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DAVID W. TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Maryland 20084

ON IRROTATIONAL FLOWS EQUIVALENT TO THE BOUNDARY LAYER AND WAKE

THE FIFTH DAVID W. TAYLOR LECTURES

by

L. Landweber

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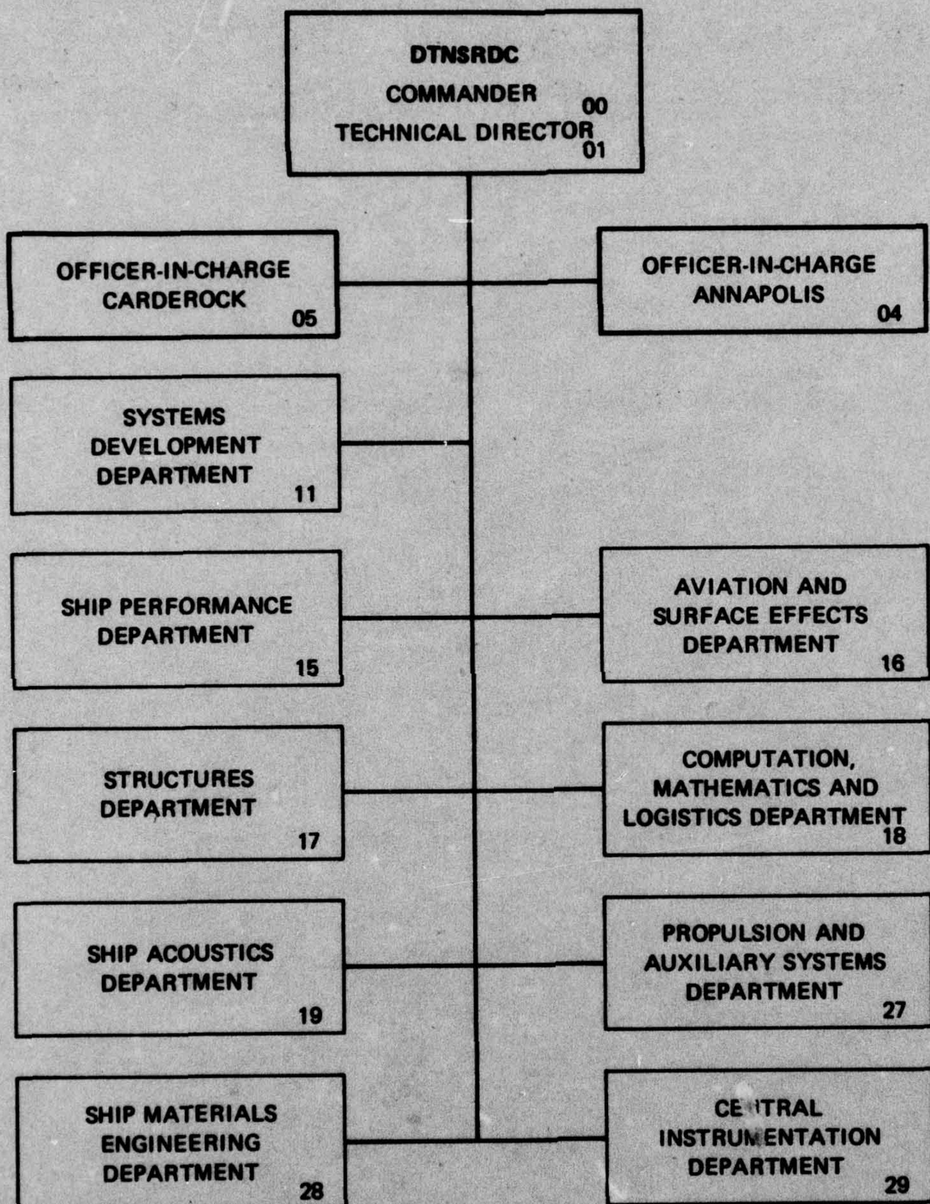

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PREFACE

The David W. Taylor Lectures were conceived to honor our founder in recognition of his many contributions to naval architecture and naval hydrodynamics. Admiral Taylor was a pioneer in the use of hydrodynamic theory and mathematics for the solution of naval problems. He established a tradition of applied scientific research at the "Model Basin" which has been carefully nurtured through the decades and which we treasure and maintain today. It is in this spirit that we have invited Prof. Louis Landweber to be a David W. Taylor Lecturer.

Prof. Landweber was born in New York in 1912. He received his Bachelor of Science degree in Physics from the City College of New York in 1932. That year he came to the U.S. Experimental Model Basin at the Washington Navy Yard as a junior physicist. When the David Taylor Model Basin opened at Carderock in 1940, Dr. Landweber headed a small research group which shortly afterwards was expanded into the Hydrodynamics Division. During his years at Carderock, he continued to head this division, and he also completed work on his Ph.D. at the University of Maryland. He was an advocate of advanced training for the staff and taught courses for the University at the Center.

After twenty-two years of distinguished research at the "Model Basin," Dr. Landweber left to become a professor at the University of Iowa. He continued his research in ship hydrodynamics at the school's Institute for Hydraulic Research. During the intervening years since leaving the Center in 1954, he has maintained close ties with his colleagues here at the Center and has returned frequently for meetings and panel sessions.

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FOREWORD

After a lapse of twenty-four years, it was surprisingly easy and pleasant to readjust for a month to a daily schedule of work in a mezzanine office at the David W. Taylor Naval Ship Research and Development Center. I found Ship Hydrodynamics to be alive and well, its problems being vigorously attacked by a dedicated and talented staff, as in the "old days."

The adjustment was probably equally easy and pleasant for another "alumnus" and David Taylor Lecturer, John Wehausen. It is remarkable that so many of the alumni have stayed in the field of Ship Hydrodynamics, an indication of the challenging nature and attractiveness of the subject.

To all who visited me, to consult, gripe, reminisce or simply to educate me, who extended generous hospitality in accordance with the fortune-cookie admonition, "Take time to play in order to have a long life," who invited me and arranged that day-to-day living would be convenient and comfortable for my wife and myself, in other words, to all who helped disprove the old adage that "one cannot go home," our deepest appreciation.

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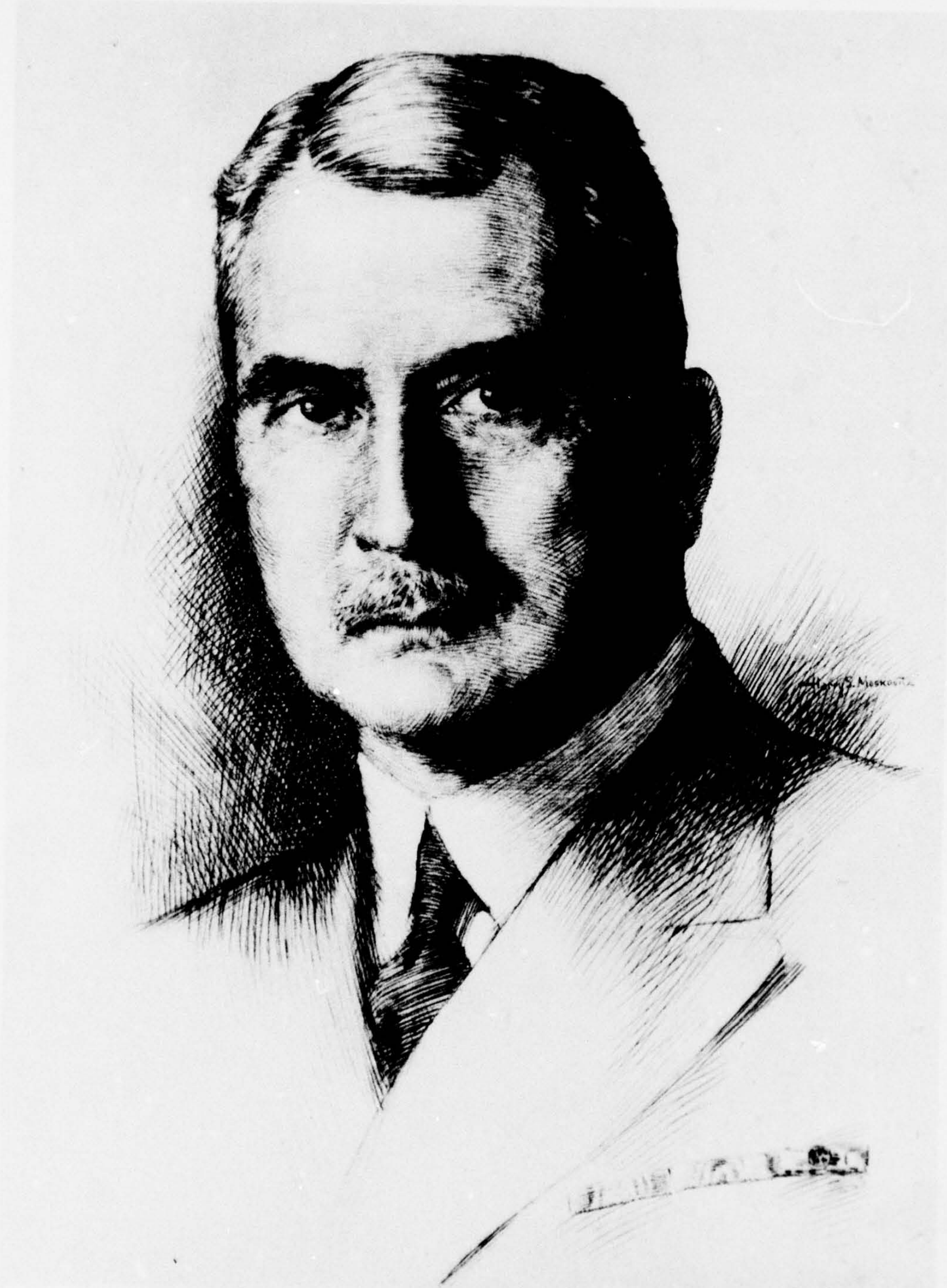


