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PERIODIC SOLUTIONS OF A HAMILTONIAN SYSTEM ON A PRESCRIBED ENER--ETC(U)

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PERIODIC SOLUTIONS OF A HAMILTONIAN
SYSTEM ON A PRESCRIBED ENERGY SURFACE

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PERIODIC SOLUTIONS OF A HAMILTONIAN SYSTEM
ON A PRESCRIBED ENERGY SURFACE

Paul H. Rabinowitz

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ABSTRACT

The existence of periodic solutions having prescribed energy for a Hamiltonian system of ordinary differential equations is studied. It is shown in particular that if the Hamiltonian is of classical type, i.e., $H(p,q) = K(p,q) + V(q)$ where $p, q \in \mathbb{R}^n$, and K and V satisfy $K(0,q) = 0$, $p \circ K_p(p,q) > 0$, $K(p,q) \rightarrow \infty$ as $|p| \rightarrow \infty$, $D = \{q \in \mathbb{R}^n \mid 0 \leq V(q) < 1\}$ is diffeomorphic to the unit ball in \mathbb{R}^n and $V_q \neq 0$ on ∂D , then Hamilton's equations,

$$(*) \quad \dot{p} = -H_q, \quad \dot{q} = H_p,$$

have a periodic solution on $H^{-1}(1)$.

Key Words: Hamiltonian system of ordinary differential equations, energy surface, periodic orbit, critical point, action integral, canonical transformation, Lagrange multipliers, index theory, equivariant mapping

Subject Classification: 34C15, 34C25

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SIGNIFICANCE AND EXPLANATION

Hamilton's equations are basic in the study of theoretical mechanics. A particular class of motions of interest for (*) are periodic ones. For Hamiltonians which are of the form $H(p,q) = K(p,q) + V(q)$, we give sufficient conditions for the kinetic and potential energies K and V to satisfy so that (*) possesses a periodic orbit on a prescribed energy surface.

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Periodic solutions of a Hamiltonian system
on a prescribed energy surface

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Paul H. Rabinowitz

§1. Introduction

Consider the Hamiltonian system of ordinary differential equation

$$(1.1) \quad \dot{z} = \mathcal{J}H_z, \quad \mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

where $z = (p, q)$, $p, q \in \mathbb{R}^n$, and $H: \mathbb{R}^{2n} \rightarrow \mathbb{R}$. A basic question of interest in the study of Hamiltonian mechanics is the existence of periodic solutions of (1.1) on a given energy surface. Simple examples show that there need not exist any periodic orbits [1]. Some recent results of a positive nature were proved simultaneously by A. Weinstein [2] and the author [3]:

Theorem 1.2: Let $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ with $H^{-1}(1)$ a manifold which is diffeomorphic to S^{2n-1} under radial projection. Then (1.1) possesses a periodic solution on $H^{-1}(1)$.

Weinstein actually obtained the special case in which $H^{-1}(1)$ bounds a convex region and $H \in C^2$. A key step in his proof is the following theorem which is of independent interest:

Theorem 1.3: Let $H(p, q) = K(p, q) + V(q)$ where $K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and satisfy:

- (V₁) If $D = \{q \in \mathbb{R}^n \mid 0 \leq V(q) < 1\}$, then $V_q \neq 0$ on ∂D .
- (V₂) There exists a C^2 diffeomorphism ψ of \mathbb{R}^n to \mathbb{R}^n such that the open unit ball, $B_1(0)$, in \mathbb{R}^n is diffeomorphic to D .
- (K₃) For each $q \in \bar{D}$, $K(0, q) = 0$ and $K(p, q)$ is even and strictly convex in p .
- (K₂) For fixed $q \in \bar{D}$ and $p \in S^{n-1}$,

$$\lim_{\alpha \rightarrow \infty} K(\alpha p, q) > 1 - V(q).$$

Then there exists a $T > 0$ and a solution $z = (p, q)$ of (1.1) such that $q(0), q(T) \in \partial D$ and $q(t) \in D$ for $0 < t < T$.

Since $H(z(t)) \equiv 1$ for this solution, $p(0) = p(T) = 0$. Extending p as an odd function and q as an even function about 0 and T and using (K₃) then gives a $2T$ periodic solution of (1.1).

Weinstein's proof of Theorem 1.3 is based in part on earlier work of Seifert [4] on the special case of $K(p, q) = \sum a_{ij}(q) p_i p_j$ where $(a_{ij}(q))$ is a positive definite matrix uniformly for $q \in \bar{D}$. Roughly speaking, the solutions of Theorem 1.3 are determined as geodesics for the Riemannian metric K of Seifert or the Finsler metric K of Weinstein.

A rather different procedure was used to prove Theorem 1.2 in [3]. (See also [5] for another proof). In this paper we will show how the method introduced in [3] can be applied to treat (1.1) under a weakened version of (K₃). To state it, for $a, b \in \mathbb{R}^n$, let $(a, b)_{\mathbb{R}^n}$ denote the usual \mathbb{R}^n inner product of a and b . If $\varphi : \mathbb{R}^u \times \mathbb{R}^v \rightarrow \mathbb{R}^Y$, $\varphi = \varphi(x, y)$, then φ_x denotes the Frechet derivative of φ with respect to x , etc.

Theorem 1.4: Let $H(p, q) = K(p, q) + V(q)$ where $K \in C^2(\mathbf{R}^{2n}, \mathbf{R})$, $V \in C^1(\mathbf{R}^n, \mathbf{R})$, V satisfies $(V_1) - (V_2)$, and K satisfies

(K_1) For $q \in \bar{D}$, $K(0, q) = 0$ and $(p, K_p(p, q))_{\mathbf{R}^n} > 0$ for $p \neq 0$,

and (K_2) . Then (1.1) possesses a periodic solution on $H^{-1}(1)$.

Remark 1.5: It is straightforward to verify that $(K_1) - (K_2)$ and $(V_1) - (V_2)$ imply that $H^{-1}(1)$ contains a compact manifold, \mathfrak{m} , which is the boundary of a neighborhood \mathfrak{O} of 0 in \mathbf{R}^{2n} . Moreover if $(p, q) \in \mathfrak{O}$ and $q \in D$, then $\mathfrak{O}_q = \{p \in \mathbf{R}^n \mid (p, q) \in \mathfrak{O}\}$ is a star-shaped region in \mathbf{R}^n . Also if $q \in D$ and $H(p, q) = 1$, then $(p, q) \in \mathfrak{m}$.

