN PHYSICAL GEODESY		5. TYPE OF REPORT & PERIOD COVERED Scientific. Interim Scientific Report No. 17
		6. PERFORMING ORG. REPORT NUMBER Dept. of Geod. Sci. No. 273
7. AUTHOR(s) Lars Sjöberg		8. CONTRACT OR GRANT NUMBER(s) F19628-76-C-0010
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Geodetic Science The Ohio State University - 1958 Neil Avenue Columbus, Ohio 43210		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62101F 760003 AG
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Geophysics Laboratory Hanscom AFB, Massachusetts 01731 Contract Monitor: Bela Szabo/LW		12. REPORT DATE July 1978
		13. NUMBER OF PAGES 88
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

A-Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

geodesy, gravity, potential theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In 1963 A. Bjerhammar solved the geodetic boundary value problem by applying Poisson's integral equation for a finite set of observed free-air gravity anomalies. Due to the relation between the number of observations (m) and the number of chosen unknowns (N) different solutions are obtained: non-singular (m = N), least squares (m > N) and minimum norm solutions (m < N). In the special case $N \to \infty$ it is shown that the Bjerhammar solution with Poisson's → over

(approaches infinity)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(cont)

kernel and a solution by collocation with the corresponding kernel are identical. Bjerhammar's method is generalized by using other kernel functions, and each minimum norm solution is shown to correspond to one specific set of degree variances in collocation.

The impulse approaches (reflexive prediction, Dirac method) of Bjerhammar are presented. The kernel function of the non-singular Dirac method is obtained from that in collocation by substituting $c_n / (2n+1) \, (r_B^2 / r_j r_i)^{n+2}$ by $\sqrt{c_n / (2n+1)} \, (r_B / r_j)^{n+2}$. Hence the Dirac kernel is not symmetric in contrast to the one in collocation. Moreover, it is shown that for a given radius r_B of the Bjerhammar sphere, the Dirac method gives a better conditioning of the equation system. It is demonstrated that the two solutions are identical for Poisson's kernel, if the depth of the Bjerhammar sphere in collocation is half of that in the application of the Dirac approach. The solution by collocation is therefore twice as sensitive to the choice of radius as is the latter method.

In the theoretical case with a continuous coverage of observations at the surface of the earth, it is shown that both the Dirac method and collocation give a unique solution for any choice of positive degree variances of the kernel functions, whenever the solutions exist. However, the intermediate solutions for Δg^* and X at the Bjerhammar sphere do not exist in general. If collocation is applied by solving the Wiener-Hopf integral equation, a convergent solution is proved outside a sphere. However, inside the bounding sphere of the earth the convergence is still not proved.

A

Delta g star

Foreword

This report was prepared by Dr. Lars Sjöberg, Research Associate, Department of Geodetic Science, The Ohio State University, under Air Force Contract No. F19628-76-C-0010, The Ohio State University Research Foundation Project No. 4214 B1 (710335) which is under the direction of Professor Richard H. Rapp. The contract covering this research is administered by the Air Force Geophysics Laboratory, Hanscom Air Force Base, Massachusetts, with Mr. Bela Szabo, Contract Monitor.

Acknowledgements

I am grateful to Dr. Richard H. Rapp for the opportunity to study this topic and for his helpful suggestions and guidance during the course of this research.

Table of Contents

1. Introduction

In 1963 A. Bjerhammar formulated the boundary value problem of physical geodesy in the following way: "A finite number of gravity data (gravity anomalies) is given for a non-spherical surface and it is required to find such a solution† that the boundary values are satisfied in all given points". (Discrete boundary value problem.)

For the solution of this problem Bjerhammar used a spherical reference surface completely embedded in the earth with its center at the earth's center of gravity and rotating with the same angular velocity as the real earth. The observed gravity anomalies (Δg_j) are reduced to the internal sphere (the Bjerhammar sphere) by means of Poisson's integral equation for the harmonic function $r\Delta g$ (see Figure 1):

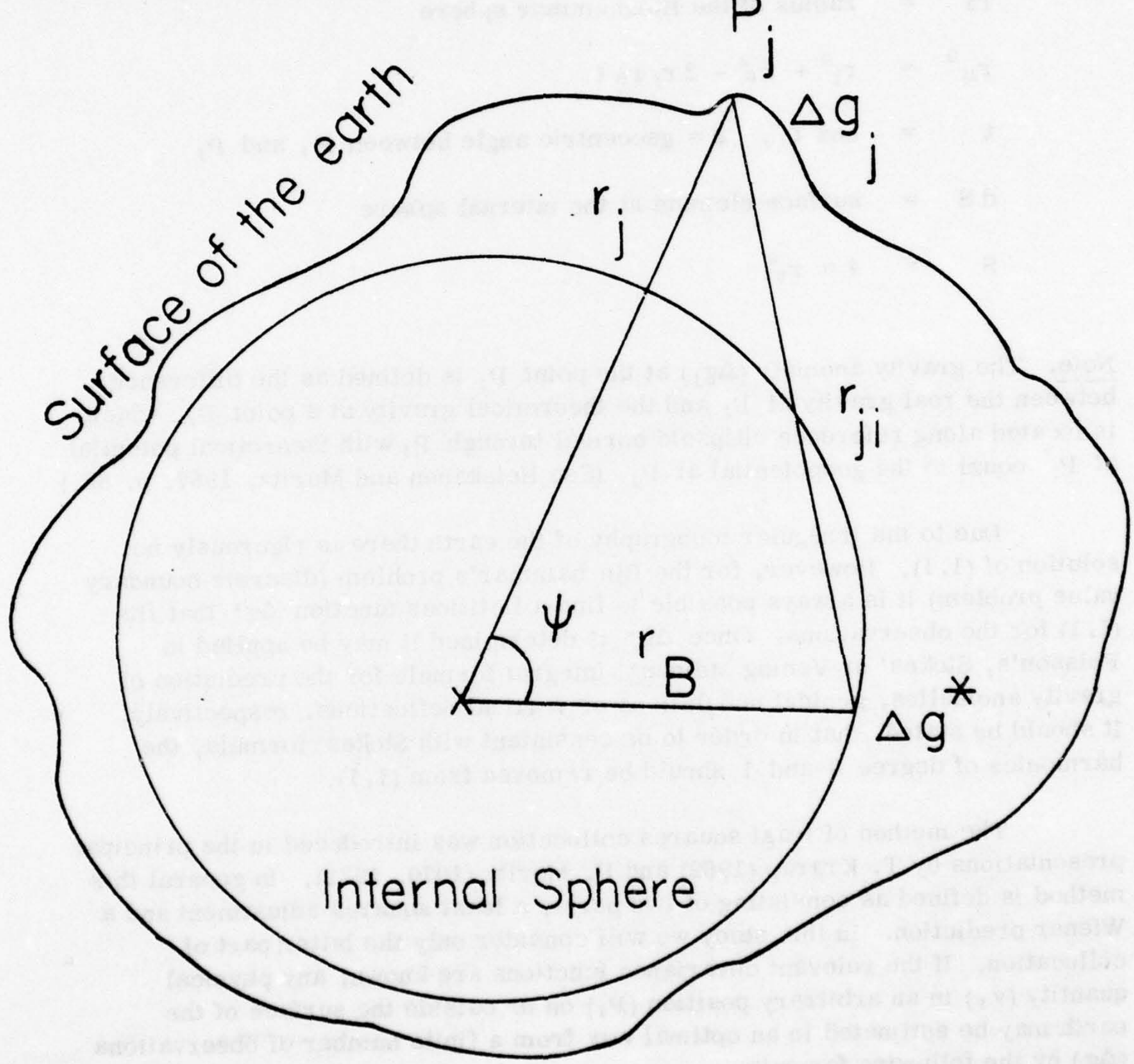


Figure 1. The observation Δg at the surface of the earth is reduced to the fictitious field Δg* on the internal sphere by means of Poisson's integral equation.

† for the height anomalies (see section 5)

$$(1.1) \qquad r_j \Delta g_j = \frac{r_j^2 - r_B^2}{4\pi r_B} \iint \frac{r_B \Delta g^*}{r_{j1}^{\ 3}} \, dS$$

where

Δg_j = real gravity - theoretical gravity

Δg^* = (fictitious) gravity anomaly at the Bjerhammar sphere

r_j = geocentric distance to the point P_j

r_B = radius of the Bjerhammar sphere

$r_{j1}^{\ 2} = r_j^2 + r_B^2 - 2 r_j r_B t$

$t = \cos\psi$, ψ = geocentric angle between P_1 and P_j

dS = surface element at the internal sphere

$S = 4\pi \ r_B^2$

Note. The gravity anomaly (Δg_j) at the point P_j is defined as the difference between the real gravity at P_j and the theoretical gravity at a point P_j', which is located along reference ellipsoid normal through P_j with theoretical potential at P_j' equal to the geopotential at P_j. (See Heiskanen and Moritz, 1967, p. 83.)

Due to the irregular topography of the earth there is rigorously no solution of (1.1). However, for the Bjerhammar's problem (discrete boundary value problem) it is always possible to find a fictitious function Δg^* that fits (1.1) for the observations. Once Δg^* is determined it may be applied in Poisson's, Stokes' or Vening Meinesz' integral formula for the prediction of gravity anomalies, geoidal undulations or vertical deflections, respectively. It should be stated, that in order to be consistent with Stokes' formula, the harmonics of degree 0 and 1 should be removed from (1.1).

The method of least squares collocation was introduced in the principal presentations by T. Krarup (1969) and H. Moritz (1970, 1972). In general this method is defined as consisting of two parts: a least squares adjustment and a Wiener prediction. In this study we will consider only the latter part of collocation. If the relevant covariance functions are known, any physical quantity (v_1) in an arbitrary position (P_1) on or outside the surface of the earth may be estimated in an optimal way from a finite number of observations (Δg) by the following formula:

(1.2) $$v_i = c_i (C + D)^{-1} \Delta g$$

where

c_i = $cov(v_i, \Delta g)$

C = $cov(\Delta g, \Delta g)$

D = error covariance matrix.

In particular we predict gravity anomalies by:

(1.3) $$\Delta g_i = c_i (C + D)^{-1} \Delta g$$

where the elements of c_i and C are given by the spatial covariance function for the gravity anomalies:

(1.4) $$C_{ij} = \sum_{n=0}^{\infty} c_n (r_B^2 / r_i r_j)^{n+2} P_n(\cos \psi_{ij})$$

where c_n are the so-called degree variances of the gravity anomalies defined according to the definition in Heiskanen and Moritz (1967, p. 259):

(1.5) $$c_n = \frac{1}{4\pi} \iint \Delta g_n^2 \, d\sigma$$

where Δg_n is the anomaly Laplace harmonic on the Bjerhammar sphere.

The purpose of this paper is to study the relations between some of Bjerhammar's solutions and collocation. We start with solving Poisson's integral equation (1.1) in different ways according to Bjerhammar.

2. Bjerhammar Solutions at the Internal Sphere

In the numerical application of (1.1) Bjerhammar used a finite set of blocks on the internal sphere with a constant value of Δg^* over each block. In this way the following matrix equation is obtained from (1.1) for m observations and N surface blocks:

(2.1) $$\underset{m\,N}{A} \; \underset{N\,1}{\Delta g^*} = \underset{m\,1}{\Delta g}$$

where

Δg = vector of observed gravity anomalies

Δg^* = vector of unknowns

A = coefficient matrix with elements $A_{\ell k}$

(2.2) $$A_{\ell k} = \frac{r_\ell^2 - r_B^2}{4\pi r_\ell} \iint_{\Delta S_k} \frac{dS}{r_{\ell k}^3}$$

ΔS_k = the surface of block k

Applying the mean value theorem of integral calculus formula (2.2) may be rewritten:

(2.2a) $$A_{\ell k} = \frac{r_\ell^2 - r_B^2}{4\pi r_\ell r_{\ell k}^3} \Delta S_k$$

where $\underline{k}$ refers to a certain mean value point P_k inside ΔS_k. In practice we may approximate P_k by the center of the block k. If the matrix A has full rank there is always a solution of (2.1) for $N \geq m$. For $N = m$ we obtain the unique solution:

(2.3) $$\Delta g^* = A^{-1} \Delta g$$

where A^{-1} is the Cayley inverse of A.

For $N > m$ the solution is not unique unless an additional condition is satisfied (condition adjustment, Section 2.2). Finally, for $N < m$ and full rank of A the least squares solution minimizes the square sum of the residuals (see Section 2.3).

2.1 Generalization

By expanding the Poisson kernel into a series of Legendre's polynomials, formula (1.1) becomes:

(2.4) $$\Delta g_j = \frac{1}{S} \iint \Delta g^* \sum_{n=0}^{\infty} (2n+1) \left(\frac{r_B}{r_j}\right)^{n+2} P_n(\cos\psi)\, dS$$

We may also expand Δg^* into a series of Laplace's harmonics (we assume that the expansion is convergent):

$$\Delta g^* = \sum_{n=0}^{\infty} \Delta g_n$$

Due to the orthogonality of the spherical harmonics when integrated over a sphere:

$$\iint \Delta g_n \; P_{n'} \quad dS = 0 \quad \text{for} \quad n \neq n'$$

(2.4) may be rewritten in the following way:

$$\Delta g_j = \frac{1}{S} \iint u^* \sum_{n=0}^{\infty} \sqrt{(2n+1)\, c_n^*} \left(\frac{r_B}{r_j}\right)^{n+2} P_n (\cos\psi) \;\; dS \tag{2.5}$$

$$u^* = \sum_{n=0}^{\infty} \frac{\Delta g_n}{\sqrt{c_n^*/(2n+1)}} \tag{2.6}$$

In contrast to the degree variances in collocation (c_n), which are a priori defined by formula (1.5), the parameters c_n^* introduced in formulae (2.5) and (2.6) of the generalized Bjerhammar method, are more or less arbitrary. Formally the conversion from (2.4) to (2.5) is valid for any c_n^* such that (2.6) and the kernel of (2.5) converge. However, from a practical point of view it will hardly be advisable to solve (2.5) for a u^* that is less smooth (with less attenuating higher degree terms) than Δg^*. Already the solution for Δg^* is questionable in the continuous case. See Section 5, Molodensky et al. (1962) and Pick (1965). In the discrete case a solution is always possible. By changing c_n^* a variety of solutions to Bjerhammar's problem are obtained. These solutions for u^* are completely in accordance with the solutions for Δg^* in (2.1) - (2.2).

Example: The inverse Stokes function. We insert:

$$\sqrt{c_n^*} = \frac{n-1}{r_B} \sqrt{2n+1}$$

into (2.5) and (2.6) (with lower limit n = 2 of the summation). Then we obtain (for $dS = r_B^2 \, d\sigma$):

$$\Delta g_j = \iint M(r_j, \psi)\, u^* \, d\sigma$$

where

$$M(r_j,\psi) = \frac{1}{4\pi r_j} \sum_{n=2}^{\infty} (2n+1)(n-1)\left(\frac{r_B}{r_j}\right)^{n+1} P_n(\cos\psi)$$

and

$$u^* = \sum_{n=2}^{\infty} \frac{r_B \Delta g_n}{n-1} = \sum_{n=2}^{\infty} T_n = T^*$$

where we have introduced the notation (Heiskanen and Moritz, ibid., p. 97):

$$T_n = r_B \Delta g_n / (n-1)$$

T_n being the Laplace harmonic of degree n of the (fictitious) disturbing potential T^* at the Bjerhammar sphere.

M may be written in a closed form (Sjöberg, 1975, p. 107):

$$M(r_j, \psi) = \frac{s}{8\pi r_j} \left[\frac{3(1-s^2)^2}{\ell^5} + \frac{s^2-5}{\ell^3} + 2 \right]$$

where

$$\ell^2 = 1 - 2st + s^2$$

$$s = r_B / r_j$$

$$t = \cos\psi$$

<u>Note</u>. In Sjöberg (ibid.) the term n = 0 is included in M.

Further examples of u^* are given in Table 2.1.

2.2 Minimum Norm Solutions

As previously stated, a matrix equation

(2.7)
$$\underset{mN}{A}\ \underset{N1}{u^*} = \underset{m1}{\Delta g} \quad , \qquad m < N$$

does not have a unique solution unless an additional condition is imposed. We may define this solution as the one satisfying the following condition:

(2.8a)
$$\| u^* \|^2 = \frac{1}{S} \sum_{k=1}^{N} (u_k^*)^2 \Delta S_k = \text{minimum}$$

or, with matrix notations

(2.8b)
$$(u^*)^T Q^{-1} u^* = \text{minimum}$$

where

$$Q^{-1} = \frac{1}{S} \begin{pmatrix} \Delta S_1 & & & \\ & \Delta S_2 & & \\ & & \ddots & \\ & & & \Delta S_N \end{pmatrix}$$

From least squares adjustment we obtain the following solution of (2.7) with the condition (2.8) (see for instance, Bjerhammar, 1973, Ch. 12):

(2.9)
$$u^* = Q A^T (A Q A^T)^{-1} \Delta g$$

with the minimum norm

(2.9a)
$$(u^*)^T Q^{-1} u^* = \Delta g^T (A Q A^T)^{-1} \Delta g$$

Subsequently, different solutions are obtained by changing the number of blocks at the internal sphere. In the original paper by Bjerhammar (1964) most studies were restricted to the non-singular case (N = m). Condition adjustment (N > m) was treated in 1968 and 1969. It is obvious that these solutions from (2.9) are generally more cumbersome to compute than (2.3). However, the computational effort is drastically reduced to that in the non-singular case, if we let N approach infinity for well-behaving surface elements. This we show next.

We rewrite (2.9) in the following way:

(2.10) $$u^* = Q A^T X$$

where

$$X = (A Q A^T)^{-1} \Delta g$$

(2.11) $$(A Q A^T)_{ij} = S \sum_{k=1}^{N} (A)_{ik} (A)_{jk} \Delta S_k^{-1}$$

The coefficients $(A)_{jk}$ are given by the generalization of (2.1) and (2.2a) by (2.5):

(2.12) $$(A)_{jk} = \frac{1}{S} \sum_{n=0}^{\infty} \sqrt{(2n+1)\, c_n^*} \left(\frac{r_B}{r_j}\right)^{n+2} P_n(\cos\psi)\, \Delta S_k$$

By letting N go to infinity in such a way that the largest diameter of all ΔS_k approaches zero it is shown in the Appendix, Proposition A.1, that $(A Q A^T)_{ij}$ becomes:

(2.13) $$C_{ij} = \lim_{N \to \infty} (A Q A^T)_{ij} = \sum_{n=0}^{\infty} c_n^* \left(\frac{r_B^2}{r_i r_j}\right)^{n+2} P_n(\cos\psi_{ij})$$

Thus we obtain in the limit:

(2.13a) $$X = C^{-1} \Delta g$$

where the elements of C are given by (2.13). Applying (2.7) for the prediction of new anomalies from the original set Δg we obtain:

$$\Delta g_i = c_i X$$

or

(2.14) $$\Delta g_i = c_i C^{-1} \Delta g$$

where

$$c_1 = \lim_{N \to \infty} A_1 Q A^T$$

The elements of c_1 are also given by (2.13). Furthermore it follows from (2.9a) that the minimum norm in the limit is given by:

$$\|u^*\|^2 = \Delta g^T C^{-1} \Delta g \qquad (2.15)$$

We notice that (2.14) and (2.13) are exactly the solutions obtained with the collocation formulae (1.3) - (1.4) with c_n substituted by c_n^*. Hence, for each type of degree variances (c_n^*) in collocation, there is an identical minimum norm solution in the generalized Bjerhammar approach. Some of these relations are given in Table 2.1. The derivations are given in Sjöberg (1975).