ON IRROTATIONAL FLOWS EQUIVALENT TO THE
BOUNDARY LAYER AND WAKE

INTRODUCTION

One of the simplest and most useful results of boundary-layer theory is that the flow exterior to the boundary layer, which will be assumed to be irrotational, is pushed outwards by an amount called the displacement thickness. This suggested, early in the development of boundary-layer theory, that the accuracy of a boundary-layer calculation for a body could be improved by adding the boundary-layer thickness to the body dimensions and using the predicted pressure distribution on the so-thickened body in a recalculation of the boundary layer. The irrotational field about the thickened body, including the displacement thickness of the wake, is itself of great interest, and numerous attempts have been made in this manner to calculate the effect of the boundary layer and wake on the outer irrotational flow.

The concept of the thickened body gives an approximate model which is usually justified by its consistency with the approximations of thin boundary-layer theory. We shall review the basis of this model and suggest ways of refining it. Such a development would be useful for several current problems of ship hydrodynamics, among them the determination of Betz sources in a method of calculating viscous drag by an analysis of wake survey data, and the investigation of the effect of the boundary layer and wake on wavemaking resistance.

We shall assume that the boundary layer and wake (BLW) are known, and seek an irrotational model which yields the actual outer flow. An approximate solution of this problem has been given by Preston [1] and Lighthill [2].

Solution of Preston and Lighthill - Two-Dimensional Case

Let us consider a two-dimensional flow about a body of small curvature in a uniform stream of velocity U_o, on which a thin boundary layer is present. Coordinates parallel and normal to the surface will be denoted by (s,n), and the corresponding velocity components by (u,v). We shall suppose that the radii of curvature are so large compared with the boundary-layer thickness δ, that the curvilinearity of the coordinate system may be ignored. The normal velocity component at $n = \delta$ is then given by

$$v(\delta) = \int_0^\delta \frac{\partial v}{\partial n}\, dn = - \int_0^\delta \frac{\partial u}{\partial s}\, dn = - \delta \frac{dU}{ds} + \frac{d}{ds} \int_0^\delta (U-u)\, dn = - \delta \frac{dU}{ds}$$

$$+ \frac{d}{ds} (U\delta_1), \quad \delta_1 = \int_0^\delta \left(1- \frac{u}{U}\right) dn \qquad (1)$$

where $U = u\,(s,\delta)$ and δ_1 is the displacement thickness. In the irrotational flow about the body, with velocity components $U_I(s,n)$, $V_I(s,n)$, we have $(\partial U_I/\partial n)_0 = 0$ where the subscript 0 denotes $n = 0$. Hence we may put $U_I(s,n) = U_I(s)$, since

$$U_I(s,n) \fallingdotseq U_I(s,0) + n \left(\frac{\partial U_I}{\partial n}\right)_0 \fallingdotseq U_I(s,0) = U_I(s), \quad 0 \le n \le \delta$$

The normal velocity in this irrotational flow, at $n = \delta$, would then be given by $V_I(s,\delta) \fallingdotseq V_I(s,0) + \delta(\partial V_I/\partial n)_0$, and hence, since $V_I(s,0) = 0$ and $\partial V_I/\partial n = - dU_I/\partial s$, we obtain

$$V_I(s,\delta) \fallingdotseq - \delta \frac{dU_I}{ds} \qquad (2)$$

We also assume that $U = u(s,\delta) = U_I(s,\delta)$ for a thin boundary layer. Thus the first term of the right member of (1) is attributable to this irrotational flow, and the second term represents an additional outward flow due to the boundary layer.

Put $V' = \frac{d}{ds} (U\delta_1)$ and consider that the "known" values of V' on the contour $n = \delta$, the edge of BLW, pose an exterior Neumann boundary-value problem for determining a source distribution $m(s)$ on the surface of the body and (if the body is symmetric) along the centerline of the wake.

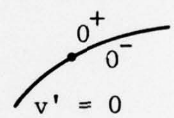
Preston and Lighthill give the solution without hesitation. Their result is

$$m = \frac{1}{2\pi} V'(s,\delta) = \frac{1}{2\pi} \frac{d}{ds} (U\delta_1) \tag{3}$$

for which the argument given by Lighthill is: "This additional outflow is exactly 'as if' the irrotational flow around the body were supplemented by the effect of a surface distribution of sources, whose strength ... is ..."

Equation (3) does not seem as obvious to the present writer. In order to derive it, one must assume that the velocity U(s) is undisturbed by the distribution m(s), an assumption which may not be consistent with the uniquely defined solution of the Neumann problem. According to this assumption, the body surface is an equipotential for the distribution m(s) (because the tangential velocity due to m(s) is zero). This immediately yields

$$m(s) = \frac{1}{2\pi} V'(s,0_+) \tag{4}$$



since $V'(s,0_-) = 0$ on the interior side of the equipotential surface and $V'(s,0^+) - V'(s,0^-) = 2\pi m(s)$. Then, from the Taylor expression $V'(s,\delta) = V'(s,0_+) + \delta \left(\frac{\partial V'}{\partial n}\right)_0 + \ldots \doteq V'(s,0_+)$, we obtain the approximate solution (3).

The special case of a flat plate, (even in a nonzero pressure gradient), however, confirms the approximate solution (3) without requiring the equipotential assumption. Since the plate is the limiting case of a

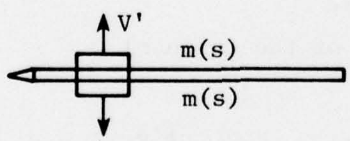


Figure 1

very thin body, the distributions m(s) are present on both sides. Application of the Gauss flux theorem (the usual proof) immediately gives Equation (3).

Another case that yields (3) is that of a constant source distribution on a circle. This gives a velocity potential due to a source of strength $2\pi ma$ at the center of the circle of radius a, and hence $V'(s,n) \doteq V'(s,0) = \frac{(2\pi ma)}{a} = 2\pi m$. But the local element about the point s contributes πm to V'. Hence, if we consider a nonuniform distribution which is large in the neighborhood of s, and negligible elsewhere, we would obtain $V'(s,n) \doteq \pi m$, half of the Preston-Lighthill formula. This suggests that, without the equipotential assumption, Equation (3) would give a good approximation only for very thin two-dimensional forms.

