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COMBINATORIAL TOPOLOGY AND TRADE OFF METHOD IN BIB DESIGNS.(U)
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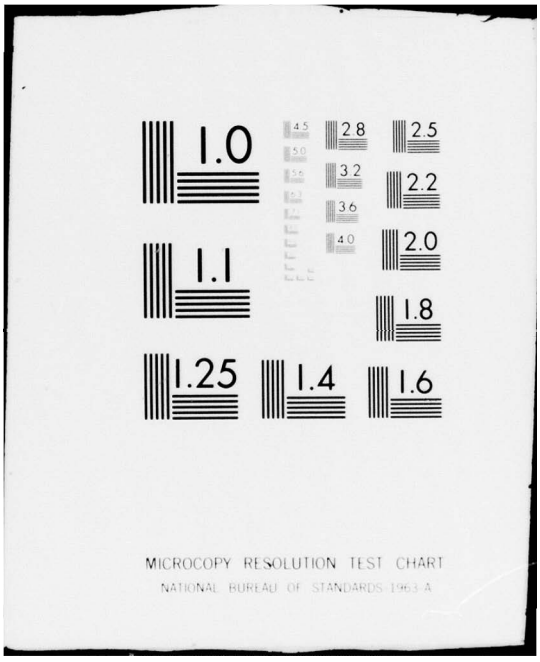
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Combinatorial Topology and the Trade Off Method in BIB Designs*

by

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July 1978

* Invited paper presented at International Symposium on Combinatorial Mathematics and Optimal Design held in Colorado State University during June 5-9, 1978.

1. Research supported in part by AFSOR 76-3050.

2. Research supported in part by NSF Grant No. MCS77-03533.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER: AFOSR-TR- 78-1457	2. GOVT ACCESSION NO. / 3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): COMBINATORIAL TOPOLOGY AND TRADE OFF METHOD IN BIB DESIGNS,	5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): A. Hedayat and Shou-Yen Robert Li (Shuo)	8. CONTRACT OR GRANT NUMBER(s): NSF-MCS77-05533 / AFOSR-76-3050
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Illinois at Chicago Circle, Department of Mathematics, Chicago, Illinois 60680	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F / 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332	12. REPORT DATE: July 1978
	13. NUMBER OF PAGES: 21
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

(12) 24 pp

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Minimal trades, Eulerian triangulation, compact connected surface, planar graphs.

(V/2)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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20. Abstract continued.

pairs of varieties is contained in the same number of blocks from both collections. A method, based on combinatorial topology, for studying the trades with block size 3 is introduced. Using this method it is proven that the volume of a trade can never be 5. Also we present a simple topological proof of the existence of a unique type of minimal trades, into which every trade can be decomposed. We also give a short proof of the fact that $BIB(v,b,r,k,\lambda)$ designs, with possibly negative frequency of blocks, exist whenever the necessary conditions $rv = bk$ and $\lambda(v-1) = r(k-1)$ hold. The technique in this proof is a simple version of Graver and Jurkat (1973) concerning null t-design.

Combinatorial Topology and the Trade Off Method in BIB Designs*

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Abstract.

When two BIB designs based on the same set of varieties have identical parameters, one can obtain either of them from the other by trading off some blocks for an equal number of blocks. Define a trade to be two collections of blocks such that each of the $\binom{v}{2}$ pairs of varieties is contained in the same number of blocks from both collections. A method, based on combinatorial topology, for studying the trades with block size 3 is introduced. Using this method it is proven that the volume of a trade can never be 5. Also we present a simple topological proof of the existence of a unique type of minimal trades, into which every trade can be decomposed. We also give a short proof of the fact that BIB(v,b,r,k,λ) designs, with possibly negative frequency of blocks, exist whenever the

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necessary conditions $rv = bk$ and $\lambda(v-1) = r(k-1)$ hold. The technique in this proof is a simple version of Graver and Jurkat (1973) concerning null t-design.