Table 2.1

The Relations Between the Degree Variances in Collocation and Some Minimum Norms in the Generalized Bjerhammar Method†

$\sqrt{c_n^*/(2n+1)}$	Norm	Explanation of Symbol
1	Δg^*	gravity anomaly
$(n-1)/r_B$	T^*	disturbing potential
$4\pi(n-1)/(2n+1)$	φ^*	density layer
$4\pi n(n-1)/r_B(2n+1)$	μ^*	double layer
$(n-1)/(n+1)$	δ^*	gravity disturbance
$\gamma(n-1)/\sqrt{n(n+1)}$	θ^*	deflection of the vertical
$(n-1)/\sqrt{(2n+1)(n+1)}$	$\bar{\delta}^*$	potential gradient
$r_B^{\nu-1}\ n!\,(n-1)/(n+\nu)!$	$\dfrac{\partial^{(\nu)} T^*}{\partial r^{\nu}}$	the ν th derivative of T

γ = normal gravity at the reference ellipsoid

† from Sjöberg, (1975).

Subsequently there is a duality between the choice of degree variances in collocation and the minimum norm in the generalized Bjerhammar technique. Such an approach with a minimum norm was also emphasized by Krarup (1969).

Remark. c_n of formula (1.4) was replaced by $(2n+1)\ c_n$ in Sjöberg (ibid.) and by $(2n+1)\ \sigma_n^2$ by Lauritzen (1973, p. 73) and Bjerhammar (1974 and 1977 a, b). See also Krarup (ibid.).

Also in the generalized Bjerhammar technique the observation errors can be taken into account [cf. formula (1.2)]. In this case we start with the model:

$$A\,u^* = \Delta g - \epsilon$$

where

$$E\,\{\epsilon\,\epsilon^T\} = D$$

Rearranging this equation as:

$$[A, I]\begin{bmatrix} u^* \\ \epsilon \end{bmatrix} = \Delta g$$

we readily obtain the solution:

$$\begin{bmatrix} u^* \\ \epsilon \end{bmatrix} = \begin{bmatrix} Q A^T \\ D \end{bmatrix} [A Q A^T + D]^{-1}\ \Delta g \tag{2.9b}$$

which minimizes

$$(u^*)^T\ Q^{-1}\ u^* + \epsilon^T\ D^{-1}\ \epsilon$$

The predictions are given by

$$v_1 = A_1\,Q\,A^T\ [A Q A^T + D]^{-1}\ \Delta g \tag{2.16}$$

and for $N \to \infty$ this solution and (1.2) are identical. In this case the minimum norm becomes:

$$\Delta g^T\ C^{-1}\ \Delta g + \epsilon^T\ D^{-1}\ \epsilon \tag{2.17}$$

2.3 Least Squares Solutions for u*

If we apply formula (2.7) with $m > N$, this matrix equation will not in general be consistent. Adding a residual vector, we obtain the following model:

(2.18)
$$\underset{mN}{A}\ \underset{N1}{u^*} = \underset{m1}{\Delta g} - \underset{m1}{\epsilon} \quad , \quad m > N \quad , \quad E\{\epsilon\epsilon^T\} = P\sigma^2$$

This system may be solved with ordinary least squares adjustment by elements with the solution:

(2.19)
$$\hat{u}^* = (A^T P A)^{-1} A^T P \Delta g$$

which minimizes the square sum of the residuals ($\epsilon^T P \epsilon$). The variance of unit weight (σ^2) is estimated by:

(2.20)
$$s^2 = \Delta g^T P (\Delta g - A\hat{u}^*) / (m - N)$$

and the covariance matrix of u* after adjustment is estimated by:

(2.21)
$$Q_{uu} = s^2 (A^T P A)^{-1}$$

All these derivations follow from elementary least squares adjustment (see for instance, Bjerhammar, 1973, Ch. 11).

The solution $\hat{u}^*$ can now be used for the prediction of derived quantities (v_1) in an arbitrary position P_1:

(2.22)
$$\hat{v}_1 = A_1 \hat{u}^*$$

where A_1 is the operator (vector) that relates v_1 to u*.

The prediction variance may be determined in the following way:

$$\epsilon_1 = v_1 - \hat{v}_1 = v_1 - A_1 (u + \epsilon_u)$$

$$\epsilon_i^2 = v_i^2 + A_i(uu^T + \epsilon_u \epsilon_u^T) A_i^T - 2A_i(u + \epsilon_u) v_i^T$$

Taking the expectation of both members we obtain the prediction variance:

$$m_i^2 = \sigma_{vi}^2 + A_i(C_{uu} + Q_{uu}\sigma^2) A_i^T - 2 A_i C_{uv}$$

where

$$\sigma_{vi}^2 = E\{v_i^2\}$$

$$C_{uu} = E\{uu^T\}$$

$$C_{uv} = E\{uv^T\}$$

Furthermore we have assumed that $E\{\epsilon_u v_i^T\} = 0$. The prediction variance may be written:

(2.23) $$m_i^2 = \sigma^2 A_i Q_{uu} A_i^T + m_A^2$$

where

$$m_A^2 = \sigma_{vi}^2 + A_i C_{uu} A_i^T - 2 A_i C_{uv}$$

The first term of (2.23), which is caused by the error of u*, is easily determined in the adjustment. The second term (m_A^2) is due to the error of A_i. The determination of m_A^2 requires that the covariance relations are known.

The least squares solutions are favorable first of all when a large number of observations (m) are available. It should be noted that the solutions for $m > N$ do not include pure prediction. A filtering of the data is always present in this type of prediction.

2.4 Impulse Approach

Bjerhammar (1974) introduced a method, where the unknowns (u*) are located in discrete points at the internal sphere. This method, called the Dirac method, gives the following model:

(2.24) $$u^*(\bar{r}) = \sum_{k=1}^{N} u^*_k \; \delta(\bar{r} - \bar{r}_k)$$

where

$\bar{r}$ = (r_B, φ, λ) = coordinates of a current point at the internal sphere

$\bar{r}_k$ = $(r_B, \varphi_k, \lambda_k)$ = coordinates of a selected carrier point P_k at the internal sphere

u^*_k = unknown associated with P_k

N = number of unknowns

$\delta()$ = Dirac's delta function, defined by

(2.24a) $$\frac{1}{4\pi} \iint u^*(\bar{r}) \; \delta(\bar{r} - \bar{r}_k) \, d\sigma = u^*(\bar{r}_k)$$

Inserting (2.24) into (2.5) we arrive at

(2.25) $$\Delta g_j = \sum_{k=1}^{N} u_k^* \; K(\bar{r}_j, \bar{r}_k)$$

where

(2.26) $$K(\bar{r}_j, \bar{r}_k) = \sum_{n=0}^{\infty} \sqrt{(2n+1)\, c_n^*} \left(\frac{r_B}{r_j}\right)^{n+a} P_n(\cos \psi_{jk})$$

For m observations (Δg_j) (2.25) gives an exact matrix equation which may be solved by means of condition adjustment, least squares adjustment or direct solution dependent on whether $m < N$, $m > N$, or $m = N$. A generalization of the method is obtained if we also allow for carrier points located outside the Bjerhammar sphere (reflexive prediction, Bjerhammar, 1974).

It is obvious that this method is a generalization of the commonly used buried mass point method. In fact, the latter is obtained in the special case (cf. Table 2.1).

$$\sqrt{c_n^*} = \text{const.}\; (n-1)/\sqrt{(2n+1)}$$

The non-singular Dirac method with carrier points located at the intersections of the internal sphere with the radius vectors of the observations is very similar to collocation. The auto-covariance function for Δg in collocation (with internal sphere radius r_B):

(2.27) $$C(j, i) = \sum_{n=2}^{\infty} c_n (r_B^2/r_j r_i)^{n+2} P_n(\cos \psi_{ji})$$

corresponds to the following kernel function in the Dirac method:

(2.28) $$C(j, i) = \sum_{n=2}^{\infty} \sqrt{(2n+1)\, c_n^*}\ (r_o/r_j)^{n+2} P_n(\cos \psi_{ji})$$

where r_o is the selected internal sphere radius (not necessarily the same as r_B). Hence, C (j, i) of (2.27) and (2.28) are identical for:

$$c_n = c_n^* = 2n+1$$

and

(2.29) $$r_o = r_B^2/r_i$$

Consider the case $r_i = r$ = radius of the mean earth sphere:

$$r_o = r - h_o$$

and

$$r_B = r - h_B$$

where h_o and h_B are the depths of the internal spheres from the mean earth sphere. By inserting these expressions for r_o and r_B into (2.29) we obtain:

$$h_o = 2h_B - h_B^2/r$$

or (after omitting the last term)

(2.30) $$h_o = 2h_B$$

Thus we have shown that in the special case $c_n = c_n^* = 2n+1$ and the radii of the observation points are constant (r), the kernel function C (j, i) of the Dirac method and collocation are identical whenever the depth to the Bjerhammar sphere in the previous method (h_o) is twice that in the latter method (h_B). This result holds also for the predictions of the two methods (see Sections 5 and 8). See also Bjerhammar (1977, a,b).

From numerical point of view it is of importance that the kernel function (2.28) is unsymmetric in contrast to (2.27).

3. Stability of the Solutions

It is of great importance for the prediction results that the matrix equations are well conditioned. In this section we are going to compare the stability of the following systems in different cases:

(3.1)
$$\underset{m\,m}{A}\; u^* = \underset{m\,1}{\Delta g}$$

and

(3.2)
$$\underset{m\,m}{C}\; X = \underset{m\,1}{\Delta g}$$

These equations were introduced in (2.7) and (2.13a). Formulae (3.1) (m = N) and (3.2) (N = ∞) may be regarded as the extreme cases of (2.7). From the derivations in Section 2.2 we may also consider (3.2) as an intermediate system of equations in collocation.

A suitable measure of the stability of a matrix system is the condition number defined as:

(3.3)
$$\kappa = \lambda_{max} / \lambda_{min}$$

where λ refers to the eigen values of the coefficient matrix of the system (A and C). The condition number has the following bounds:

$$1 \le \kappa \le \infty$$

where the lower bound is the ideal situation and ∞ is a completely unstable (singular) system. Our main problem is therefore to determine the eigen values λ_{min} and λ_{max}.

We start with a two-dimensional example. We assume that Δg is observed at m regularly distributed points on a circle of radius r. The unknowns (u^*) are located on a circle of radius r_B. Applying the impulse approach [formula (2.25) with m = N] the system of equations becomes:

$$\sum_{k=1}^{N} K(\theta_j - \theta_k)\, u_k^* = \Delta g_j \;; \quad j = 1, 2, \ldots, N \tag{3.4}$$

where $K(\theta_j - \theta_k)$ is the two-dimensional kernel function corresponding to u^* and

$$\theta_p = p\, 2\pi/N$$

The Fourier series of K is:

$$K(\theta) = \sum_{n=-\infty}^{\infty} b_n\, e^{in\theta} \tag{3.5}$$

where

$$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\theta)\, e^{-in\theta}\, d\theta \tag{3.6}$$

Furthermore, the eigen values (λ_ℓ) of (3.4) are given by:

$$\sum_{k=1}^{N} K(\theta_j - \theta_k)\, X_k^{(\ell)} = \lambda_\ell\, X_j^{(\ell)} \;; \quad \ell, j = 1, 2, \ldots, N \tag{3.7}$$

where $X_j^{(\ell)}$ is the j th element of the ℓ th eigen vector. Let us try:

$$X_j^{(\ell)} = \cos x = (e^{ix} + e^{-ix})/2 \tag{3.8}$$

where

$$x = j\,\ell\, 2\pi/N$$

Using the relation

$$\sum_{j=1}^{N} e^{ij\frac{2\pi}{N}(\ell - q)} = \begin{cases} N & \text{if } \ell = q + Np \;;\; p = 0, \pm 1, \pm 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

we can easily arrive at:

$$\sum_{j=1}^{N} X_j^{(\ell)} X_j^{(q)} = \begin{cases} N/2 & \text{if} \quad \ell = q \\ 0 & \text{otherwise} \end{cases}$$

Thus (3.8) satisfies the necessary property of eigen vectors being orthogonal to each other. We have also shown that (3.8) should be normed by $\sqrt{2}/N$ in order to be the eigen vectors of (3.7). Finally, $X_j^{(\ell)}$ must satisfy (3.7) for some eigen values λ_ℓ. Inserting (3.8) and (3.5) into (3.7) we obtain from the left member:

$$\sum_{n=-\infty}^{\infty} b_n e^{inj\frac{2\pi}{N}} \sum_{k=1}^{N} e^{i(\ell-n)\frac{2\pi}{N}k} = N e^{i\ell j\frac{2\pi}{N}} \sum_{p=-\infty}^{\infty} b_{\ell+Np}$$

Here we have used (3.9). Thus we obtain:

$$\lambda_\ell = N \sum_{p=-\infty}^{\infty} b_{\ell+Np}$$

where b_n is given by (3.6). If K is Poisson's kernel, then (Seeley, 1966, p. 14):

$$b_n = s^{|n|} = (r_B/r)^{|n|}$$

and

$$\lambda_\ell = N \sum_{p=-\infty}^{\infty} s^{|\ell+Np|}$$

The sum is given in a closed form in the Appendix, Corollary A.2. We obtain:

$$\lambda_\ell = N(s^\ell + s^{N-\ell})/(1-s^N), \quad \ell = 1, 2, \ldots, N$$

The condition number is finally given by:

$$\varkappa_1 = (1+s^N)/(s^\beta + s^{N-\beta})$$

where

$$\beta = \left(\frac{N}{2}\right) = \text{integer part of } N/2$$

An analogous derivation of the condition number of (3.2) gives:

$$\varkappa_2 = (1 + s^{2N})/(s^{2\beta} + s^{2N-2\beta})$$

so that for large N

$$\varkappa_2/\varkappa_1 \approx s^{-N/2}$$

It is obvious that the system (3.1) is more stable than (3.2) for a given depth to the Bjerhammar circle.

Next we proceed to the planar cases of (3.1) and (3.2). We are going to use an approximate similarity transformation of the coefficient matrix to diagonal form.

The operator C is said to be unitarily equivalent to S, if:

$$S = U\,C\,U^{-1}$$

where $S_{ij} = \lambda_{ij}\,\delta_{ij}$, δ_{ij} is the Kronecker's symbol and U and U^{-1} are unitary transformations, inverse to one another. For a matrix it is a similarity transformation to diagonal form. According to Moritz (1966) and Schwarz (1971) such a unitary equivalence exists between c(x, y) and its Fourier transform:

$$S(u,v) = U \cdot c(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(x,y)\, e^{-i(xu+yv)}\, dx\,dy \tag{3.10}$$

This formula may therefore be used for an approximate determination of the eigen values of (3.1) - (3.2). If m_0 and n_0 are the numbers of blocks in the x- and y- directions, we obtain (cf. Schwarz, ibid.):

$$\lambda_{nm} = a\; S\left(\frac{2\pi n}{2n_0+1}, \frac{2\pi m}{2m_0+1}\right) \tag{3.11}$$

where

$$0 \le n \le n_o \qquad\qquad 0 \le m \le m_o$$

and

$$a = \|C\|$$

The covariance functions will be used in the planar approximation. Furthermore, equal area blocks are assumed with $n_o = m_o$.

We are now going to determine the condition number for the Poisson kernel. This kernel corresponds to $c_n = 2n+1$ and the minimum norm of Δg^* (see Table 2.1). The planar equivalence to the operator A of (2.1) is:

(3.12)
$$A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x,y)\,dx\,dy$$

where

(3.12´)
$$a(x,y) = \frac{h}{2\pi} \,/\, (x^2 + y^2 + h^2)$$

$h = z + b/2$

$b = 2\,(R - r_B)$

R = mean earth radius

r = radius of the computation point

r_B = radius of the Bjerhammar sphere

$z = r - R$

Formula (3.12´) has the following Fourier transform:

(3.13)
$$S(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x,y)\, \exp\{-i\,(ux + vy)\}\,dx\,dy$$

$$= 2\pi h \int_0^{\infty} \frac{r}{(r^2+h^2)^{3/2}}\, J_o(\omega r)\,dr = 2\pi \exp\{-h\omega\}$$

where

$$r = \sqrt{x^2 + y^2} \quad , \quad \omega = \sqrt{u^2 + v^2}$$

and

$J_0(\omega r)$ = zero order Bessel function.

Hence

$$\lambda_{nm} = 2\pi a \exp\{-2\pi h \sqrt{n^2 + m^2}/(2n_0 + 1)\}$$

and

$$\varkappa_a = \exp\{2\pi\sqrt{2}\, h\, n_0/(2n_0+1)\} \approx \exp\{\pi\sqrt{2}\, h\} \tag{3.14}$$

The planar approximation to the Poisson covariance function is (Moritz, 1976, p. 41):

$$C(x, y) = C(r) = \frac{B/Z^2}{(1 + r^2/Z^2)^{3/2}} \tag{3.15}$$

where $Z = z_i + z_j + b$ and $B = 2R^2$.

The Fourier transform of the covariance function is:

$$S(\omega) = 2\pi B \exp\{-Z\omega\} \tag{3.16}$$

Thus we obtain:

$$\varkappa_c = \exp\{2\pi\sqrt{2}\, Z\, n_0/(2n_0 + 1)\}$$

Assuming that $z_i = z_j = z$ we have $Z = 2h$ and

$$\varkappa_c = \exp\{4\sqrt{2}\pi h\, n_0/(2n_0 + 1)\} \approx \exp\{2\sqrt{2}\pi h\} \tag{3.17}$$

so that

$$\varkappa_c \approx \varkappa_a^2 \tag{3.18}$$

From formula (3.18) it is obvious that the condition number ($\varkappa_a$) of the "original" equation (3.1) is squared by letting the number of blocks on the internal sphere approach infinity ($N \rightarrow \infty$) (and then solving the system (3.2)). A considerable loss of stability is achieved. Again we emphasize that the equation system (3.1) is generally unsymmetric in contrast to (3.2), which fact has to be considered in the numerical solution of the system. It is not recommended to form normal equations in (3.1) in order to obtain a symmetric system, because this procedure will increase the condition number to that of system (3.2). (3.1) should be solved directly, for instance, by Gauss elimination or in an iterative way by successive approximations.

3.1 The Effect of Smoothing (Noise)

We assume that the matrices A and C of formulae (3.1-2) are substituted by:

$$\overline{A} = A + D$$

and

$$\overline{C} = C + B\,D$$

where the elements of D are given by:

$$d_{ij} = \begin{cases} k & \text{if} \quad i = j \quad , \quad k > 0 \\ 0 & \text{otherwise} \end{cases}$$

and $B = 2\,R^2$ {see formula (3.15)}.

Then $\overline{A}$ and $\overline{C}$ correspond to the kernel functions:

$$\overline{a}\,(x,y) = a\,(x,y) + k\;\delta\,(x,y)$$

and

$$\overline{c}\,(x,y) = c\,(x,y) + B\,k\;\delta\,(x,y)$$

with the spectra

$$\overline{S}_a\,(u,v) = S_a\,(u,v) + k$$

$$\overline{S}_c\,(u,v) = S_c\,(u,v) + B\,k$$

and the eigen values

$$\bar{\lambda}_{nm}^{(a)} = \lambda_{nm}^{(a)} + a\,k$$

$$\bar{\lambda}_{nm}^{(c)} = \lambda_{nm}^{(c)} + a\,B\,k$$

Here δ is the delta function and S, λ and a are the same as in the previous section. In this case we obtain for the Poisson kernel {cf. (3.14) and (3.17)}:

$$\varkappa_a \approx \frac{k + 2\pi}{k + 2\pi \exp\{-\pi h\sqrt{2}\}} \quad < \exp\{\sqrt{2}\,\pi h\}$$

and

$$\varkappa_c \approx \frac{k + 2\pi}{k + 2\pi \exp\{-2\sqrt{2}\,h\}} \quad < \exp\{2\sqrt{2}\,\pi h\}$$

We notice that the smoothing stabilizes both systems of equations (cf. Section 6.3).