The distribution m(x) on the surface of the body y = f(x), which satisfies the boundary condition $\partial\phi/\partial n_s = v'(s,0)$ can be determined as the solution of the Fredholm integral equation of the second kind, (see Figure 2)

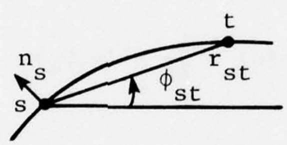


Figure 2

$$\pi m(s) + \oint m(t) \frac{\partial}{\partial n_s} \ln r_{st} \, dt = V'(s) \tag{5}$$

in which t also denotes arc length along the body contour. The indeterminacy of the kernel $\frac{\partial}{\partial n_s} \ln r_{st}$ at s = t can be removed by applying, the identity (at a smooth point of the contour)

$$\oint \frac{\partial}{\partial n_t} \ln r_{st} \, dt = \oint \frac{\partial}{\partial t} \phi_{st} \, dt = \pi \tag{6}$$

where the complex vector $r_{st} e^{i\phi_{st}}$, extending from s to t, has given the pair of conjugate functions

$$\ell n \left(r_{st} \, e^{i\phi_{st}} \right) = \ell n \, r_{st} + i\phi_{st}$$

to which the Cauchy-Riemann Equations have been applied. We then obtain from (5) and (6)

$$2\pi m(s) + \oint \left[m(t) \frac{\partial}{\partial n_s} \ell n \, r_{st} - m(s) \frac{\partial}{\partial n_t} \ell n \, r_{st} \right] dt = V'(s) \tag{7}$$

Since the new integrand vanishes at the point $s = t$, where the largest contribution to the integral in (5) occurs, the form in (7) suggests that the Preston-Lighthill formula should give a good first approximation.

From formula (3) for the source distribution, Lighthill immediately concludes that the displaced surface, obtained by the addition of the displacement thickness, is a stream surface. This again requires the assumption that all the flux from the source distribution m(s) be outward, i.e., that the given contour be an equipotential for the distribution. For then, the Gauss flux theorem gives the relation

$$2\pi \int_0^s m \, ds = U \, n(s)$$

where n(s) denotes the stream surface generated by m(s). Comparison with (3) then shows that $n(s) = \delta_1$.

A Second Approximation for the Source Distribution - Two-Dimensional Case

It appears to be easier to derive a formula for a source distribution M(x), equivalent to the BLW, for a thin, symmetric body, when M(x) is distributed on the axis of symmetry, the x-axis. The given profile will be denoted by $y = f(x)$, the edge of the boundary layer, EBLW, by $y = g(x)$, and the profile thickened by the displacement thickness by $y = h(x)$. Put

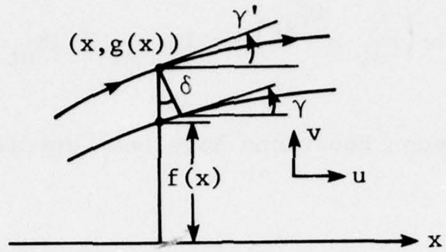


Figure 3

$df/dx = \tan\gamma$. For the irrotational flow, the velocity components in the x- and y-directions will be denoted by $u(x,y)$ and $v(x,y)$.

We wish to determine the source distribution $M(x)$ such that, on $y = g(x)$, the stream function $\psi(x,y)$ assumes the values

$$\psi = U(\delta - \delta_1) \tag{8}$$

We have

$$\left(\frac{\partial\psi}{\partial s}\right)_g = \left(\psi_x \frac{dx}{ds} + \psi_y \frac{dy}{ds}\right)_g = \left(-v \frac{dx}{ds} + u \frac{dy}{ds}\right)_g \tag{9}$$

But

$$v(x,g) \doteqdot v(x,0) + g\, v_y(x,0) = \pi M(x) - g\, u_x(x,0) \tag{10}$$

$$u(x,g) \doteqdot u(x,0) + g\, u_y(x,0) \doteqdot u(x,0) \tag{11}$$

since $u_y(x,0) = 0$ by symmetry. Also, by (8), we have

$$\left(\frac{\partial\psi}{\partial s}\right)_g = \frac{d}{dx}\,[U(\delta-\delta_1)]\left(\frac{dx}{ds}\right)_g \tag{12}$$

and hence, substituting (10), (11), and (12) into (9) and solving for M(x), we obtain

$$\pi M(x) \doteq \frac{d}{dx} [g(x)\ u(x,0)-U(\delta-\delta_1)] \tag{13}$$

Since, as is seen from Figure 3,

$$g(x) \doteq f(x) + \delta \sec \gamma \tag{14}$$

and

$$U \doteq u(x,g) \sec \gamma \doteq u(x,0) \sec \gamma \tag{15}$$

we can write (13) in the form

$$\pi M(x) \doteq \frac{d}{dx} \{u(x,0)[f(x)+\delta_1 \sec \gamma]\} \tag{16}$$

Here we have assumed that $f(x)$, δ, δ_1, $(u-U_0)$, v and γ are small quantities of the first order and terms of third and higher order have been neglected.

For the irrotational flow about the body without a boundary layer, (16) yields the well-known, second-order approximation for a centerplane distribution,

$$\pi M_0(x) \doteq \frac{d}{dx} [u(x,0)\ f(x)] \tag{17}$$

Hence, the additional source strength due to the boundary layer is given by

$$M(x) - M_0(x) \doteq \frac{1}{\pi}\frac{d}{dx} [u(x,0)\ \delta_1 \sec \gamma] \tag{18}$$

The result in (18) resembles the Preston-Lighthill formula (3), especially when the body slope is small. The distribution strength is approximately doubled because a pair of distributions on the body surface has coalesced on the centerplane, as was done in the example of the boundary layer on a flat plate.

Similarly to (14), the displacement-thickness profile, $y = h(x)$, may be expressed by

$$h(x) \doteqdot f(x) + \delta_1 \sec \gamma \tag{19}$$

and (17) becomes

$$M_1(x) \doteqdot \frac{1}{\pi} \frac{d}{dx} [u(x,0)\ h(x)] \tag{20}$$

Comparison with the form of (16) now shows that, to the second order of accuracy, the irrotational flow about the displaced surface satisfies the specified boundary condition (8) at the edge of the boundary layer and wake.

Solutions for Axisymmetric Flow - First Approximation

Let $r_0(s)$ denote the radius of a body of revolution, where s is arc length along a meridian section of the body, and r the radial distance to an arbitrary point. Let n denote distance from the body along the outward normal to its surface. We shall also employ cylindrical coordinates, (x,r) where the axis of symmetry is taken as the x-axis, and put $r_0(s) = f(x)$ and $\tan \gamma = df/dx$. The edge of the boundary layer is defined by the surface

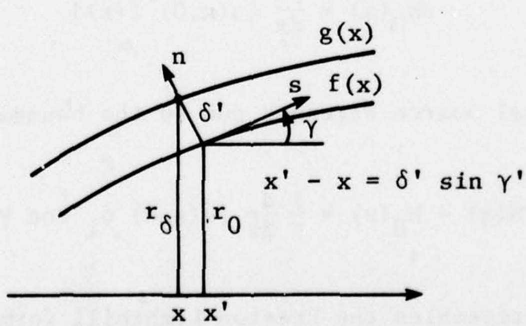


Figure 4

$$r_\delta(s) = r_0(s) + \delta \cos \gamma \text{ or } g(x) = f(x') + \delta (x') \cos \gamma (x') \quad (21)$$

By a Taylor expansion about x, with $d\gamma/ds = - k$, where k is the curvature of the meridian profile, and with $x' - x = \delta(x') \cos \gamma(x')$, g(x) becomes

$$g(x) = f(x) + \delta \sec \gamma + \delta\delta' \tan \gamma - \frac{1}{2} k\delta^2 \tan^2 \gamma \sec \gamma + \ldots \quad (22)$$

The element of arc along a curve n = constant is $h_1\, ds = (1+kn)\, ds$. The equation of continuity is then

$$\frac{\partial(ru)}{\partial s} + \frac{\partial}{\partial n} [(1+kn)\, rv] = 0 \quad (23)$$

where u(s,n) and v(s,n) are the actual velocity components in the directions of increasing values of s and n. We shall also require the velocity components U(s,n) and V(s,n) of the "equivalent" irrotational flow.

