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QUASIPLANAR AND PSEUDOPLANAR GRAPHS.(U)
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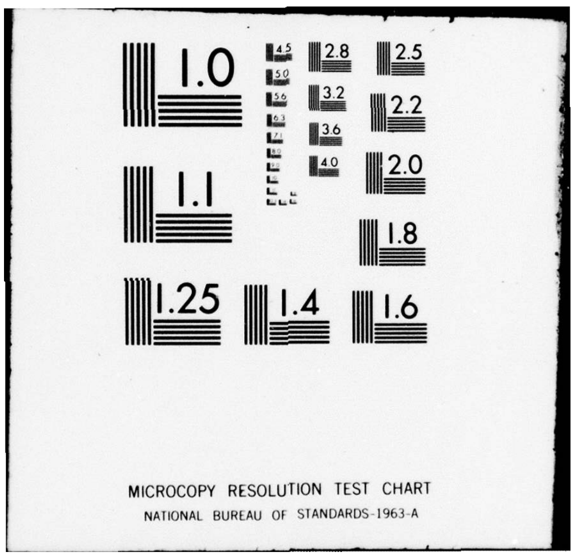
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BY

RANEL E. ERICKSON

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TECHNICAL REPORT NO. 34

1 MAY 1, 1978

12 34p.

PREPARED UNDER
OFFICE OF NAVAL RESEARCH CONTRACT
N00014-75-C-0493, (NR-042-264)
NSF-ENG76-12266

DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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Technical Report No. 34

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Prepared Under
Office of Naval Research Contract N00014-75-C-0493*

Department of Operations Research
Stanford University
Stanford, California

*Also partially supported by National Science Foundation Grant ENG-76-12266.

ABSTRACT

Two new generalizations of planar graphs, called quasiplanar and pseudoplanar graphs, are introduced and discussed. A graph is called quasiplanar if for each node t, the set of nodes incident to t can be labeled $1, \ldots, m$ so that for each $1 \leq h < i < j < k \leq m$, each pair of paths not containing t and having respective endnodes h,j and i,k share a common node. A (directed) graph is called pseudoplanar if for each pair of nodes s,t, the set of nodes adjacent to t can be labeled $1, \ldots, m$ so that for each maximal arborescence not containing t and having root s and each endnode adjacent to t, the endnode descendants of each node in the arborescence are either $j, \ldots, k$ or $k, \ldots, m, 1, \ldots, j$ for some $1 \leq j \leq k \leq m$. Planar graphs are quasiplanar and they in turn are pseudoplanar. Conversely, a pseudoplanar graph that contains with each arc its reverse arc is quasiplanar. And a quasiplanar graph that excludes subgraphs that are refinements of the complete bipartite graph K_{33} with three nodes in both sets is planar. Kuratowski (1930) characterized planar graphs as those that exclude subgraphs that are refinements of either the complete graph on five nodes or K_{33}. An analogous characterization of quasiplanar graphs is given in this paper in which the excluded subgraphs differ from Kuratowski's only by adding an edge in K_{33}. In a companion paper with Veinott, an algorithm given for finding minimum-concave-cost flows in single-source networks is shown to run in polynomial time when the associated graph is pseudoplanar.

QUASIPLANAR AND PSEUDOPLANAR GRAPHS

1. Introduction.

The purpose of this paper is to introduce and discuss quasiplanar and pseudoplanar graphs, two new generalizations of planar graphs. To do this we define t-quasiplanar and st-pseudoplanar graphs where s and t are nodes in the graph. These generalizations are obtained by abstracting two properties of planar graphs, viz, the "intersecting-staggered-paths property" and the "consecutive-endnodes property." These properties are introduced in Section 2.

The motivation for studying these graphs comes from a companion paper with Veinott [1]. There we give an algorithm for finding minimum-concave-cost flows in single-source networks and show it runs in polynomial time when the associated graph is st-pseudoplanar where s is the source, t is the sink, and the flows in all arcs adjacent to the sink are fixed.

The main result of this paper, given in Section 3, is a characterization of quasiplanar graphs which is analogous to the Kuratowski Theorem for planar graphs. Kuratowski characterized planar graphs by excluding subgraphs that are refinements of either the complete graph on five nodes or the complete bipartite graph with three nodes in both sets. The subgraphs excluded from quasiplanar graphs differ from the Kuratowski graphs only by adding an edge in the bipartite graph.

In Section 4 it is shown that a quasiplanar graph is pseudoplanar, and that the converse holds in bidirected graphs. Hence the two extensions coincide in bidirected graphs.

2. Preliminaries.

The following notation and definitions will be used throughout this paper. A (directed) graph G consists of a nonempty finite set G_N of nodes and a set G_A of ordered pairs of nodes called arcs. Call H a subgraph of G if $H_N \subset G_N$ and $H_A \subset G_A$. Define the subgraph $G(H)$ of G induced by H as the maximal subgraph of G using only the nodes in H. Usually the subscripts N and A are clear from context and will be suppressed. We say that a node u is adjacent to (resp., adjacent from) a node v if (u,v) (resp., (v,u)) is an arc in G. Let a path (resp., chain) be an alternating sequence of distinct nodes and arcs $u_0, a_1, u_1, \ldots, a_n, u_n$ beginning and ending with nodes u_0 and u_n, called endnodes, such that $a_k \in \{(u_{k-1},u_k), (u_k,u_{k-1})\}$ (resp., $a_k = (u_{k-1},u_k)$). A cycle (resp., circuit) is a path (resp., chain) with an additional arc $a_0 \in \{(u_0,u_n),(u_n,u_0)\}$ (resp., $a_0 = (u_n,u_0)$). We interpret a chain as a "directed" path and a circuit as a "directed" cycle. A node u is said to be accessible from node v if there is a chain from u to v (i.e., $u_0 = u$, $u_n = v$ in definition).

An arborescence is a graph containing no cycles and all nodes are accessible from a distinguished node called the source. Call the nodes accessible from a node in an arborescence descendants of that node. An endnode in an arborescence is a node without descendants. Given a node s and a set M of nodes in a graph G, define an arborescence from s to M as a subgraph of G that is an arborescence with source s and with endnodes in M. A maximal arborescence from s to M is an arborescence T from s to M with a maximal number of nodes from M in T.

Let t be a node in a graph G. Denote by A_t the set of nodes in G adjacent to t. Let the degree d_t of a node t be the number of nodes in A_t. Call a graph planar if it can be embedded in the plane so that paths intersect only at nodes. After embedding, choose an arbitrary node t and label by $1, 2, \ldots, m$ the nodes in A_t cyclically, say clockwise, around t. It will be shown in the sequel that the following two properties are satisfied with such a labeling. Consider s and t as fixed nodes in an arbitrary graph G.

Intersecting-Staggered-Paths Property.

There exists a labeling $1, 2, \ldots, m$, called consistent, of A_t such that for each ordered quadruple $h < i < j < k$ and for each pair of paths, called staggered, not containing t with endnodes h, j and i, k, respectively, the paths share a common node.

The following is an example of staggered paths where the solid node is the common node and $h = 2$, $i = 3$, $j = 4$, and $k = 6$.