In §2, by making a canonical transformation, the proof of Theorem 1.4 will be reduced to the special case of $D = B_1(0)$. This transformation reduces the smoothness of H by one derivative, but fortunately if $D = B_1(0)$, H need only satisfy $H \in C^1(\mathbf{R}^{2n}, \mathbf{R})$ for our proof. Some further modifications are made to H in §2 converting it to a more suitable Hamiltonian for our methods. Finally in §3 we prove Theorem 1.4 for the modified problem. Following [3], the basic idea is to obtain a solution of (1.1) on $H^{-1}(1)$ as a critical point of the action integral

$$(1.6) \quad A(z) = \int_0^{2\pi} (p, \dot{q})_{\mathbf{R}^n} dt$$

subject to the constraint

$$(1.7) \quad \frac{1}{2\pi} \int_0^{2\pi} H(z) dt = 1 .$$

Here $z(t) = (p(t), q(t)) \in E \equiv (W^{1,2}(S^1))^{2n}$, i.e. E is the Hilbert space of $2n$ tuples of 2π periodic functions which together with their first derivatives are square integrable. Any critical point z of A subject to (1.7) and having a non-zero Lagrange multiplier λ will be a 2π periodic solution of

$$(1.8) \quad \dot{z} = \lambda \mathcal{J} H_z .$$

Since (1.8) is a Hamiltonian system, $H(z(t)) \equiv \text{constant}$ so (1.7) shows $z(t) \in H^{-1}(1)$. After rescaling t , z becomes a $2\pi\lambda$ periodic solution of (1.1).

We do not know a direct way to find critical points of A in E under the constraint (1.7). Therefore we use the approximation procedure of [3]. This involves restricting A to finite dimensional submanifolds of (1.7) and finding critical points of the approximate problem together with strong enough estimates to pass to a limit and find a solution of (1.8).

An interesting open question in the setting of Theorems 1.2, 1.3, or 1.4 is whether stronger statements can be made about the number of distinct periodic orbits on level sets of H . E.g. if H consists of a positive definite quadratic part + higher order terms at $z = 0$, a theorem of Weinstein ([1] or [6]) asserts the existence of at least n distinct periodic solutions on $H^{-1}(c)$ for all sufficiently small $c > 0$.

§2. The modified problem

We begin this section by reducing the proof of Theorem 1.4 to the following result:

Theorem 2.1: Let $H(p, q) = K(p, q) + V(q)$ where $K \in C^1(\mathbb{R}^{2n}, \mathbb{R})$, $V \in C^1(\mathbb{R}^n, \mathbb{R})$ and K, V satisfy

$$(V_1') \quad 0 \leq V < 1 \text{ in } B_1(0) \text{ and } V_q \neq 0 \text{ and } \partial B_1(0)$$

$$(K_1') \quad \text{For } q \in \overline{B_1(0)}, \quad K(0, q) = 0 \text{ and } (p, K_p(p, q))_{\mathbb{R}^n} > 0 \text{ if } p \neq 0$$

$$(K_2') \quad \text{For } q \in \overline{B_1(0)} \text{ and } p \in S^{n-1},$$

$$\lim_{\alpha \rightarrow \infty} K(\alpha p, q) > 1 - V(q) .$$

Then (1.1) possesses a periodic solution on $H^{-1}(1)$.

To carry out the reduction of Theorem 1.4 to Theorem 2.1, suppose the hypotheses of Theorem 1.4 are satisfied. We extend ψ to a canonical transformation of \mathbb{R}^{2n} to \mathbb{R}^{2n} in a standard fashion [7]. Set $q = \psi(Q)$ and $p = (\psi_Q(Q))^{-T} P$ where C^T denotes the transpose of the matrix C . The transformation $(P, Q) \rightarrow (p, q)$ is canonical if and only if

$$(2.2) \quad \mathfrak{L}^T \mathfrak{J} \mathfrak{L} = \mathfrak{J}$$

where

$$\mathbf{f} = \begin{pmatrix} \mathbf{G} & \mathbf{B} \\ \mathbf{C} & \mathbf{K} \end{pmatrix}, \quad \mathbf{G} = \frac{\partial \mathbf{q}}{\partial \mathbf{Q}}, \quad \mathbf{B} = \frac{\partial \mathbf{q}}{\partial \mathbf{P}}, \quad \mathbf{C} = \frac{\partial \mathbf{p}}{\partial \mathbf{Q}}, \quad \mathbf{K} = \frac{\partial \mathbf{p}}{\partial \mathbf{P}}.$$

Due to our choice of \mathbf{p} , this will be the case if and only if $(\psi_Q(\mathbf{Q}))^T \frac{\partial \mathbf{p}}{\partial \mathbf{Q}}$ is a symmetric matrix. To verify the symmetry, set $\mathbf{Q} = \varphi(\mathbf{q})$ where $\varphi = \psi^{-1}$ and $\mathbf{p} = \varphi_Q^T \mathbf{P}$. Let G_{ij} denote the element in the i^{th} row and j^{th} column of \mathbf{G} . Then

$$\begin{aligned} (2.3) \quad ((\psi_Q)^T \frac{\partial \mathbf{p}}{\partial \mathbf{Q}})_{ij} &= \sum_k \frac{\partial \psi_k}{\partial Q_i} \frac{\partial p_k}{\partial Q_j} \\ &= \sum_{k, \ell} \frac{\partial \psi_k}{\partial Q_i} \frac{\partial \psi_\ell}{\partial Q_j} \frac{\partial}{\partial q_\ell} \sum_m \left(\frac{\partial \varphi}{\partial \mathbf{q}} \right)_{km}^T P_m \\ &= \sum_{k, \ell} \frac{\partial^2 \Phi}{\partial q_\ell \partial q_k} \frac{\partial \psi_k}{\partial Q_i} \frac{\partial \psi_\ell}{\partial Q_j} = \left(\Phi'' \frac{\partial \psi}{\partial Q_i}, \frac{\partial \psi}{\partial Q_j} \right)_{\mathbb{R}^n} \end{aligned}$$

where $\Phi = (\varphi(\mathbf{q}), \mathbf{P})_{\mathbb{R}^n} = (\mathbf{Q}, \mathbf{P})_{\mathbb{R}^n}$ and Φ'' is the corresponding matrix of second partial derivatives. The symmetry of Φ'' then shows i and j can be interchanged in (2.3) and therefore the transformation is canonical.