4. On the Convergence of an Iterative Solution

In this section we are going to investigate a condition for convergence of an iterative solution of the matrix equation (3.2):

$$C\,X = \Delta g \tag{4.1}$$

Using the ordinary method of successive approximation, we obtain:

$$X^{(k)} = \Delta g + (I - C)\,X^{(k-1)}; \; k = 1, 2, \ldots \tag{4.2}$$

and

$$\Delta X^{(k)} = X^{(k)} - X^{(k-1)} = (I - C)(X^{(k-1)} - X^{(k-2)}) = (I - C)^{k-1}(X^{(1)} - X^{(0)})$$

Hence

$$\|\Delta X^{(k)}\| \leq \|I - C\|^{k-1} \|X^{(1)} - X^{(0)}\|$$

A sufficient condition for the convergence of (4.2), i.e.

$$\lim_{k \to \infty} \|\Delta X^{(k)}\| = 0$$

is therefore

$$\|I - C\| < 1$$

or

$$\max_i \sum_j |\delta_{ij} - c_{ij}| < 1$$

Now

$$|\delta_{ij} - c_{ij}| = \begin{cases} |1 - c_{ii}| \, , & \text{if } i = j \\ |c_{ij}| \, , & \text{if } i \neq j \end{cases}$$

so that the condition may be written:

(4.3a) $$\sum_{j=1}^{m} |c_{ij}| < 2 \; , \quad \text{if } c_{ii} > 1$$

and

(4.3b) $$2\, c_{ii} > \sum_{j=1}^{m} |c_{ij}| \, , \quad \text{if } c_{ii} \leq 1$$

For finite c_{ii} we may always divide each row of the equation system with a sufficiently large number to ensure that the diagonal elements of C are equal to or less than unity. Hence we can limit our study to the condition (4.3b).

First, we study this condition for Poisson's integral equation for the circle (Sealey, 1966, p. 14):

(4.4)
$$\Delta g_j = \int_{-\pi}^{\pi} a\,(j, i)\,\Delta g^*\,d\,\theta_i$$

where

$$a\,(j, i) = \frac{1}{2\pi}\;\frac{1 - s^2}{1+s^2 - 2s\cos\psi_{ij}} = \frac{1}{2\pi}\;\frac{1 - s^2}{|s-e^{i\psi_{ij}}|^2}$$

$$s = r_B/r_j\ (< 1)$$

$$\psi_{ij} = \theta_j - \theta_i$$

Suppose that we apply (4.4) in a discrete approach with equal spacing between the N unknowns (Δg^*_k). We obtain:

(4.5)
$$\underset{m\,N}{A}\;\underset{N\,1}{\Delta g^*} = \underset{m\,1}{\Delta g}$$

where

(4.6)
$$A_{jk} = a\,(j,k)\,\Delta\theta = (1 - s^2)/N\,|s - e^{i(\theta_j - \theta_k)}|^2$$

$$\Delta\theta = 2\pi/N$$

$$\theta_k = 2\pi k/N\;;\; k = 1, 2, \ldots, N$$

In the non-singular Dirac approach (N = m) we arrive at the following condition from (4.3b) and (4.6) (for $\theta_j = 0$):

$$\frac{2}{m}\;\frac{1-s^2}{(1-s)^2} > \frac{1}{m}\sum_{k=1}^{m}\frac{1 - s^2}{|s-e^{i\theta_k}|^2}$$

or

(4.7)
$$S\,(m, s) = \frac{1}{m}\sum_{k=1}^{m}\frac{1 - s^2}{|s-e^{i\theta_k}|^2} - \frac{2(1+s)}{m(1-s)} < 0$$

The Poisson kernel has the following Fourier series:

$$\frac{1}{m}\ \frac{1-s^2}{|s-e^{i\theta_k}|^2} = \frac{1}{m}\sum_{n=-\infty}^{\infty} s^{|n|}\ e^{in\theta_k}$$

Furthermore

$$\sum_{k=1}^{m} e^{in2\pi k/m} = \begin{cases} m \text{ if } n = m \times j\ , & j = 0, \pm 1, \pm 2, \ldots \\ 0 \text{ otherwise} \end{cases}$$

so that (4.7) becomes

$$S(m,s) = 1 + 2\sum_{j=1}^{\infty} s^{jm} - \frac{2(1+s)}{m(1-s)} = \frac{1+s^m}{1-s^m} - \frac{2(1+s)}{m(1-s)}$$

Subsequently, for large values of m the critical relation between m and s with

$$S(m_0, s) = 0$$

is given by

(4.8a) $$m_0 = 2\frac{1+s}{1-s}$$

or

(4.8b) $$s = \frac{m-2}{m+2}$$

The corresponding relation for collocation is obtained by substituting s by s^2 in the previous formulae. The results are shown in Table 4.1.

For comparison we also derive the relation between m and s in the case of condition adjustment ($N > m$). Then we arrive at the following condition from (4.3b):

$$S(m, N, s) = \max_i \sum_{j=1}^{m} (AQA^T)_{ij} - 2(AQA^T)_{ii} < 0$$

where

$$(AQA^T)_{ij} = N\sum_{k=1}^{N} A_{ik}\ A_{jk}$$

Table 4.1

The Relation Between the Number of Observations (m_0) and the Ratio h/a Compared to the Dirac Method and Collocation*

	Dirac Method		Collocation	
m_0	s	$\frac{h}{a} = \frac{1+s}{\pi s}$	s	$\frac{h}{a}$
10	0.667	0.796	0.816	0.708
100	0.961	0.650	0.961	0.643
1000	0.996	0.638	0.998	0.637
10000	0.9996	0.637	0.9998	0.637
∞	1	$2/\pi$	1	$2/\pi$

* $h = r - r_B$ and $a = 2\pi\, r/m_0$

$$A_{ik} = \frac{1 - s^2}{N(1-2\,s\cos\psi_{ik} + s^2)}$$

$$\psi_{ik} = \theta_i - \theta_k = \frac{2\pi i}{m} - \frac{2\pi k}{N} = 2\,\pi(ip-k)/N$$

$$p = N/m \text{ (integer)}$$

The sum S (m, N, s) is given in the Appendix, Proposition A. 3. For large N we have, approximately:

$$S(m,s) \approx 2 - m + \frac{2m}{1-s^{2m}} - \frac{4}{1-s^2} < 0$$

or

$$m < 2\,\frac{1-s^{2m}}{1+s^{2m}}\,\frac{1+s^2}{1-s^2}$$

For large m we finally arrive at:

$$m < m_0 = 2\,\frac{1+s^2}{1-s^2}$$

and

$$s > \sqrt{\frac{m-2}{m+2}}$$

These are the limiting relations between s and m when using collocation [cf. (4.8 a-b)].

What is the relation between s and m that satisfies the condition (4.3 b) for the generalized integral equation:

(4.4́) $$\Delta g_j = \int_0^{2\pi} a\,(j,k)\; u^*\; d\,\theta_k$$

where

(4.9) $$a\,(j,k) = \sum_{n=-\infty}^{\infty} \sqrt{c_n^*}\; s^{|n|}\; e^{in(\theta_j-\theta_k)}$$

Following the previous derivations, we obtain in the non-singular case of the Dirac method [cf. (4.7)]:

$$S\,(m,s) = \sum_{j=-\infty}^{\infty} \sqrt{c_{jm}^*}\; s^{|j|m} - \frac{2}{m} \sum_{n=-\infty}^{\infty} \sqrt{c_n^*}\; s^{|n|} < 0$$

Thus the sufficient relation between s and m for the convergence according to (4.3b) is very much dependent on c_n^*.

Example: $\sqrt{c_n^*} = |n| + 1$

$$S\,(m,s) = \sum_{j=-\infty}^{\infty} |j|\,m\; s^{|j|m} + \sum_{j=-\infty}^{\infty} s^{|j|m} - \frac{2}{m}\Big(\sum_n |n|\, s^{|n|} - \sum_n s^{|n|}\Big) =$$

$$= \Big(s\,\frac{d}{ds} + 1\Big)\Big(\sum_{j=-\infty}^{\infty} s^{|j|m} - \frac{2}{m}\sum_{n=-\infty}^{\infty} s^{|n|}\Big)$$

$$= \left(s\frac{d}{ds}+1\right)\left(1+2\,\frac{s^m}{1-s^m}-\frac{2}{m}-\frac{4}{m}\,\frac{s}{1-s}\right)=$$

$$= 2m\,\frac{s^m}{(1-s^m)^2}-\frac{4}{m}\,\frac{s}{(1-s)^2}+1+\frac{2s^m}{1-s^m}-\frac{2}{m}-\frac{4}{m}\,\frac{s}{1-s}<0$$

For large m we have approximately

$$m-2-\frac{4s}{(1-s)^2}-\frac{4s}{1-s}<0$$

or

$$m < m_0 = 2+\frac{4s\,(2-s)}{(1-s)^2}$$

and

$$s > s_0 = 1-\left\{1-\left(\frac{m_0-2}{m_0+2}\right)^2\right\}^{\frac{1}{2}}$$

For $m_0\to\infty$, $s_0\to 1$ and $h/a\to 2/\pi$. Again we obtain 2π as a sufficient ratio between the height of the observations and the spacing of the observations for convergence in the continuous case (cf. Table 4.1).

We now investigate Poisson's integral equation in the planar approximation (see formula (3.12)). We assume that a square of side B is divided into N surface elements of side $b = B/N'$, where $N' = \sqrt{N}$. Then we obtain the following coefficients:

$$A_{jk} = \frac{b^2\,h}{2\pi\,[(x_j-x_k)^2+(y_j-y_k)^2+h^2]^{3/2}} =$$

$$= c/2\pi\,[(j_x-k_x)^2+(j_y-k_y)^2+c^2]^{3/2}$$

where

$c = h/b$

$x_j = j_x\,b$

$y_j = j_y\,b$

In the non-singular case ($N' = m'$) the condition (4.3b)becomes (for $j_x = j_y = m'/2$):

$$S(m', c) = \frac{c}{2\pi} \sum_{k_x=1}^{m'} \sum_{k_y=1}^{m'} \frac{1}{[(m'/2 - k_x)^2 + (m'/2 - k_y)^2 + c^2]^{3/2}} - \frac{1}{\pi c^2} < 0$$

The critical relation between m' and c is shown in Table 4.2.

Table 4.2

The Ratio $c = h/b$ Satisfying $S(m', c) = 0$ as Given for Different m'. b = block size, h = height.

m'	Dirac Method	Collocation
50	0.57	0.285
100	0.54	0.270
500	0.52	0.260
1000	0.52	0.260
5000	0.51	0.255
10000	0.50	0.250

The result of the computations is that the ratio h/b should be less than 0.5 for an uncritical convergence of the non-singular Dirac solution with Poisson's kernel (see also Koch, 1968). If the method of collocation is used, the ratio should be less than 0.25. In this study we have based the results on the condition (4.3b) for a solution by successive approximations. However, Schwarz (1971) emphasized that more generous ratios (h/b) may be allowed by using other numerical methods. The figures given in this section should therefore, first of all, be used to compare the stability of different methods (Dirac method, condition adjustment, collocation) with each other.

In conclusion, the Dirac method with ($m = N$) gives a more stable solution than collocation for a given radius r_B. In the case of condition adjustment ($N > m$) the stability of the Dirac method is roughly the same as for collocation. This study was restricted to the Poisson kernel functions, and the conditioning changes with the choice of degree variances (c_n and c_n^*). In general, however, the above tendency can be expected for an arbitrary type of kernel function.

5. Predictions According to Bjerhammar

Using the collocation formula (1.2) any geophysical quantity v_i can be predicted from a vector Δg of gravity anomalies by:

(5.1a) $$v_i = c_i X$$

where

(5.1b) $$X = (C + D)^{-1} \Delta g$$

In the generalized Bjerhammar methods the unknowns X of collocation are replaced by the unknowns u^* at the internal sphere (see section 2). The prediction of v_i becomes accordingly:

(5.2) $$v_i = F_i u^*$$

or

(5.2′) $$v_i = \sum_{k=1}^{N} F_{ik} u_k^*$$

where F_i is the relevant coefficient matrix (with elements F_{ik}) relating u^* to v_i. Formula (5.2′) is the discrete form of the integral formula:

(5.2″) $$v_i = \frac{1}{S} \iint f_{ik} u^* \, dS$$

and the kernel function f_{ik} and the coefficient F_{ik} are related by {cf. formulae (2.2) and (2.2a)}:

(5.3) $$F_{ik} = \frac{1}{S} \iint_{\Delta S_k} f_{ik} \, dS = f_{i\underline{k}} \, \Delta S_k / S$$

where k corresponds to some mean value point inside ΔS_k. In practice this point is approximated by the center of ΔS_k. The function f_{ik} can be derived from formula (2.5) for each specific quantity v_i. For $v_i = \Delta g_i$ we have:

(5.4) $$f_{ik} = \sum_{n=0}^{\infty} \sqrt{(2n+1) c_n^*} \, (r_B / r_i)^{n+2} P_n (\cos \psi_{ik})$$

and $F_{ik} = A_{ik}$ according to (2.12). Further examples are given below.

Prediction of Disturbing Potentials

In the spherical approximation the disturbing potential (T_i) is related to the gravity anomaly (Δg_i) by (Heiskanen and Moritz, ibid., p. 89):

$$\Delta g_i = -\frac{\partial T_i}{\partial r_i} - \frac{2T_i}{r_i} \tag{5.5}$$

By replacing T_i of (5.5) by:

$$f_{ik} = r_B \sum_{n=2}^{\infty} \frac{\sqrt{(2n+1)c_n^*}}{n-1} \left(\frac{r_B}{r_i}\right)^{n+1} P_n(\cos\psi_{ik}) \tag{5.6}$$

the right hand side of (5.5) becomes (5.4) (we exclude terms of degree less than 2). This result implies that f_{ik} according to (5.6) is the kernel function in (5.2´´) for $v_i = T_i$.

Prediction of Height Anomalies

From Brun's theorem (Heiskanen and Moritz, ibid., p. 293) the kernel function for the height anomaly ($v_i = \zeta_i$) becomes:

$$f_{ik} = \frac{r_B}{\gamma} \sum_{n=2}^{\infty} \frac{\sqrt{(2n+1)\, c_n^*}}{n-1} \left(\frac{r_B}{r_i}\right)^{n+1} P_n(\cos\psi_{ik}) \tag{5.7}$$

where γ is the theoretical gravity at the reference ellipsoid. In the special case $r_i = r_B$ and $c_n^* = 2n+1$ ($u^* = \Delta g^*$) (5.7) inserted into (5.2´´) yields Stokes' formula (Heiskanen and Moritz, ibid., p. 94).

Prediction of Deflections of the Vertical

The two components of the deflections of the vertical (ξ, η) are related to the height anomaly ζ according to (Heiskanen and Moritz, ibid., p. 312):

$$\begin{Bmatrix} \xi_i \\ \eta_i \end{Bmatrix} = -\frac{1}{r_i} \begin{Bmatrix} \dfrac{\partial}{\partial \varphi_i} \\ \dfrac{1}{\cos\varphi_i} \dfrac{\partial}{\partial \lambda_i} \end{Bmatrix} \zeta_i = -\frac{1}{r_i} \begin{Bmatrix} \dfrac{\partial \psi}{\partial \varphi_i} \\ \dfrac{1}{\cos\varphi_i} \dfrac{\partial \psi}{\partial \lambda_i} \end{Bmatrix} \frac{\partial \zeta_i}{\partial \psi}$$

where φ_i and λ_i are the geodetic latitude and longitude. By using the following relations from Heiskanen and Moritz (ibid., p. 113) and Abramowitz and Stegun (1964, p. 334):

$$\frac{\partial \psi}{\partial \varphi} = -\cos \alpha \qquad \frac{\partial \psi}{\partial \lambda} = -\sin \alpha \cos \varphi$$

and

$$\frac{\partial P_n(\cos \psi)}{\partial \psi} = -\sin \psi \frac{d P_n(\cos \psi)}{d \cos \psi} =$$

$$= \frac{n(n+1)}{(2n+1)\sin \psi} \left\{ P_{n+1}(\cos \psi) - P_{n-1}(\cos \psi) \right\}$$

where α is the azimuth from the fixed (computation) point to the moving point, we obtain the following kernel functions for ξ_i and η_i:

$$\text{(5.8)} \qquad f_{ik} = \begin{Bmatrix} \cos \alpha_{ik} \\ \sin \alpha_{ik} \end{Bmatrix} \frac{1}{\gamma \sin \psi_{ik}} \times$$

$$\times \sum_{n=2}^{\infty} \frac{n(n+1)}{n-1} \sqrt{\frac{c_n^*}{2n+1}} \left(\frac{r_B}{r_i}\right)^{n+2} \{P_{n+1}(\cos \psi) - P_{n-1}(\cos \psi)\}$$

where $\cos \alpha_{ik}$ refers to ξ_i and $\sin \alpha_{ik}$ to η_i. Formula (5.8) inserted into (5.2˝) gives a generalization of Vening Meinesz' formula (Heiskanen and Moritz, ibid., p. 114).

Prediction of the Vertical Gradient of Gravity

The kernel function for $v_i = \partial \Delta g_i / \partial r_i$ is easily obtained from (5.4). The result is:

$$\text{(5.9)} \qquad f_{ik} = -\frac{1}{r_i} \sum_{n=0}^{\infty} \sqrt{(2n+1) c_n^*} \, (n+2) \, (r_B/r_i)^{n+2} P_n(\cos \psi_{ik})$$

For certain sets of c_n^* the above kernel functions may be given in closed forms (cf. below and Sjöberg, 1975, section A.2).

Note. In the impulse methods (section 2.4) $F_{ik} = f_{ik}$.

5.1 Minimum Norm Solutions

The prediction of a quantity v_i according to any of the methods of Bjerhammar described in section 2 is given by formulae (5.2´) and (5.3) for any finite number of unknowns (N). For example, in the minimum norm solutions (section 2.2) the prediction formula becomes from (5.2) and (2.9):

$$v_i = F_i Q A^T (A Q A^T)^{-1} \Delta g$$

From this formula we obtain in the limit $(N \to \infty)$(cf. 2.14):

$$v_i = k_i C^{-1} \Delta g \tag{5.10}$$

where the elements of the matrix C are given by formula (2.13):

$$C_{ij} = \sum_{n=0}^{\infty} c_n^* (r_B^2/r_i r_j)^{n+2} P_n(\cos\psi_{ij})$$

and

$$k_i = \lim_{N \to \infty} F_i Q A^T$$

The following elements k_{ik} of k_i are obtained in accordance with formulae (2.13), (5.4) and (5.6)-(5.9). The coefficients c_n^* are given by the selected minimum norm (see Table 2.1).