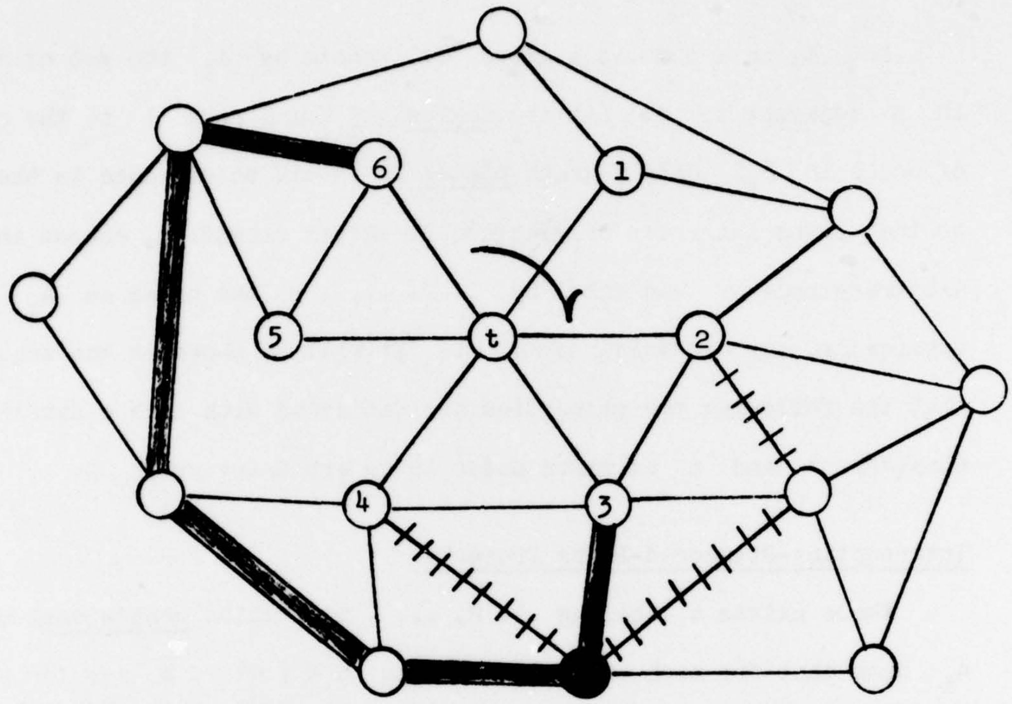

Figure 1.

Consecutive-Endnodes Property.

There exists a labeling 1, 2, ... , m, called compatible, of A_t such that for each maximal arborescence H from s to A_t not containing t and each node i in H the endnode descendants of i are consecutively labeled, i.e., for some $1 \leq j \leq k \leq m$, the endnodes are either j, j+1, ... , k or k, ... , m, 1, 2, ... , j. These two properties will be used to define quasiplanar and pseudoplanar graphs in the next two sections.

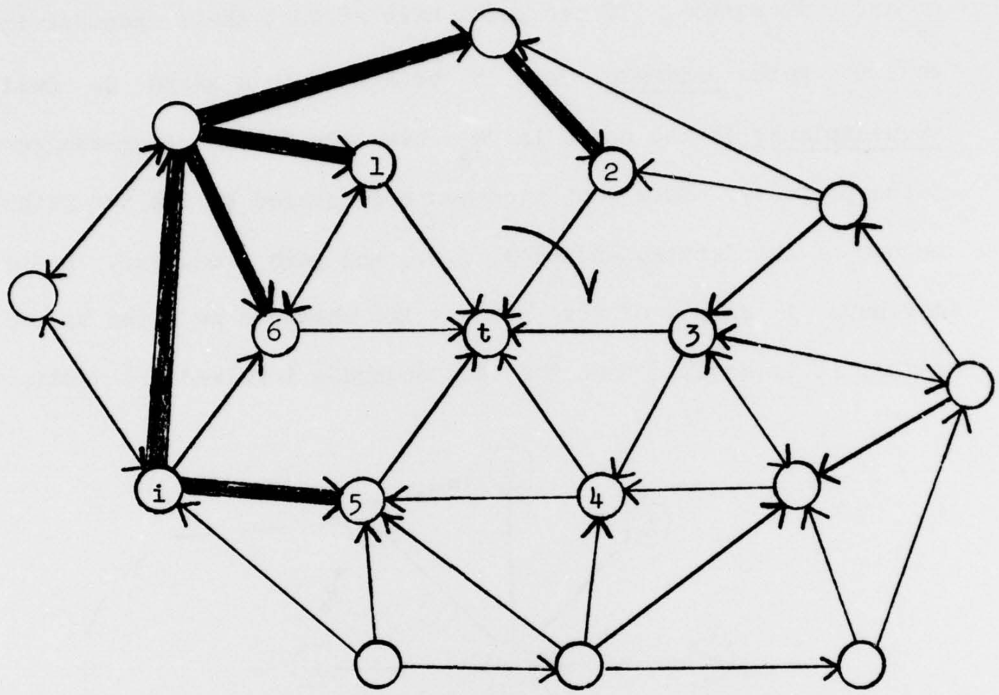

Figure 2.

3. Quasiplanar Graphs.

The concept of planarity does not depend on the direction of the arcs and is usually defined for undirected graphs. In order to simplify the presentation of quasiplanarity, assume throughout this section that the graphs are undirected, i.e., consider the ordered pairs of nodes (often called edges) as unordered. Extend other concepts in the natural way, for example, a path is an alternating sequence of distinct nodes and edges $u_0, a_1, u_1, \ldots, a_n, u_n$ beginning and ending with nodes such that $a_k = (u_{k-1}, u_k)$ for $k = 1, \ldots, n$. Let H and G be (undirected) graphs and call G a refinement of H if G can be obtained from H by successively replacing arcs by paths. Denote by $|S|$ the cardinality of a set S.

Two paths are said to be disjoint if they have no nodes (and thus no arcs) in common. If two paths have at most their endnodes in common, call the paths separate. Let t be a node in a graph G. Call G t-quasiplanar if the nodes in A_t have the intersecting-staggered-paths property. Note that a common node shared by the two paths may be one of the labeled endnodes, i.e., the path from, say, h to j may have i as one of its nodes. But whenever we refer to two staggered paths, it is assumed that the four endnodes involved are distinct.