0. Introduction.

Let v,b,r,k,λ be positive integers related by the equations $rv = bk$ and $\lambda(v-1) = r(k-1)$. Let Σ denote the set $\{1,2,\ldots,v\}$. A 2-element subset of Σ shall be called a pair and a k-element subset shall be called a block.

Let P denote the incidence matrix of pairs versus blocks, which is a $\binom{v}{2}$ by $\binom{v}{k}$ zero-one matrix. A $\binom{v}{k}$ dimensional column vector D is called a $BIB(v,b,r,k,\lambda)$ design, or simply a $BIB(v,k,\lambda)$ design, if

$$PD = \lambda \underline{1} ,$$

where $\underline{1}$ is the vector with all entries equal to 1. Unless otherwise mentioned, we require the entries in the design D to be non negative. An entry in D represents the multiplicity (frequency) that the corresponding block appears in the design. An integer vector T of the same dimension is called a (v,k) trade if

$$PT = \underline{0}.$$

The sum of all positive entries in a trade is called its volume.

Let D be a $BIB(v,k,\lambda)$ design. For every (v,k) trade T, the vector $D + T$ is another $BIB(v,k,\lambda)$ design provided that all of its entries are non negative. Conversely every $BIB(v,k,\lambda)$ design can be written in the form of $D + T$.

As explained in Foody and Hedayat (1977) and Hedayat and Li (1978), it is desirable to search for different BIB designs with identical parameters for the purpose of experimental designing or controlled sampling. Given a design D to start with, in order to search for designs with the same parameters it suffices to investigate the trades. The purpose of this article is to introduce a combinatorial topological method for studying trades on blocks of size 3. We shall demonstrate the convenience of this method in constructing trades and then use this method to prove two theorems characterizing the trades. Theorem 4.1 in Hedayat and Li (1978) asserts that the volume of a trade can be any non negative integer except 1,2,3, and 5. There the proof of nonexistence of trades of volume 5 depends on a statement in Foody and Hedayat (1977) which follows from an argument of exhaustive search. But this argument was not presented, because it would be too long and tedious. In Section 2, we shall give a topological proof of the non existence of trades of volume 5. The smallest volume of a trade is 4, and there exists a unique trade with volume 4 up to isomorphism. In Section 3, we prove that every trade is a linear combination of finitely many trades of this minimal type. This fact is a special case of Theorem 5.1 in Graver and Jurkat (1973) about t-designs.

1. The Compact Surface Associated With a Trade.

For convenience the block consisting of the unordered elements $x_1,\ldots,x_k$ will be denoted by $(x_1 \ldots x_k)$. This is

equivalent to a $\binom{v}{k}$ dimensional column vector with one of the entries equal to 1 and all others 0. Similarly the typical notation for a pair will be (xy).

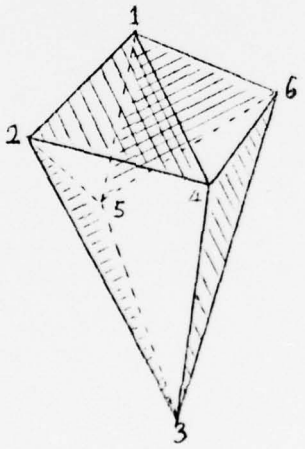
We now direct our attention to studying (v,3) trades.

Example 1.1. (125) + (146) + (234) + (356) - (124) - (156) - (235) - (346) represents a trade. When this trade is added to the design (124) + (137) + (156) + (235) + (267) + (346) + (457), we obtain another design (125) + (137) + (146) + (234) + (267) + (356) + (457). In other words, from the first design the four blocks (124), (156), (235), and (346) have been traded for the blocks (12), (146), (234), and (356) to obtain the second design.

Now we introduce a geometric representation of the (v,3) trades. Given a trade T, construct a compact surface without boundary as follows. First create two collections of 2-simplexes (triangles) with their vertices labeled by elements of V. The 2-simplexes in one collection will be called the positive triangles and those in the other collection will be called the negative triangles. For every term +(xyz) in T, there corresponds a positive triangle with vertices labeled by x, y, and z. If the coefficient of (xyz) in T is $m > 1$, then there are m copies of such a triangle. On the other hand, for every term -(xyz) in T, there corresponds a negative triangle in the similar manner. So every pair (xy) appears on the same number of triangles in both collections.