(5.4´) $$k_{ik} = \sum_{n=0}^{\infty} c_n^* s^{n+2} P_n(t) \quad ; \quad \text{for } v_i = \Delta g_i$$

(5.6´) $$k_{ik} = \frac{r_i}{\gamma} \sum_{n=2}^{\infty} \frac{c_n^*}{n-1} s^{n+2} P_n(t) \quad ; \quad \text{for } v_i = \zeta_i$$

(5.8) $$k_{ik} = \frac{1}{\gamma \sin\psi_{ik}} \sum_{n=2}^{\infty} \frac{c_n^* n(n+1)}{n-1} \left\{ P_{n+1}(t) - P_{n-1}(t) \right\}$$

$$x \; s^{n+2} \begin{Bmatrix} \cos\alpha_{ik} \\ \sin\alpha_{ik} \end{Bmatrix} \quad ; \quad \text{for } v_i = \begin{Bmatrix} \xi_i \\ \eta_i \end{Bmatrix}$$

(5.9') $$k_{ik} = -\frac{1}{r_i}\sum_{n=0}^{\infty} c_n^* (n+2)\, s^{n+2} P_n(t)\ ; \quad \text{for } v_i = \frac{\partial \Delta g_i}{\partial r_i}$$

where

$$s = r_B^2/r_i r_k \quad \text{and} \quad t = \cos\psi_{ik}$$

The above predictions from formula (5.10) are identical with the predictions using collocation (without noise). Erroneous observations can be considered by adding a noise covariance matrix to C of (5.10) before inversion {see section 2, formula (2.16)}. The only type of observations considered in Bjerhammar's methods are free-air gravity anomalies. However, as in collocation, heterogeneous data might be included in the predictions.

Finally, we mention that for many norms the above kernel functions can be written in closed forms. As an example we give the kernel functions for $\|\Delta g^*\| = \min.$ ($c_n^* = 2n+1$):

(5.4'') $$k_{ik} = s^2(1-s^2)/\ell^3\ ; \quad \text{for } v_i = \Delta g_i$$

(5.6'') $$k_{ik} = r_i s^2 \{2/\ell + 1 - 3\ell - s\,t(5 + 3\ln\varphi/2)\}; \quad \text{for } v_i = T_i$$

(5.8'') $$k_{ik} = \frac{s^3}{\gamma}\sin\psi_{ik}\begin{Bmatrix}\cos\alpha_{ik}\\ \sin\alpha_{ik}\end{Bmatrix}\Big\{-\frac{2}{\ell^3} - \frac{3}{\ell} + 5 - \frac{3st(1+\ell)}{\ell\varphi} + 3\ln\varphi/2\Big\}; \quad \text{for } v_i = \begin{Bmatrix}\xi_i\\ \eta_i\end{Bmatrix}$$

(5.9'') $$k_{ik} = \frac{-s^2}{2r_i\ell^3}\{1 - 5s^2 + 3(1-s^2)^2/\ell^2\}; \text{for } v_i = \partial\Delta g_i/\partial r_i$$

where

$$s = r_B^2/r_i r_k \quad , \quad t = \cos\psi_{ik}$$

and

$$\ell = (1 - 2st + s^2)^{\frac{1}{2}}\ , \quad \varphi = 1 - st + \ell$$

More examples with derivations are given in Sjöberg (ibid., Appendices A.2 and A.3).

5.2 Model Studies

In Sjöberg (1975, section 19) some minimum norm solutions {formula (5.10)} for ζ, Δg and θ $(=\sqrt{\xi^2+\eta^2})$ are computed for two simple earth models. Some of these results are reported below. For details we refer to Sjöberg (ibid.). The following norms are used:

Method	Norm	$c_n^*/(2n+1)$
1	Δg^*	1
2	T^*	$(n-1)^2/r_B^2$
3	Lauritzen[1)]	$(n-1)/((2n+1)(n-2))$
4	θ^* [2)]	$(n-1)^2/(n(n+1))$

1) empirical norm from Lauritzen (1973, section 10.3)
2) $(\theta^*)^2 = (\xi^*)^2 + (\eta^*)^2$

5.2.1 Bjerhammar's Model

Bjerhammar's model consists of a homogeneous sphere with a spherical and homogeneous mass disturbance M (Figure 2), where $M = 8.37758 \times 10^{15}$ kg and a = 6362 km. The radius of the main sphere is 6370 km.

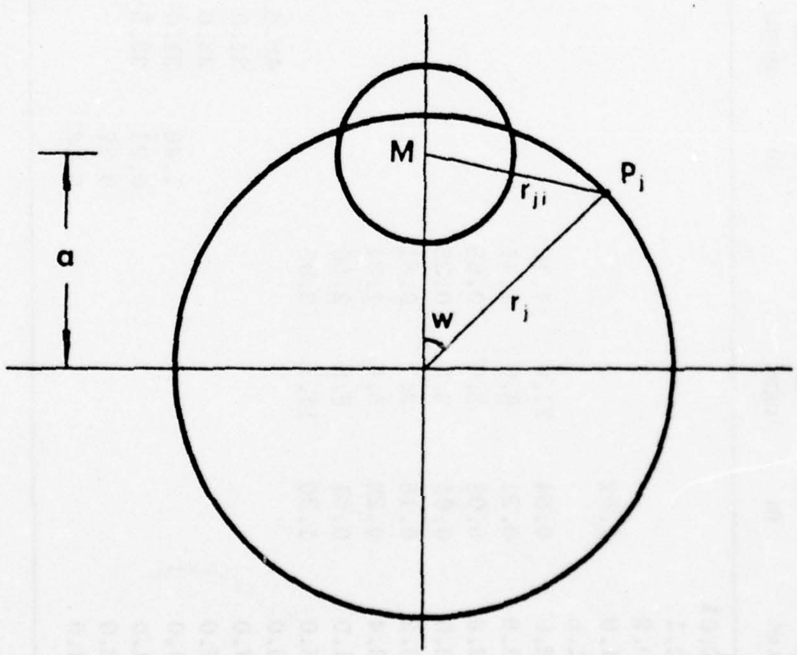


Figure 2. Bjerhammar's Model. The disturbing sphere M is located with its center inside the main sphere.

Table 5.1

RMS Prediction Errors for Bjerhammar's Model

Method	1			2			3			4		
Depth* km	ζ m	Δg mgal	θ''	ζ m	Δg mgal	θ''	ζ m	Δg mgal	θ''	ζ m	Δg mgal	θ''
0.01							11.90	34.2				
0.1							12.22	35.0	8.48			
0.2							12.55	36.0	7.92			
1.0	1.52						15.02	44.0	6.39			
1.5							16.47	47.2	7.61			
2.0	0.64	21.0	4.10				17.88	49.0	9.16			
2.6	0.21	6.0	1.41							0.32	6.4	1.44
2.8	0.08	3.0	0.68							0.19	3.3	0.71
3.0	0.05	2.3	0.35							0.08	2.2	0.32
3.2	0.16	3.7	0.81							0.06	3.5	0.77
3.4	0.28	5.4	1.34							0.15	5.1	1.29
4.0	0.61	8.8	2.50									
5.0	1.20	11.1	2.90									
6.0					48.6	8.02						
7.0					34.0	6.56						
8.0					28.6	6.25						
10.0				1.40	31.0	9.42						
11.0				0.91	33.5							
12.0				0.37								
14.0				5.12								

* from surface of mean earth sphere to surface of Bjerhammar sphere.
No. of observations = 15, no. of prediction points = 24 (all points located within the spherical distance 7!30" from the axis through M of Figure 2). Reference: Table A.6 of Sjöberg (1975).

The "observed" Δg values (due to the disturbance M) were selected at the surface of the model. The first set of observations (15 observations) was selected for $\omega \leq 7'30''$, the additional set of 35 observations was selected for $8' \leq \omega \leq 25'$ (all 35 points were located at the surface of the main sphere). The predictions of ζ, Δg and θ were carried out at the surface of the model (between the observation points) for different depths to the Bjerhammar sphere (Table 5.1). The RMS errors at the optimal depths are summarized in Table 5.2. Methods 1 and 4 give good agreements with the theoretical values. The methods 2 and 3 are not suitable for this local model. The importance of the choice of radius of the Bjerhammar sphere is obvious from Table 5.1.

Table 5.2

RMS Prediction Errors at the Optimal Depths
to the Bjerhammar Sphere

	15 observations				50 observations			
Method[1]	Depth[2] /km/	ζ /m/	Δg /mgal/	θ''	Depth[2] /km/	ζ /m/	Δg /mgal/	θ''
1	3	0.04	22.7	0.33	3	0.03	2.27	0.33
2	8-12	0.37	27.5	6.20	-	--	--	--[3]
3	-	11.9	34.2	6.25	-	--	--	--[3]
4	3	0.05	2.23	0.32	3	0.03	2.21	0.33

[1] Norm used
[2] Difference of the radii of main sphere and Bjerhammar sphere
[3] Not computed
No. of predicted points = 24, RMS (anomaly - mean value) = 182 mgal.
Reference: Sjöberg (1975, Table 19.1)

5.2.2 Molodenskii's Model

The surface of Molodenskii's model is a cone. Two spheres, whose centers are located on the axis of the cone, are taken as the anomalous masses (see Figure 3). The attractions of the spheres m_2 and m_1 on the axis of the cone 4050 m above the reference plane are 150 and 100 mgal, respectively ($m_2 \approx 1.458 \times 10^{15}$ kg, $m_1 \approx 6.334 \times 10^{13}$ kg). As there is a disturbing mass (m_1) above the reference plane, it is clear that this model is much rougher than Bjerhammar's model. The computations revealed that for all methods (norms) the depth to the Bjerhammar sphere should be selected as close to the reference plane as possible (10 meters was used in the computations).

All observation and prediction points were selected on the surface of the cone. A summary of the RMS prediction errors is given in Tables 5.3 and 5.4. We notice that the predictions are essentially better for a more regular distribution of the observations (Table 5.4). Again, methods 2 and 3 are less favorable than 1 and 4. We also notice that these RMS errors are of a magnitude larger than those obtained for the smoother model of Bjerhammar.

A comparison between a minimum norm solution and the Dirac method for Molodenskii's model is given in section 8.

Figure 3. Molodenskii's Model

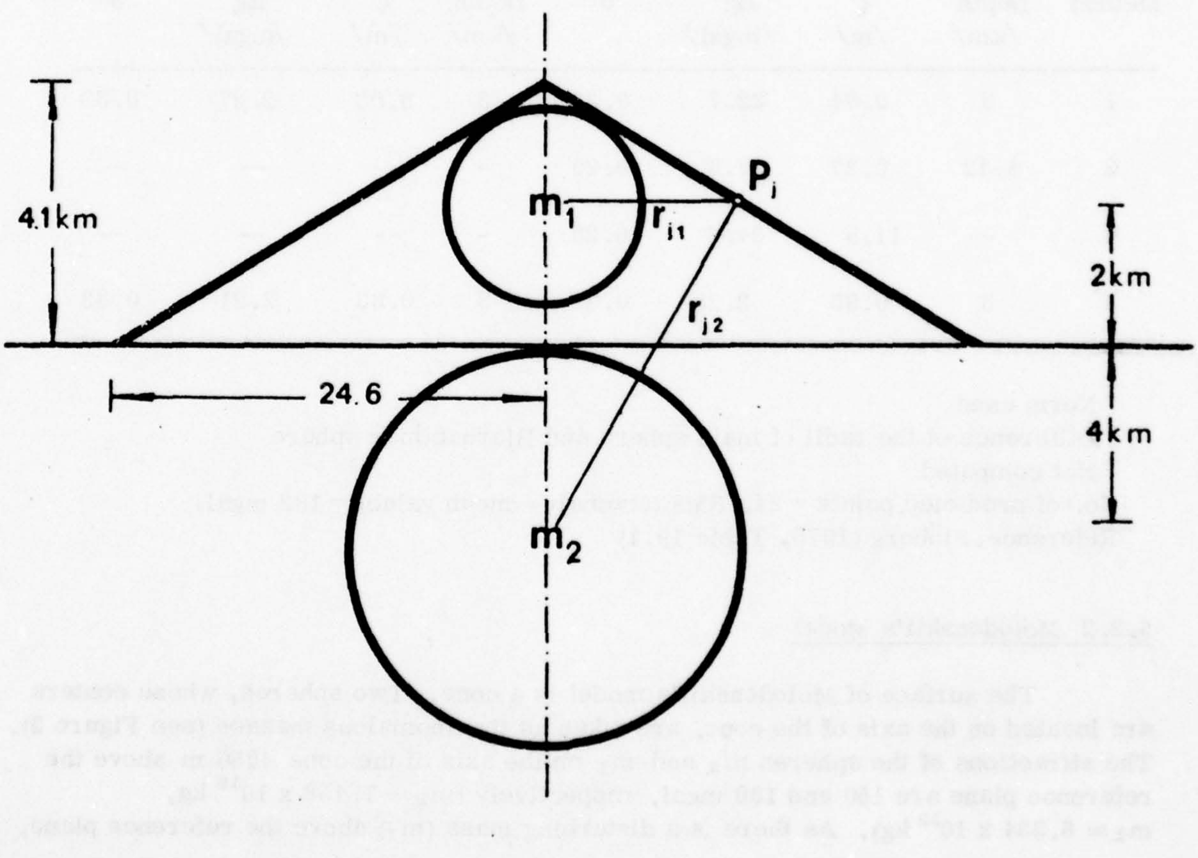


Table 5.3

Molodenskii's Model. RMS Prediction Errors at the Optimal Depths to the Bjerhammar Sphere

Method[1]	Depth /km/	ζ /m/	Δg /mgal/	ξ''	θ''
1	10	0.67	25.0	7.58	13.78
2	200-3500	0.23	33.3	5.48	6.25
3	10	13.5	45.5	14.24	28.25
4	10	0.63	25.0	7.58	13.76

[1] Norm used
(10 m is the minimum depth used in the computations.) No. of observations = 20 (irregular distribution), no. of predictions = 19 {RMS (anomaly - mean) = 81.7 mgal}. From Sjöberg (1975, Table 19.2).

Table 5.4

Molodenskii's Model. RMS Prediction Errors at the Depth 10 m to the Bjerhammar Sphere

Method[1]	ζ /m/	Δg /mgal/	ξ''	θ''
1	0.09	15.4	2.45	2.70
3	5.2	18.9	3.11	5.80
4	0.06	15.4	2.45	2.70

[1] Norm used
No. of observations = 24 (regular distribution), no. of predictions = 19 {RMS (anomaly - mean) = 81.7 mgal}. From Sjöberg (1975, Table 19.3).

6. Integral Formulas as Limiting Cases

In this section we are going to study the solutions by collocation and the Dirac method for a continuous field of observations at the surface of the earth. Two problems are discussed: the uniqueness of the solutions and the existence of the solutions.

6.1 The Uniqueness of the Solutions

The non-singular Dirac method was presented in (2.25) for a finite set of observations (we exclude terms of degree less than 2):

$$\Delta g_P = \sum_{k=1}^{N} u^*_k \, A(P,k) \quad ; \qquad P = 1, 2, \ldots, \tag{6.1}$$

where

$$A(P,k) = \sum_{n=2}^{\infty} \sqrt{(2n+1)\, c_n^*}\, (r_B/r_P)^{n+2}\, P_n(\cos \psi_{Pk})$$

Using the solutions for u_k^* of (6.1) the disturbing potential (T) may be estimated by formulae (5.2´) and (5.6) (for $F_{1k} = f_{1k}$):

$$\hat{T}_P = r_B \sum_{k=1}^{N} u_k^* \, S(P,k)$$

where

$$S(P,k) = \sum_{n=2}^{\infty} \frac{\sqrt{(2n+1)c_n^*}}{n-1} \left(\frac{r_B}{r_P}\right)^{n+2} P_n(\cos \psi_{Pk})$$

and the height anomaly is given by {cf. (5.7)}:

$$\zeta_P = \hat{T}_P / \gamma \tag{6.2}$$

Let N go to infinity with a well-behaving distribution of the carrier points. Then we arrive at the following integral formulas from (6.1) and (6.2) in the continuous case:

(6.1´) $$\Delta g\,(P) = \frac{1}{4\pi} \iint A\,(P, Q)\; u^*\,(Q)\; d\,\sigma_Q$$

and

(6.2´) $$\zeta\,(P) = \frac{1}{4\pi} \iint S\,(P, Q)\; u^*\,(Q)\; d\,\sigma_Q$$

In the special case $c_n^* = 2n + 1$ ($u^* = \Delta g^*$) (6.1´) and (6.2´) become Poisson's and the generalized Stokes' formula, respectively.

We now study the solution by collocation in the same way. From (1.2) we get the following intermediate step (for $D = 0$):

(6.3a) $$\Delta g_P = C\,X \quad , \qquad P = 1, 2, \ldots, N$$

or

(6.3b) $$\Delta g_P = \sum_{k=1}^{N} c\,(P,k)\; X_k \;, \qquad P = 1, 2, \ldots, N$$

where

(6.3c) $$c\,(P,k) = \sum_{k=2}^{\infty} c_n\,(r_B^2/r_P\, r_k)^{n+2}\; P_n\,(\cos\psi_{Pk})$$

and

$$X = \text{vector of unknowns } (X_k)$$

The prediction of ζ_P is given by:

(6.4) $$\hat{\zeta}_P = \frac{r_P}{\gamma} \sum_{k=1}^{N} X_k\; S'\,(P,k)$$

where

$$S'(P,k) = \sum_{n=2}^{\infty} \frac{c_n}{n-1} \left(\frac{r_B^2}{r_P\, r_k}\right) P_n\,(\cos\psi_{Pk})$$

Formula (6.3b) and (6.4) may be written:

$$\Delta g_P = \frac{1}{4\pi} \iint c\,(P,Q)\; X\,(Q)\, d\,\sigma_Q\,, \quad P = 1, 2, \ldots, N$$

and

$$\zeta_P = \frac{r_P}{4\pi\gamma} \iint S'(P,Q)\; X\,(Q)\; d\,\sigma_Q$$

where

$$X\,(Q) = \sum_{k=1}^{N} X_k\, \delta\,(Q - Q_k)$$

δ = Dirac's delta function {see (2.24a)}

Q_k = the foot point at the internal sphere of the normal through the observation in P_k

In the limit $N \to \infty$ (with the maximum distance among the observations approaching 0) these equations become the following integral equations:

(6.3´)
$$\Delta g\,(P) = \frac{1}{4\pi} \iint c\,(P,Q)\; X\,(Q)\; d\,\sigma_Q$$

and

(6.4´)
$$\hat{\zeta}\,(P) = \frac{r_P}{4\pi\gamma} \iint S'(P,Q)\; X\,(Q)\; d\,\sigma_Q$$

Formulae (6.2´) and (6.4´) are estimates of ζ (P). Furthermore, the estimates seem to differ for various choices of $c_n{}^*$ and c_n of the kernel functions. However, next we are going to show that these differences are merely apparent whenever the solutions exist.

First, we expand c (P, Q) of (6.3c) into spherical harmonics {cf. formula (A.1)}:

$$c\,(P,Q) = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} \frac{c_n}{2n+1}\, Y_{nm}(P)\, Y_{nm}(Q) \left(\frac{r_B{}^2}{r_P\, r_Q}\right)^{n+2}$$

By inserting this expansion into (6.3´) we obtain:

$$\Delta g\,(P) = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} B_{nm} \left(\frac{r_B}{r_P}\right)^{n+2} Y_{nm}\,(P)$$

where

$$B_{nm} = \frac{c_n}{4\pi(2n+1)} \iint \left(\frac{r_B}{r_Q}\right)^{n+2} Y_{nm}(Q)\, X(Q)\, d\sigma_Q$$

In the same way (6.4') and $\Delta g(P)$ may be developed into the following series:

$$\hat{\zeta}(P) = r_P \sum_{n=2}^{\infty} \sum_{m=-n}^{n} B_{nm} \frac{1}{n-1} \left(\frac{r_B}{r_P}\right)^{n+2} Y_{nm}(P)$$

and

$$\Delta g(P) = \sum_n \sum_m A_{nm} \left(\frac{r_B}{r_P}\right)^{n+2} Y_{nm}(P)$$

where

$$A_{nm} = \frac{1}{4\pi} \iint \Delta g^*(Q)\, Y_{nm}(Q)\, d\sigma_Q$$

From the above expansions we obtain for $c_n > 0$:

$$B_{nm} = A_{nm}$$

and

$$\hat{\zeta}(P) = \frac{r_P}{\gamma} \sum_n \sum_m \frac{1}{n-1} \left(\frac{r_B}{r_P}\right)^{n+2} A_{nm} Y_{nm}(\underline{P}) =$$

$$= \frac{r_B}{4\pi\gamma} \iint S(r_P, \underline{\psi}_{PQ})\, \Delta g^*(Q)\, d\sigma_Q$$

where

$$S(r_P, \underline{\psi}_{PQ}) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{r_B}{r_P}\right)^{n+1} P_n(\cos \psi_{PQ})$$

$S(r_P, \psi_{PQ})$ is the generalized Stokes' function. Thus we have shown that each collocation solution for ζ (for $c_n > 0$) in the continuous case equals Stokes' integral formula.