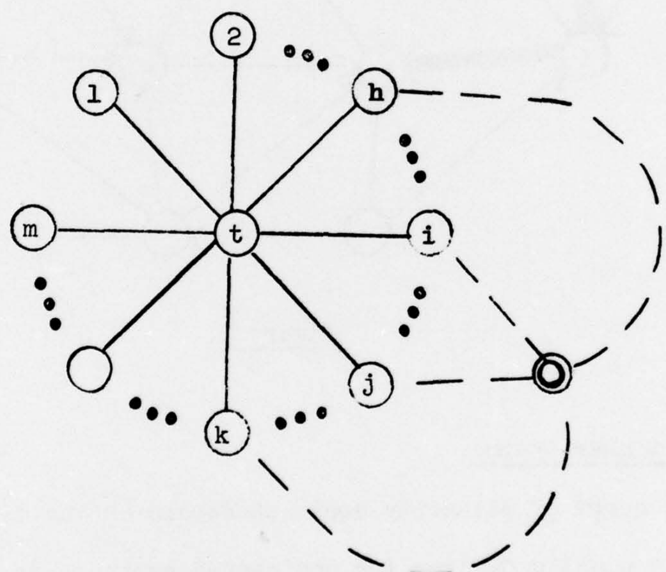

Figure 3. Staggered Paths*

If t is not in the graph G, then let G be t-quasiplanar by definition. Call G quasiplanar if it is t-quasiplanar for each node t in G. For notational convenience, let K_{mn} denote the complete bipartite graph, i.e., a graph consisting of two sets of nodes S_1

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* In the figures dashed lines represent paths, solid lines represent arcs, and circles represent nodes.

and S_2, where $|S_1| = m$ and $|S_2| = n$ with edges joining each node in S_1 to each node in S_2. Also denote by K_n the <u>complete graph</u> consisting of n nodes with a maximum number of edges.

Note that if the degree of node t is three or less, then the graph is trivially t-quasiplanar since there do not exist two staggered paths having endnodes in A_t. This implies that K_{33} is quasiplanar.

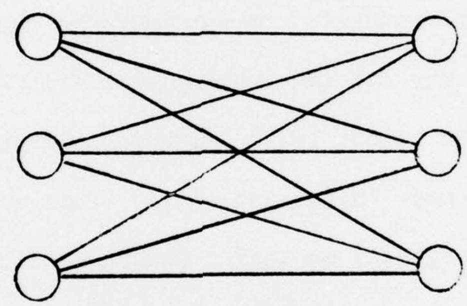


Figure 4. Quasiplanar Nonplanar Graph

Lemma 3.1.

A subgraph of a quasiplanar (resp., t-quasiplanar) graph is quasiplanar (resp., t-quasiplanar).

Proof.

Let t be a node, G a t-quasiplanar graph, and H be a subgraph of G. If $t \notin H$, then H is t-quasiplanar by definition. If $t \in H$, then $t \in G$, $A_{Ht} \subseteq A_{Gt}$, and the consistent ordering of A_{Gt} imposes a consistent ordering of A_{Ht}.

The next result indicates the relationship between quasiplanar graphs and planar graphs, justifying the use of the term "quasiplanar."

Theorem 3.2.

A planar graph is quasiplanar. Conversely a quasiplanar graph containing no subgraph that is a refinement of K_{33} is planar.

Proof.

Let G be a planar graph embedded in the plane so that paths intersect only at nodes, i.e., arcs meet only at endpoints. For each node t in G, label the adjacent set A_t in a cyclic manner, say clockwise, around node t corresponding to the edges that emanate from node t. Given any labeled quadruple $h < i < j < k$ in A_t and any pair of staggered paths (P,Q) with these endnodes. Then $P \cup \{t\}$ together with edges (h,t) and (j,t) form a cycle which divides the plane into two regions, an inside and an outside. But by the cyclic ordering, nodes i and k lie in different regions; hence by planarity, P and Q share a common node (see Figure 5).

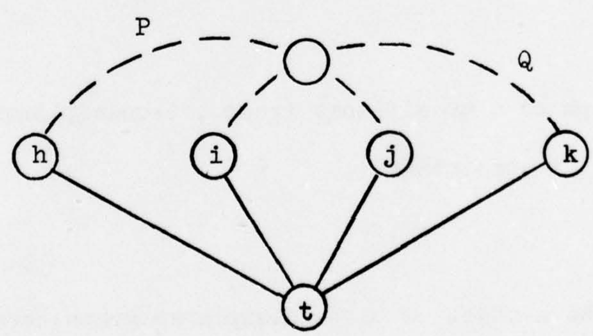


Figure 5. Planar Staggered Paths

The converse is proved by contraposition. Let G be a quasiplanar graph containing no subgraph that is a refinement of K_{33}. Suppose G is not planar. Then by Kuratowski's Theorem [4], there must be a subgraph H of G that is a refinement of K_5. Let t be a node in H. The adjacent set of t consists of four nodes for which each labeling results in nonintersecting staggered paths. Thus H is not t-quasiplanar and by Lemma 3.1 neither is G. Q.E.D.

Thus K_{33} is an example of a graph that is quasiplanar but, by Kuratowski's Theorem, not planar. On the other hand, K_5 is neither quasiplanar nor planar.

The main result of this section is Theorem 3.7. It is an analog of the Kuratowski Theorem in that it characterizes quasiplanar graphs by excluding two particular graphs. Several definitions and lemmas will now be presented which lead to the main result.

If G is a graph and S is a subset of the nodes, let $G \setminus S$ denote the subgraph of G induced by the nodes not in S. For notational convenience let $G \setminus t$ denote $G \setminus \{t\}$. A graph is <u>connected</u> if there is a path between each pair of nodes. A graph G is <u>biconnected</u> if $G \setminus t$ is connected for each node t in G. A <u>component</u> of a graph is a maximal connected subgraph. The components of a graph are unique.

Let G be a graph and t be a node such that $G \setminus t$ is not biconnected. A pair of graphs (B_1, B_2) is called a <u>t-bisection</u> of G if it is obtained in the following manner. Let v be a node other than t such that $G \setminus \{t,v\}$ has at least two components. The node v exists because $G \setminus t$ is not biconnected. Partition these components into two nonempty collections of components and let S_1 and S_2 denote respectively the sets of nodes in each collection. Consider the subgraphs of G induced by $S_k \cup \{t,v\}$ for k = 1, 2. If (t,v) is not in G but there are nodes in both S_1 and S_2 which are adjacent to t in G, then add edge (t,v) to the induced subgraphs to obtain B_1 and B_2 respectively. Otherwise B_1 and B_2 are just the induced subgraphs. The following properties are immediate: (1) The number of nodes in B_k is strictly less than the number of nodes in G for k = 1, 2; and (2) If $G \backslash t$ is connected, then so is $B_k \backslash t$ for k = 1, 2. See Figure 6.