The boundary-layer thickness of an axisymmetric boundary layer is usually defined as

$$\delta_1 = \int_0^\delta \frac{r}{r_0} \left[1- \frac{u(s,n)}{U(s,\delta)}\right] dn \quad (24)$$

An alternative definition, in which a higher-order term is retained, is given by

$$\int_0^{\delta^*} r\, U(s,\delta)\, dn = \int_0^\delta r[U(s,\delta)-u(s,n)]\, dn = r_0\, U\, \delta_1$$

Since $r = r_0 + n \cos \gamma$, this yields the quadratic equation

$$r_0\, \delta^* + \frac{1}{2} \delta^{*2} \cos \gamma = r_0\, \delta_1 \quad (25)$$

and the relation

$$\delta^* = \frac{2\delta_1}{1+\left[1+\frac{2\delta_1 \cos\gamma}{r_0}\right]^{1/2}} \tag{26}$$

The equivalent irrotational flow is determined by the values of $v(s,\delta)$. By (23), we have

$$(1+k\delta)\, r_\delta\, v(s,\delta) = -\int_0^\delta \frac{\partial}{\partial s}(ru)\, dn = \int_0^\delta \frac{\partial}{\partial s}[r(U-u)-rU]\, dn$$

$$= \frac{d}{ds}(r_0 U \delta_1) - \delta \frac{d}{ds}(r_0 U) - \frac{\delta^2}{2}\frac{d}{ds}(U \cos\gamma) \tag{27}$$

where $U = U(s,\delta)$. In the irrotational flow about the body without a boundary layer, with velocity components (U_I, V_I), (27) becomes

$$(1+k\delta)\, r_\delta\, V_I(s,\delta) = -\int_0^\delta \frac{\partial}{\partial s}[rU_I(s,n)]\, dn$$

$$\doteq -\delta \frac{d}{ds}(r_0 U_I) - \frac{\delta^2}{2}\frac{d}{ds}(U_I \cos\gamma)$$

since $(\partial U_I/\partial n)_{n=0} = 0$, and $U_I(s,n) = U_I(s,0) + O(\delta^2) \doteq U_I(s)$. If the boundary layer is thin, then $U_I(s) \doteq U$ and the first term of the right member of Equation (27) can be interpreted as that contributing an additional flux due to the boundary layer. We define the additional normal velocity V' due to the boundary layer by

$$(1+k\delta)\, r_\delta V' = \frac{d}{ds}(r_0 U \delta_1) \tag{28}$$

If the body were a circular cylinder of constant radius r_0, we could immediately determine the source strength m(s) equivalent to the effect of the boundary layer. Application of the Gauss flux theorem gives

$$2\pi r_\delta V' = 4\pi(2\pi r_0 m)$$

and hence by (28),

$$m = \frac{1}{4\pi}\frac{d}{ds}(U\delta_1) \tag{29}$$

A generalization of (29) can be derived from the integral-equation formulation of the Neumann problem for the prescribed boundary-layer flux. If P and Q are points on the body surface, we have

$$2\pi m(P) = \int_S m(Q)\frac{\partial}{\partial n_Q}\frac{1}{r_{PQ}}\,dS_Q + V'(P) \tag{30}$$

The singularity of the kernel when Q coincides with P can be removed by means of the relation, valid at smooth points of the surface,

$$\int_S \frac{\partial}{\partial n_Q}\frac{1}{r_{PQ}}\,dS_Q = -2\pi \tag{31}$$

Thus (30) may be written in the form

$$4\pi m(P) = \int_S \left[m(Q)\frac{\partial}{\partial n_P}\frac{1}{r_{PQ}} - m(P)\frac{\partial}{\partial n_Q}\frac{1}{r_{PQ}}\right]dS_Q + V'(s,0) \tag{32}$$

which suggests the first approximation

$$m(P) = \frac{1}{4\pi} V'(s,0) \doteq \frac{1}{4\pi r} \frac{d}{ds} (r_0 U \delta_1) \tag{33}$$

a generalization of (29).

In order to improve upon the approximate source distribution (33), we shall now consider an axial source distribution M(x). For irrotational flow about the given body, r = f(x), the modified Munk formula of Landweber [3] gives the approximate solution, in terms of the free-stream velocity U_0,

$$M_0(x) \doteq \frac{1}{4} (1+k_1) U_0 \frac{d}{dx} f^2(x) \tag{34}$$

where k_1 is the longitudinal added-mass coefficient. For the thickened body, with $r = r_1(x) = f(x) + \delta_1 \sec \gamma$, formula (34) gives

$$M_1(x) = \frac{1}{4} (1+k_1) U_0 \frac{d}{dx} r_1^{\,2}(x)$$

$$\doteq M_0 + \frac{1}{2} (1+k_1) U_0 \frac{d}{dx} \left(f\delta_1 \sec \gamma + \frac{1}{2} \delta_1^{\,2} \sec^2 \gamma\right) \tag{35}$$

If the alternative displacement thickness δ^* is used, the same expression is applicable, with δ_1 replaced by δ^*.

<u>Axisymmetric Flow - Second Approximation</u>

We shall now derive a second-order approximation for an axial source distribution for slender bodies. The given profile is $r = r_0(x) = f(x)$, the edge of the boundary layer is $r = g(x)$, and the displacement-thickness profile is $r = h(x)$. The velocity components in the x- and r-directions for the equivalent irrotational model are u(x,r), v(x,r).

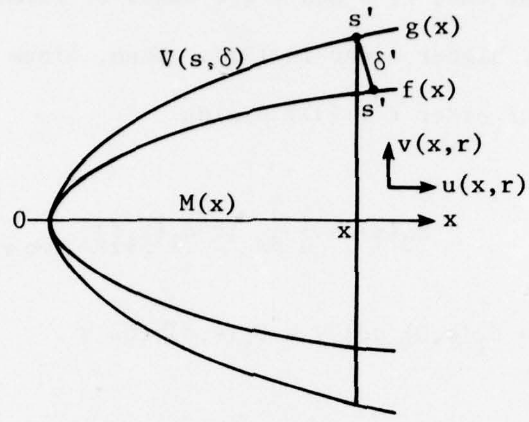


Figure 5

Let us apply the Gauss flux theorem to the volume of the body of revolution generated by the profile $r = f(x)$, extending from the nose to a transverse section at x, assuming that the flow about the body is generated by an axial source distribution $M_0(x)$. Since $r = f(x)$ is a stream surface, the flux theorem gives

$$4\pi \int_0^x M_0 \, dx = \int_0^f 2\pi r u_I(x,r) \, dr \tag{36}$$

Here $[u_I(x,r),\ v_I(x,r)]$ denote the velocity components of the irrotational flow when the given profile is a stream surface, as distinct from the components (u,v) of the equivalent irrotational flow. Differentiating (36) and integrating by parts, we then obtain

$$M_0(x) = \frac{1}{4} \frac{d}{dx} \left[f^2 \, u_I(x,f) - \int_0^f r^2 \frac{\partial u_I}{\partial r} \, dr \right] \tag{37}$$

We shall assume that f, δ and γ are small of first order, and neglect terms of fourth and higher order in (37). Then, since $r^2 \frac{\partial u_I}{\partial r} = r \frac{\partial(rv_I)}{\partial x} \doteqdot 2r \frac{dM_0}{dx}$ is small of order f^3, (37) yields

$$M_0(x) \doteqdot \frac{1}{4} \frac{d}{dx} [f^2 u_I(x,f)]$$

or, since $u_I(x,f) = U_I(s,0) \cos \gamma \doteqdot U_I(s,\delta) \cos \gamma$

$$M_0(x) = \frac{1}{4} \frac{d}{dx} [f^2 U_I(s,\delta) \cos \gamma] \tag{38}$$

Similarly, the source distribution $M_1(x)$ for the irrotational flow about the surface displaced by the displacement thickness δ^* is given by

$$M_1(x) = \frac{1}{4} \frac{d}{dx} [(f+\delta^* \sec \gamma)^2 U_I^*(s,\delta) \cos \gamma]$$

where U_I^* denotes the s-component of the velocity. Then, assuming that $U_I(s,\delta) = U_I^*(s,\delta)$, we obtain

$$M_1(x) - M_0(x) = \frac{1}{4} \frac{d}{dx} [(2f\delta^* + \delta^{*2} \sec \gamma) U_I(s,\delta)] \tag{39}$$

Similarly, when the surface of revolution is generated by $r = g(x)$ and the axial distribution is $M(x)$, the Gauss flux theorem gives

$$4\pi \int_0^x M(x) = \int_0^s 2\pi g(x) V(s,\delta)(1+k\delta) dS + \int_0^g 2\pi r u(x,r) dr$$

or, by (27), and integration-by-parts of the last integral,

$$2M(x) \doteq \frac{d}{dx}(Uf\delta_1) - \delta\frac{d}{dx}(Uf) - \frac{\delta^2}{2}\frac{d}{dx}(U\cos\gamma) + \frac{1}{2}\frac{d}{dx}[g^2u(x,g)]$$

The same analysis, applied to the source distribution $M_0(x)$, yields

$$2M_0(x) \doteq -\delta\frac{d}{dx}(U_If) - \frac{\delta^2}{2}\frac{d}{dx}(U_I\cos\gamma) + \frac{1}{2}\frac{d}{dx}[g^2u_I(x,g)]$$

and hence, assuming $U(s,\delta) = U_I(s,\delta)$ and $u(x,g) = u_I(x,g)$, we obtain

$$M(x) - M_0(x) = \frac{1}{2}\frac{d}{dx}(Uf\delta_1) \tag{40}$$

Comparison of (39) and (40) now shows that $M_1(x) = M(x)$ if

$$f\delta^* + \frac{1}{2}\delta^{*2}\sec\gamma = f\delta_1 \tag{41}$$

This agrees with the definition of δ^* in (25) when γ is small. Otherwise, the displaced surface should be taken in accordance with (41) rather than (25).