Thus (1.1) transforms to a new Hamiltonian system

$$(2.4) \quad \dot{\mathbf{P}} = -\hat{\mathbf{H}}_{\mathbf{Q}}, \quad \dot{\mathbf{Q}} = \hat{\mathbf{H}}_{\mathbf{P}}$$

where

$$\begin{aligned} \hat{\mathbf{H}}(\mathbf{P}, \mathbf{Q}) &= \mathbf{H}(\psi_Q(\mathbf{Q})^{-T} \mathbf{P}, \psi(\mathbf{Q})) \\ &= \mathbf{K}(\psi_Q(\mathbf{Q})^{-T} \mathbf{P}, \psi(\mathbf{Q})) + \mathbf{V}(\psi(\mathbf{Q})) = \hat{\mathbf{K}}(\mathbf{P}, \mathbf{Q}) + \hat{\mathbf{V}}(\mathbf{Q}). \end{aligned}$$

Note in particular that $\widehat{V} \in C^1(\mathbb{R}^n, \mathbb{R})$ and satisfies (V_1') . Moreover $\widehat{K} \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ —observe the loss of a derivative—, $\widehat{K}(0, Q) = 0$, and by (K_1) for $|Q| < 1$ and $P \neq 0$,

$$\begin{aligned} (P, \widehat{K}_P)_{\mathbb{R}^n} &= \sum_i P_i \widehat{K}_{P_i} = \sum_{i,j} P_i K_{P_j} \frac{\partial p_j}{\partial P_i} \\ &= \sum_{i,j} P_i K_{P_j} \left(\frac{\partial \varphi}{\partial q} \right)_{ij} = (p, K_P)_{\mathbb{R}^n} > 0 \end{aligned}$$

so \widehat{K} satisfies (K_1') . Lastly (K_2') obtains since

$$\lim_{\alpha \rightarrow \infty} \widehat{K}(\alpha P, Q) = \lim_{\alpha \rightarrow \infty} K(\alpha \psi_Q^{-T} P, \psi(Q)) > 1 - \widehat{V}(Q)$$

for $|Q| < 1$ and $P \neq 0$ via (K_2) .

Thus to prove Theorem 1.4, it suffices to prove Theorem 2.1.

Henceforth we assume the hypotheses of Theorem 2.1 are satisfied. Let \mathfrak{m} be as in Remark 1.5 with D replaced by $B_1(0)$ so \mathfrak{m} is a compact C^1 manifold in \mathbb{R}^{2n} .

Lemma 2.5: Let $\overline{H} \in C^1(\mathbb{R}^{2n}, \mathbb{R})$, $\overline{H}^{-1}(1) = \mathfrak{m}$, and $\overline{H}_z \neq 0$ on \mathfrak{m} .

If $\zeta(t)$ satisfies

$$(2.6) \quad \dot{\zeta} = \mathcal{J} \overline{H}_z(\zeta)$$

and $\zeta(0) \in \mathfrak{m}$, then there is a reparameterization $z(t)$ of ζ which satisfies (1.1). In particular if $\zeta(t)$ is periodic, so is $z(t)$.

Proof: This is Lemma 3.1 of [5]. If $H, \bar{H} \in C^2$, the proof follows immediately from the facts that (1.2) and (2.6) are Hamiltonian systems so the corresponding flows remain on \mathfrak{m} and that $H_z(z) = \beta(z) \bar{H}_z(z)$ for $z \in \mathfrak{m}$ where $0 \neq \beta \in C^1(\mathfrak{m}, \mathbb{R})$. For the C^1 case, a bit more care must be taken. See [5].

On the basis of Lemma 2.5, to find a periodic solution of (1.1) on \mathfrak{m} , it suffices to find a suitable \bar{H} and find a periodic solution for (2.6) on \mathfrak{m} . Such a function \bar{H} will be constructed next. It possesses properties which are more amenable to the variational approach taken in §3 than does H .

By (V_1') , there are constants $\delta, \alpha > 0$ such that

$$(2.7) \quad (V_q(q), q) \geq \alpha V(q) \geq \frac{\mu}{2}$$

if $1 - 2\delta \leq |q| \leq 1 + 2\delta$. Moreover by (V_1') again, there is a $\mu = \mu(\delta) > 0$ such that if

$$(2.8) \quad V(q) > 1 \quad \text{and} \quad |q| \leq 1 + 2\delta, \quad \text{then} \quad |q| > 1 - \delta.$$

Next observe that since $K(0, q) = 0$, there exists $\mu_1 = \mu_1(\delta) > 0$ such that

$$(2.9) \quad K(p, q) < \mu/2 \quad \text{if} \quad |p| < \mu_1 \quad \text{and} \quad |q| \leq 1 + 2\delta.$$

Finally note that there is a constant $M_1 > 0$ such that

$$(2.10) \quad m \subset \{z \in \mathbb{R}^{2n} \mid |z| < M_1\}.$$

We can further assume that if $|p| > M_1$ and $|q| \leq 1$,

$$(2.11) \quad \min(K(p, q), |p|^2) > i.$$

For $a, b \in \mathbb{R}$ and $a < b$, let $\chi(s; a, b) \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(s; a, b) = 1$ if $s \leq a$; $= 0$ if $s \geq b$; and $\frac{d\chi}{ds} < 0$ if $a < s < b$. Set

$$(2.12) \quad \begin{cases} \chi_1(p) = 1 - \chi(|p|; \frac{\mu_1}{2}, \mu_1) \\ \chi_2(q) = \chi(|q|; 1 - 2\delta, 1 - \delta) \\ \chi_3(q) = \chi(|q|; 1, 1 + \delta) \\ \chi_4(p) = \chi(|p|; M_1 + 1, M_1 + 2) \end{cases}$$

Now we define functions \bar{V} , \bar{K} , \bar{H} as follows

$$(2.13) \quad \begin{cases} \bar{V}(p, q) = \chi_1(p)\chi_2(q)V(q) + (1 - \chi_2(q))\chi_3(q)V(q) + \rho_1(1 - \chi_3(q))|q|^2 \\ \bar{K}(p, q) = \chi_4(p)\chi_3(q)K(p, q) + \rho_2(1 - \chi_4(p))|p|^2 \\ \bar{H}(p, q) = \bar{K}(p, q) + \bar{V}(p, q) \end{cases}$$

where $\rho_1, \rho_2 > 0$ are free for the moment. The functions $\chi_2 - \chi_4$ insure that $\bar{H}(z)$ grows like $|z|^2$ for large $|z|$ and the χ_1 term forces $\bar{V}(p, q) = 0$ if $|p| < \mu_1/2$ and $|q| < 1 - 2\delta$.

Lemma 2.14: For ρ_1, ρ_2 sufficiently large,

- 1° $(p, \overline{H}_p(z))_{\mathbb{R}^n} \geq 0$ with strict inequality if $p \neq 0$ and $|q| < 1 + \delta$.
- 2° $(p, \overline{H}_p(z))_{\mathbb{R}^n} \geq 2\rho_2|p|^2$ if $|p| \geq M_1 + 2$.
- 3° If $p = 0$, $(q, \overline{H}_q(0, q))_{\mathbb{R}^n} = (q, \overline{V}_q(0, q))_{\mathbb{R}^n} \geq 0$ with strict inequality if $|q| \geq 1 - \delta$.