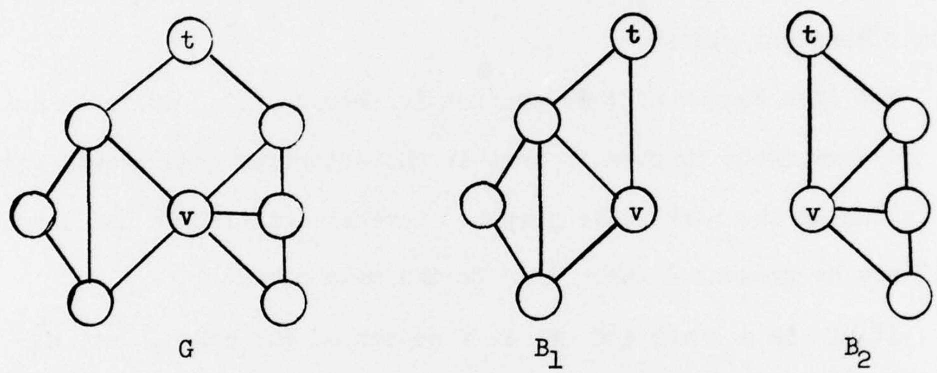


Figure 6. t-Bisection

A t-decomposition of a graph G is a set of graphs, each member H of which contains t and has the property that $H \backslash t$ is biconnected, that is constructed in the following manner. If $C_1, C_2, \ldots, C_q$ are the components of $G \backslash t$, then let H_k be the subgraph of G induced by the nodes in C_k together with the node t for $1 \leq k \leq q$. Let $\mathcal{H} \equiv \{H_k\}_{k=1}^{q}$. Repeatedly replace each graph H in $\mathcal{H}$ for which $H \backslash t$ is not biconnected by the two graphs in a t-bisection of H. By property (1) of t-bisections, the number of such replacements is bounded by the number of nodes in G. The resultant set $\mathcal{H}$ is the t-decomposition of G.

The following lemma will be used in the proof of Theorem 3.7 to justify considering only those graphs G where $G \setminus t$ is biconnected.

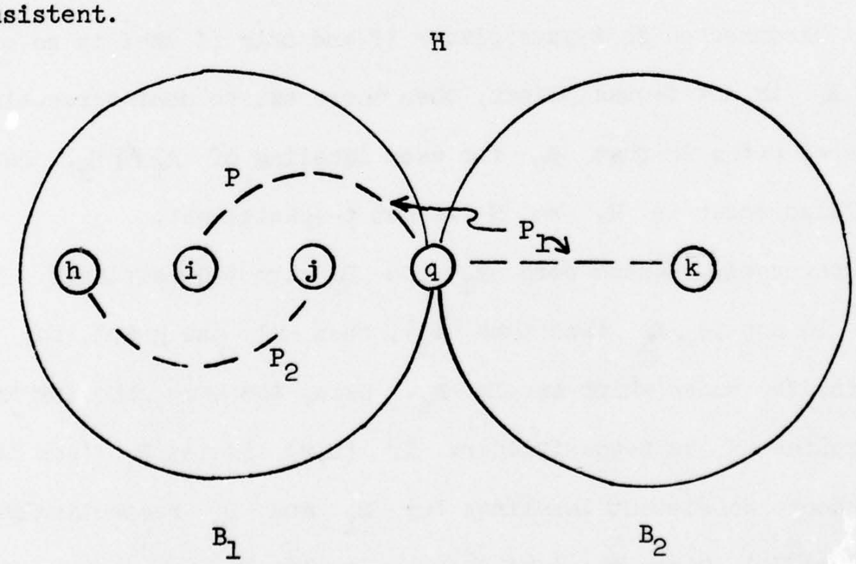
Lemma 3.3.

A graph is t-quasiplanar if and only if all the graphs in a t-decomposition thereof are t-quasiplanar.

Proof.

Let t be a node in a graph G and $\mathcal{C}$ be the set of components C in $G \setminus t$. Clearly $H \equiv G(C \cup t)$ is t-quasiplanar for each $C \in \mathcal{C}$ if and only if G is t-quasiplanar. Since t-decompositions are formed by repeated t-bisections, it is sufficient to show that each graph B_1 and B_2 obtained in a t-bisection of a graph H such that $H \setminus t$ is not biconnected is t-quasiplanar if and only if that is so of H. If a B_k is not t-quasiplanar, then there exists nonintersecting staggered paths in that B_k for each labeling of $A_t \cap B_k$. But these paths also occur in H, so H is not t-quasiplanar.

Conversely, assume both B_1 and B_2 are t-quasiplanar. If (t,v) is not in B_1 (and thus B_2), then only one graph, say B_1, contains the nodes which are in A_t. Using the same labeling as for B_1 implies H is t-quasiplanar. If (t,v) is in B_1 (and thus B_2), then choose consistent labelings for B_1 and B_2 respectively. (Refer to Figure 7.) Since the labelings are cyclic, we may arrange the labels so that v receives the greatest label, say q, in B_1. Add $q-1$ to the labels in B_2 after the labels therein have been arranged so v has the label one. This results in a consistent labeling for H. To prove this, suppose there is a pair (P_1,P_2) of disjoint paths in H. Four cases must be considered. If P_1 and P_2 are contained in the same subgraph, then they are not staggered since the labeling is consistent in that subgraph. If they are in different subgraphs, then the ordering would not allow them to be staggered either. They cannot both be in both subgraphs since they would intersect at the node v. Finally, suppose one path, say P_1, intersects both subgraphs and the other is in only one subgraph, say B_1. Since v is the only node in $H \setminus t$ common to both subgraphs, P_1 contains v. Let P be the maximal subpath of P_1 contained in B_1. The node v is an endnode of P with label q. But q was chosen to be the greatest label in B_1. Hence if P_1 and P_2 are staggered in H, P and P_2 are also staggered paths in B_1 and so share a common node which is impossible. We conclude that P_1 and P_2 are not staggered, and thus the labeling is consistent. Q.E.D.



<u>Figure 7.</u>

By a <u>simple contraction</u> of a graph G we mean that two adjacent nodes, say s and u, are replaced by a single node w which is adjacent to those nodes adjacent to s or u. When this is so call s and u the parents of w. Call the graph H a contraction of G if it can be obtained from G by a sequence of simple contractions, and call a node s in G an ancestor of a node w in H if there is a sequence $(s,G) = (w_1,H_1), \dots , (w_q,H_q) = (w,H)$ such that H_k is a simple contraction of H_{k-1}, and w_{k-1} is a parent of w_k for $2 \leq k \leq q$. In particular, a node is an ancestor of itself. Call a contraction H of G t-avoiding if t is the only ancestor of itself. Let the inverse operation of a simple contraction (resp., contraction) be called a simple expansion (resp., expansion).

Lemma 3.4.

A t-avoiding contraction of a t-quasiplanar graph is t-quasiplanar.

Proof.

It suffices to prove the result for a t-avoiding simple contraction G of a t-quasiplanar graph G'. Let s, u be the nodes in G' which are replaced by w in G. If P is a path in G, denote by P' its unique extension in G' formed by replacing w, if it appears in P, by s, u and the edge (s,u). Since G' is t-quasiplanar, A_t' has a consistent labeling. This induces a natural labeling on A_t, where if $w \in A_t$, then either s or u is in A_t' and w receives the label of one such node. Let P and Q be staggered paths in G with that labeling. Then P' and Q' are staggered paths in G', and so, because G' is t-quasiplanar, share a common node p. Thus, P and Q must share a common node, viz., one for which p is a parent, completing the proof.