The same result may be shown for Poisson's and Vening Meinesz' integrals. Moreover, the same proof holds for the non-singular Dirac method in the continuous case {formula (6.1') and (6.2')}. The uniqueness of all solutions may be regarded as a consequence of Stokes' theorem (Heiskanen and Moritz, 1967, p. 17), which states that a function harmonic outside a closed surface S is uniquely determined by its values on S.

However, the conversion from one set of c_n (c_n^*) to another is, of course, not valid unless the unknowns (X and u*) exist and the kernel functions are bounded. This question is discussed in the next section.

6.2 The Existence of the Solutions

In sections 3 - 4 we have studied the stability of the coefficient matrix for different methods, which is of importance for the practical application with a finite set of observations. Now we ask under which circumstances there exists a solution in the continuous case (infinite number of observations). As will be shown, the existence of a solution to any of the integral equations is considerably dependent on the smoothness of the observed field.

In Moritz (1975), a proof of the convergence of least squares interpolation is presented for an element T of a certain Hilbert space (with a given kernel function). However, Tscherning (1977a) showed that the disturbing potential of the earth (T) is not an element of the Hilbert space, associated with the empirical covariance function. Subsequently, Moritz' proof is not applicable in this case.

In our study we start with Poisson's integral equation for the exterior of a circle of radius r_B. All observations are assumed to be located on a circle of radius r. Then we have {cf. formula (4.4)}:

$$\Delta g\,(\theta_j) = \int_{-\pi}^{\pi} k\,(\theta_j, \theta_k)\, \Delta g^*\,(\theta_k)\, d\,\theta_k \qquad (6.5)$$

where

$$k\,(\theta_j, \theta_k) = \frac{1}{2\pi}\, \frac{1 - s^2}{|s - e^{i(\theta_j - \theta_k)}|^2} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} s^{|n|}\, e^{in(\theta_j - \theta_k)} \qquad (6.6)$$

Now we assume that Δg may be expanded into a Fourier series and we try a corresponding series for Δg^*:

$$\Delta g\,(\theta_j) = \sum_{m=-\infty}^{\infty} a_m\, e^{im\theta_j} \qquad (6.7a)$$

and

(6.7b) $$\Delta g^* (\theta_k) = \sum_{\ell=-\infty}^{\infty} b_\ell \, e^{i\ell\theta_k}$$

Furthermore, we assume that both the unknowns (Δg^*) and the observations (Δg) are uniformly distributed with:

$$\theta_j = \frac{2\pi j}{m} \qquad j = 1, 2, \ldots, m$$

where m is the number of observations (Dirac method). This implies that Δg^* may be written:

$$\Delta g^*(\theta) = \sum_{k=1}^{m} \Delta g^*(\theta_k)\, \delta(\theta - \theta_k)$$

and

(6.8) $$\Delta g\,(\theta_j) = \sum_{k=1}^{m} k\,(\theta_j, \theta_k)\, \Delta g^*(\theta_k) \quad , \quad j = 1, 2, \ldots, m$$

Let us now regard (6.8) as a linear filter with each "tone" $b_\ell e^{i\ell\theta}$ of Δg as input and $a_\ell e^{i\ell\theta}$ of Δg as output. Then we obtain from (6.6) - (6.8):

$$a_\ell e^{i\ell 2\pi j/m} = \frac{b_\ell}{2\pi} \sum_{k=1}^{m} \sum_{n=-\infty}^{\infty} s^{|n|} e^{in2\pi(j-k)/m} \; e^{i\ell 2\pi k/m}$$

$$= \frac{b_\ell}{2\pi} \sum_{n=-\infty}^{\infty} s^{|n|} e^{inj2\pi/m} \sum_{k=1}^{m} e^{i(\ell-n)2\pi k/m}$$

$$= \frac{b_\ell}{2\pi} \; e^{i\ell 2\pi j/m} \; m \; u_{\ell,m}(s)$$

where (see Appendix, Corollary A.2):

$$u_{\ell,m}(s) = \sum_{p=-\infty}^{\infty} s^{|\ell + mp|} = \left(s^{|\ell| - m\alpha} + s^{m - |\ell| + m\alpha}\right) / (1 - s^m)$$

$$\alpha = \left[\frac{|\ell|}{m}\right] = \text{integer part of } \frac{|\ell|}{m}$$

Hence

$$b_\ell = 2\pi a_\ell / (m\, u_{\ell,m}(s)) \tag{6.9}$$

Moreover we notice

$$\lim u_{\ell,m}(s) = s^{|\ell|}, \quad m \to \infty$$

so that for large m we have, approximately:

$$b_\ell \approx 2\pi a_\ell / (m\, s^{|\ell|}) \tag{6.9'}$$

We conclude that Δg^* does not generally exist for a dense distribution of the observations, and the convergence for Δg^* as $m \to \infty$ is very much dependent on the behavior of a_ℓ as $\ell \to \infty$.

Let us assume that the solution for Δg^* exists. The predictions of $\Delta g(R, \theta)$ are then given by:

$$\hat{\Delta g}(R, \theta) = \sum_{\ell=-\infty}^{\infty} \sum_{k=1}^{m} k(\theta, \theta_k)\, \Delta g^*_\ell(\theta_k)$$

$$= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} b_\ell \sum_{k=1}^{m} \sum_{n=-\infty}^{\infty} s_R^{|n|} e^{in(\theta - \theta_k)} e^{i\ell\theta_k}$$

$$= \frac{m}{2\pi} \sum_{\ell=-\infty}^{\infty} b_\ell\, e^{i\ell\theta}\, v_{\ell,m}(s_R, \theta)$$

where

$$s_R = r_B / R$$

and

$$v_{\ell, m}(s_R, \theta) = \sum_{j=-\infty}^{\infty} s_R^{|\ell + mj|} e^{imj\theta}$$

This sum is given in a closed form in the Appendix, Proposition A.2. Inserting (6.9) we finally obtain:

$$\hat{\Delta g}(R, \theta) = \sum_{\ell=-\infty}^{\infty} a_\ell \frac{v_{\ell, m}(s_R, \theta)}{u_{\ell, m}(s)} e^{i\ell\theta} \tag{6.10}$$

Furthermore it follows from Corollary A.3 that:

$$v_{\ell, m} \to s_R^{|\ell|} , \quad m \to \infty$$

so that

$$\lim_{m \to \infty} \hat{\Delta g}(R, \theta) = \sum_{\ell=-\infty}^{\infty} a_\ell \left(\frac{r}{R}\right)^{|\ell|} e^{i\ell\theta}$$

Thus $\hat{\Delta g}$ is convergent in the limit for $R \geq r$. (The convergence for $R = r$ was proved by Hörmander. See Bjerhammar, 1974.) Subsequently, we have shown that although the predictions are uniformly convergent for $R \geq r$, the intermediate solution Δg^* does not generally exist in the continuous case ($m \to \infty$). This result is of great practical importance.

Let us now substitute (6.5) by the following general integral equation [cf. formula (2.5)]:

$$\Delta g(\theta_j) = \int_{-\pi}^{\pi} k(\theta_j, \theta_k)\, u^*(\theta_k)\, d\theta_k \tag{6.11}$$

where

$$k(\theta_j, \theta_k) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sqrt{c_n^*}\ s^{|n|}\ e^{in(\theta_j - \theta_k)}$$

To ensure that the kernel k is bounded, we suppose that the magnitude of c_n^* satisfies:

(6.12) $$\sqrt{c_n^*}\ s^{|n|} < 1/n \text{ for } n > M$$

where M is an integer. The solution of (6.11) becomes:

$$u^*(\theta) = \sum_{\ell=-\infty}^{\infty} b_\ell\, e^{i\ell\theta}$$

where {cf. (6.9) and (6.9′)} :

$$b_\ell = 2\pi\, a_\ell / (m\sqrt{c_\ell^*}\ u_{\ell,m}(s))$$

or, approximately for large m:

(6.13) $$b_\ell \approx 2\pi\, a_\ell / (m\sqrt{c_\ell^*}\ s^{|\ell|})$$

From (6.12) and (6.13) it is obvious that it is generally not possible to select a set c_n^* such that u^* is convergent for arbitrary a_ℓ.

Next, we proceed to the global equivalence of the above derivations. We assume that the gravity field of the earth can be expanded into a series of spherical harmonics $[Y_{nm}(P)]$ for each point (P) on the surface of the earth (cf. section 6.1):

$$\Delta g(P) = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} A_{nm} \left(\frac{r_B}{r_P}\right)^{n+2} Y_{nm}(P)$$

where $Y_{nm}(P)$ is defined in the Appendix, formula A.1. Furthermore, we expand u^* into a corresponding series:

(6.14) $$u^*(Q) = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} u_{nm}\, Y_{nm}(Q)$$

where u_{nm} are unknown coefficients. By inserting these two series into (6.1´), we arrive at the following identity for each spherical harmonic:

$$\sqrt{\frac{c^*_{nm}}{2n+1}}\, u_{nm} \equiv A_{nm}$$

or

(6.15)
$$u_{nm} = A_{nm} \sqrt{\frac{2n+1}{c^*_n}}$$

Here we have expanded $P_n(\cos \psi_{PQ})$ of the kernel function A (P, Q) according to the Addition theorem (A.1):

$$P_n(\cos \psi_{PQ}) = \frac{1}{2n+1} \sum_{m=-n}^{n} Y_{nm}(P)\, Y_{nm}(Q)$$

and we have also used the orthogonality property for the spherical harmonics.

The same technique may be used to solve for the intermediate solution X (Q) in collocation. We insert the expansion for Δg (P) and:

(6.16)
$$X(Q) = \sum_{n,m} X_{nm}\, Y_{nm}(Q)$$

into (6.3´), and we generalize the auto-covariance function by substituting the empirical set c_n of (1.5) by an arbitrary set of positive definite parameters c^*_n. The result is:

(6.17)
$$\sum_{n,m} \sum_{n',m'} \frac{c_n^*}{2n+1} \left(\frac{r_B}{r_P}\right)^{n+2} Y_{nm}(P)\, X_{n'm'} \iint \left(\frac{r_B}{r_Q}\right)^{n+2} Y_{nm}(Q)\, Y_{n'm'}(Q)\, d\sigma_Q =$$

$$= \sum_{n,m} A_{nm} \left(\frac{r_B}{r_P}\right)^{n+2} Y_{nm}(P)$$

In this case we can not simply apply the orthogonality property of the spherical harmonics to identify the unknowns, because r_Q is a variable. However, if we limit ourselves to the special case $r_Q = r = \text{constant}$ (all observations on a sphere) we obtain:

$$\left(\frac{r_B}{r}\right)^{n+2} \frac{c_n^*}{2n+1} X_{nm} = A_{nm}$$

or

(6.18) $$X_{nm} = \frac{2n+1}{c_n^*} A_{nm} \left(\frac{r}{r_B}\right)^{n+2}$$

In summary, the solutions for u* and X are:

(6.19) $$u^* (Q) = \sum_{n,m} \sqrt{\frac{2n+1}{c_n^*}} A_{nm} Y_{nm} (Q)$$

and

(6.20) $$X (Q) = \sum_{n,m} \frac{(2n+1)}{c_n^*} A_{nm} \left(\frac{r}{r_B}\right)^{n+2} Y_{nm} (Q)$$

If we now consider that the choice of c_n^* is restricted by the condition of bounded kernel (covariance) functions, the convergence of (6.19) and (6.20) is essentially dependent on the smoothness of A_{nm} for higher degrees.

Let us for the moment assume that (6.19) and (6.20) converge. By inserting these solutions into the integrals (6.1′) and (6.3′), we obtain in both cases:

(6.21) $$\hat{\Delta g} (P) = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} A_{nm} \left(\frac{r_B}{r_P}\right)^{n+2} Y_{nm} (P)$$

This series is identical with the exterior gravity anomaly of the earth. Thus the predictions converge to the true values, whenever the unknowns u* and X exist in the continuous case (cf. section 6.1).

According to the Hilbert-Schmidt theorem (Chambers, 1976, p. 50), the series solution (6.21) for $\hat{\Delta g}$ (P) of the left member of the integral equation (6.3′) is valid if and only if:

$$1)\quad c\,(P,Q) = \sum_{n=2}^{\infty} c_n^*\,(r_B/r)^{2(n+2)}\; P_n\,(\cos\psi_{PQ}) < \infty$$

and

$$2)\quad \|X\|^2 = \frac{1}{4\pi}\iint X^2\,(Q)\; d\,\sigma_Q < \infty$$

where r is the minimum radius to any point at the surface of the earth. Using (6.20), the condition 2) may be written:

$$2')\quad \|X\|^2 = \sum_{n=2}^{\infty}\left[\frac{(2n+1)\sigma_n}{c_n^*}\left(\frac{r}{r_B}\right)^{2(n+2)}\right]^2 < \infty$$

where we have used the notation:

$$\sigma_n^2 = \sum_{m=-1} A_{nm}^2\left(\frac{r_B}{r}\right)^{2(n+2)} = c_n\left(\frac{r_B}{r}\right)^{2(n+2)}$$

σ_n^2 is the anomaly degree variance at the sphere of radius r (cf. formula (1.5)). From 1) and 2') we arrive at the following inequality for the magnitude of c_n^*:

(6.22) $$n\sqrt{n}\;\sigma_n < c_n^*\left(\frac{r_B}{r}\right)^{2(n+2)} < n^{-1}$$

In the same way we obtain the following conditions from the integral equation (6.1') with the solution (6.19):

$$1)\quad A\,(P,Q) = \sum_{n=0}^{\infty}\sqrt{(2n+1)\,c_n^*}\;(r_B/r)^{n+2}\; P_n\,(\cos\psi_{PQ}) < \infty$$

and

$$2)\quad \|u^*\|^2 = \sum_{n=0}^{\infty}(2n+1)\,\frac{\sigma_n^2}{c_n^*}\left(\frac{r}{r_B}\right)^{2(n+2)} < \infty$$

and the corresponding magnitude bounds for c_n^* are given by:

(6.23) $$n^2\,\sigma_n^2 < c_n^*\left(\frac{r_B}{r}\right)^{2(n+2)} < n^{-3}$$

The inequalities (6.22) and (6.23) should be regarded as necessary bounds for c_n^* in a convergent solution of collocation and the Dirac method, respectively. We notice that both (6.22) and (6.23) are satisfied for:

$$\sigma_n < n^{-5/2} \qquad (6.23')$$

However for the gravity field of the earth, we have according to Kaula's rule of thumb for the variation of the potential coefficients, that σ_n is on the order of $n^{-1/2}$. Thus we conclude that the solutions for u* and X are not convergent in the continuous case. The same negative result is obtained if we replace the observations Δg of (6.1′) and (6.3′) by the disturbing potential. In this case, we arrive at the same inequalities (6.22) and (6.23). The degree variances of T (σ_n^2(T)) are of magnitude n^{-3} according to Kaula's rule, and it is clear that (6.23′) is not satisfied.

In all the methods of Bjerhammar, the idea is to determine a fictitious field u* (or Δg^*) on the internal sphere, and then to use u* (Δg^*) in the classical integral equations for estimating geophysical quantities such as Δg, T, ξ, and ζ. In collocation, we are not restricted to the determination of the corresponding intermediate solution X as was suggested previously. For example, the prediction of Δg_P may be expressed directly as a linear combination of the observations (discrete case):

$$\hat{\Delta g}_P = \sum_{i=1}^{m} h_{Pi}\, \Delta g_i \qquad (6.24)$$

where h_{Pi} are the weights, which are given by the following discrete Wiener-Hopf equations:

$$c_{Pk} = \sum_{i=1}^{m} h_{Pi}\, c_{ik} \;\; ; \quad k = 1, 2, \ldots, m \qquad (6.25)$$

where c is the covariance function of Δg. In the continuous case (6.24) and (6.25) become:

$$\hat{\Delta g}\,(P) = \frac{1}{4\pi} \iint h\,(P,Q)\, \Delta g\,(Q)\, d\sigma_Q \qquad (6.24')$$

and

$$c\,(P,Q) = \frac{1}{4\pi} \iint h\,(P,Q')\, c\,(Q',Q)\, d\sigma_{Q'} \qquad (6.25')$$

If P is a point at the surface of the earth, (6.25´) has the solution:

$$h(P,Q) = \delta(P - Q)$$

where δ is Dirac's delta function defined in (2.24a). If P is a point outside the surface of the earth, we may assume that the covariance function c is spatially homogeneous and isotropic:

$$c(P,Q) = \sum_{n=2}^{\infty} c_n (r_B/r_P r_Q)^{n+2} P_n(\cos\psi_{PQ})$$

In order to solve for h we try the following expansion:

$$h(P,Q) = \sum_{n=2}^{\infty} h_n P_n(\cos\psi_{PQ})$$

where h_n are unknown coefficients to be determined. Let us restrict ourselves to the spherical approximation of the earth, i.e. $r_Q = r_Q' = r =$ constant. Then we obtain from (6.25´):

$$h_n = (2n+1)(r/r_P)^{n+2}$$

and the solution for h becomes a modified Poisson's kernel function:

$$h(P,Q) = \sum_{n=2}^{\infty} (2n+1)(r/r_P)^{n+2} P_n(\cos\psi_{PQ}) =$$

$$= \frac{r^2}{r_P} \frac{(r_P^2 - r^2)}{(r_P^2 + r^2 - 2rr_P\cos\psi_{PQ})^{3/2}} - \left(\frac{r}{r_P}\right)^2 - 3\left(\frac{r}{r_P}\right)^3 \cos\psi_{PQ}$$

and (6.24´) becomes Poisson's integral (without the terms of degrees less than 2). Thus we have found that by carrying out the collocation solution by solving the Weiner-Hopf integral equation, the correct gravity anomaly is recovered on and outside the surface of a spherical earth.

In the same way we can solve for the disturbing potential in the exterior of a sphere. Let us use the estimator:

(6.26a)
$$\hat{T}(P) = \frac{1}{4\pi} \iint h(P,Q)\, \Delta g(Q)\, d\sigma_Q$$

where h (P, Q) is given by:

(6.26b)
$$k(P,Q) = \frac{1}{4\pi} \iint h(P,Q')\, c(Q',Q)\, d\sigma_{Q'}$$

and

$$k(P,Q) = r_P \sum_{n=2}^{\infty} \frac{c_n}{n-1} (r_B^2/r_P\, r)^{n+2}\, P_n(\cos\psi_{PQ})$$

$$c(P,Q) = \sum_{n=2}^{\infty} c_n\, (r_B/r)^{2(n+2)}\, P_n(\cos\psi_{PQ})$$

By trying an expansion for h in (6.26b) (cf. the previous example), the coefficients h_n are easily identified and the solution for h becomes:

$$h(P,Q) = r\, S(r_P, \psi_{PQ})$$

where

$$S(r_P, \psi_{PQ}) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{r}{r_P}\right)^{n+1} P_n(\cos\psi_{PQ})$$

Thus the solution for T/γ from (6.26a) with $r_P = r$ is Stokes' formula (cf. section 6.1).