A Vorticity Theorem

Consider a steady mean flow of an incompressible fluid about a body at rest in an otherwise unbounded fluid, with mean vorticity present in the boundary layer and wake, BLW, bounded internally by the body surface S and externally by the surface T; see Figure 6. The boundary conditions to be

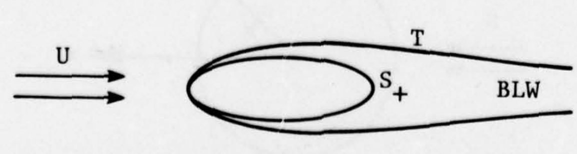


Figure 6

satisfied are:

1. The velocity vector $\bar{v}$ on S_+, the exterior side of S, satisfies the nonslip condition, $\bar{v} = 0$.

2. The flow exterior to T is irrotational.

Then we have the following <u>theorem</u>:

<u>The disturbance flow field exterior to S can be generated by the vorticity in BLW alone</u>.

<u>Proof</u>:

First let us suppose that distributions of vorticity and irrotational singularities are present in B, the interior of the region bounded by S. The velocity components normal to S induced by these distributions define a Neumann problem for a source distribution M on S, equivalent to the interior distributions.

We now have a source distribution M on S and vorticity $\bar{\omega}$ in BLW. The tangential velocity v_t is continuous across S, and hence, by the nonslip condition, $v_t =$ [illegible] on the interior side of S. Also the flow induced by the distrib[illegible] is irrotational within B. Hence $\bar{v} = 0$ within B, and, consequently, the [illegible]rmal component of the velocity on the interior side of S is zero. But, because of the nonslip condition, v_n is zero on the exterior side of S also, hence the strength of the source distribution must be zero. This leaves only the vorticity distribution $\bar{\omega}$, as we wished to show.

<u>Verification with Stokes Solution for a Sphere</u>

Stokes solution for a sphere of radius <u>a</u> in a uniform stream in the x-direction, expressed in spherical coordinates (R,θ,ϕ), is given by

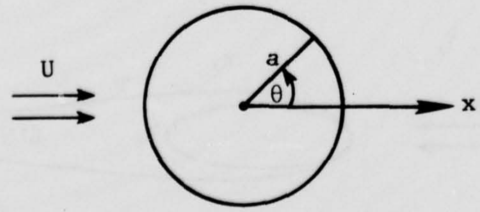


Figure 7

$$v_R = U \cos\theta + 2\left(\frac{A}{R^3} + \frac{B}{R}\right)\cos\theta \tag{44}$$

$$v_\theta = -U \sin\theta + \left(\frac{A}{R^3} - \frac{B}{R}\right)\sin\theta \tag{45}$$

$$A = \frac{1}{4}Ua^3, \; B = -\frac{3}{4}Ua$$

Here the first and second terms represent components due to the uniform stream and a doublet at the center of the sphere, and the last terms are due to vorticity. We shall now show that the velocity at points of the x-axis, ($\theta = 0$, $R \geq a$), can be obtained from the vorticity outside the sphere, together with the uniform stream.

The vorticity, determined from the last terms of (44) and (45), is given by

$$\omega_R = \omega_\theta = 0, \; \omega_\phi = \frac{2B}{R^2}\sin\theta \tag{46}$$

At constant R and θ, a vortex ring of unit strength induces a velocity (by the Biot-Savart Law), at a point x of the x-axis, given by

$$\frac{R^2 \sin^2\theta}{2(x^2-2xR\cos\theta+R^2)^{3/2}}$$

The velocity due to the vorticity in the space exterior to the sphere is then given by

$$u = B\int_a^\infty \int_0^\pi \frac{R\sin^3\theta \, d\theta \, dR}{(x^2-2xR\cos\theta+R^2)^{3/2}} \tag{47}$$

With the substitution $\mu = \cos\theta$, integration with respect to μ yields

$$u = -\frac{B}{4x^3}\int_a^\infty \frac{1}{R^2}\left[(x^2-R^2)^2\left(\frac{1}{|R-x|}-\frac{1}{|R+x|}\right)+2(x^2+R^2)(|R-x|-R-x)\right.$$

$$\left.-\frac{1}{3}(|R-x|)^3-(R+x)^3\right]dR$$

This is readily evaluated by considering separately the integrations from a to x, and from x to ∞. The former yields

$$u_1 = \frac{4}{3}\frac{B}{x^3}\int_a^x RdR = -\frac{Ua}{2x^3}(x^2-a^2)$$

and the latter gives

$$u_2 = \frac{4}{3}B\int_x^\infty \frac{dR}{R^2} = -\frac{Ua}{x}$$

Hence

$$u = u_1 + u_2 = \frac{Ua^3}{2x^3} - \frac{3Ua}{2x} \tag{48}$$

in agreement with the distrubance velocity in (1) when $\theta = 0$, given in (44).

This result indicates that the doublet at the center of the sphere is an irrotational equivalent of the negative of the internal vorticity, since the sum of their fields is zero exterior to the sphere.

Some Applications of the Vorticity Theorem

We have previously investigated irrotational equivalents of the boundary layer and wake such that the outer irrotational flow was preserved. It was seen that this could be accomplished in various ways, such as a source distribution on the given surface and along the wake, a source distribution on the edge of the boundary layer and wake, or by the irrotational flow about the displacement-thickness surface. In none of these irrotational models was it possible to retain the given body as a stream surface.

Intuitively, it appears desirable to match the boundary conditions on both the body and the edge of the boundary layer and wake, in order to obtain a more realistic irrotational model. We have seen that the vorticity distribution alone yields both the nonslip condition on the body and the boundary condition on EBLW. In the previous models only the boundary condition on EBLW was employed. An additional source distribution, on the body or in its interior is required in order to satisfy the condition that the body remain a stream surface. The condition of zero tangential velocity would not be satisfied, but this seems to be physically less important in an irrotational model.

The boundary conditions on the body surface S and on EBLW define a Neumann problem which can be readily formulated as a pair or integral equations to be solved simultaneously for a pair of source distributions. The locations of the source distributions may vary, even for a given body, as has already been illustrated. If these are taken to be distributions $m(P)$ on S and $\mu(Q)$ on T; one obtains the integral equations

$$2\pi m(P) - \int_S m(P') \frac{\partial}{\partial n_P} \frac{1}{r_{PP'}} dS_{P'} - \int_T \mu(Q') \frac{\partial}{\partial n_P} \frac{1}{r_{PQ'}} dS_{Q'}$$

$$= - U_0 \frac{\partial x}{\partial n_P} \tag{49}$$

$$2\pi\mu(Q) - \int_T \mu(Q') \frac{\partial}{\partial n_Q} \frac{1}{r_{QQ'}} dS_{Q'} - \int_S m(P') \frac{\partial}{\partial n_Q} \frac{1}{r_{P'Q}} dS_{P'} = v(Q) \tag{50}$$

where P and P' denote points on S, and Q and Q' on EBLW, and $v(Q)$ denotes the normal component of the velocity at T, which is assumed to be known.

The Poincaré Transformation [4]

It is well known that the velocity field induced by a vortex ring is identical to that of a doublet sheet on a surface capping the ring. The Poincaré transformation, derived below, enables more general relationships between fields of vorticity and irrotational distributions to be obtained. Since the entire disturbance flow field is induced by the vorticity alone, the velocity field of the equivalent irrotational distributions would be identical to that induced by the vorticity in the regions exterior to the vorticity domain. For the case where the vorticity lies in the boundary layer and wake of a body, the vorticity induces not only an irrotational field exterior to its outer boundary T, but also an irrotational field within the body. Because of the nonslip condition, the induced velocity within the body must be zero. Thus, in contrast with the distributions previously considered, the Poincare transformation offers the possibility of matching the boundary conditions on both the interior and exterior boundaries of BLW.