Proof:

$$(2.15) \quad (p, \overline{H}_p(z))_{\mathbb{R}^n} = \chi_4(p)\chi_3(q)(p, K_p(z))_{\mathbb{R}^n} + 2\rho_2(1-\chi_4(p))|p|^2 \\ + (\chi_3(q)K(z) - \rho_2|p|^2)(p, \chi_{4p})_{\mathbb{R}^n} + \chi_2(q)V(q)(p, \chi_{1p})_{\mathbb{R}^n}$$

from which 1° and 2° follow via (2.12), (V_1') , and (K_1') provided that ρ_2 is sufficiently large. To prove 3°, note that for $p = 0$,

$$(2.16) \quad (p, \overline{H}_q(0, q))_{\mathbb{R}^n} = (q, \overline{V}_q(0, q))_{\mathbb{R}^n} = (1-\chi_2(q))(q, V_q(q))_{\mathbb{R}^n} \\ + ((1-\chi_2(q))V(q) - \rho_1|q|^2)(q, \chi_{3q})_{\mathbb{R}^n} + 2\rho_1(1-\chi_3(q))|q|^2$$

so 3° follows as above if ρ_1 is sufficiently large.

Henceforth we assume ρ_1, ρ_2 are large enough so that the inequalities in Lemma 2.14 are valid.

Lemma 2.17: $\overline{H}^{-1}(1) = \mathbb{R}$.

Proof: Suppose first that $|p| \geq \mu_1$. Then $\chi_1(p) = 1$. If in addition $\overline{H}(z) = 1$, then 3° of Lemma 2.14 and the form of \overline{H} implies that $|q| \leq 1$. Hence $\chi_3(q) = 1$ and

$$(2.18) \quad 1 = \chi_4(p)K(z) + \rho_2(1-\chi_4(p))|p|^2 + V(q).$$

We can assume $\rho_2 \geq 1$ so by (2.11), (2.18) cannot be satisfied if $|p| > M_1$. Hence $|p| \leq M_1$, $\chi_4(p) = 1$, and $H(z) = 1$. Since $|q| \leq 1$, it follows that $z = (p, q) \in \mathfrak{M}$. Conversely if $|p| \geq \mu_1$ and $z \in \mathfrak{M}$, then $\chi_4(p) = 1 = \chi_3(q)$. Therefore $\overline{H}(z) = H(z) = 1$.

Next suppose that $|p| < \mu_1$. Hence $\chi_4(p) = 1$. If $\overline{H}(z) = 1$, $|q| \leq 1$ so $\chi_3(q) = 1$ and

$$(2.19) \quad 1 = K(z) + (\chi_1(p)\chi_2(q) + (1-\chi_2(q)))V(q).$$

By (2.9),

$$V(q) > \frac{1 - \mu/2}{\chi_1(p)\chi_2(q) + (1-\chi_2(q))} \geq 1 - \mu.$$

Therefore $|q| > 1 - \delta$ by (2.8) and $\chi_2(q) = 0$. Hence $H(z) = 1$ and $z \in \mathfrak{M}$ as above. Conversely if $|p| < \mu_1$ and $z \in \mathfrak{M}$, $\chi_4(p) = 1 = \chi_3(q)$ and

$$\overline{H}(z) = K(z) + (\chi_1(p)\chi_2(q) + (1-\chi_2(q)))V(q).$$

As above $K(z) < \mu/2$ implies $\chi_2(q) = 0$ and $\overline{H}(z) = H(z) = 1$.

Lemmas 2.5 and 2.17 reduce the proof of Theorem 2.1 to finding a periodic solution of (2.6) on \mathbb{M} with \bar{H} defined by (2.13). We shall determine such a solution but before doing so it is convenient to make one further modification to \bar{H} . Set

$$(2.20) \quad \bar{H}_\varepsilon(z) = \bar{H}(z) + \varepsilon|p|^2.$$

Note that

$$(2.21) \quad (p, \bar{H}_\varepsilon p(z))_{\mathbb{R}^n} \geq 2\varepsilon|p|^2$$

for all $z \in \mathbb{R}^{2n}$. Our goal now is to find a periodic solution of

$$(2.22) \quad \dot{z} = \mathcal{J} \bar{H}_\varepsilon z$$

on $H_\varepsilon^{-1}(1)$ for all small $\varepsilon > 0$ and let $\varepsilon \rightarrow 0$ to get a periodic solution of (2.6) on \mathbb{M} .

§3. The existence proof

Suppose $z(t)$ is a periodic solution of (2.22). Since its period T is a priori unknown, it is convenient to make the change of time variable $t \rightarrow \frac{2\pi}{T} t \equiv \lambda^{-1} t$ so that (2.22) becomes

$$(3.1) \quad \dot{z} = \lambda \mathcal{J} \bar{H}_{\varepsilon z}$$

with z now a 2π periodic function. The parameter λ must be determined in the course of the proof.

Set $E = (W^{1,2}(S^1))^n$ and for $z \in E$, let

$$\psi(z) = \frac{1}{2\pi} \int_0^{2\pi} \bar{H}_{\varepsilon}(z) dt .$$

Define

$$S = \{z \in E \mid \psi(z) = 1\} .$$

As was noted in the introduction, any critical point of

$$A(z) = \int_0^{2\pi} (p, \dot{q})_{\mathbf{R}^n} dt$$

on S with a nonzero Lagrange multiplier provides a solution of (3.1) on $H_{\varepsilon}^{-1}(1)$. Since we know of no direct method to find such critical points, we begin with an approximation argument.

Let e_1, \dots, e_{2n} denote the usual orthonormal basis in \mathbf{R}^{2n} and set

$$E_m = \text{span} \{ (\sin jt) e_k, (\cos jt) e_k \mid 0 \leq j \leq m, 1 \leq k \leq 2n \}$$

and $S_m = S \cap E_m$.

Lemma 3.2: S_m is a compact C^1 manifold which bounds a neighborhood of 0 in E_m .

Proof: For $z, \zeta \in E_m$, if $\Psi'(z)\zeta$ denotes the Frechet derivative of Ψ at z acting on ζ ,

$$(3.3) \quad \Psi'(z)\zeta = \frac{1}{2\pi} \int_0^{2\pi} (\overline{H}_{\varepsilon z}(z), \zeta)_{\mathbf{R}^{2n}} dt.$$

Hence by (2.21),

$$(3.4) \quad \Psi'(z)(p, 0) \geq \frac{\varepsilon}{\pi} \|p\|_{L^2}^2.$$

If $p = 0$, by 3° of Lemma 2.14,

$$(3.5) \quad \Psi'(z)(0, q) > 0$$

unless $\|q(t)\|_{L^\infty} < 1 - \delta$. But then

$$\Psi(0, q) = \frac{1}{2\pi} \int_0^{2\pi} (1 - \chi_2(q)) V(q) dt < 1$$

so $(0, q) \notin S_m$. Hence $\Psi'(z) \neq 0$ for $z \in S_m$ and by the implicit function theorem, S_m is a C^1 manifold in E_m .