6.3 The Effect of Smoothing (Noise)

The solutions for u* and X in the methods of Bjerhammar and collocation may be smoothed by adding a noise covariance function d (j,k) to the auto-covariance function c (j, k) [see formula (1.2) and (2.16)]. Consider the example of the previous section with a continuous field of observed Δg on a sphere of radius r. Let us replace the covariance function of (6.3′) by:

(6.27) $$\bar{c}(P,Q) = c(P,Q) + d(P,Q)$$

where

$$d(P,Q) = d\,\delta(\psi_{PQ}) \quad , \qquad d > 0$$

$\delta(\psi_{PQ})$ = Dirac's delta function, defined in (2.24a). This noise covariance function corresponds to pure white noise. By inserting (6.27) into (6.3′), we obtain in accordance with (6.17) and (6.18) (for $r_P = r_Q = r$):

(6.28) $$X_{nm} = \frac{A_{nm}(r_B/r)^{n+2}}{d + (r_B/r)^{2(n+2)} c_n/(2n+1)}$$

For large degrees (n) the coefficients X_{nm} can be approximated by:

$$X_{nm} \approx \frac{1}{d} A_{nm} \left(\frac{r_B}{r}\right)^{n+2}$$

Subsequently

$$X(Q) = \sum_{n,m} X_{nm} Y_{nm}(Q)$$

is convergent whenever the spherical harmonic expansion of Δg has the radius of convergence r.

The prediction of new anomalies from the intermediate solution X is given by (6.16) and (6.3′). The result is:

$$\hat{\Delta g}(P) = \sum_{n,m} X_{nm} (r_B^2/rr_P)^{n+2} \frac{c_n}{2n+1} Y_{nm}(P)$$

By inserting (6.28) we arrive at the following predictor and prediction error:

(6.29) $$\hat{\Delta g}(P) = \sum_{n,m} \frac{c_n (r_B/r)^{2(n+2)}}{(2n+1)\,d + c_n (r_B/r)^{2(n+2)}} A_{nm} \left(\frac{r_B}{r_P}\right)^{n+2} Y_{nm}(P)$$

and

$$(6.30) \qquad \hat{\Delta g}(P) - \Delta g(P) = -\sum_{n,m} \frac{d}{d + (r_B/r)^{2(n+2)} c_n/(2n+1)} A_{nm} \left(\frac{r_B}{r}\right)^{n+2} Y_{nm}(P)$$

Subsequently, by selecting a sufficiently small $d > 0$, the prediction error becomes arbitrarily small. However, in practice the constant d has to exceed a certain minimum value due to the requirement of X to be within the range of the computer.

In a similar way we may approximate the solution for the Dirac method.

The smoothing is first of all justified when the observations are erroneous. [Cf. formula (1.2) and (1.3).] These predictions are statistically biased, which we conclude from the following application of (1.3). Let:

$$\hat{\Delta g}_P = c_P (C + D)^{-1} \tilde{\Delta g}$$

be an estimator of the anomaly Δg_P, which is included in the vector of observations $(\tilde{\Delta g})$. Furthermore, $\tilde{\Delta g}$ consists of a signal Δg and its error ϵ:

$$\tilde{\Delta g} = \Delta g + \epsilon$$

where Δg is the true anomaly and:

$$E\{\epsilon\} = 0 \qquad \text{and} \qquad E\{\epsilon \epsilon^T\} = D$$

$$E\{\ \} = \text{the probabilistic expectation}$$

Now it follows immediately that:

$$E\{\hat{\Delta g}_P\} = c_P (C + D)^{-1} E\{\tilde{\Delta g}\} = c_P (C + D)^{-1} \Delta g$$

Hence

$$E\{\hat{\Delta g}_P\} \neq c_P C^{-1} \Delta g = \Delta g_P$$

and we have proved that $\hat{\Delta g}_P$ is biased.

In the continuous case we may prove the bias in the following way. Let the observed anomaly field $(\widetilde{\Delta g})$ consist of the signal Δg and the noise ϵ:

$$\widetilde{\Delta g} = \Delta g + \epsilon$$

where

$$\Delta g\,(P) = \sum_{n,m} A_{nm} \left(\frac{r_B}{r}\right)^{n+2} Y_{nm}\,(P)$$

$$\epsilon\,(P) = \sum_{n,m} \epsilon_{nm} \left(\frac{r_B}{r}\right)^{n+2} Y_{nm}\,(P)$$

$$E\,\{\epsilon\} = E\,\{\epsilon_{nm}\} = 0$$

The estimator of $\Delta g(P)$ when including the noise in the observations is in accordance with (6.29):

$$\hat{\Delta g}\,(P) = \sum_{n,m} \frac{c_n\,(r_B/r)^{2(n+2)}}{(2n+1)\,d + c_n\,(r_B/r)^{2(n+2)}}\,(A_{nm} + \epsilon_{nm}) \left(\frac{r_B}{r_P}\right)^{n+2} Y_{nm}\,(P)$$

and the bias becomes

$$E\,\{\hat{\Delta g}(P)\} - \Delta g\,(P) = -\sum_{n,m} \frac{d}{d + (r_B/r)^{2(n+2)} c_n/(2n+1)}\,A_{nm} \left(\frac{r_B}{r_P}\right)^{n+2} Y_{nm}(P)$$

The bias is due to the fact that Δg itself is not a random stochastic process, only its errors, ϵ. (The expectation should not be interchanged with the global average.) A way to diminish the bias is to subtract a low degree spherical harmonic reference expansion from the observations prior to the prediction. Even though we cannot correct for all the bias, the resulting residuals are, hopefully, more random than the original observations. The general prediction formula (1.2) should therefore be modified in the following way [cf. Sjöberg, 1975, formula (16.1) and Koch, 1977, formula (18)]:

(6.31) $$v_i = \hat{v}_i + c_i\,(C + D)^{-1}\,(\Delta g - \hat{\Delta g})$$

where

$$\hat{v}_i,\ \hat{\Delta g} = \text{ low degree spherical harmonic expansions for } v_i \text{ and } \Delta g.$$

This is the well-known collocation with the inclusion of systematic terms represented by $\hat{v}_i$ and $\hat{\Delta g}$. We conclude that this formula should be used not only in local studies, but also in global applications of collocation to reduce the bias in the model.

6.4 The Limiting Case of a Least Squares Solution

Finally, we are going to study the convergence of the least squares solution of Poisson's equation for the circle. We assume that we are using the Dirac method with $m > N$. The residuals may be written:

$$\epsilon_j = \Delta g_j - \sum_{k=1}^{N} k(\theta_j, \theta_k)\, \Delta g_k^* \quad , \quad j = 1, 2, \ldots, m$$

The minimum of the square sum of the residuals is obtained for the normal equations:

$$\sum_{j=1}^{m} k(\theta_j, \theta_q)\, \Delta g_j = \sum_{j=1}^{m} \sum_{k=1}^{N} k(\theta_j, \theta_q)\, k(\theta_j, \theta_k)\, \Delta g_k^*$$

where

$$q = 1, 2, \ldots, N$$

Inserting

$$\Delta g(\theta_j) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta_j} \quad , \quad \theta_j = 2\pi j/m = 2\pi r j/N$$

and

$$\Delta g^*(\theta_k) = \sum_{\ell=-\infty}^{\infty} b_\ell e^{i\ell\theta_k} \quad , \quad \theta_k = 2\pi k/N$$

where r is an integer (m is a whole multiple of N) into the normal equations it can be shown that:

$$b_\ell = a_\ell \sum_p s^{|\ell + Np|} / N \sum_p \sum_q s^{|\ell + Np + mq| + |\ell + Np|}$$

where all summation intervals are $[-\infty, +\infty]$. Specially, for $m \to \infty$ we arrive at:

$$b_\ell \to a_\ell \frac{\sum_p s^{|\ell + Np|}}{N \sum_p s^{2|\ell + Np|}} \tag{6.27}$$

where the sums are given in a closed form in the Appendix, Corollary A.2. It follows from (6.32) that Δg^* will generally not exist when N approaches infinity, because for large N we have approximately:

$$b_\ell \approx a_\ell \, s^{-|\ell|}/N$$

The prediction of an arbitrary point $\Delta g(\theta, R)$ from Δg^* and (6.32) gives:

$$\hat{\Delta g}(\theta, R) = \sum_\ell a_\ell \, e^{i\ell\theta} \, v_{\ell,N}(\theta, R) \frac{\sum_p s^{|\ell + Np|}}{\sum_p s^{2|\ell + Np|}}$$

where

$$v_{\ell,N}(\theta, R) = \sum_p s_R^{|\ell + Np|} e^{iNp\theta}, \quad s_R = r_B/R$$

Specially for $\theta = \theta_j = 2\pi j/N$ and $R = r$ we obtain:

$$\hat{\Delta g}(\theta_j, r) = \sum_\ell a_\ell \, e^{i\ell\theta_j} \frac{\left(\sum_p s^{|\ell + Np|}\right)^2}{\sum_p s^{2|\ell + Np|}}$$

and finally

$$\lim_{N \to \infty} \hat{\Delta g}(\theta_j, r) = \sum_\ell a_\ell \, e^{i\ell\theta_j} = \Delta g(\theta_j, r)$$

Hence, the least squares prediction converges to the true value when N approaches infinity.

7. On the Choice of Radius

In the studies of the Bjerhammar problem little attention has been paid to the choice of radius of the Bjerhammar sphere (r_B). However, in Sjöberg (1975) the importance of this choice is demonstrated for different covariance functions. In this section we are going to use the following one-dimensional prediction estimate on a circle as a basis (cf. the previous section):

(7.1) $$\hat{\Delta g}(r,\theta) = \sum_{\ell=-\infty}^{\infty} a_\ell \frac{v_{\ell,m}}{u_{\ell,m}} e^{i\ell\theta}$$

where

$$v_{\ell,m} = v_{\ell,m}(s,\theta) = \sum_{j=-\infty}^{\infty} s^{|\ell+mj|} e^{imj\theta}$$

$$u_{\ell,m} = v_{\ell,m}(s,0)$$

and m is the number of observations. The corresponding prediction errors are:

(7.2a) $$\epsilon(\theta) = \hat{\Delta g}(r,\theta) - \Delta g(r,\theta) = \sum_{\ell=-\infty}^{\infty} \epsilon_\ell(\theta)$$

where

(7.2b) $$\epsilon_\ell(\theta) = a_\ell \left[\frac{v_{\ell,m}}{u_{\ell,m}} - 1\right] e^{i\ell\theta}$$

We define the mean prediction error in the following way:

(7.3) $$\bar{\epsilon} = \frac{1}{2\pi} \int_0^{2\pi} \epsilon(\theta)\, d\theta$$

First we prove the following proposition.

Proposition 7.1: If $a_0 = 0$ then

$$0 < |\bar{\epsilon}| < \left|\sum_{j=-\infty}^{\infty} a_{jm}\right|$$

and

$$\bar{\epsilon} \to 0 \quad , \quad s \to 1$$

$$\bar{\epsilon} \to \sum_{j} a_{jm} \quad , \quad s \to 0$$

where m is the number of observations.

Proof: Using

$$\frac{1}{2\pi} \int_0^{2\pi} \Delta g(\theta)\, d\theta = a_0 = 0$$

it follows from (7.1):

$$\bar{\epsilon} = \frac{1}{2\pi} \int_0^{2\pi} \Delta\hat{g}(\theta)\, d\theta = \sum_{\ell} \frac{a_\ell}{u_{\ell,m}} \sum_{j} s^{|\ell + mj|} \frac{1}{2\pi} \int_0^{2\pi} e^{i(\ell + mj)\theta}\, d\theta =$$

$$\sum_{j} \frac{a_{jm}}{u_{jm,m}} = \frac{1-s^m}{1+s^m} \sum_{j} a_{jm}$$

Here we have used Corollary A.2 with $\ell = jm$:

$$u_{jm,m} = \frac{1+s^m}{1-s^m}$$

From this expression, the proposition readily follows.

Although $s \to 1$ gives the mean prediction error 0 and has also proved to give the most stable solutions (section 3), the "best" choice of radius r_B must be determined in some mean square sense. Let us therefore study the prediction of the signal:

(7.4)
$$\Delta g_\ell(\theta) = b \cos(\ell\theta) + c \sin(\ell\theta)$$

at the circle of radius r. We define the relative variance in the following way:

(7.5)
$$R_\ell = \int_0^{2\pi} \epsilon^2(\theta)\, d\theta \Big/ \int_0^{2\pi} \Delta g_\ell^2\, d\theta$$

where

$$\epsilon(\theta) = \hat{\Delta g_\ell}(\theta) - \Delta g_\ell(\theta) \quad \text{(prediction error)}$$

It is shown in the Appendix, Proposition A.4, that (7.5) may be written (for $\ell > 0$):

(7.6)
$$R_\ell \begin{cases} S_\ell & \text{if } \ell \neq \frac{m}{2}p, \quad p = 1, 2, 3, \ldots \\ \frac{1+2\beta}{2(1+\beta)} + \frac{1}{2(1+\beta)} S_\ell & \text{if } \ell = \frac{m}{2} p \end{cases}$$

where

(7.6´)
$$S_\ell = 1 + \frac{1-s^m}{1+s^m} \frac{1+s^{2m-4n}}{(1+s^{m-2n})^2} - \frac{1-s^m}{1+s^{m-2n}} 2 s^{\ell-n}$$

$$n = \ell - \left[\frac{\ell}{m}\right] m$$

$$\beta = (c/b)^2$$

From (7.6) and (7.6´) we draw the following conclusions:

a) $\lim R_\ell = 0$, $m \to \infty$
b) $\lim S_\ell = 1$, $s \to 1$
c) if $\ell < m/2$ then $\lim R_\ell = 0$, $s \to 0$
d) if $\ell = m/2$ then $S_\ell > 1/2$ and $\lim S_\ell = 1/2$, $s \to 0$
e) if $\ell > m/2$ then $S_\ell > 1/2$
f) if $\ell = mp$ then $S_\ell < 2$ and $\lim S_\ell = 2$, $s \to 0$
g) if $\ell = m(p + 1/2)$ then $S_\ell < 3/2$ and $\lim S_\ell = 3/2$, $s \to 0$
h) if $\ell > m/2$ and $\ell \neq mp/2$ then $\lim S_\ell = \infty$, $s \to 0$

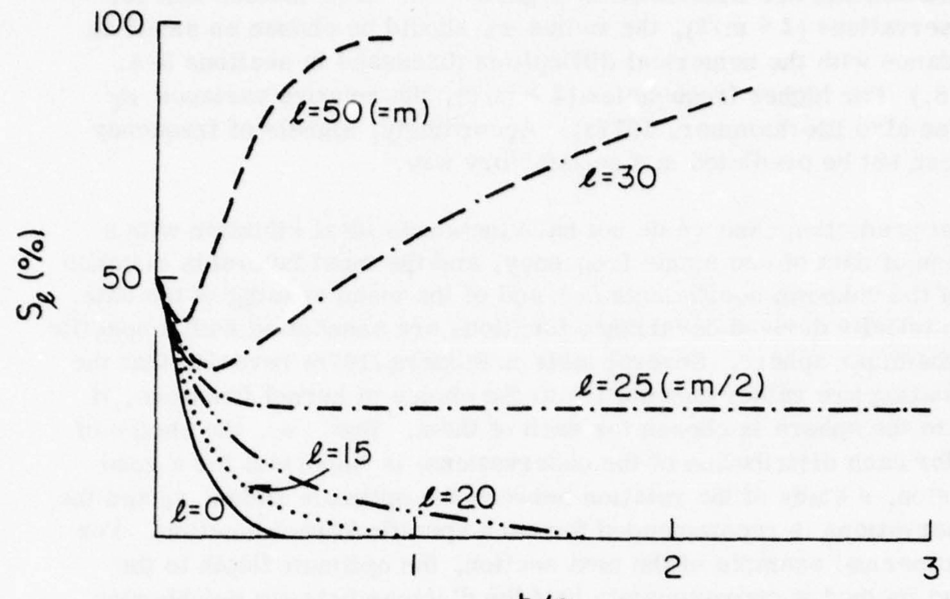


Figure 4. The relative variance S_1 in the Dirac method for the circle, using Poisson's kernel, is given for signals of various frequencies (1) as a function of h/a, where h= distance from the Bjerhammar circle to the circle of observations and a= spacing of the data ($2\pi r/m$). m=50.

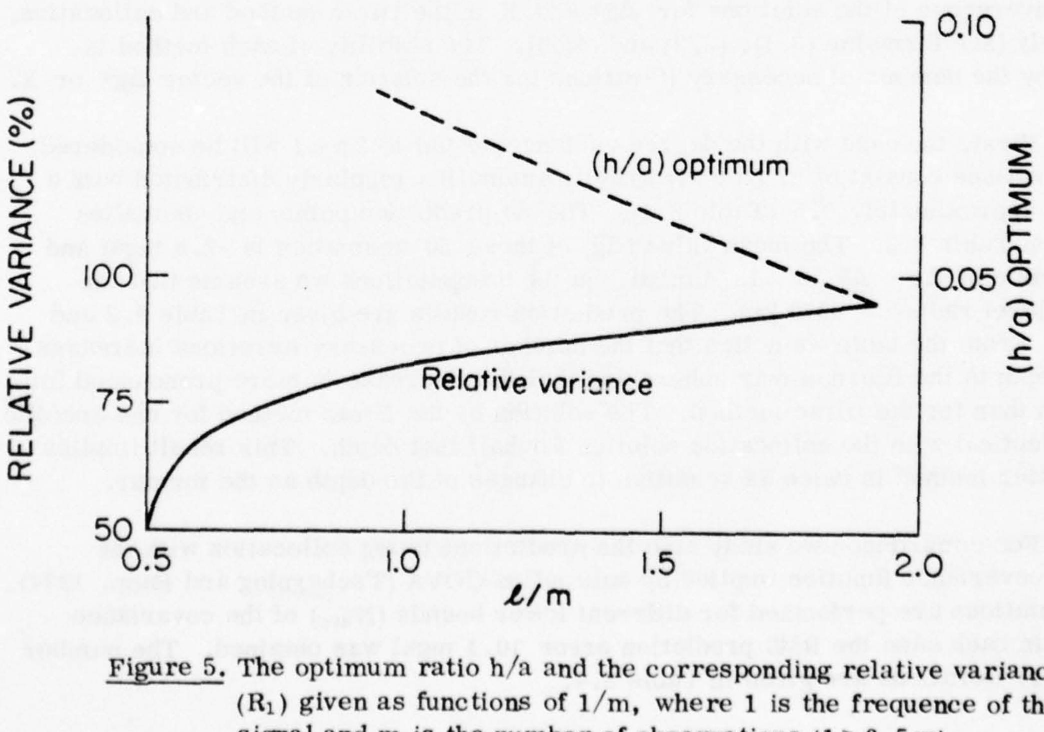


Figure 5. The optimum ratio h/a and the corresponding relative variance (R_1) given as functions of 1/m, where 1 is the frequence of the signal and m is the number of observations ($1 \geq 0.5$ m).

These statements are illustrated in Figures 4-5. It is obvious that for low-frequency observations ($\ell \leq m/2$), the radius r_B should be chosen as small as possible in accordance with the numerical difficulties discussed in sections 3-4. (See also section 8.) For higher frequencies ($\ell > m/2$), the relative variance R_ℓ is at least 50% (see also Bjerhammar, 1977a). Accordingly, signals of frequency higher than m/2 can not be predicted in a satisfactory way.