Let us suppose that vorticity $\bar{\omega}$ = curl $\bar{v}$ is present in a domain D, bounded by a closed surface S, and denote the domain exterior to D by E. Here $\bar{v}$ denotes the velocity vector of the fluid flow. We shall distinguish between a fixed point P(x,y,z) at which induced velocities are calculated, and variable points of integration, $Q(\xi,\eta,\zeta)$. The position vector from P to Q is $\bar{r}_{PQ}$ and has the magnitude r_{PQ}.

The velocity induced by a vorticity distribution can be expressed either by means of the Biot-Savart Law,

$$\bar{v}_P = \frac{1}{4\pi} \int_D \frac{\bar{\omega} \times \overline{r_{PQ}}}{r_{PQ}^3} \, d\tau \qquad (51)$$

or in terms of the vector potnetial,

$$\bar{v}_P = \frac{1}{4\pi} \nabla_P \times \int_D \frac{\bar{\omega}}{r_{PQ}} \, d\tau \qquad (52)$$

We shall need to distinguish between the vector operators

$$\nabla_P \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} \text{ and } \nabla \equiv \bar{i}\frac{\partial}{\partial \xi} + \bar{j}\frac{\partial}{\partial \eta} + \bar{k}\frac{\partial}{\partial \zeta}$$

The Poincaré transformation is as follows:

$$\nabla_P \times \left[\int_D \frac{\nabla\times\bar{v}}{r_{PQ}}\,d\tau - \int_S \frac{\bar{n}\times\bar{v}}{r_{PQ}}\,dS\right] = \nabla_P\left[\int_D \frac{\nabla\cdot\bar{v}}{r_{PQ}}\,d\tau - \int_S \frac{\bar{v}\cdot\bar{n}}{r_{PQ}}\,dS\right]$$

$$+ \begin{Bmatrix} 4\pi \\ 2\pi \\ 0 \end{Bmatrix} \bar{v}_P, \quad \begin{Bmatrix} P \text{ in } D \\ P \text{ on } S \\ P \text{ in } E \end{Bmatrix} \qquad (53)$$

where $\bar{n}$ is the unit vector in the direction of the outward normal to S. First suppose that P is in E. Then we have

$$\nabla_P \times \int_D \frac{\nabla\times\bar{v}}{r_{PQ}}\,d\tau = \nabla_P \times \int_D \left[\nabla\times\left(\frac{\bar{v}}{r_{PQ}}\right) - \nabla\left(\frac{1}{r_{PQ}}\right)\times\bar{v}\right] d\tau$$

$$= \nabla_P \times \int_S \frac{\bar{n}\times\bar{v}}{r_{PQ}}\,dS + \int_D \left[\bar{v}\cdot\nabla_P\nabla_P \frac{1}{r_{PQ}} - \bar{v}\nabla_P\cdot\nabla_P \frac{1}{r_{PQ}}\right] d\tau$$

$$= \nabla_P \times \int_S \frac{\bar{n}\times\bar{v}}{r_{PQ}}\,dS + \nabla_P \int_D \bar{v}\cdot\nabla_P \frac{1}{r_{PQ}}\,d\tau \qquad (54)$$

But

$$\int_D \bar{v} \cdot \nabla_P \frac{1}{r_{PQ}} \, d\tau = - \int_D \left[\nabla \cdot \left(\frac{\bar{v}}{r_{PQ}} \right) - \frac{\nabla \cdot \bar{v}}{r_{PQ}} \right] d\tau$$

$$= \int_D \frac{\nabla \cdot \bar{v}}{r_{PQ}} \, d\tau - \int_S \frac{\bar{v} \cdot \bar{n}}{r_{PQ}} \, dS \tag{55}$$

Substituting (55) and (54) now yields (53) for P in E.

Next consider the case that P is in D, and let $\forall_0$ denote the volume and S_0 the surface of a small sphere of radius r_0 about P. Then we have

$$\nabla_P \times \int_D \frac{\nabla \times \bar{v}}{r_{PQ}} \, d\tau = \nabla_P \times \int_{D'} \frac{\nabla \times v}{r_{PQ}} \, d\tau + \nabla_P \times \int_{\forall_0} \frac{\nabla \times \bar{v}}{r_{PQ}} \, d\tau \tag{56}$$

where $D' = D - \forall_0$. The last integral in (6) is proportional to the velocity induced by the vorticity within $\forall_0$, according to (52) and hence must vanish as the radius of the sphere approaches zero. Also, by Equations (54) and (55), we have

$$\nabla_P \times \int_{D'} \frac{\nabla \times \bar{v}}{r_{PQ}} \, d\tau = \nabla_P \times \int_S \frac{\bar{n} \times \bar{v}}{r_{PQ}} \, dS + \nabla_P \times \int_{S_0} \frac{\bar{n} \times v}{r_{PQ}} \, dS$$

$$+ \nabla_P \int_{D'} \bar{v} \cdot \nabla \frac{1}{r_{PQ}} \, d\tau \tag{57}$$

and

$$\int_{D'} \bar{v} \cdot \nabla_P \frac{1}{r_{PQ}}\, d\tau = \int_{D'} \frac{\nabla \cdot \bar{v}}{r_{PQ}}\, d\tau - \int_{S} \frac{\bar{v} \cdot \bar{n}}{r_{PQ}}\, dS - \int_{S_0} \frac{\bar{v} \cdot \bar{n}}{r_{PQ}}\, dS \tag{58}$$

The convention on the positive sense of $\bar{n}$ requires that it be directed inwards on the sphere S_0. For the term in (57), we have

$$\nabla_P \times \int_{S_0} \frac{\bar{n} \times \bar{v}}{r_{PQ}}\, dS = \int_{S_0} \nabla_P \left(\frac{1}{r_{PQ}}\right) \times (\bar{n} \times \bar{v})\, dS = - \int_{S_0} \frac{\bar{n} \times (\bar{n} \times \bar{v})}{r_0^{\,2}}\, r_0^{\,2}\, d\Omega$$

$$\approx - \int_{S_0} \bar{n}\, \bar{n} \cdot \bar{v}\, d\Omega + 4\pi \bar{v}_P \tag{59}$$

where $dS = r_0^{\,2}\, d\Omega$; and in (58),

$$\nabla_P \int_{S_0} \frac{\bar{v} \cdot \bar{n}}{r_{PQ}}\, dS = - \int_{S_0} \bar{n}\, \bar{n} \cdot \bar{v}\, d\Omega \tag{60}$$

Also we have

$$\lim_{r_0 \to 0} \nabla_P \int_{D'} \frac{\nabla \cdot \bar{v}}{r_{PQ}}\, d\tau = \nabla_P \int_{D} \frac{\nabla \cdot \bar{v}}{r_{PQ}}\, d\tau \tag{61}$$

since the velocity field of a volume distribution of sources is continuous. Collecting the results in Equations (56) to (61) now yields (53) for P in D.

Finally, when P is a "smooth" point of S, i.e., a point at which the tangent to S is continuous, we introduce a small hemisphere about P, of radius r_0, and apply the Poincare transformation to the so-diminished volume D', and to the bounding surface, consisting of the hemispherical surface S_0 and the remainder S' of S. The proof is similar to that for

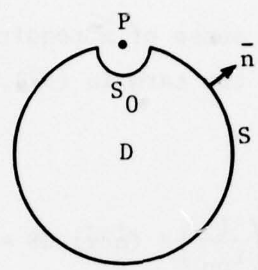


Figure 8

P in D. In Equations (57) and (58), we need only to replace S by S', and in Equation (59) $4\pi\bar{v}_P$ becomes $2\pi\bar{v}_P$, or, indeed, $\alpha\bar{v}_P$ if P is a corner point of S of solid angle α. Instead of (61), we need

$$\lim_{r_0\to 0} \nabla_P \times \int_{S'} \frac{\bar{n}\cdot\bar{v}}{r_{PQ}}\, dS = \nabla_P \times \int_S \frac{\bar{n}\times\bar{v}}{r_{PQ}}\, dS$$

and

$$\lim_{r_0\to 0} \nabla_P \int_{S'} \frac{\bar{v}\cdot\bar{n}}{r_{PQ}}\, dS = \nabla_P \int_S \frac{\bar{v}\cdot\bar{n}}{r_{PQ}}\, dS$$

which are verified by observing that the velocity fields of the surface distributions of vorticity, $\bar{n} \times \bar{v}$, and of sources, $\bar{v} \cdot \bar{n}$, are continuous.