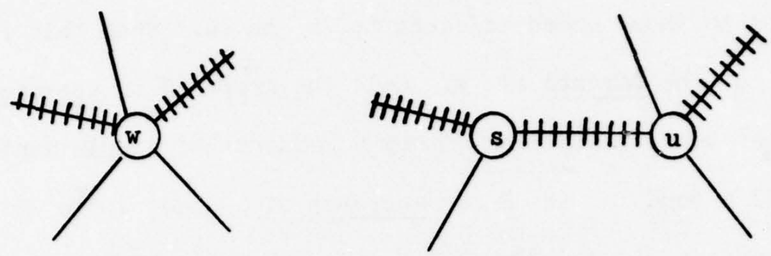


Figure 8. Extending Paths in Extensions

The following graph illustrates the need for the contraction in Lemma 3.4 to be t-avoiding.

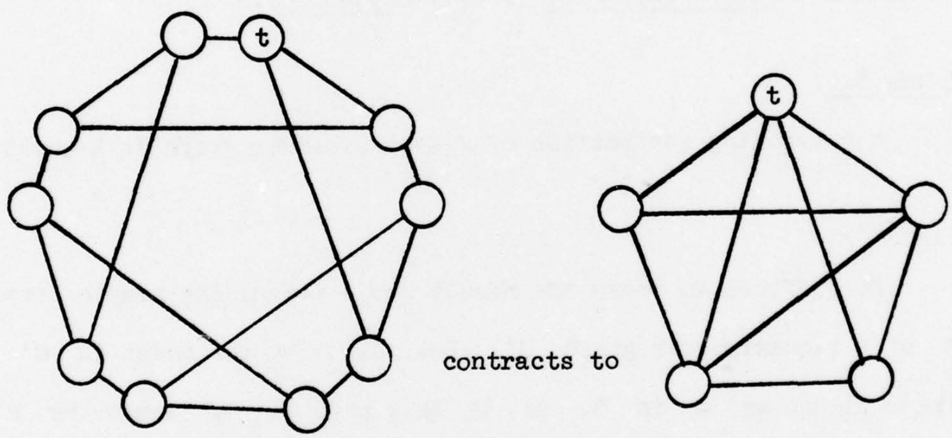


Figure 9. A Quasiplanar Graph
With a Nonquasiplanar Contraction

Note that the graph G in Figure 9 has only nodes of degree three and so is quasiplanar. Yet if we contract each pair of nodes as indicated, we obtain the nonquasiplanar graph K_{55}.

Given a node t, denote by K_t the complete graph with five nodes, one being the node t. Denote by E_t the graph K_{33} augmented with an additional edge and t being one of the two nodes having degree four. (See Figure 10.)

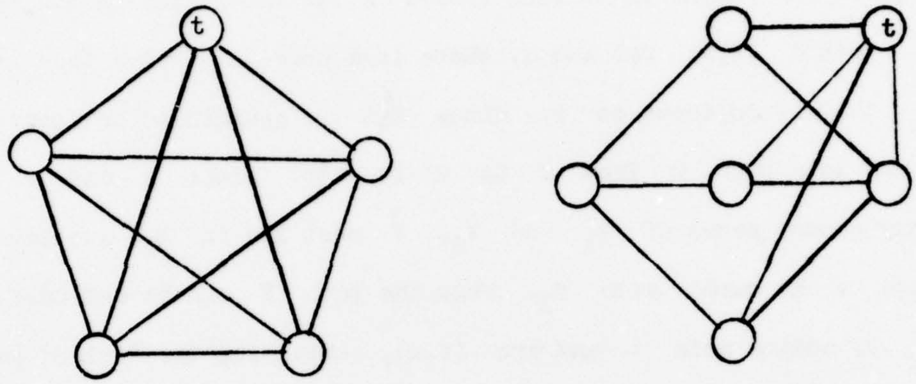


Figure 10. K_t and E_t

Lemma 3.5.

If G contains no subgraph with a t-avoiding contraction that is K_t or E_t, then neither does each graph in a t-decomposition of G.

Proof.

Clearly if G contains no subgraph with a t-avoiding contraction that is K_t or E_t, then neither does a component H of $G\backslash t$. It suffices to show that if H contains no such subgraphs and has the property that $H\backslash t$ is connected but not biconnected, then the graphs in the t-bisection (B_1,B_2) of H also contain no such subgraphs. Since the only arc which might be added is (t,v), the result follows immediately if (t,v) is in H. Assume (t,v) is not in H and refer to Figure 11. Suppose that there is a subgraph J in say B_1 with a t-avoiding contraction that is K_t or E_t. Then J must contain (t,v) otherwise J is contained in H which is impossible. We will show that there is a path from t to v in B_2 not using arc (t,v) which may replace (t,v) in J obtaining a new subgraph in H that has a t-avoiding contraction that is K_t or E_t. This contradiction implies that there is no such subset J in the graphs of the t-bisection.

Since (t,v) was added, there is a node u (other than t) in B_2 that is adjacent to t. Since $H\backslash t$ is assumed to be connected, there is a path P from v to u in $H\backslash t$. Since t and v are the only common nodes of B_1 and B_2, P must lie in B_2 and have only node v in common with B_1. Thus the path P can be extended in B_2 by adding node t and arc (t,u), obtaining the desired path and completing the proof.

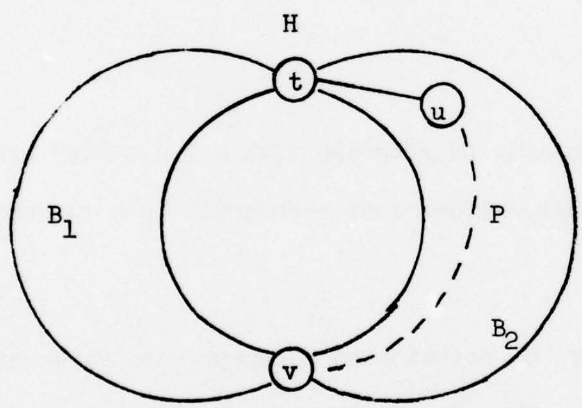


Figure 11.

Neither K_t nor E_t are t-quasiplanar. This is easily verified by deleting the node t and noting that there exists a path between each pair of nodes in A_t that is disjoint from another path between the other two nodes in A_t. Thus each labeling of the nodes in A_t results in a pair of disjoint staggered paths. Furthermore, neither

K_t nor E_t is a contraction of the other. The next lemma leads to the main result.

Lemma 3.6.

It t is a node in a graph G with $d_t \geq 4$, $G \setminus t$ biconnected, and E_t not a t-avoiding contraction of any subgraph of G, then there is a cycle in $G \setminus t$ containing all the nodes adjacent to t in G.

Proof.

Since $G \setminus t$ is biconnected, Menger's Theorem [5] asserts that there exists a cycle in $G \setminus t$ containing a given pair of nodes. Thus, if $d_t \leq 2$, the result follows. Assume by induction that there is a cycle C containing $k \geq 2$ nodes in A_t. If $k = d_t$, we are done. Otherwise, orient the cycle and label the nodes in $A_t \cap C$ 1, 2, ... , k in the order traversed in the cycle. Let C_j be the segment of the cycle from the node $j \in A_t \cap C$ to the next labeled node ($j+1$ if $j < k$ and 1 if $j = k$). Let v be a node in A_t but not on C.