In a real prediction case we do not have the above ideal situation with a uniform distribution of data of one single frequency, and the most favorable distance r_B is a function of the unknown coefficients (a_ℓ) and of the mean spacing of the data. In most cases empirically derived covariance functions are associated with a specific radius of the Bjerhammar sphere. Several tests in Sjöberg (1975) revealed that the solutions by collocation are rather unsensitive to the choice of kernel functions, if the optimal depth to the sphere is chosen for each of them. However, the choice of radius (changing for each distribution of the observations) is important for a good result. In conclusion, a study of the relation between the optimum radius r_B and the spacing of the observations is recommended for each specific kernel function. For instance, in the numerical example of the next section, the optimum depth to the sphere in the Dirac method is approximately half the distance between neighboring observations.

8. Computations

The iterative method described in section 4 will be used to demonstrate the rate of convergence of the solutions for Δg^* and X in the Dirac method and collocation, respectively [see formulae (3.1), (3.2) and (4.2)]. The stability of each method is reflected by the number of necessary iterations for the solution of the vector Δg^* or X.

First, the case with the degree variances equal to $2n+1$ will be considered. The observations consist of 87 free air gravity anomalies regularly distributed with a spacing of approximately $0\overset{\circ}{.}5$ (Table 8.1). The 50 prediction points and anomalies are given in Table 8.2. The mean value ($\overline{\Delta g}$) of these 50 anomalies is -2.8 mgal and the RMS value of $\Delta g - \overline{\Delta g}$ is ± 13.4 mgal. In the computations we assume that the mean sea level radius is 6370 km. The prediction results are given in Table 8.3 and Figure 6. From the table we notice that the number of necessary iterations increases with the depth to the Bjerhammar sphere and that this increase is more pronounced for collocation than for the Dirac method. The solution by the Dirac method for one specific depth is identical with the collocation solution for half that depth. This result implies that the latter method is twice as sensitive to changes of the depth as the former.

For comparison we study also the predictions using collocation with the empirical covariance function implied by subroutine COVA (Tscherning and Rapp, 1974). The computations are performed for different lower bounds (N_{min}) of the covariance function. In each case the RMS prediction error 10.1 mgal was obtained. The number of necessary iterations are given in Table 8.4.

Table 8.1

Observation points. 87 free-air gravity anomaly stations in Manitoba, Canada.

NO	LONGITUDE DEG	LATITUDE DEG	ALTITUDE METER	ANOMALY MGAL
13676	48.10878	90.18959	501.396	8.12
13675	48.10654	90.39793	465.429	5.72
13668	48.12804	91.19112	421.843	4.32
13661	48.26439	91.47469	431.901	-11.41
15691	48.23920	91.95221	372.161	-16.01
15793	48.52264	90.11258	430.377	-26.16
13650	48.52742	90.55254	470.916	-24.76
13655	48.54848	90.98921	450.799	-8.36
13659	48.47208	91.45769	374.904	-21.45
15688	48.46989	92.10847	362.407	-18.25
10261	48.56660	92.46326	361.797	-18.07
10252	48.60010	92.81689	337.718	-18.97
10248	48.55446	93.62741	346.862	3.74
10245	48.72795	93.91647	352.654	-0.20
13543	49.03152	90.02982	473.049	-28.75
15754	49.04839	90.47089	492.252	-11.56
15745	49.02661	91.06570	441.960	-8.25
15702	49.07143	91.61995	439.217	-24.96
13575	49.14218	91.92789	421.843	-19.37
13610	49.50134	90.03465	449.885	5.06
13606	49.59610	90.58807	446.837	-15.16
13604	49.61888	90.92754	437.998	-12.16
15729	49.53860	91.64418	407.213	-17.95
196	49.49666	92.03333	436.778	1.16
15717	49.92172	90.08134	468.173	11.08
15713	49.96526	90.56398	415.137	0.08
15380	49.96666	90.91167	410.261	14.58
10267	49.01868	92.57602	385.267	-17.55
11423	49.02036	92.98222	408.432	21.05
10013	49.53339	92.40990	383.743	6.16
254	49.95332	91.51166	400.812	-22.72
13565	49.92413	91.95653	358.445	-5.62
11409	49.07217	93.54082	352.349	9.74
1554	49.02666	93.91666	356.921	-13.75
12014	49.47910	93.06764	363.626	-32.63
10100	49.52834	93.51404	359.664	-14.54
10111	49.99075	93.49245	348.082	4.17
209	49.03166	94.51999	323.698	-9.55
10157	49.03590	95.21919	323.698	4.95
10165	49.17368	95.47781	344.119	1.62
10161	49.07393	95.96857	330.098	15.24
10227	49.50439	94.00826	344.119	-15.24
10219	49.45992	94.53586	323.088	-13.23
10173	49.48048	95.01280	323.088	-4.64
10170	49.45171	95.51802	343.814	-10.53
10188	49.48811	95.99078	318.211	-4.44
4507	49.05833	96.53999	299.923	4.74
1130	49.57166	96.57333	270.053	1.35
4616	49.58833	96.98000	241.402	0.14
10040	50.04314	92.56009	400.812	-11.24
10064	50.01875	92.93719	395.935	-5.13
10125	50.01784	94.12364	405.079	11.47
10147	50.07710	94.45625	359.054	-1.44
11465	50.29559	95.05542	323.698	9.21
12043	50.05191	95.55070	309.067	11.66
4513	50.00499	96.02499	271.881	2.57
4544	50.04498	96.59000	249.631	-7.54
4552	50.04666	96.51999	248.412	-7.04
5743	50.03166	97.04166	228.295	18.06
16028	50.50333	90.14667	387.401	-15.52
16025	50.57166	90.46832	405.384	1.26
15335	50.46999	91.15166	392.887	-4.72
16048	50.55499	91.42500	396.849	14.87
15344	50.52834	92.05000	356.616	12.27
10050	50.49197	92.57567	357.530	-6.72
10059	50.46123	92.90984	357.835	7.38
10135	50.52052	93.48242	340.461	9.77
12177	50.49446	94.12927	369.417	21.78
12189	50.50020	94.42551	340.157	16.58
12185	50.49776	95.11777	361.188	16.48
11471	50.51193	95.95348	252.679	-11.92
10435	50.58276	95.50301	313.334	7.06
9541	50.53166	96.53833	264.566	1.97
16021	51.02333	90.13666	379.781	-5.32
16016	51.12833	90.45332	373.990	-23.74
15989	51.42500	90.42500	360.273	-33.89
16019	51.42999	90.03166	387.401	-15.39
15323	51.89499	90.14999	350.825	1.52
15310	51.89999	90.93999	365.760	-0.58
16004	51.00833	91.07832	373.990	5.38
16044	51.05000	91.52998	376.123	-22.12
15348	51.06833	92.16666	410.565	-18.53
636	51.06999	92.57166	392.887	3.17
10473	51.16444	92.88702	413.004	-19.14
15986	51.53999	91.08665	378.257	-12.81
15983	51.54832	91.52998	379.781	-11.01
10463	51.56641	92.07666	399.288	-16.12

Table 8.2

Prediction points. 50 free air gravity anomaly stations in Manitoba, Canada.

NO	LONGITUDE DEG	LATITUDE DEG	ALTITUDE METER	ANOMALY MGAL
16009	50.59499	91.14833	386.181	[illegible].76
16011	50.74666	90.96165	379.476	[illegible].63
10499	50.79678	92.36719	425.501	8.82
16042	50.95000	91.75833	402.031	-9.20
16012	50.88333	91.01666	378.257	1.71
10052	50.59193	92.61716	428.854	12.66
10047	50.22284	92.84720	358.445	-3.57
10038	50.21497	92.37631	357.835	-3.37
15372	50.49666	91.87166	357.225	12.68
273	50.40833	91.50833	370.027	3.99
15370	50.75499	91.88333	391.973	7.53
16047	50.68333	91.41998	382.219	13.54
10496	50.80524	92.14000	396.545	-2.18
10081	50.21280	93.23965	364.845	7.53
625	50.30666	93.17999	361.493	4.41
16002	51.12833	90.86833	380.390	-21.34
16050	50.29498	91.38998	373.685	-14.68
15331	50.74500	90.75665	390.449	7.83
15334	50.42332	90.71666	391.058	2.09
16056	50.18333	90.68832	404.774	-15.16
15384	50.12999	90.23999	417.576	-3.35
9164	48.91499	90.60001	457.809	-11.23
13576	49.02196	91.96181	425.501	-21.95
15715	49.84846	90.40294	442.265	-2.80
15723	49.74696	90.75410	435.559	-5.18
15675	49.86766	92.09277	382.524	-23.51
10019	49.84630	92.39220	366.674	-18.90
15378	49.92999	91.38333	388.925	-29.92
10203	49.76701	94.87744	359.969	2.11
11455	49.62477	94.02734	373.990	-10.26
10222	49.59979	94.35619	323.088	5.44
5710	49.43166	96.27499	357.225	5.27
5546	49.72333	95.24666	338.328	16.62
10198	49.62842	95.49942	318.211	-1.36
5531	49.71666	94.93666	359.664	10.92
5076	49.71500	94.80666	345.643	14.12
10078	49.90807	93.14622	372.770	-8.51
10104	49.68629	93.87337	379.171	-18.87
10098	49.30528	93.51170	338.023	6.60
5084	49.81332	92.97501	355.092	-10.10
10014	49.62167	92.44067	385.572	18.04
12002	49.49670	92.69481	403.555	11.76
10007	49.22151	92.46306	405.384	-23.49
213	49.14166	92.70332	388.620	-22.67
15725	49.70807	91.09705	435.559	-8.48
15736	49.66466	91.81667	419.405	-24.77
15738	49.24759	91.46384	445.617	-13.99
10249	48.66917	93.27843	337.718	-6.09
13652	48.56082	90.73390	447.751	-23.06
15748	48.83151	90.96706	445.922	-18.72

Figure 6. RMS prediction errors compared for various depths to the Bjerhammar sphere. $c_n^* = 2n + 1$, no. of observations = 87, no. of predictions = 50.

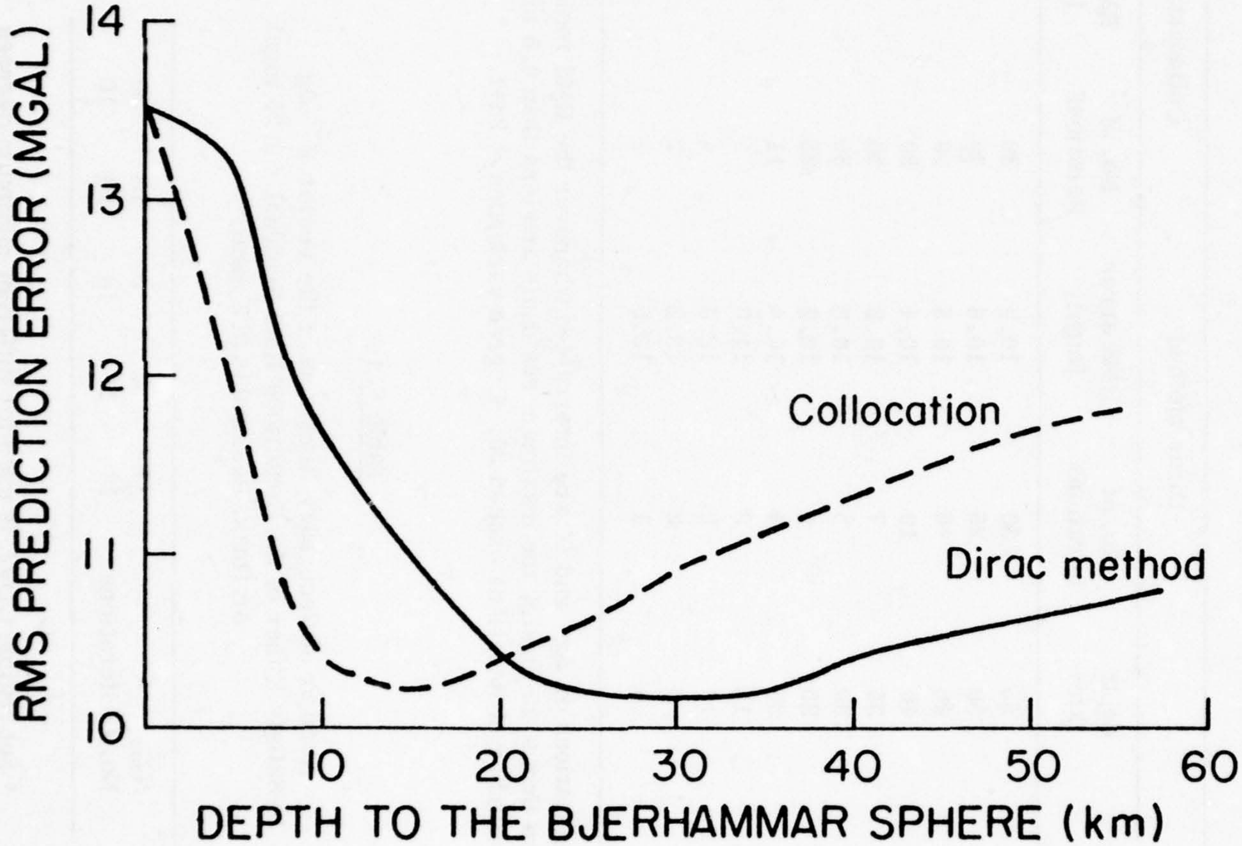


Table 8.3

Comparison Between the Predictions of 50 Free Air Gravity Anomalies From 87 Observed Anomalies Using the Dirac Method and Collocation*

		Dirac Method		Collocation	
r_s (km)	depth (km)	No. of iterations	RMS error [mgal]	No. of iterations	RMS error [mgal]
6315	55	30	10.7	30	11.8
6320	50	25	10.6	30	11.7
6325	45	16	10.5	30	11.5
6330	40	10	10.4	30	11.3
6335	35	7	10.2	30	11.1
6340	30	5	10.2	30	10.9
6345	25	4	10.2	25	10.6
6350	20	4	10.4	11	10.4
6355	15	3	11.0	5	10.2
6360	10	2	12.0	4	10.4
6365	5	2	13.2	2	11.9
6370	0	1	13.5	1	13.5

* The iterations of Δg^* and X are interrupted whenever the RMS residuals are less than 0.25 mgal., the maximum-residuals are less than 0.5 mgal. or the number of iterations exceed 30. Degree variances = 2n+1.

Table 8.4

Number of Necessary Iterations of the Vector $C^{-1}\,\Delta g$ to Satisfy Either of the Conditions RMS Residual < 0.25 mgal or $|$max. Residual$|$ < 0.5 mgal.*

N_{min}	3	5	3	13	20
No. of iterations	18	16	14	13	10

* Subroutine COVA is used for different minimum degrees (N_{min})

The empicical covariance function in collocation gives a slightly better prediction result than the Dirac method for its optimum radius of r_B. This gain is obtained at the cost of at least twice as many iterations. For a denser spacing of the data, the number of necessary iterations will increase and the difference between the two methods becomes more pronounced. With this reasoning, the non-iterative solution may fail, when using collocation (due to numerical singularity), while a solution by the Dirac method may still be useful. For example, the prediction result reported in Table 8.5 shows small differences between collocation and the Dirac method (93 well distributed observations). However, when a few more observation points were included (99 points) between the previous ones (with minimum spacing 675 meters), the RMS prediction error of ξ increased to 11."4 for the collocation type of solution, while the RMS error of the Dirac method was still useful (1."3). (From Sjöberg, 1975.)

Table 8.5

RMS Prediction Errors for Molodenskii's Model

No. of obs.	Collocation ζ /m/	Δg /mgal/	ξ''	Dirac Method ζ /m/	Δg /mgal/	ξ''
93	0.01	1.4	0.59	0.12	1.8	0.90
99			11.4			1.3

* Depth to the Bjerhammar Sphere = 10m. No. of prediction points = 37. $c_n^* = 2n+1$. Reference: Sjöberg (1975, section 19.2.3).

9. Conclusions

The purpose of this report has been to compare some methods of A. Bjerhammar with collocation for the solution of the boundary value problem in physical geodesy. In practice, the number of observations are finite ("discrete boundary value problem"). In the present application collocation is most frequently identified as Wiener-Hopf prediction of stochastic processes, in which case the covariance functions are assumed to be known (homogeneous and isotropic). The main problem is therefore to find the appropriate covariance functions. Rigorously this has been proved to be an impossible task, because of the non-ergodicity of the empirical covariance functions (Lauritzen, 1973).

Bjerhammar's methods are based on Poisson's integral equation and Stokes' formula. For a finite set of observations there is always a fictitious field (Δg^*) at an internal sphere that satisfies the integral equation. In Bjerhammar's applications Δg has originally been considered as mean anomalies over blocks of certain sizes at the internal sphere (the Bjerhammar sphere). Dependent on the number of such blocks, different types of solutions of the problem are obtained. If the number of blocks (N) are less than the number of observations (m), a unique solution is given from adjustment by elements. For $N > m$ a unique solution is given by condition adjustment, which solution minimizes the norm of the unknowns (Δg^*). In the special case $N \to \infty$ and well-behaving surface elements we have shown that the minimum norm solution of Bjerhammar ($\| \Delta g^* \| =$ minimum) approaches the solution by collocation with the degree variances equal to 2n+1. Furthermore, this proof has been generalized to yield, that for each set c_n in collocation, there is a corresponding minimum norm $\| u^* \|$ in the generalized Bjerhammar approach (section 2.2). Thus the problem of selecting the degree variances in collocation is identical with the problem of selecting the minimum norm in the Bjerhammar method. It should be stated that already Krarup (1969) regarded collocation as a generalized approximation for a specified minimum norm.

A different type of solution, called reflexive prediction, was introduced by Bjerhammar (1974). In this method the external gravity field is assumed to be generated at a priori selected fictitious carrier points, on or outside the Bjerhammar sphere. Once the carrier points and the sphere are defined, Poisson's integral equation can be applied rigorously, and the result is a set of linear equations. If all carrier points are located at the internal sphere, the method is called the Dirac approach. Due to the arbitrary location and number of carrier points, a wide variety of solutions are possible. Of special interest are filtering (less number of carrier points than observations) and the non-singular Dirac method (see below).

In this report we have, first of all, compared collocation and the non-singular Dirac approach with carrier points located at the intersections of the internal sphere with the radius vectors of the observations. The coefficient matrix of the latter method is non-symmetric in contrast to the former. Moreover it was shown in section 2 that for a constant geocentric radius of all observations in an area (and $c_n = 2n + 1$) the two methods give identical prediction results for:

$$h_0 = 2\, h_B \tag{9.1}$$

where h_0 and h_B are the depths to the Bjerhammar sphere in the Dirac method and collocation, respectively. This result has been verified to hold also for approximately constant geocentric radius of the observations (section 8).

It was demonstrated in sections 3 and 4 that the numerical stability of the solutions differs for the two methods (collocation and Dirac). For a given radius of the Bjerhammar sphere the Dirac method is more stable. In the numerical example of section 8 it was found that for Poisson's kernel ($c_n^* = 2n+1$), the conditionings of the two methods are equal provided that formula (9.1) is satisfied. For this kernel function the solution by collocation is twice as sensitive to the choice of Bjerhammar sphere as the solution by the impulse method, a fact that is important for the solution of a large system with a dense distribution of the observations.

In the continuous case (with observations covering all the surface of the earth) we have found (section 6) that all predictions with the generalized Dirac method and collocation are unique for various sets of positive c_n^* (c_n), whenever the solutions exist. This result may be regarded as a consequence of Stokes' theorem. In general, however, the intermediate solutions u^* and X do not exist in the continuous case. The existence of the solutions requires that the degree variances of the observations are at most of magnitude n^{-5}, a condition which is not satisfied for the gravity field of the earth according to Kaula's rule. Approximate solutions may be found, for example, by adding a positive constant to the kernel function. In the same way it was found in section 6.3 that a solution may exist when considering that the observations are erroneous. However, these solutions are statistically biased. If collocation is carried out by solving the Wiener-Hopf integral equation, a convergent solution is obtained outside a sphere (all observations on the sphere). However, inside the bounding sphere of the real earth, the convergence is still not proved.