Next observe that if $z \in E_m$ and $\|z\|_E = 1$,

$$(3.6) \quad \Psi(rz) \geq \frac{r^2}{2\pi} \int_0^{2\pi} [\rho_2(1-\chi_4(rp))|p|^2 + \rho_1(1-\chi_3(rq))|q|^2] dt \rightarrow \infty$$

as $r \rightarrow \infty$. Hence S_m is compact. Lastly observe that for $(0, q) \in E_m$ with $\|q\|_{L^2} = 1$, by (3.5) - (3.6) there is a unique $r = r(q)$ such that $\Psi(0, rq) = 1$. Since $\Psi(p, q) \geq \Psi(0, q)$, (3.4) then shows if

$\|q\|_{L^2} < r(q/\|q\|_{L^2})$ (if $q \neq 0$) and $\|p\|_{L^2} = 1$, there is a unique $s = s(p, q) > 0$ such that $\Psi(sp, q) = 1$. It follows that S_m bounds a neighborhood of 0 in E_m which for fixed q is star-shaped with respect to p . The lemma is proved.

Next we exploit some invariance properties inherent in our spaces and operators. Let E_m^+ , E_m^- , E^0 denote respectively the subspaces of E_m on which A is positive definite, negative definite, and null. It is easy to represent these spaces explicitly, namely

$$E^0 = \text{span} \{e_k \mid 1 \leq k \leq 2n\},$$

$$E_m^+ = \text{span} \{(\sin jt)e_k - (\cos jt)e_{k+n}, (\cos jt)e_k + (\sin jt)e_{k+n} \mid 1 \leq j \leq m, \\ 1 \leq k \leq n\}$$

$$E_m^- = \text{span} \{(\sin jt)e_k + (\cos jt)e_{k+n}, (\cos jt)e_k - (\sin jt)e_{k+n} \mid 1 \leq j \leq m, \\ 1 \leq k \leq n\}.$$

These subspaces are orthogonal under the L^2 inner product and are invariant under the family of mappings $z(t) \rightarrow z(t+\tau)$ for all $\tau \in [0, 2\pi]$.

This family of mappings induces an S^1 action on E_m (see [8] or [3]). A cohomological index theory for compact Lie group actions developed in [8] can be applied to our setup here. Let \mathcal{E}_m denote the family of subsets of $E_m \setminus \{0\}$ which are invariant under the above S^1 action, i.e. $B \in \mathcal{E}_m$ implies $z(t+\tau) \in B$ for all $\tau \in [0, 2\pi]$ whenever $z(t) \in B$. A mapping $f: E_m \rightarrow E_m$ will be called equivariant if $f: \mathcal{E}_m \rightarrow \mathcal{E}_m$. Similarly a mapping $g: E_m \rightarrow \mathbf{R}$ will be called equivariant if $g(z(t+\tau)) = g(z(t))$ for all $\tau \in [0, 2\pi]$ and $z \in E_m$. Note that A and Ψ are equivariant mappings.

Lemma 3.7: There is an index theory, i.e. a mapping $i: \mathcal{E}_m \rightarrow \mathbf{N} \cup \{\infty\}$ which possesses the following properties: If $B, \hat{B} \in \mathcal{E}_m$,

- 1° $i(B) < \infty$ if and only if $B \cap E^0 = \emptyset$.
- 2° If there is an $f \in C(B, \hat{B})$ with f equivariant, then $i(B) \leq i(\hat{B})$.
- 3° $i(B \cup \hat{B}) \leq i(B) + i(\hat{B})$.
- 4° If F is an invariant subspace of $E_m^+ \oplus E_m^-$ and \mathcal{S} is the boundary of a bounded open invariant neighborhood of 0 in F , then $i(\mathcal{S}) = \frac{1}{2} \dim F$.
- 5° If $B \in \mathcal{E}_m$ with $i(B) \geq mn$, F is an invariant subspace of E_m containing E^0 and having $\dim F \geq 2mn + 2n + 2$, then $B \cap F \neq \emptyset$.

Proof: The definition of the index and proofs of 1° - 4° as well as other properties of index can be found in §6-7 of [8] and 5° is Lemma 1.24 of [3].

An immediate consequence of Lemma 3.7 is

Lemma 3.8: $i(S_m \cap E_m^-) = mn$.

Proof: Since $\dim E_m^- = 2mn$ and E_m^- is an invariant subspace of $E_m^+ \oplus E_m^-$, Lemma 3.2 implies $S_m \cap E_m^-$ is the boundary of an open neighborhood Ω of 0 in E_m^- . The equivariance of Ψ implies $\Omega \in \mathcal{E}_m$. Hence the result follows from 4° of Lemma 3.7.

At this point it is convenient to further assume that H (and therefore \overline{H}_ε) $\in C^2(\mathbb{R}^{2n}, \mathbb{R})$. We will return to the C^1 case later. Since A is an equivariant mapping defined on $S_m \in \mathcal{E}_m$, there is a standard method for trying to find critical values of $A|_{S_m}$. Define

$$(3.9) \quad Y_{j,m} = \inf_{\substack{B \subset S_m \\ i(B) \geq j}} \max_{z \in B} A(z) \quad 1 \leq j \leq mn$$

Lemma 3.10: $Y_{j,m}$ is a negative critical value of $A|_{S_m}$, $1 \leq j \leq mn$.

Proof: Since $i(S_m \cap E_m^-) = mn$,

$$Y_{1,m} \leq \dots \leq Y_{mn,m} \leq \max_{S_m \cap E_m^-} A < 0.$$

Since $H \in C^2$, the remainder of the proof is standard. See e.g. Lemma 1.16 in [3].

Not all of the critical values obtained in Lemma 3.10 are of use to us. E.g. since $A|_S$ is not bounded from below, $\gamma_{1,m} \rightarrow -\infty$ as $m \rightarrow \infty$. Accordingly we focus our attention on $c_m \equiv \gamma_{mn,m}$. Heuristically c_m is the largest negative critical value that $A|_{S_m}$ possesses. Let z_m be a critical point corresponding to c_m , i.e. $A(z_m) = c_m$ and

$$(3.11) \quad A'(z_m)\zeta - \lambda_m \Psi'(z_m)\zeta = 0$$

for all $\zeta \in E_m$. To obtain a solution of (3.1), we will find upper and lower bounds for c_m , for λ_m , and finally bounds for $\|z_m\|_E$. These estimates will be determined in the following series of lemmas.

Lemma 3.12: There are constants $\alpha_1, \alpha_2 < 0$ and independent of m such that

$$(3.13) \quad \alpha_1 \leq c_m \leq \alpha_2.$$

Moreover there is an $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0]$, α_1 and α_2 are independent of ε .