Thus, there exists a one-to-one matching between the edges of positive triangles and the edges of negative triangles so that every matched pair share the same two labels. When we identify every matched pair of edges in the natural way, we obtain a compact surface without boundary. Here we emphasize the possible nonuniqueness of the matching. Different matchings may lead to different geometric configurations. (See Examples 1.4 and 1.6 below). Also the labels on the vertices are not necessarily all distinct.

Example 1.2. The trade in Example 1.1 is represented by the diamond-shaped topological sphere



Here in the picture the shaded regions are the negative triangles.

In general, a trade give rise to a compact surface that is partitioned into positive triangles and negative triangles with the following two properties.

(1) Any two positive triangles can not intersect each other

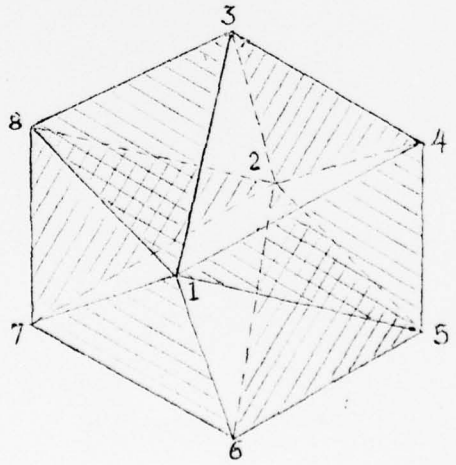
except possibly at their vertices. Neither can any two negative triangles.

(2) The intersection of a positive triangle with a negative triangle is vaccum, or one vertex, or two vertices, or an edge.

We shall refer to such a partition of surfaces, with or without boundary, as an Eulerian triangulation, although it is not quite a triangulation in the usual sense of algebraic topology. The edges of the triangles form an Eulerian graph* on the surface, i.e., a graph such that the degree (valency) of every vertex is an even integer. Also no vertex can have degree equal to two, because then there would be two triangles sharing two common edges.

The following example of trade is also obtained by triangulating a sphere.

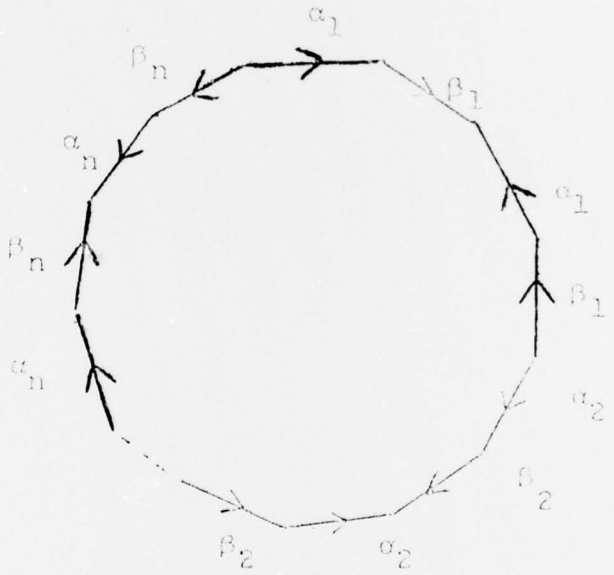
Example 1.3:



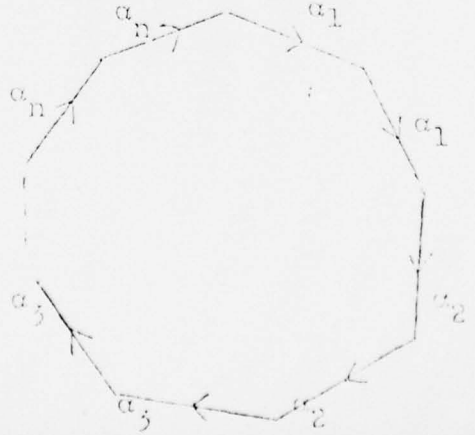
* A more precise terminology would be Eulerian multigraph than Eulerian graph according to Harary (1969).