It should be noted that the deterministic approaches by Bjerhammar through Poisson's and Stokes' formulae do not provide estimated prediction errors, as is the case in the stochastic process approach (i.e. collocation according to Moritz). However, as the uncertainty of the covariance functions has a direct impact on the error estimates, these estimates might be of limited value.

For large systems the free choice of carrier points in reflexive prediction might be advantageous (filtering is possible). However, experiences by Bjerhammar (1977b) indicate that numerical difficulties may occur in the filtering process, due to ill-conditioning.

Finally, we like to mention that the original approaches of Bjerhammar are designed to solve problems in physical geodesy, first of all the geodetic boundary problem, for gravity anomalies as the only source of data. However, it is not difficult to modify the methods in order to take other types of geophysical information into account. Thus, both collocation and Bjerhammar's methods possess a flexibility for the processing of heterogeneous data.

10. Extensions and Recommendations

This study has shown that reflexive prediction (the Dirac method) is less sensitive to changes of the radius of the internal sphere than collocation for the particular kernel and covariance functions with $c_n^* = c_n = 2n+1$. The prediction results and the stability of the two methods are the same, if collocation is applied with half the depth to the Bjerhammar sphere used in reflexive prediction. We recommend these comparisons to be carried out for other covariance functions. Of special interest would be to compare an empirical covariance function with a best fitting kernel function (cf. Sjöberg, 1975, section 18).

Special attention should be paid to the relation between the distribution of the observations and the optimum radius of the Bjerhammar sphere for various covariance (kernel) functions.

A procedure to estimate the prediction errors in reflexive prediction is of interest for the user of the method. In order to reduce the bias of the predictions, we recommend the use of formula (6.31) in all applications of collocation to the geodetic boundary problem and related problems in physical geodesy. A corresponding formula should be used in reflexive prediction, if the kernel function is modified to take noise into account.

Theoretically, it is of interest to reveal whether the Wiener-Hopf type of predictions [formulae (6.24´), (6.25´), (6.26 a-b)] converges when applied to a continuous field of observations at the surface of the earth.

References

Bjerhammar, A., A New Theory of Gravimetric Geodesy, Royal Institute of Technology, Division of Geodesy, Stockholm, 1963.

Bjerhammar, A., A New Theory of Geodetic Gravity, Royal Institute of Technology, Division of Geodesy, Stockholm, 1964.

Bjerhammar, A., On Gravity, Royal Institute of Technology, Division of Geodesy, Stockholm, 1968.

Bjerhammar, A., On the Boundary Value Problem of Physical Geodesy, Royal Institute of Technology, Division of Geodesy, Stockholm, Tellus XII, No. 4, 1969.

Bjerhammar, A., A New Theory of Errors and Generalized Matrix Inverses, Elsevier, Amsterdam, 1973.

Bjerhammar, A., Discrete Approaches to the Solution of the Boundary Value Problem in Physical Geodesy, International School of Geodesy, Erice, Italy, 1974.

Bjerhammar, A., A Review of Discrete Methods in Physical Geodesy, paper presented at the summer school, Ramsau, Austria, 1977a.

Bjerhammar, A., Collocation and Reflexive Prediction, Royal Institute of Technology, Division of Geodesy, Stockholm, 1977b.

Chambers, Ll. G., Integral Equations: A Short Course, International Textbook Co., Ltd., London, 1976.

Koch, K-R, Numerical Examples for Downward Continuation of Gravity Anomalies, Department of Geodetic Science Report No. 112, The Ohio State University, Columbus, 1968.

Koch, K-R, Least Squares Adjustment and Collocation, Bulletin Geodesique, Vol. 51, No. 2, 1977.

Krarup, T., A Contribution to the Mathematical Foundation of Physical Geodesy, Meddelelse No. 44, Geodetisk Institut, Copenhagen, 1969.

Lauritzen, S. L., The Probabilistic Background of Some Statistical Methods in Physical Geodesy, Meddelelse No. 48, Geodetisk Institut, Copenhagen, 1973.

Pick, M., On the Shape of the Earth by Using Analytical Continuation of Function, Studia Geophysica et Geodetica, Vol. 9, 1965.

Molodensky, M. S., V. F. Eremeev and M. I. Yurkina, Methods for the Study of the External Gravitational Field and Figure of the Earth, Trans. from Russian by Israel Program for Scientific Translations, Jerusalem, 1962.

Moritz, H., Methods for Downward Continuation of Gravity, Department of Geodetic Science Report No. 67, The Ohio State University, Columbus, 1966.

Moritz, H., Least-Squares Estimation in Physical Geodesy, Department of Geodetic Science Report No. 130, The Ohio State University, Columbus, 1970.

Moritz, H., Advanced Least-Squares Methods, Department of G odetic Science Report No. 175, The Ohio State University, Colum us, 1972.

Moritz, H., Integral Formulas and Collocation, Department of Geodetic Science Report No. 234, The Ohio State University, Columbus, 1975.

Moritz, H., Covariance Functions in Least-Squares Collocation, Department of Geodetic Science Report No. 240, The Ohio State University, Columbus, 1976.

Schwarz, K.-P., Numerische Untersuchung zur Schwerefortsetzung, Deutsche Geodätische Kommission, Series C, No. 171, 1971.

Sjöberg, L., On the Discrete Boundary Value Problem of Physical Geodesy with Harmonic Reduction to an Internal Sphere, Royal Institute of Technology, Division of Geodesy, Stockholm, 1975.

Seeley, J. R., An Introduction to Fourier Series and Integrals, W. A. Benjamin, Inc., New York, 1966.

Tscherning, C. C. and R. H. Rapp, Closed Covariance Expressions for Gravity Anomalies, Geoid Undulations and Deflections of the Vertical Implied by Anomaly Degree Variance Models, Department of Geodetic Science Report No. 208, The Ohio State University, Columbus, 1974.

Tscherning, C. C., A Note on the Choice of Norm When Using Collocation for the Computation of Approximations to the Anomalous Potential, Bulletin Geodesique, Vol. 51, No. 2, 1977a.

Tscherning, C. C., On the Convergence of Least Squares Collocation, paper submitted to Bulletino di Geodesia e Science Affini, 1977b.

Appendix

The Addition Theorem for Spherical Harmonics:

(A.1)
$$P_n(\cos\psi_{ik}) = \frac{1}{2n+1}\sum_{m=-n}^{n} Y_{nm}(\theta_i,\lambda_i)\,Y_{nm}(\theta_k,\lambda_k)$$

where

(A.1´)
$$Y_{nm}(\theta,\lambda) = \overline{P}_{n|m|}(\cos\theta)\begin{cases}\cos m\lambda & m \geq 0\\ \sin|m|\lambda & m < 0\end{cases}$$

(θ,λ) = spherical coordinates (colatitude and longitude)

(A.2)
$$\frac{1}{4\pi}\iint Y_{nm}\,Y_{n'm'}\,d\sigma = \delta_{nn'}\,\delta_{mm'}$$

σ = the unit sphere
δ = Kronecker's delta

Corollary A.1:

(A.3)
$$\frac{1}{4\pi}\iint P_n(\cos\psi_{ik})\,P_{n'}(\cos\psi_{jk})\,d\sigma_k = \frac{1}{2n+1}\,P_n(\cos\psi_{ij}) \times \delta_{nn'}$$

The corollary follows from (A.1) and (A.2).

Proposition A.1:

If A is a matrix of dimensions (m x N), where

$$(A)_{jk} = \frac{1}{4\pi}\sum_{n=0}^{\infty}\sqrt{(2n+1)}\,\sqrt{c_n}\left(\frac{r_B}{r_j}\right)^{n+2} P_n(\cos\psi_{jk})\,\Delta\sigma_k$$

and

$$Q = 4\pi \begin{pmatrix} \Delta\sigma_1 & & & \\ & \Delta\sigma_2 & & \\ & & \ddots & \\ & & & \Delta\sigma_N \end{pmatrix}^{-1}$$

and

$$\sum_{k=1}^{N} \Delta\sigma_N = 4\pi$$

then

$$\lim_{\substack{N\to\infty \\ \max_k \Delta\sigma_k \to 0}} (A Q A^T)_{ij} = \sum_{n=0}^{\infty} c_n \left(\frac{r_B^2}{r_i r_j}\right)^{n+2} P_n(\cos\psi_{ij})$$

Proof:

$$(A Q A^T)_{ij} = 4\pi \sum_{k=1}^{N} (A)_{ik} (A)_{jk} \Delta\sigma_k^{-1} =$$

$$= \frac{1}{4\pi} \sum_{k=1}^{N} \sum_{n=0}^{\infty} \sqrt{(2n+1)} \sqrt{c_n} \left(\frac{r_B}{r_i}\right)^{n+2} P_n(\cos\psi_{ik}) \sum_{n'=0}^{\infty} \sqrt{(2n'+1)} \sqrt{c_{n'}} \left(\frac{r_B}{r_j}\right)^{n'+2} P_{n'}(\cos\psi_{jk}) \Delta\sigma_k$$

For $N \to \infty$ in such a way that max $\Delta\sigma_k \to 0$ the summation over k becomes the corresponding integral, so that:

$$\lim_{N\to\infty} (A Q A^T)_{ij} = \frac{1}{4\pi} \sum_{n} \sum_{n'} \sqrt{(2n+1)(2n'+1)} \sqrt{c_n c_{n'}} \left(\frac{r_B}{r_i}\right)^{n+2} \left(\frac{r_B}{r_j}\right)^{n'+2} \times$$

$$\times \iint P_n(\cos\psi_{ik}) P_{n'}(\cos\psi_{jk}) \, d\sigma_k$$

Corollary A.1 finally yields:

$$\lim_{N\to\infty} (AQA^T)_{ij} = \sum_{n=0}^{\infty} c_n \left(\frac{r_B^2}{r_i r_j}\right)^{n+2} P_n(\cos\psi_{ij})$$

Proposition A.2

(A.4) $$\sum_{j=-\infty}^{\infty} s^{|m+Nj|} e^{i(m+Nj)\theta} = (se^{i\varphi})^{|m|-\alpha N} \left[\frac{1}{1-s^N e^{iN\varphi}} + \frac{s^{N-2|m|+2N\alpha} e^{-iN\varphi}}{1-s^N e^{-iN\varphi}}\right]$$

for $0 \leq s < 1$ and $\alpha = \left[\frac{|m|}{N}\right]$ = integer part of $\frac{|m|}{N}$, and $\varphi = \begin{cases} \theta & \text{if } m \geq 0 \\ -\theta & \text{if } m < 0 \end{cases}$

Proof:

Let us substitute j of (A.4) by $k - \alpha$. Then we obtain in the left member:

$$\sum_{k=-\infty}^{\infty} s^{|m+Nk-N\alpha|} e^{i(m+Nk-N\alpha)\theta} = S_1 + S_2$$

where

$$S_1 = (se^{i\varphi})^{|m|-\alpha N} \sum_{k=0}^{\infty} (se^{i\varphi})^{Nk} = (se^{i\varphi})^{|m|-\alpha N} / [1-(se^{i\varphi})^N]$$

$$S_2 = (se^{-i\varphi})^{N\alpha-|m|} \sum_{k=1}^{\infty} (se^{-i\varphi})^{Nk} = (se^{-i\varphi})^{N\alpha-|m|} / [1-(se^{-i\varphi})^N]$$

Corollary A.2:

(A.5) $$\sum_{j=-\infty}^{\infty} s^{|m+Nj|} = (s^{|m|-\alpha N}) / (1-s^N)$$

The proof follows directly from the proposition for $\theta = 0$.

Corollary A.3:

(A.6)
$$\lim_{N\to\infty} \sum_{j=-\infty}^{\infty} s^{|m+Nj|} e^{i(m+Nj)\varphi} = s^{|m|} e^{im\varphi}$$

The proof follows directly from (A.4) when considering that $\alpha \to 0$, $N \to \infty$.

Proposition A.3:

If

$$A_{jk} = \frac{1}{N} \sum_{n=-\infty}^{\infty} s^{|n|} e^{in(\theta_j - \theta_k)}$$

$$\theta_k = 2\pi k/N$$
$$\theta_j = 2\pi j/m$$
$$N = pm \qquad p = \text{integer}$$

and

$$\theta_q = 0$$

then

$$S(m,N,s) = N \sum_{j=1}^{m} \sum_{k=1}^{N} A_{qk} A_{jk} - 2N \sum_{k=1}^{N} A_{qk}^2$$

equals

$$S(m,N,s) = \Big[(2-m)(1+s^N) + 2m \frac{1-s^{2N-2m}}{(1-s^{2m})(1-s^{2N-m})} - 4 \frac{1-s^{2N-2}}{(1-s^2)(1-s^{2N-1})} +$$

$$+ \frac{2Ns^N}{1-s^m} - \frac{4Ns^N}{1-s}\Big] / (1-s^N)$$

Proof:

For $\theta_q = 0$ we have:

(A.7) $$B_j = N^2 \sum_{k=1}^{N} A_{qk} A_{jk} = \sum_{n=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} s^{|n|+|\ell|} e^{i\ell\theta_j} \sum_{k=1}^{N} e^{-i(n+\ell)\theta_k} =$$

$$= N \sum_{n=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} s^{|n|+|n+Nr|} e^{i(n+Nr)\theta_j} = N \sum_{n=-\infty}^{\infty} s^{|n|} e^{in\theta_j} \sum_{r=-\infty}^{\infty} s^{|n+Nr|}$$

and

(A.8) $$\sum_{j=1}^{m} B_j = Nm \sum_{t=-\infty}^{\infty} s^{m|t|} \sum_{r=-\infty}^{\infty} s^{m|t+pr|}$$

Inserting (A.5) into (A.7) for $\theta_j = 0$, we obtain:

$$N \sum_{k=1}^{N} A_{1k}^2 = \sum_{n=-\infty}^{\infty} s^{|n|} \sum_{r=-\infty}^{\infty} s^{|n+Nr|} = \frac{1}{1-s^N} \sum_{-\infty}^{\infty} (s^{2|n|-\alpha N} + s^{N+\alpha N}) =$$

$$= \frac{1}{1-s^N} \left[-1 - s^N + 2 \sum_{k=0}^{\infty} \sum_{n=Nk}^{Nk+N-1} (s^{2n-k} + s^{N+k}) \right] =$$

$$= \frac{1}{1-s^N} \left[-1 - s^N + 2 \sum_{k=0}^{\infty} \frac{s^{(2N-1)k} - s^{(2N-1)k+2N-2}}{1-s^2} + 2N \sum_{k=0}^{\infty} s^{N+k} \right] =$$

$$= \frac{1}{1-s^N} \left[-1 - s^N + \frac{2(1-s^{2N-2})}{(1-s^2)(1-s^{2N-1})} + \frac{2Ns^N}{1-s} \right]$$

In the same way we obtain from (A.8):

$$\frac{1}{N} \sum_{j=1}^{m} B_j = \frac{m}{1-s^N} \left[-1 - s^N + 2 \frac{1-s^{2N-2m}}{(1-s^{2m})(1-s^{2N-m})} + \frac{2ps^N}{1-s^m} \right]$$

so that

$$S(m,N,s) = \frac{1}{N}\sum_{j=1}^{m} B_j - 2N\sum_{k=1}^{N} A_{1k}^2$$

$$= \left[-m - ms^N + 2m\,\frac{1-s^{2N-2m}}{(1-s^{2m})(1-s^{2N-m})} + \frac{2Ns^N}{1-s^m} + 2 + 2s^N - \frac{4(1-s^{2N-2})}{(1-s^2)(1-s^{2N-1})} - \frac{4Ns^N}{1-s}\right] / (1-s^N)$$

Proposition A.4 :

If

$$\Delta g(\theta) = b\cos(\ell\theta) + c\sin(\ell\theta)\ ,\ \ell > 0$$

and

$$\hat{\Delta g}(\theta) = \frac{a_\ell}{u_{\ell,m}}\, v_{\ell,m}(\theta)\, e^{i\ell\theta} + \frac{a_{-\ell}}{u_{\ell,m}}\, v_{\ell,m}(\theta)\, e^{-i\ell\theta}$$

where

$$a_\ell = \tfrac{1}{2}(b - ic) \qquad , \qquad a_{-\ell} = \tfrac{1}{2}(b + ic)$$

$$v_{\ell,m}(\theta) = \sum_j s^{|mj+\ell|}\, e^{imj\theta}$$

and

$$u_{\ell,m} = v_{\ell,m}(0)$$

then

$$R_\ell = \frac{\int_0^{2\pi} (\hat{\Delta g} - \Delta g)^2\, d\theta}{\int_0^{2\pi} \Delta g^2\, d\theta} = \begin{cases} S_\ell & \text{if } \ell \neq \frac{m}{2}p\ ,\ p = 1, 2, \ldots \\ \frac{1+2\beta}{2(1+\beta)} + \frac{1}{2(1+\beta)}\, S_\ell & \text{if } \ell = \frac{m}{2}p \end{cases}$$

where

$$S_\ell = 1 + \frac{1-s^m}{1+s^m} \cdot \frac{1+s^{2m-4n}}{(1+s^{m-2n})^2} - \frac{1-s^m}{1+s^{m-2n}}\, 2\, s^{\ell-n}$$

$$n = \ell - \left[\frac{\ell}{m}\right] m$$

$$\beta = (c/b)^2$$

Proof:

$$I_1 = \frac{1}{2\pi} \int_0^{2\pi} \Delta g^2 \, d\,\theta = (b^2 + c^2)/2$$

$$I_2 = \frac{1}{2\pi} \int (\Delta\hat{g} - \Delta g)^2 \, d\,\theta = \frac{1}{2\pi} \int (\epsilon_\ell^2 + \epsilon_{-\ell}^2 + 2\,\epsilon_\ell\,\epsilon_\ell) d\,\theta = I_{21} + I_{22} + I_{23}$$

where

$$\epsilon_\ell = a_\ell \left(\frac{v_\ell}{u_\ell} - 1\right) e^{i\ell\theta}$$

$$\epsilon_{-\ell} = a_{-\ell} \left(\frac{v_\ell}{u_\ell} - 1\right) e^{-i\ell\theta}$$

and

$$I_{21} = \frac{1}{2\pi} \int \epsilon_\ell^2 \, d\,\theta = \begin{cases} a_\ell^2 \,[u_\ell^{(2)}/u_\ell^2 - 2\,s^\ell/u_\ell] & \text{if } \ell = \frac{m}{2}\,p \\ 0 & \text{otherwise} \end{cases}$$

$$I_{22} = \frac{1}{2\pi} \int \epsilon_{-\ell}^2 \, d\,\theta = \begin{cases} a_{-\ell}^2 \,[u_\ell^{(2)}/u_\ell^2 - 2s^\ell/u_\ell] & \text{if } \ell = \frac{m}{2}\,p \\ 0 & \text{otherwise} \end{cases}$$

$$I_{23} = \frac{1}{2\pi} \int \epsilon_\ell\,\epsilon_{-\ell} \, d\,\theta = 2\,a_\ell a_{-\ell}\,[1 + u_\ell^{(2)}/u_\ell^2 - 2\,s^\ell/u_\ell]$$

$$u_\ell^{(2)}(s) = u_\ell(s^2)$$

The proposition readily follows for:

$$R_\ell = (I_{21} + I_{22} + I_{23})/I_1$$

and

$$(a_\ell + a_{-\ell})^2 = b^2/4 \quad , \quad a_\ell \, a_{-\ell} = (b^2 + c^2)/4$$