Proof: Since $\bar{H}_\varepsilon(0) = 0$, (2.13) shows there is a constant $M_2 > 0$ and independent of ε such that

$$(3.14) \quad \bar{H}_\varepsilon(z) \leq \frac{1}{2} + (M_2 + \varepsilon)|z|^2$$

for all $z \in \mathbf{R}^{2n}$. Choosing $z = z(t) \in S$, (3.14) yields

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \bar{H}_\varepsilon(z) dt \leq \frac{1}{2} + \frac{M_2 + \varepsilon}{2\pi} \|z\|_{L^2}^2.$$

Hence

$$(3.15) \quad \|z\|_{L^2} \geq \left(\frac{\pi}{M_2 + \varepsilon} \right)^{\frac{1}{2}} \geq \left(\frac{\pi}{2M_2} \right)^{\frac{1}{2}} \equiv M_3$$

for all $z \in S$ provided that $0 \leq \varepsilon \leq \varepsilon_0 \leq M_2$ which choice we make.

Similarly (2.13) shows if $\rho_3 = \min(\rho_1, \rho_2)$, there is a constant $\alpha_3 > 0$ and independent of ε such that

$$(3.16) \quad \bar{H}_\varepsilon(z) \geq \frac{\rho_3}{2} |z|^2 - \alpha_3$$

for all $z \in \mathbf{R}^{2n}$. Again choosing $z = z(t) \in S$ in (3.16) gives

$$(3.17) \quad \|z\|_{L^2} \leq \left(\frac{4\pi(1 + \alpha_3)}{\rho_3} \right)^{1/2} \equiv M_4.$$

Thus we have upper and lower bounds for $z \in S$ independent of $\varepsilon \in (0, \varepsilon_0]$.

To get the bounds for c_m , note first that the form of A and E_m^- imply

$$(3.18) \quad c_m \leq \max_{z \in S_m \cap E_m^-} A(z) \leq \max \{A(z) \mid z \in E_m^-, \|z\|_{L^2} = M_3\} \\ = -\pi M_3^2 \equiv \alpha_2.$$

To get the lower bound, let

$$F = E_m^+ \oplus E^0 \oplus \text{span} \{(\sin t)e_1 + (\cos t)e_{k+1}, (\cos t)e_1 - (\sin t)e_{k+1}\}.$$

Then $\dim F = 2mn + 2n + 2$, $F \supset E^0$, and F is an invariant subspace of E_m . Thus by 5° of Lemma 3.7, if $B \in S_m$ and $i(B) \geq mn$, then $B \cap F \neq \emptyset$. Choosing any $\tilde{z} \in B \cap F$ shows

$$(3.19) \quad \min_{F \cap S_m} A \leq A(\tilde{z}) \leq \max_B A.$$

Since (3.19) holds for all $B \subset S$ in satisfying $i(B) \geq mn$, (3.17) and (3.19) yield

$$(3.20) \quad c_m \geq \min_{F \cap S_m} A \geq \min \{A(z) \mid z \in F, \|z\|_{L^2} = M_4\} \\ = -\pi M_4^2 \equiv \alpha_1.$$

Remark 3.21: We now show how to treat the case in which H and therefore \bar{H}_ε is merely assumed to belong to $C^1(\mathbb{R}^{2n}, \mathbb{R})$. Let $V_k(q), K_k(z)$ be sequences of functions which converge uniformly in the C^1 norm to V and K respectively on $\{(p, q) \in \mathbb{R}^{2n} \mid |p| \leq M_2 + 2, |q| \leq 1 + \delta\}$. Replacing V and K by V_k, K_k respectively in (2.13) and (2.20) defines functions $\bar{H}_k(z), \bar{H}_{k\varepsilon}(z)$ where $\bar{H}_{k\varepsilon}(z) \rightarrow \bar{H}_k(z)$ uniformly in the C^1 norm on \mathbb{R}^{2n} as $k \rightarrow \infty$. Set

$$\Psi_k(z) = \frac{1}{2\pi} \int_0^{2\pi} \bar{H}_{k\varepsilon}(z) dt .$$

Then Ψ_k is equivariant on E_m and $\Psi_k \rightarrow \Psi$ in the C^1 norm uniformly on E_m . Moreover (3.4) - (3.6) show that for large k , $\Psi_k^{-1}(1)$ is a C^2 compact manifold in E_m which is the boundary of a bounded neighborhood of 0. By the C^2 case, for each such k ,

$$Y_{j,m}^k \equiv \inf_{\substack{B \subset \Psi_k^{-1}(1) \\ i(B) \geq j}} \max_{z \in B} A(z), \quad 1 \leq j \leq mn$$

is a negative critical value of $A|_{\Psi_k^{-1}(1) \cap E_m}$. We will only study

$c_m^k \equiv Y_{mn,m}^k$. Let z_m^k be a corresponding critical point. Since these critical points are uniformly bounded in the finite dimensional space E_m and $\Psi_k \rightarrow \Psi$ uniformly in C^1 on E_m , a subsequence of z_m^k converges to a critical point z_m of A on S_m . Define $c_m = A(z_m)$ so by definition c_m is a critical value of A on S_m . An inspection of the proof of Lemma 3.12 shows the estimates (3.14) - (3.17) with S replaced where appropriate by $\Psi_k^{-1}(1)$ can be assumed independent of k as well as m and ε . Hence the bounds (3.18) and (3.20) hold for c_m^k independently of m , ε , and large k . Letting $k \rightarrow \infty$, we see (3.13) holds for c_m .

For the remainder of the proof of Theorem 2.1, we assume we are in the $H \in C^1$ case. The next lemma provides ε dependent bounds for λ_m .

Lemma 3.22: There are constants $\mu_1, \mu_2 > 0$ and independent of m such that

$$(3.23) \quad \beta_1 \leq \lambda_m \leq \beta_2 .$$

Proof: From (3.11) with $z_m = (p_m, q_m)$ and $\xi = (p_m, 0)$, we find

$$(3.24) \quad c_m = \int_0^{2\pi} (p_m, \dot{q}_m)_{\mathbf{R}^n} dt = \lambda_m \int_0^{2\pi} (p_m, \bar{H}_{\varepsilon p}(z_m))_{\mathbf{R}^n} dt .$$

Since $c_m < 0$, (2.21) shows $\lambda_m < 0$. Moreover by (3.24), (2.15), and (3.17)

$$\frac{c_m}{\lambda_m} \leq M_5 + 2(\rho_2 + \varepsilon) M_4^2$$

which together with (3.13) gives the upper bound for λ_m .

A lower bound for λ_m is more difficult to obtain. In fact we will only obtain an indirect estimate. Thus suppose $\lambda_m \rightarrow -\infty$ as $m \rightarrow \infty$ along some subsequence. Since by (2.21),

$$\frac{c_m}{\lambda_m} = \int_0^{2\pi} (p_m, \bar{H}_{\varepsilon p}(z_m))_{\mathbf{R}^n} dt \geq 2\varepsilon \|p_m\|_{L^2}^2 ,$$

the integrals

$$(3.25) \quad \int_0^{2\pi} (p_m, \bar{H}_{\varepsilon p}(z_m))_{\mathbf{R}^n} dt$$

tend to 0 as $m \rightarrow \infty$ along some subsequence. But (3.25) consists of five non negative terms (see 2.25), so each term tends to 0 as $m \rightarrow \infty$.