Equivalent Irrotational Flow From Poincare Transformation

Let us apply the Poincaré transformation (53) to a case in which the flow is entirely due to the vorticity in a domain D, bounded by a closed surface S. We shall seek to express the velocity at a point P of E as the gradient of a velocity potential.

The first term of (53) is seen to give the velocity $\bar{v}_{VP}$ due to vorticity in the form of the vector potential (52), as $4\pi\bar{v}_{VP}$. Since the assumed flow is solenoidal, the third term of (53) is zero. We then have, from (53),

$$4\pi\bar{v}_{VP} = \nabla_P \times \int_S \frac{\bar{n}\times\bar{v}}{r_{PQ}}\, dS - \nabla_P \int_S \frac{\bar{v}\cdot\bar{n}}{r_{PQ}}\, dS \tag{62}$$

This expresses the velocity in terms of that induced by a source distribution of strength $\bar{v}_n/4\pi$ and by a vortex sheet of strength $\bar{n} \times \bar{v} = v_s\bar{t}$, both on S. Here we have expressed the velocity vector $\bar{v} = \bar{n}v_n + \bar{s}v_s$, where $\bar{s}$ is the unit vector in the direction of the projection of $\bar{v}$ on the tangent plant at Q. Then, putting $\bar{n} \times \bar{s} = \bar{t}$, we obtain the form given above.

In order to express the field of the vortex sheet as the gradient of a potential, let us define a function ϕ_1, harmonic in D, which on S satisfies

$$\bar{n} \times \bar{v}_1 = \bar{n} \times \bar{v}, \quad \bar{v}_1 = \nabla\phi_1 \tag{63}$$

We have then

$$\frac{\partial\phi_1}{\partial s} = v_{1s} = v_s \quad \text{on S} \tag{64}$$

and hence, by integration along s, the values of ϕ_1 on S may be presumed to be given in terms of the known values of v_s. Thus the boundary condition (64) sets up a Dirichlet problem for determining ϕ_1.

Since $\nabla \times \bar{v}_1 = 0$, application of the Poincaré transformation to $\bar{v}_1$, taking (63) into account, yields

$$\nabla_P \times \int_S \frac{\bar{n}\times\bar{v}}{r_{PQ}}\, dS = \nabla_P \int_S \frac{\bar{v}_1\cdot\bar{n}}{r_{PQ}}\, dS \tag{65}$$

Hence (62) becomes

$$\bar{v}_{VP} = \frac{1}{4\pi} \nabla_P \int_S \frac{\bar{n}\cdot(\bar{v}_1-\bar{v})}{r_{PQ}} dS \tag{66}$$

a velocity field due to a source distribution of strength

$$\sigma = \bar{n} \cdot (\bar{v}-\bar{v}_1)/4\pi \tag{67}$$

The form of σ shows that $\bar{n} \cdot \bar{v}_1$ is the normal component of the velocity on the interior side of the source distribution on S. Uniqueness of solutions of Neumann problems on S then shows that $\bar{v}_1$ is the irrotational velocity field in D associated with the source distribution σ on S.

An alternative source distribution on S can also be found directly as the solution of the exterior Neumann problem for the given values of $\bar{v} \cdot \bar{n}$ on S, as was done previously in considering the displacement effects of the boundary layer. Applied to the vorticity field BLW in the flow about a body, the present approach requires that a composite bounding surface,

$$S = S_B + T + A$$

be used, where S_B is the surface of the given body, and A is the transverse surface of the truncated wake. If the transverse section is taken sufficiently far downstream, the effects of the source distribution on A may be neglected, and the Neumann problem could be formulated as a pair of simultaneous Fredholm integral equations of the second kind. The resulting source distribution, however, would not coincide with that given in (67). In the present treatment, the value $\bar{v} \cdot \bar{n} = 0$ on S_B is preserved, whereas, in deriving (67), the tangential component $\bar{v} \times \bar{n} = 0$ on S_B, in accordance with the nonslip condition, was preserved. In the former case, S_B remains a stream surface, in the latter, an equipotential.

Betz Method for Determining Viscous Drag

Betz [5] and Landweber and Wu [6] use equivalent irrotational flows to derive formulas for the viscous drag of a body in terms of measured values of pressure and velocity at a transverse section of the wake. A refinement of these derivations, in which additional wake characteristics are taken into account, will now be presented.

The body is taken at the center of a circular channel of large radius, and is at rest in a uniform stream of velocity U in the positive x-direction. The disturbance velocity components in a rectangular (x,y,z) coordinate system are (u,v,w), and p denotes the pressure.

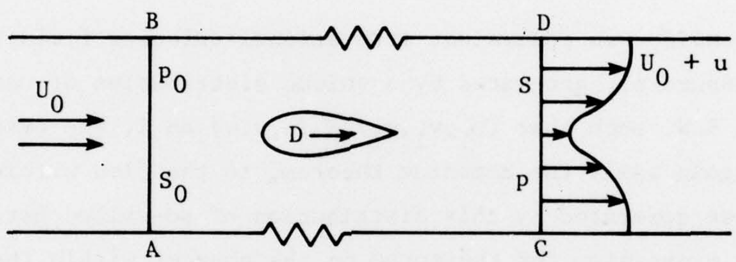


Figure 9

We select a control surface consisting of the transverse sections AB, far ahead of the body, CD or S a moderate distance behind it, and the portion of the channel wall lying between these sections. On the section AB, designated S_0, the pressure is the constant p_0, and the velocity is $(U_0, 0, 0)$. Application of the momentum theorem to this control surface yields the expression for the body drag D,

$$D = \int_S \{p_0 - p - \rho\ [(U_0+u)^2 - U_0^2]\}\ dS \tag{68}$$

in which ρ is the mass density of the fluid. If the wake is turbulent, Reynolds stresses will be present, but these can be taken into account most efficiently by averaging the resulting expression derived for the drag. In terms of the total heads, defined by

$$\rho g\ H_0 = p_0 + \frac{1}{2}\ \rho\ U_0{}^2, \quad \rho g\ H = p + \frac{1}{2}\ \rho\ [(U_0+u)^2+v^2+w^2] \tag{69}$$

where g is the acceleration of gravity, (68) becomes

$$D = \int_S \left\{ \rho g\ (H_0-H) - \frac{1}{2}\ \rho\ [(U_0+u)^2-U_0{}^2-v^2-w^2] \right\} dS \tag{70}$$

We now consider an equivalent irrotational velocity field $(U_0 + u_1, v_1, w_1)$, with pressure p_1, generated by a volume distribution of sources of strength μ in BLW, such that $(u_1, v_1, w_1) \equiv (u,v,w)$ on T, the outer boundary of BLW. We again apply the momentum theorem, to the flow within the same control surface generated by this distribution of so-called Betz sources, to obtain the expression for the force on the sources within the control volume,

$$D_s = -\frac{1}{2}\ \rho \int_S [(U_0+u_1)^2-U_0{}^2-v_1{}^2-w_1{}^2]\ dS \tag{71}$$

the term corresponding to the difference in heads vanishing since the field is irrotational.

Another expression for the force on the Betz sources is given by the Lagally formula

$$D_s = -\ 4\pi\rho \int_{\mathcal{V}} \mu(U_0+u_1)\ d\tau \tag{72}$$

where $\forall$ denotes the control volume. This is not the usual application of the Lagally theorem, which gives the force on a closed body. Here it represents the reaction on the Betz sources due to the flux of momentum of the source-generated discharge.

Since the flux through the bounding surface T is the same with Betz sources as in the solenoidal (divergenceless) real flow, the difference in flux for the flows through the area A of S intersected by the wake is attributable to the Betz sources. The Gauss flux theorem then yields the formula

$$4\pi \int_{\forall} \mu \, d\tau = \int_{A} (u_1 - u) \, dS \tag{73}$$

Far downstream, the section S will be denoted by S_∞ and the wake area by A_∞. In terms of the flux Q across this wake area, we have the well-known formula for the drag,

$$D = \rho \, U_0 Q, \quad Q = - \int_{A_\infty} u \, dS \tag{74}$$

A similar expression for the force on the Betz sources, obtained from the asymptotic form of (71), is

$$D_{s_\infty} = - \rho \, U_0 \int_{S_\infty} u_1 \, dS$$

But

$$\int_{S_\infty} u_1 \, dS = \int_{S_\infty} (u_1 - u) \, dS + \int_{S_\infty} u \, dS$$

and since $u_1 \equiv u$ except within the wake and, by continuity, $\int_{S_\infty} u \, dS = 0$, we obtain

$$D_{s\infty} = - \rho U_0 \int_{A_\infty} (u_1 - u)\, dS \tag{75}$$

and hence, from the asymptotic form of (73),

$$D_{s\infty} = - 4\pi\rho U_0 \int_{BLW} \mu\, d\tau \tag{76}$$

Comparison with the asymptotic form of (72) now yields

$$\int_{BLW} \mu\, u_1\, d\tau = 0 \tag{77}$$

From the asymptotic form of (73), we can also show that

$$4\pi \int_{BLW} \mu\, d\tau = Q \tag{78}$$

which implies, by (74) and (76), that

$$D_{s\infty} = - D \tag{79}$$

This is derived by neglecting u_1 in comparison with u in (6), since the Betz sources are concentrated near the body, so that u_1 diminishes as the inverse square of the distance, while, for a turbulent wake, $|u|$ diminishes as the inverse 2/3-power of the distance.