This figure represents the trade (134) + (156) + (178) + (238) + (245) + (267) - (138) - (145) - (167) - (234) - (256) - (278). Again the shaded regions are the negative triangles.

It is well-known that a compact connected surface is either a sphere, or a connected sum of tori, or a connected sum of projective planes (see, for example, Theorem 5.1 in Massey (1967)). The standard presentation of the connected sum of n tori is by identifying edges of a 4n-gon in pairs.

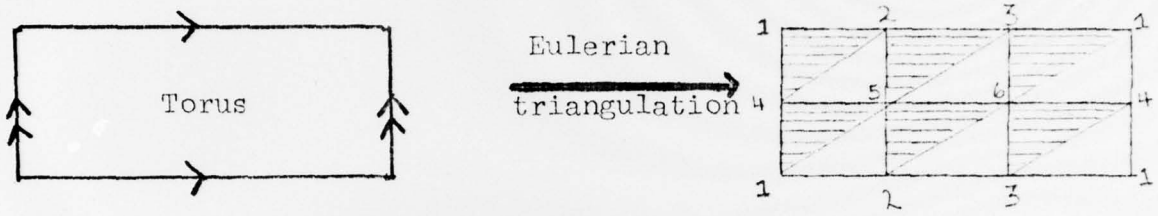


Similarly for the connected sum of n projective planes we have the following figure.

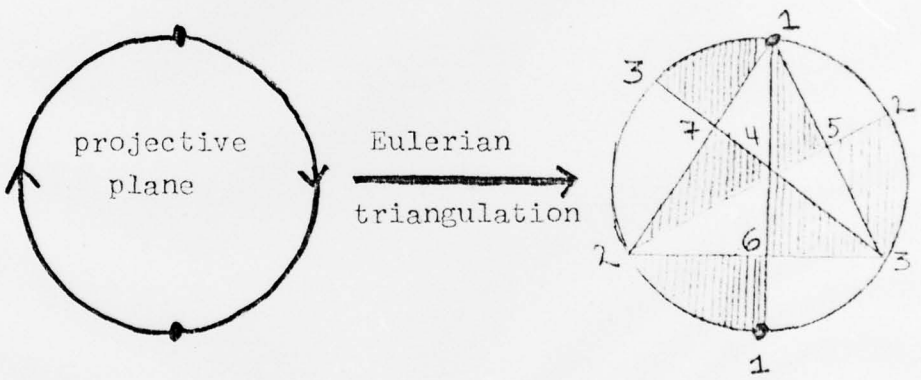


Using these standard presentation of surfaces, we can easily construct more trades.

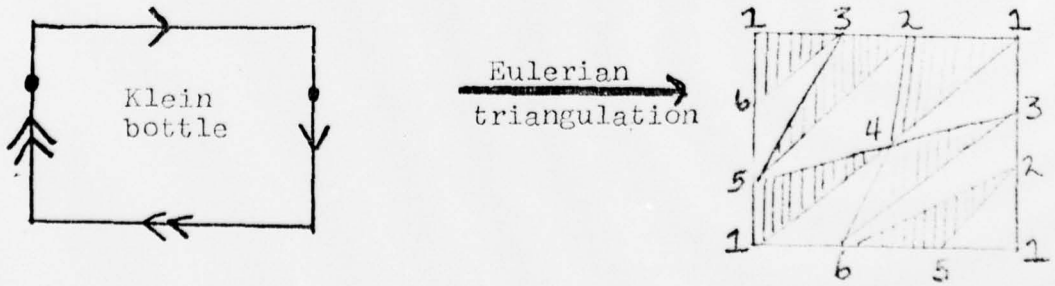
Example 1.4.



Example 1.5.