In particular $\varepsilon \|p_m\|_{L^2}^2 \rightarrow 0$ and if

$$\tau_m(\sigma) = \text{measure} \{t \in [0, 2\pi] \mid p_m(t) \geq \sigma\},$$

then for all $\sigma > 0$, $\tau_m(\sigma) \rightarrow 0$ as $m \rightarrow \infty$ along our subsequence. Since

$$(3.26) \quad \begin{aligned} 1 = \Psi(z_m) &= \frac{1}{2\pi} \int_0^{2\pi} [\chi_4(p_m) \chi_3(q_m) K(z_m) + \\ &+ \rho_2(1-\chi_4(p_m)) |p_m|^2 + \varepsilon |p_m|^2 + \chi_1(p_m) \chi_2(q_m) V(q_m) + \\ &+ (1-\chi_2(q_m)) \chi_3(q_m) V(q_m) + \rho_1(1-\chi_3(q_m)) |q_m|^2] dt, \end{aligned}$$

it easily follows that the first four terms on the right hand side of (3.26) tend to 0 as $m \rightarrow \infty$. Moreover choosing $\zeta = (0, q_m)$ in (3.11) yields

$$(3.27) \quad \begin{aligned} \frac{c_m}{\lambda_m} &= \int_0^{2\pi} (q_m, \bar{H}_{\varepsilon q}(z_m))_{\mathbb{R}^n} dt = \int_0^{2\pi} [(q_m, \chi_4(p_m) \chi_3(q_m) K_q(z_m))_{\mathbb{R}^n} \\ &+ (q_m, \chi_4(p_m) K(z_m) \chi_3(q_m))_{\mathbb{R}^n} + (q_m, \chi_1(p_m) \chi_2(q_m) V_q(q_m))_{\mathbb{R}^n} \\ &+ (q_m, \chi_1(p_m) V(q_m) \chi_2(q_m))_{\mathbb{R}^n} + (q_m, (1-\chi_2(q_m)) \chi_3(q_m) V_q(q_m))_{\mathbb{R}^n} \\ &- (q_m, \chi_3(q_m) V(q_m) \chi_2(q_m))_{\mathbb{R}^n} + (q_m, ((1-\chi_2(q_m)) V(q_m) - \rho_1 |q_m|^2) \chi_3(q_m))_{\mathbb{R}^n} \\ &+ 2 \rho_1(1-\chi_3(q_m)) |q_m|^2] dt. \end{aligned}$$

Since $K(0, q) = 0$, $K_q(0, q) = 0$ and the first two terms on the right hand side of (3.27) go to 0 as $m \rightarrow \infty$ along our subsequence. Likewise as in (3.26) the next two terms in (3.27) tend to 0 as $m \rightarrow \infty$. The remaining four terms are non negative and since the left hand side of (3.27) tends to 0, each of these terms also goes to 0 as $m \rightarrow \infty$. In particular

$$(3.28) \quad \int_0^{2\pi} (1-\chi_3(q_m)) |q_m|^2 dt \rightarrow 0$$

and

$$(3.29) \quad \int_0^{2\pi} (1-\chi_2(q_m)) \chi_3(q_m) (q_m, V_q(q_m))_{\mathbb{R}^n} dt \rightarrow 0$$

as $m \rightarrow \infty$. On examining (3.26) again and using (3.28), we conclude that

$$(3.30) \quad \frac{1}{2\pi} \int_0^{2\pi} (1-\chi_2(q_m)) \chi_3(q_m) V(q_m) dt \rightarrow 1$$

as $m \rightarrow \infty$. But $(1-\chi_2(s)) \chi_3(s)$ is nonzero only for $1-2\delta < s < 1+\delta$. Hence by (2.7) and (3.30),

$$(3.31) \quad \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} (1-\chi_2(q_m)) \chi_3(q_m) (q_m, V_q(q_m))_{\mathbb{R}^n} dt &\geq \\ &\geq \frac{\epsilon}{2\pi} \int_0^{2\pi} (1-\chi_2(q_m)) \chi_3(q_m) V(q_m) dt \rightarrow \frac{\epsilon}{2\pi} \end{aligned}$$

as $m \rightarrow \infty$ along our subsequence, contrary to (3.29). Consequently there must exist ϵ_1 as in the statement of the lemma.

One final estimate is required:

Lemma 3.32: There is a constant $M_\epsilon > 0$ and independent of m such that

$$z_m \in M_\epsilon.$$

Proof: Using (3.11) with $\zeta = (\dot{q}_m, -\dot{p}_m)$ and the Schwarz inequality yields

$$(3.33) \quad \|\dot{z}_m\|_{L^2} \leq |\lambda_m| \|\bar{H}_{\epsilon z}(z_m)\|_{L^2}.$$

Since $\bar{H}_{\epsilon}(z)$ grows at most linearly in z , the bound for $\|z_m\|_E$ now follows from (3.17), (3.33), and (3.23).

Combining the estimates given in the above lemmas now gives an existence result.

Lemma 3.34: There exists a solution $(\lambda_{\epsilon}, z_{\epsilon})$ of (3.1) with $z_{\epsilon} \in C^1(S^1, \mathbf{R}^{2n})$ and $\bar{H}_{\epsilon}(z_{\epsilon}(t)) \equiv 1$.

Proof: Lemmas 3.22 and 3.32 and the Sobolev imbedding theorem imply that along some subsequence $z_m \rightarrow z_{\epsilon}$ weakly in E and strongly in L^{∞} and $\lambda_m \rightarrow \lambda_{\epsilon} < 0$ where $(\lambda_{\epsilon}, z_{\epsilon})$ satisfies (3.11) for all $\zeta \in \bigcup_{m \in \mathbf{N}} E_m$. Hence $z_{\epsilon} \in S$ and is a weak solution of (3.1) with $\lambda = \lambda_{\epsilon}$. It readily follows that z_{ϵ} satisfies (3.1) pointwise a.e. But since $\bar{H}_{\epsilon z}(z_{\epsilon})$ is continuous, z_{ϵ} must be continuously differentiable. Hence z_{ϵ} is a classical solution of (3.1). Lastly $\bar{H}_{\epsilon}(z_{\epsilon}) \equiv \text{constant}$ since (3.1) is a Hamiltonian system. Therefore $z_{\epsilon} \in S$ implies that $\bar{H}_{\epsilon}(z_{\epsilon}) = 1$.