Menger's Theorem implies, by adding an artificial node adjacent to the nodes in C, that there exists two separate paths in $G \setminus t$ from v to the artificial node each intersecting C at unique distinct nodes u and w respectively. If u and w lie on a common segment C_j, then the cycle C can be immediately extended to contain v completing the induction. Thus it remains to show that all separate paths from v to C must intersect C on a common segment. Two cases will be considered, viz., $k = 2$ and $k > 2$.

If $k = 2$, then the subgraph (not containing t) in Figure 12 is obtained, where we denote v and the nodes in $A_t \cap C$ by solid nodes.

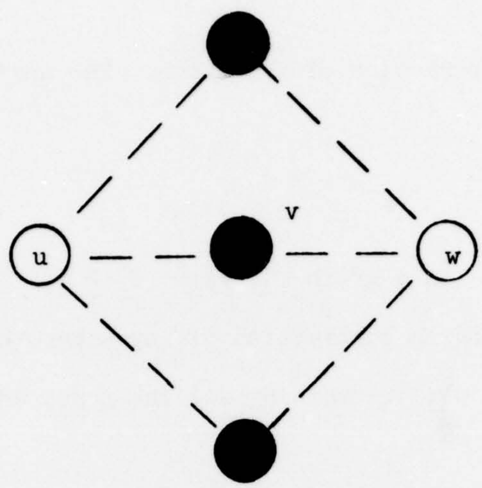


Figure 12.

Since $d_t \geq 4$, there exists another node in A_t, say x. Also since $G \setminus t$ is connected, there is a path P from x to the subgraph in Figure 12. If P first meets the subgraph in Figure 12 at a nonsolid node, then by contracting along that path, appending node t, and possibly deleting a few arcs, we obtain E_t, which is a contradiction. Thus using Menger's Theorem as above, we obtain two separate paths from x to distinct solid nodes in Figure 12. The resultant subgraph is shown in Figure 13 taking into account obvious symmetries. A cycle with at least three solid nodes is easily observed in Figure 13.

Consider now the case $k \geq 3$. Then u and w divide the cycle C into two segments with each containing at least one solid node that is not an endnode thereof, as illustrated in Figure 14.

On appending t, deleting certain edges, and contracting the subgraph of Figure 14 to that of Figure 15, we obtain E_t which is impossible. Therefore all separate paths from v to C must intersect C on a common segment. Q.E.D.

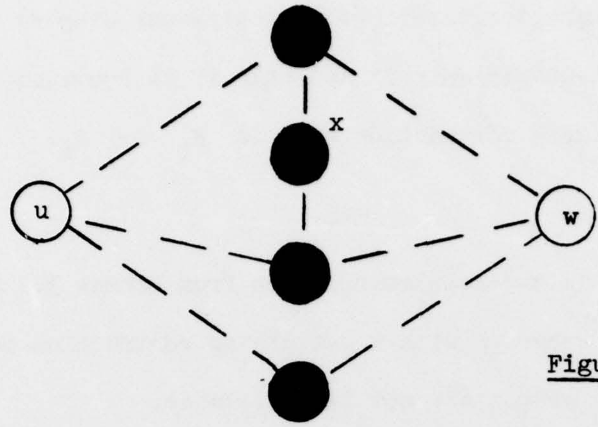


Figure 13.

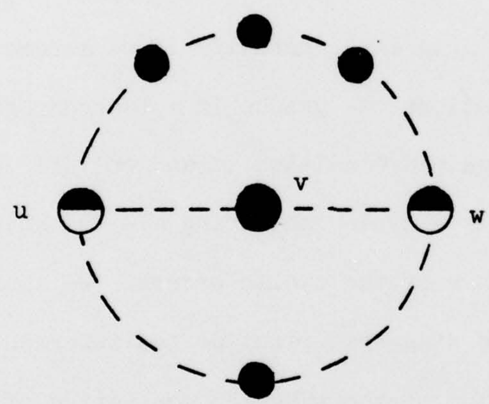


Figure 14.

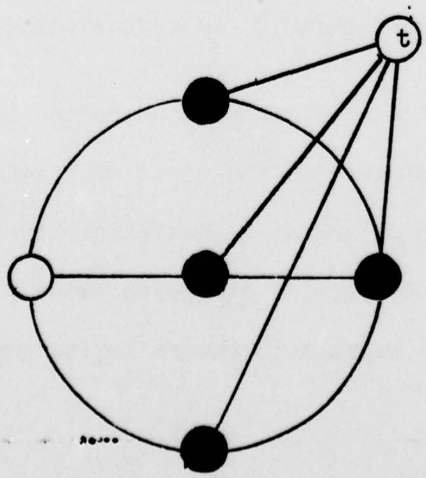


Figure 15.

Theorem 3.7. (Characterization of t-Quasiplanar Graphs)

A graph is t-quasiplanar if and only if it contains no subgraph that has a t-avoiding contraction that is K_t or E_t.

Proof.

Suppose G is t-quasiplanar. Then from Lemmas 3.1 and 3.4, G cannot contain a subgraph with a t-avoiding contraction being K_t or E_t because these graphs are not t-quasiplanar.

Conversely, assume G contains no subgraph with a t-avoiding contraction that is K_t or E_t where t is a node in G. If $d_t < 4$, then G is trivially t-quasiplanar. Thus suppose $d_t \geq 4$. By Lemmas 3.3 and 3.5 we do not lose any generality if we assume $G \setminus t$ is biconnected, since otherwise we consider the graphs in a t-decomposition, each of which satisfies the desired condition if and only if G does. Using Lemma 3.6, there exists a cycle containing all the nodes adjacent to t. Now label A_t in either of the cyclic orders. We show that this labeling is consistent. If two staggered paths do not intersect, then K_t is a t-avoiding contraction of the subgraph consisting of the cycle, the staggered paths, and t as illustrated in Figure 16. This is a contradiction and establishes that G is t-quasiplanar as desired. Q.E.D.

The cycle containing A_t can be constructed in polynomial time by using the method indicated in the proof of Lemma 3.6. The above proof indicates that to determine t-quasiplanarity it remains to show that there are no nonintersecting staggered paths. As we shall see, this problem is closely related to the following problem.

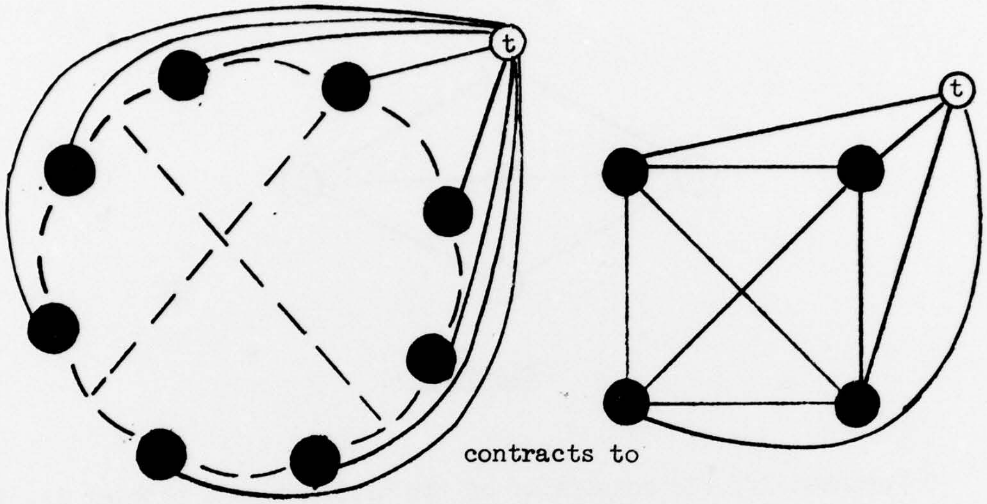


Figure 16.