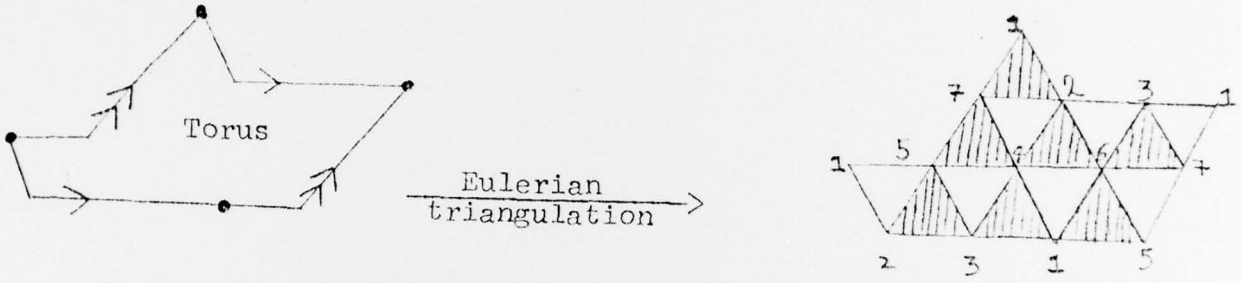


Example 1.6.

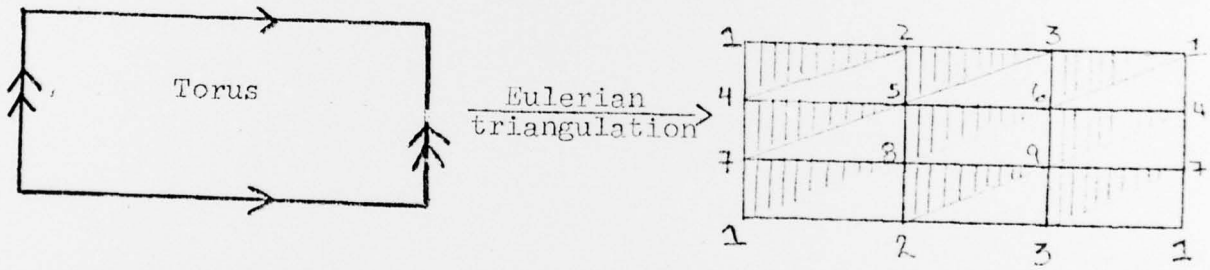


Note that the figures in Example 1.4 and 1.6 represent the same trade.

Example 1.7.



Example 1.8.



2. Nonexistence of Trades of Volume 5.

We have seen the convenience in constructing trades from the concept of Eulerian triangulation. In the proof of Theorem 2.1 below, we shall also find the same concept useful in establishing negative results. First we state a couple of self-evident lemmas.

Lemma 2.1. For every Eulerian triangulation of a compact surface with boundary, the number of boundary edges that are on positive triangles differs from the number of those on negative triangles by a multiple of 3.

Lemma 2.2. There exist no trades of volume 1, 2 or 3; therefore the minimum trade volume is 4.

Lemma 2.3. If a disc is Eulerian triangulated with exactly 2 boundary edges, then

(i) exactly one boundary edge is on a positive triangle and the other is on a negative triangle, and

(ii) there are at least 4 positive and 4 negative triangles.

Proof: Statement (i) follows directly from Lemma 2.1. From this, we know the Eulerian triangulation represents a trade, even though the surface has a boundary. The second statement now follows from Lemma 2.2.

Theorem 2.1. There exist no trades of volume 5.

Proof: Assuming there exists compact surface without boundary that has been Eulerian triangulated by exactly 5 positive and

5 negative triangles, we want to derive a contradiction. First, we know that the triangulation on every connected component of the surface represents a trade. So the surface must be connected by Lemma 2.2. There are 10 triangles in total, so there are 15 edges. Let n be the number of vertices. The Euler characteristic of this surface is

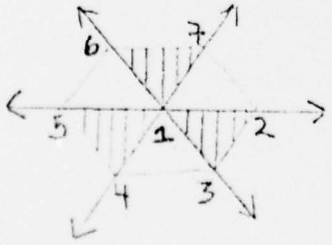
$$\begin{aligned} v &= n - 15 + 10 \\ &= n - 5 \\ &\leq 2. \end{aligned}$$

The inequality has been due to the connectedness. We label the vertices by $1,2,\ldots,n$, respectively. There are three cases to examine.