It remains to let $\epsilon \rightarrow 0$ and obtain a solution of

$$\dot{z} = \lambda \mathcal{J} \bar{H}_z$$

on \mathbb{M} which via previous remarks will then complete the proof of Theorem 2.1. We begin with ε independent a priori bounds for (λ, z) for 2π periodic solutions of (3.1) on \mathbb{M} . Recall that ε_0 was defined in Lemma 3.12, μ in (2.8), and M_1 in (2.10).

Lemma 3.35: Let (λ, z) be a solution of (3.1) with $z \in C^1(S^1, \mathbb{R}^{2n})$, $\lambda(z) = c < 0$, and $\bar{H}_\varepsilon(z) = 1$. If $\varepsilon_1 = \min(\varepsilon_0, \mu/2M_1^2)$ and $\varepsilon \in [0, \varepsilon_1]$, then there are constants $M_7 - M_9 > 0$ and independent of ε such that

$$(3.36) \quad \begin{cases} \frac{c}{M_7} \leq \lambda \leq \frac{c}{M_8} \\ \|z\|_{C^1} \leq M_9 \end{cases}$$

Proof: As earlier if $z = (p, q)$,

$$(3.37) \quad \frac{c}{\lambda} = \int_0^{2\pi} (p, \bar{H}_{\varepsilon p}(z))_{\mathbb{R}^n} dt.$$

If $\bar{H}_\varepsilon(z) = 1$, $|q| \leq 1$ via e.g. Lemma 2.17 and the definition of \bar{H}_ε . Moreover the form of \bar{H}_ε implies $|p| < M_1$. Hence the integrand on the right hand side of (3.37) is bounded from above by a constant independent of ε from which the upper bound for λ follows.

To obtain the lower bound, let

$$\tau(\sigma) = \text{measure} \{ t \in [0, 2\pi] \mid |p(t)| \geq \sigma \}.$$

Then

$$(3.38) \quad \frac{c}{\lambda} = \int_0^{2\pi} (q, \bar{H}_{\varepsilon q}(z))_{\mathbf{R}^n} dt = I_1 + I_2$$

where I_1 (resp. I_2) denotes the integral over the set in which $|p(t)| < \sigma$ (resp. $|p(t)| \geq \sigma$) and σ is free for now. The bounds for $|p|$, $|q|$ on $H_\varepsilon^{-1}(1)$ imply that there is a constant $M > 0$ such that

$$(3.39) \quad |(q, \bar{H}_{\varepsilon q}(z))_{\mathbf{R}^n}| \leq M$$

for all $\varepsilon \in [0, \varepsilon_0]$ and $z = (p, q) \in H_\varepsilon^{-1}(1)$. Therefore (3.38) - (3.39) imply that

$$(3.40) \quad \frac{c}{\lambda} \geq I_1 - \tau(\sigma) M .$$

Choosing

$$(3.41) \quad \sigma < \mu_1/2 ,$$

$\chi_1(p) = 0$ and

$$(3.42) \quad \bar{H}_\varepsilon(z) = 1 = K(p, q) + (1 - \chi_2(q)) V(q) + \varepsilon |p|^2 .$$

Hence by (2.9), our choice of ε , (3.42), and (2.8), $|q| > 1 - \delta$.

Consequently $\chi_2(q) = 0$ and the integrand in the I_1 term is

$$(q, K_q(z))_{\mathbf{R}^n} + (q, V_q(q))_{\mathbf{R}^n} .$$

Set

$$\omega(\sigma) = \max \{ (q, K_q(z))_{\mathbf{R}^n} \mid |q| \leq 1, |p| \leq \sigma \} .$$

Since $K_q(0, q) = 0$, $\omega(\sigma) \rightarrow 0$ as $\sigma \rightarrow 0$. Thus by (2.8) and (2.7) for $\sigma < \mu/2$,

$$I_1 \geq (2\pi - \tau(\sigma)) \left(\frac{c}{2} - \omega(\sigma) \right) .$$

Further choose σ such that

$$(3.43) \quad \omega(\sigma) \leq \frac{c}{4} .$$

Then

$$I_1 \geq (2\pi - \tau(\sigma)) \frac{c}{4}$$

and by (3.40)

$$\frac{c}{\lambda} \geq \frac{c\pi}{2} - \tau(\sigma) \left(M + \frac{c}{4} \right) .$$

Thus if

$$(3.44) \quad \tau(\sigma) < \xi \equiv \frac{\alpha \pi}{4M + \alpha} ,$$

$$(3.45) \quad \frac{c}{\lambda} \geq \frac{\alpha \pi}{4} .$$

If, on the other hand, $\tau(\sigma) \geq \xi$, by (3.37) and (3.42),

$$(3.46) \quad \frac{c}{\lambda} \geq \int_0^{2\pi} (p, K_p(z))_{\mathbf{R}^n} dt \geq \tau(\sigma) \gamma(\sigma) \geq \xi \gamma(\sigma)$$

where

$$\gamma(\sigma) = \min \{ (p, K_p(z))_{\mathbf{R}^n} \mid \sigma \leq |p| \leq M_1, |q| \leq 1 \} > 0 .$$

In any event, (3.45) - (3.46) show

$$(3.47) \quad \frac{c}{\lambda} \geq \min \left(\frac{\alpha \pi}{4}, \xi \gamma(\sigma) \right) \equiv M_7$$

where σ is now fixed and satisfies (3.41) and (3.43). Thus (3.47) provides the desired lower bound on λ .

Finally the differential equation (3.1) coupled with our upper and lower bounds for λ and the pointwise bounds for $z(t)$ yield the bound for $\|z\|_{C^1}$.

Completion of proof of Theorem 2.1: Let $\varepsilon \rightarrow 0$. The bounds (3.36) and (3.13) which hold for $c_\varepsilon = A(z_\varepsilon)$ together with the differential equation

(3.1) show a subsequence of $(\lambda_\varepsilon, z_\varepsilon)$ converge to a classical solution (λ, z) of

$$(3.48) \quad \dot{z} = \lambda \mathcal{J} \bar{H}_z$$

with z 2π periodic and $z \in \mathbb{m}$.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The existence of periodic solutions having prescribed energy for a Hamiltonian system of ordinary differential equations is studied. It is shown in particular that if the Hamiltonian is of classical type, i.e., $H(p,q)=K(p,q) + V(q)$ where $p,q \in \mathbb{R}^n$, and K and V satisfy $K(0,q)=0$, $p \cdot K_p(p,q) > 0$, $K(p,q) \rightarrow \infty$ as $ p \rightarrow \infty$, $D=\{q \in \mathbb{R}^n 0 < V(q) < 1\}$ is diffeomorphic to the unit ball in \mathbb{R}^n and $V_q \neq 0$ on ∂D , then Hamilton's equations $(*) \dot{p} = -H_q, \dot{q} = H_p$, have a periodic solution on $H^{-1}(1)$.		