Disjoint Path Problem: Given a graph G and two pairs of nodes (h,j), (i,k), does there exist a pair of disjoint paths in G having the given pairs of nodes as their respective endpoints?

Corollary 3.8.

The problem of determining whether or not a graph is t-quasiplanar can be solved in polynomial time if and only if the disjoint-path problem can.

Proof.

From the above, the t-quasiplanar problem has been transformed to the disjoint path problem. Conversely, given two pairs of nodes (h,j) and (i,k) in a graph G we construct a new graph H as follows. Add a new node t adjacent to h, i, j, and k and also add (if necessary) the edges (h,i), (i,j), (j,k), and (k,h). See Figure 17.

Then there exists disjoint paths from h to j and from i to k if and only if H is not t-quasiplanar. Q.E.D.

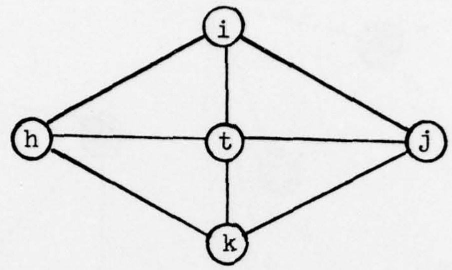


Figure 17.

Unfortunately, the complexity of the disjoint path problem is, I understand from Professor Even,an unsolved problem and has been an open question for the last few years. Some related results have been presented by Knuth (see Karp [3]) and Even, Itai, and Shamin [2].

Another immediate consequence of Theorem 3.7 is the following result.

Theorem 3.9. (Characterization of Quasiplanar Graphs)

A graph is quasiplanar if and only if for all nodes t there is no subgraph that has a t-avoiding contraction which is K_t or E_t.

Associated with each directed graph is a unique undirected graph obtained by considering the ordered pairs as unordered. Call a directed graph planar (resp., quasiplanar) if the associated undirected graph is planar (resp., quasiplanar). In the next section a property of planar graphs will be defined in such a manner as to depend on the direction of the arcs.

4. Pseudoplanar Graphs.

Let G be a directed graph, M a set of nodes in G, and s a node in G. Recall that an arborescence from s to M is an arborescence with source s and endnodes in M. Also recall that a maximal arborescence T from s to M is an arborescence from s to M such that a maximum number of nodes of M are in T.

If there is an arborescence T from a node s to a set M, then there always exists a maximal arborescence from s to M. This is easily verified from the definitions since a node t in M is accessible from s if and only if there is a chain C from s to t. This chain can be used to enlarge an arborescence T, augmenting it by the subchain of C which begins with a node q in T, ends with t, and contains no other node in T. Successively enlarging the arborescence will in at most p steps obtain a maximal arborescence where p is the number of nodes in M not in T.

Let s and t be nodes in a (directed) graph. Call the graph st-pseudoplanar if the set A_t of nodes adjacent to t can be labeled 1, 2, ... , m so as to have the consecutive-endnodes property. Note that the labeling depends only on s and t. Call a graph pseudoplanar if for each ordered pair of nodes s, t the graph is st-pseudoplanar.

A graph in which $d_t \leq 3$ is trivially st-pseudoplanar for all s. Thus, the graph in Figure 18 is pseudoplanar but not quasiplanar.

Let the arc (t,s) be called the complement of the arc (s,t) and a graph be called bidirected if the complement of each arc in the graph is also in the graph. The following theorem implies that quasiplanarity and pseudoplanarity are equivalent in bidirected graphs.

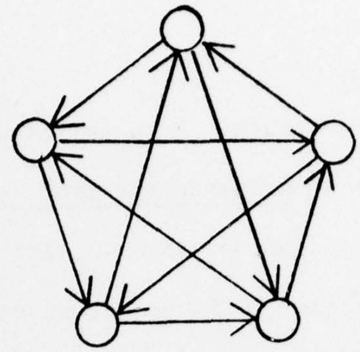


Figure 18. Pseudoplanar, Not Quasiplanar

Theorem 4.1.

A t-quasiplanar graph with every node in A_t accessible from s is st-pseudoplanar. Conversely, a graph that is bidirected and st-pseudoplanar for each s is t-quasiplanar.

Proof.

Let G be t-quasiplanar and label the nodes adjacent to t consistently. Suppose the labeling is not compatible for some s. Then there exists a maximal arborescence F from s to A_t not containing t and a node u in F such that the endnode descendents of u are not consecutively labeled. Hence there exist nodes in A_t labeled $h < i < j < k$ such that (without loss of generality) i and k are endnode descendents of u while h and j are not. Let P and Q be the unique paths in F between h,j and i,k respectively. The path P is the union of two chains, P_1 from a node s' (accessible from s) to h and P_2 from s' to j. Similarly, Q is the union of two chains, Q_1 from a node u' (accessible from u) to i and Q_2 from u' to k (see Figure 19). We will show that P and Q are disjoint which contradicts the consistent labeling assumption. Thus the labeling is compatible and G is st-pseudoplanar.

Suppose P and Q intersect at g say. Then P_m and Q_n meet at g for some $1 \leq m, n \leq 2$. By symmetry fix $n = 1$ and consider $m = 1, 2$. If $m = 1$, there is a chain from u to g to h, contradicting the inaccessibility of h from u. Similarly, if $m = 2$, there is a chain from u to g to j, contradicting the inaccessibility of j from u. We conclude that $P \cap Q = \emptyset$ as claimed.

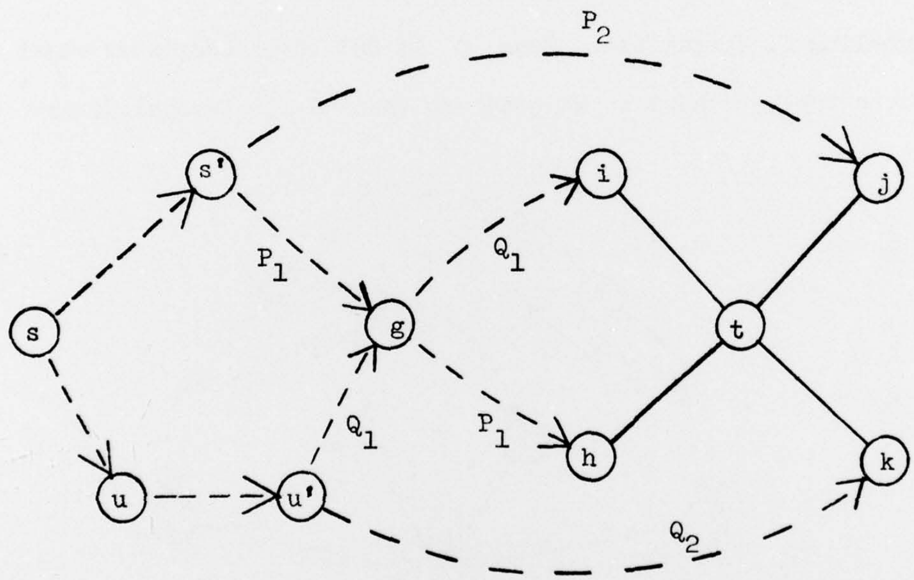


Figure 19.