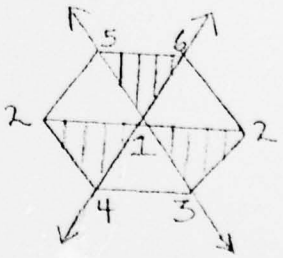
Case 1. $v = 2$. Then $n = 7$ and the surface is a topological sphere. The edges in the triangulation form a planar graph and its degree sequence is

$$(6,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4\ ).$$

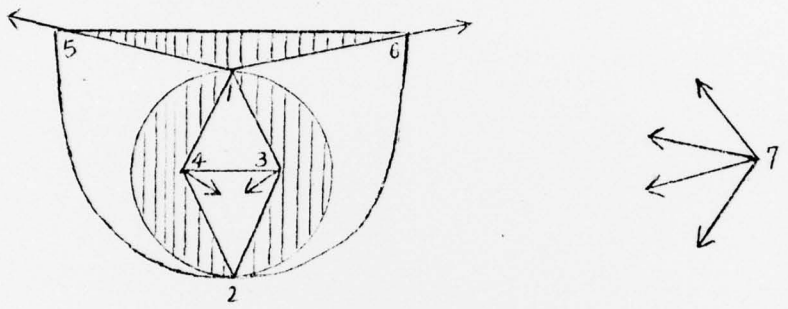
With a suitable relabeling, the neighborhood around the vertex of degree 6 is as in either graph below.



or



In the first graph, the six arrows are supposed to be linked in pairs to form a planar graph, but this is obviously impossible. After identifying the two points labeled as 2, the second graph lead to the following configuration.



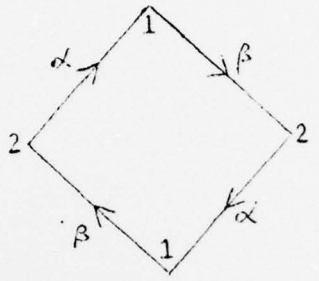
Again the arrows can not be linked in pairs to form a planar graph.

Case 2. $\chi = 1$. Then $n = 6$ and the surface is a projective plane. The degree sequence has to be one of the following three:

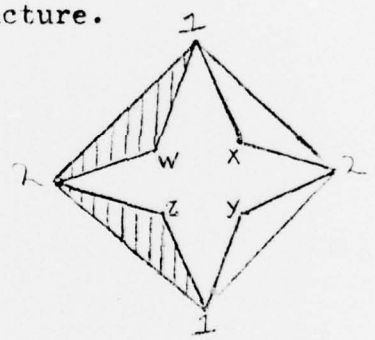
$$(6, 6, 6, 4, 4, 4)$$
$$\text{or } (8, 6, 4, 4, 4, 4)$$
$$\text{or } (10, 4, 4, 4, 4, 4).$$

Since in any case some vertex has degree at least 6, we may assume that there are two edges α and β joining between vertices 1 and 2. These two edges form a cycle. Since the fundamental group of a projective plane is $Z/2Z$, this cycle is either trivial or is the generator of the fundamental group.

First we assume that the cycle generates the fundamental group. Then the projective plane can be drawn as a square with edges identified in pairs as in below.



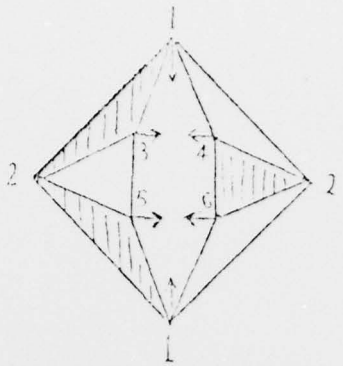
So we have an Eulerian triangulation of the square disc based on the following picture.