Put $\forall + \forall' = \forall_{BLW}$ and define a mean value $\bar{u}_1$ of u_1 by

$$\bar{u}_1 \int_{V'} \mu\, d\tau = \int_{V'} \mu\, u_1\, d\tau \tag{80}$$

We then obtain, from (77),

$$-\int_{\forall} \mu\, u_1\, d\tau = \int_{\forall'} \mu\, u_1\, d\tau = \bar{u}_1 \int_{\forall'} \mu\, d\tau$$

But, by (73) and (78),

$$\int_{\forall'} \mu\, d\tau = \int_{BLW} \mu\, d\tau - \int_{\forall} \mu\, d\tau = \frac{Q}{4\pi} - \frac{1}{4\pi} \int_{A} (u_1 - u)\, dS$$

Then, by (74),

$$-\int_{\forall} \mu\, u_1\, d\tau = \frac{\bar{u}_1}{4\pi} \left[\frac{D}{\rho U_0} - \int_{A} (u_1 - u)\, dS \right] \qquad (81)$$

Hence, by (73) and (81), (72) becomes

$$D_s = -\rho \int_{A} (u_1 - u)(U_0 + \bar{u}_1)\, dS + \frac{\bar{u}_1}{U_0} D \qquad (82)$$

A formula for the viscous drag D can now be obtained by subtracting the expression for D_s in (71) from that for D in (70) and then substituting for D_s from (82). Observing that the resulting integrand is nonzero only over the wake area A, we obtain the result

$$D = \frac{\rho/2}{1 - \dfrac{\bar{u}_1}{U_0}} \int_{A} \left[2g(H_0 - H) + (u_1 - u)(u_1 + u - 2\bar{u}_1) + v^2 + w^2 - v_1^{\,2} - w_1^{\,2} \right] dS \qquad (83)$$

In applying (83), total head tubes which measure a total head H_m, given by

$$\rho g\, H_m = p + \frac{1}{2}\rho\, [(U_0+u)^2+\lambda(v^2+w^2)],\quad 0 < \lambda < 1 \tag{84}$$

where λ is a calibration constant, (see Reference [9]), and velocities

$$u_m = \left[\frac{2}{\rho}\,(H_m-p)\right]^{1/2} - U_0 \tag{85}$$

are used for H and u. In terms of H_m and u_m, with the small difference between u_m and u neglected in higher-order terms, (83) becomes

$$D = \frac{\rho/2}{1-\bar{u}_1/U_0}\int_A [2g(H_0-H_m)+(u_1-u_m)(u_1+u_m-2\bar{u}_1) -v_1^{\,2}-w_1^{\,2}+\lambda(v^2+w^2)]\; dS \tag{86}$$

When the wake is turbulent, the mean value of D, obtained by replacing (u,v,w) by (u+u',v+v',w+w') and averaging, where (u',v',w') denote the components of the turbulent velocity fluctuations, becomes

$$D = \frac{\rho/2}{1-\bar{u}_1/U_0}\int_A \Big[2g(H_0-H_m)+(u_1-u_m)(u_1+u_m-2\bar{u}_1)-v_1^{\,2}-w_1^{\,2} +\lambda\,(v^2+w^2)\,-\overline{u'^2}+\lambda\,(\overline{v'^2}+\overline{w'^2})\Big]\; dS \tag{87}$$

where H_m and u_m now refer to their mean values. The Reynolds stress terms combine into $(2\lambda-1)\overline{u'^2}$ for isotropic turbulence, and would hence be negligible for $\lambda = 1/2$. Jin Wu's measurements [7] indicate that the turbulent stresses would contribute about 2 percent to the calculated drag with $\lambda = 0$; but his actual value was $\lambda = 0.5$, with which the turbulence terms became negligible. We shall assume that the turbulence stresses in (87) may be neglected. We shall also neglect the terms $-v_1^2 - w_1^2 + \lambda(v^2+w^2)$ since these are small and partly cancel each other.

Still unknown are the terms u_1 and $\bar{u}_1$ in (87). For estimating μ_1, we shall assume that u_1 depends upon z alone, and is given by $u_1 = u_E(z)$, the measured value of u at the wake boundary T. According to the definition of $\bar{u}_1$ as a mean of u_1 in $\forall'$, we observe that the mean is weighted by the value of the source strength μ which, together with u_1, diminishes to zero as $x \to \infty$. This suggests that the values $\bar{u}_1 = 0$ and $\bar{u}_1 = \bar{u}_E$ at A can be used to obtain bounds for the drag formula, the "true" value lying closer to the bound given by $\bar{u}_1 = \bar{u}_E$. Because $\bar{u}_1$ occurs both in the integrand and in the denominator of the expression for the drag, it is not immediately evident which of the two bounds is the larger. Denoting these bounds by D_1 when $\bar{u}_1 = 0$ and D_2 when $\bar{u}_1 = \bar{u}_E$, and applying the aforementioned approximations, we obtain from (87),

$$D_1 = \frac{\rho}{2} \int_A [2g(H_0-H_m)+u_E^2-u_m^2]\, dS \tag{88}$$

and, with $u_E + u_m - 2\bar{u}_E$ replaced by $u_m - u_E$ in the last term,

$$D_2 = \frac{\rho/2}{1-\bar{u}_E/U_0} \int_A [2g(H_0-H_m)-(u_E-u_m)^2]\, dS \tag{89}$$

The latter form was given by Tzou and Landweber [8].

That $D_1 < D_2$ is indicated by the following argument. Since the source strength μ represents the displacement effect of the boundary layer and wake, it is a positive quantity of total strength given by (78). Consequently, according to (77), u_1 cannot be of one sign throughout BLW. Over most of the boundary-layer edge of T, $\bar{u}_E > 0$. In the region of the rear of the body and the near wake, there would be some pressure recovery (especially if separation has not occurred) and u_E on T would there become negative, and gradually approach zero with increasing downstream distance. This suggests that $\bar{u}_E$ in (89) and $\bar{u}_1$ in (86) are negative. Since the difference in heads, $H_0 - H_m$, contributes about 90 percent of the magnitude of the integrand in (88) and (89), and $\bar{u}_E << U_0$, (89) may be written as

$$D_2 \doteqdot \frac{\rho}{2}\left(1+\frac{\bar{u}_E}{U_0}\right)\int_A 2g(H_0-H_m)\,dS - \frac{\rho}{2}\int_A (u_E-u_m)^2\,dS$$

which yields

$$D_2 - D_1 \doteqdot \rho\,\frac{\bar{u}_E}{U_0}\int_A g(H_0-H_m)\,dS - \rho\int_A u_E(u_E-u_m)\,dS$$

$$\doteqdot \rho\,\frac{\bar{u}_E}{U_0}\int_A [g(H_0-H_m)-U_0(u_E-u_m)]\,dS \qquad (90)$$

Applying the expression for H_m in (85) and

$$\rho g\,H_0 = p_E + \frac{1}{2}\,\rho(U_0+u_E)^2$$

in (90) and neglecting the term $u_E^2 - u_m^2$, we obtain

$$D_2 - D_1 \doteqdot \frac{\bar{u}_E}{U_0}\int_A (p_E-p_m)\,dS > 0 \qquad (91)$$

since $\bar{u}_E < 0$ at A and, according to Jin Wu's data [10], p_m is a maximum at the center of the wake, so that $p_m - p_E \geq 0$. The data given in Reference [8] indicate, however, that, at a section at 0.6 of the length behind the stern of a ship model, $\bar{u}_E$ is zero within the accuracy of the measurements. This indicates that the simpler expression (88) is suitable for computing the viscous drag from wake-survey data.

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