Conversely, let t be a (fixed) node in G and assume G is bidirected and st-pseudoplanar for each node s in G. If G is not t-quasiplanar, then there is a component H of $G \setminus t$ such that the induced subgraph $G(H \cup \{t\})$ is also not t-quasiplanar. For each labeling of A_t, there exist two disjoint staggered paths P_1 and P_2 in H with endnodes h,j and i,k respectively, $h < i < k < j$

(see Figure 20). Since H is connected, there is a path Q between P_1 and P_2. Let $s \equiv P_1 \cap Q$ and $u \equiv P_2 \cap Q$ and direct arcs away from s in these paths to obtain an arborescence from s to the end-nodes h,j and i,k of P_1 and P_2 respectively. Extend this arborescence to a maximal arborescence T from s to A_t not containing t. Note that by construction, h and j are not accessible from u in T. Otherwise a cycle would be formed in T which is impossible. Thus u does not have consecutively labeled endnode descendents for this labeling. Each labeling of G imposes a labeling on H and so no labeling is compatible. Thus G is not st-pseudoplanar which contradicts the hypothesis. We conclude that G is t-quasiplanar. Q.E.D.

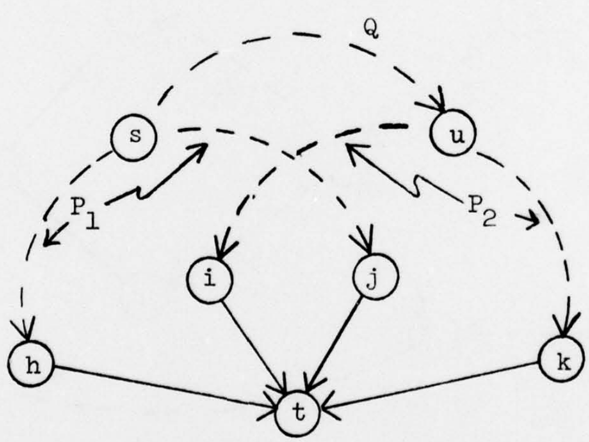


Figure 20.

An immediate consequence of the above result is the following.

Theorem 4.2.

A quasiplanar graph is pseudoplanar. Conversely, a bidirected pseudoplanar graph is quasiplanar.

Figure 21 summarizes the relationship among planar, quasiplanar, and pseudoplanar graphs.

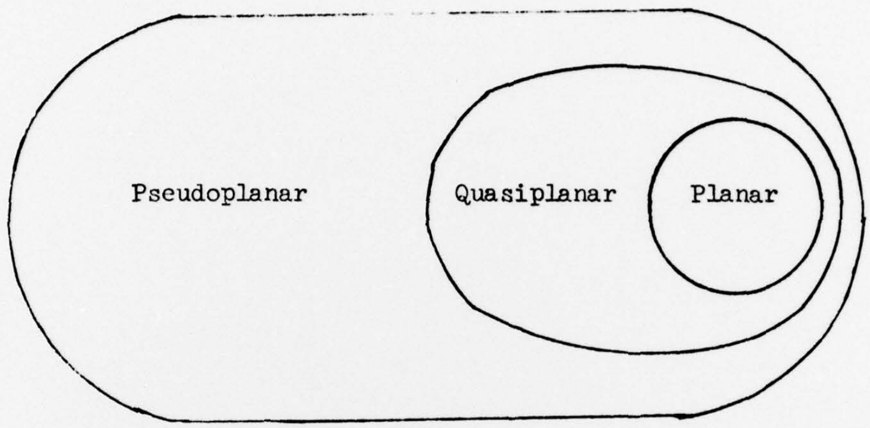


Figure 21.

Remark. The computational complexity of determining whether or not a graph is pseudoplanar is at least as great as that for the quasiplanarity problem since the latter can be transformed (by considering the graph as bidirected) to the present problem.

In conclusion we note that the above relationships imply that the intersecting-staggered-paths property and the consecutive-endnodes property do indeed hold for planar graphs. It is st-pseudoplanar networks for which a polynomial-running-time algorithm is developed in [1].

Acknowledgments.

My sincere gratitude is given to Professor Václav Chvátal who suggested that I look for a Kuratowski type theorem for quasiplanar graphs. I also thank Professor Arthur F. Veinott, Jr. for refining the statements and proofs contained herein.

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4. TITLE (and Subtitle) Quasiplanar and Pseudoplanar Graphs		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ranel E. Erickson		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0493
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Operations Research Stanford University Stanford, California 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (NR-042-264)
11. CONTROLLING OFFICE NAME AND ADDRESS Logistics and Mathematical Statistics Branch Office of Naval Research Arlington, Virginia		12. REPORT DATE May 1, 1978
		13. NUMBER OF PAGES 29
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

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Also partially supported by National Science Foundation Grant ENG-76-12266.

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Planar graphs, quasiplanar graphs, pseudoplanar graphs, aborescences, Kuratowski-type characterizations, minimum concave-cost network flows.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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ABSTRACT

Two new generalizations of planar graphs, called quasiplanar and pseudoplanar graphs, are introduced and discussed. A graph is called quasiplanar if for each node t, the set of nodes incident to t can be labeled $1, \dots, m$ so that for each $1 \le h < i < j < k \le m$, each pair of paths not containing t and having respective endnodes h,j and i,k share a common node. A (directed) graph is called pseudoplanar if for each pair of nodes s,t, the set of nodes adjacent to t can be labeled $1, \dots, m$ so that for each maximal arborescence not containing t and having root s and each endnode adjacent to t, the endnode descendants of each node in the arborescence are either $j, \dots, k$ or $k, \dots, m, 1, \dots, j$ for some $1 \le j \le k \le m$. Planar graphs are quasiplanar and they in turn are pseudoplanar. Conversely, a pseudoplanar graph that contains with each arc its reverse arc is quasiplanar. And a quasiplanar graph that excludes subgraphs that are refinements of the complete bipartite graph K_{33} with three nodes in both sets is planar. Kuratowski (1930) characterized planar graphs as those that exclude subgraphs that are refinements of either the complete graph on five nodes or K_{33}. An analogous characterization of quasiplanar graphs is given in this paper in which the excluded subgraphs differ from Kuratowski's only by adding an edge in K_{33}. In a companion paper with Veinott, an algorithm given for finding minimum-concave-cost flows in single-source networks is shown to run in polynomial time when the associated graph is pseudoplanar.

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