Here $w, x, y, z \in \{3,4,5,6\}$ and $w \neq x$, $w \neq z$, $x \neq y$, $y \neq z$. From Lemma 2.3, we also have $w \neq y$ and $x \neq z$. So w, x, y, and z are all distinct. By symmetry, let $w = 3$, $x = 4$, $y = 5$, and $z = 6$. Observe that vertex 1 must have degree more than 6 and vertex 2 has degree at least 6. Therefore the degree sequence is

$$(8, 6, 4, 4, 4, 4),$$

and the arrows in the following graph should be linked in pairs to form the triangulation.



But this is obviously impossible.

We now assume that α and β form a trivial cycle. The cycle then cuts the projective plane into two parts: a <u>disc</u> and a <u>Möbius band</u>. From Lemma 2.3 the Eulerian triangulation on the disc part takes at least 4 positive and 4 negative triangles. So the Möbius band is Eulerian triangulated by at most 1 positive and 1 negative triangles. This is a contradiction.

<u>Case 3</u>. $v \leq 0$. Then $n \leq 5$. We need to show the nonexistence of a trade T of volume 5 on 5 or less symbols. First we may assume that T is of the form

$$(123) - (124) - (134) - (23x) + - \ldots,$$

where $x = 4$ or 5. Then the coefficient of the block (145) in T must be at least 2. Thus

$$T = (123) + 2(145) - (124) - (134) - (23x)$$
$$- (1y5) - (1z5) - (u45) - (v45) + - \ldots$$

But this implies that T has volume at least 7, a contradiction.

The theorem is proved.

3. Decomposition of a Trade into Minimal Trades.

The minimum volume of a trade is 4. One observes the following two easy facts.

Proposition 3.1. If $v \leq 5$, there exists no nontrivial trade.

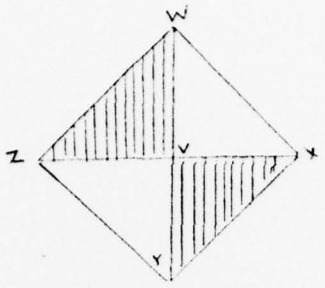
Proposition 3.2. If $v \geq 6$, there exists a unique trade of volume 4 up to isomorphism. This trade is represented by a diamond-shaped topological sphere as in Example 1.2.

Let an Eulerian triangulated compact surface without boundary be given. We shall prove that by properly attaching diamond-shaped topological spheres to the surface one can obtain an Eulerian triangulation which represents the trivial trade. This is equivalent to the following.

Theorem 3.1. Every $(v,3)$ trade is a linear combination of trades of volume 4.

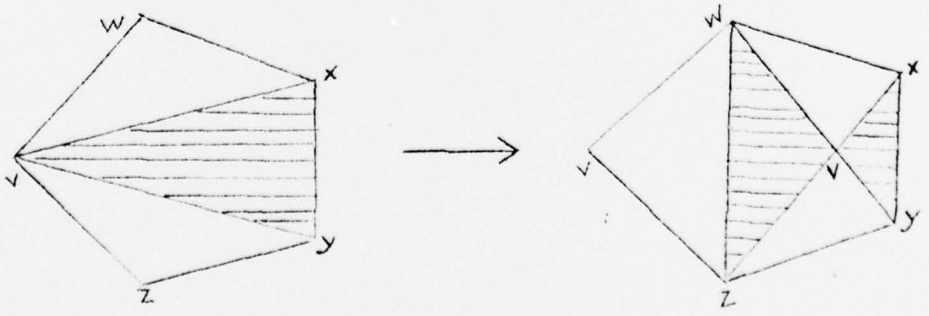
The proof is by induction on v. When $v \leq 5$, the Eulerian triangulation to start with is representing the trivial trade because of Proposition 3.1. So we shall assume that $v \geq 6$ and at least one vertex on the surface is labeled by v. It suffices to show that the total degree of all vertices labeled by v can be reduced when a diamond-shaped topological sphere is properly attached to the surface. Consider two cases.

Case 1: There exists a vertex of degree 4 which is labeled by v. Say, the neighborhood around this vertex is as in the following picture.



Choose $u < v$ such that $u \neq w, x, y, z$. Take the diamond-shaped topological sphere in Example 1.2. Replace the vertices 1, 2, 3, 4, 5, 6 in it by 3, v, x, y, w, u, respectively, and then attach it to the surface by identifying the four triangles on the surface around the vertex v with the corresponding four triangles on the topological sphere. The result is a surface with the same Eulerian triangulation except that the vertex originally labeled by v receives the new label u.

Case 2: There exists a vertex of degree more than 4 which is labeled by v. We may modify the triangulation according to the following picture, and the resulting triangulation represents the same trade.



This creates a vertex of degree 4 which is labeled by v. The procedure in the previous case now applies and reduces the total degree of vertices which have the label v. The proof is now completed by induction.

In an algebraic setting Graver and Jurkat (1973) proved that every (v,k) trade is a linear combination of trades of volume 4, where $k \geq 3$ is arbitrary.

4. BIB Designs with Possibly Negative Frequency of Blocks

Graver and Jurkat (1973) and Wilson (1973) showed that t-designs, with prescribed parameters satisfying standard necessary conditions, always exist if negative frequencies of blocks in the design are allowed. For 2-designs, i.e., BIB designs, the standard necessary conditions on the parameters are the equations $rv = bk$ and $\lambda(v-1) = r(k-1)$.

Assuming these are true we shall in the following paragraphs, give a short proof the existence of a BIB(v,k,λ) design when negative frequency of blocks are allowed. From this one can construct BIB(v,k,λ) designs for a sufficiently large λ by super imposing copies of complete designs to small designs (see Wilson (1973)).

Given parameters v, b, r, k, λ satisfying the two standard necessary condition, we want to construct a BIB design with possibly negative frequencies of blocks. First we take a collection of b blocks so that every variety is repeated r times in the collection. Represent this collection by a $\binom{v}{k}$ dimensional column vector V. As before let P be the incidence matrix of pairs versus blocks. Also let Q be the incidence matrix of varieties versus pairs. Then $\frac{1}{k-1}QP$ is the incidence matrix of varieties versus blocks. Therefore

$$\begin{aligned} QPV &= (k-1)r\, 1_v \\ &= \lambda(v-1)\, 1_v \\ &= \lambda Q\, 1_{\binom{v}{2}} . \end{aligned}$$

Here 1_v and $1_{\binom{v}{2}}$ are the v- and $\binom{v}{2}$ -dimensional vectors with all entries equal to 1. Thus PV and $\lambda 1_{\binom{v}{2}}$ represent two collections of pairs, each of them covering every variety $\lambda(v-1)$ times. In other words, $PV = \lambda_{\binom{v}{2}}$ is a trade off between 1-designs. We may assume that $PV - \lambda_{\binom{v}{2}}$ is a linear combination of alternating sums of the form

$(x_1,y_1) - (y_1,x_2) + (x_2,y_2) - (y_2,x_3) + -\ldots+(x_n,y_n) - (y_n,x_1)$

Moreover one may assume that n is equal to 2 in every alternating sum because of further decomposition in the straightforward manner.

To avoid trivial cases, let us assume that $v \geq k + 2$. Let W be the $\binom{v}{k}$ dimensional vector representing $(x_1\ y_1\ x_3 \ldots x_k) - (y_1\ x_2\ x_3 \ldots x_k) + (x_2\ y_2\ x_3 \ldots x_k) -$ $(y_2\ x_1\ x_3 \ldots x_k)$, where $x_1, x_2, \ldots x_k, y_1, y_2$ are distinct varieties. Then the vector PW represents $(x_1\ y_1) - (y_1\ x_2) + (x_2\ y_2) - (y_2\ x_1)$. By summing vectors of the type of W, we obtain a vector U such that $PU = PV - \lambda\ 1_{\binom{v}{2}}$. The vector V - U is a BIB(v,b,r,k,λ) design. Some entries in V - U may be negative.

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