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LEVEL II

DYNAMICS OF AN IMAGE VIEWED THROUGH A ROTATING MIRROR

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August 1978

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Interim rept.

31 p.

NAMRL-1252

MR04101, RR04101

Bureau of Medicine and Surgery
MR041.01.03.0154

Office of Naval Research
RR 041.01.02.NR 201.038

MR0410103,
RR0410102

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SUMMARY PAGE

THE PROBLEM

It is frequently assumed that the virtual image of a target viewed through a rotating mirror moves with respect to the observer at twice the angular rate of mirror rotation. This assumption is false, and leads to imprecise treatment of open loop tracking systems. Of particular interest is a class of dynamic visual acuity experiments in which acuity targets are viewed through a rotating mirror, where control of image velocity, exposure time, and image dimensions are of critical importance.

FINDINGS

Expressions are derived which describe the direction of the target image with respect to the observer as a function of mirror position. This relationship is nonlinear, and depends upon the distances from the center of rotation of the mirror (A) to the observer (C), and to the target (B), and upon the included angle ($\angle BAC$). Expressions are further derived for image velocity, acceleration, mirror intercept, and image dimensions as functions of mirror position.

ACKNOWLEDGMENTS

The author wishes to acknowledge the valued assistance of LT Tommy R. Morrison, MSC, USN in collecting data to corroborate calculations derived herein, and of Mr. Efrain Molina in confirming the derivations, and in pointing out the utility of identities represented by equations (17) and (18).

INTRODUCTION

A majority of the visual and optical scientists who have toured our vision laboratories during the past three years have been surprised at our statement that the ocular movement required to track an image viewed through a rotating mirror is a nonlinear function of the mirror movement. Although the surprise is quickly abated by a brief explanation, the frequency with which this fact is neglected, even in laboratory instrumentation, suggests the need for an explicit statement of the relationships involved.

The laws of reflection predict that the direction of a ray reflected by a rotating plane mirror will change at twice the rate of mirror rotation in a plane normal to the axis of rotation. Changes in mirror orientation will result in equal changes in both the angle of incidence and the angle of reflection of the ray. However, the sample of rays reflected from a target to an arbitrary tracker position changes as a function of mirror orientation. The angle of incidence of these samples is not a linear function of mirror orientation, but depends upon the relative positions of the target, mirror and tracker, as well as the orientation of the mirror.

These considerations are of importance, in general, to the design of open loop tracking systems in which a sensor views the virtual image of a target through a gimbaled mirror (1, 2) and, in particular, to the configuration of a class of dynamic visual acuity experiments in which the subject views an acuity target through a rotating mirror (3).

The purpose of this paper is to derive expressions of the relationships which determine the viewing angle as a function of mirror position, and to discuss some of the implications for stimulus control in visual experiments.

IMAGE DIRECTION: $\omega = f(\theta)$

In a Cartesian coordinate system, let $A(x_A, y_A)$, $B(x_B, y_B)$, and $C(x_C, y_C)$ represent the positions of the center of rotation of a plane mirror, a target, and the center of rotation of the eye, respectively, such that the plane of incidence is normal to the axis of rotation of the mirror. Let θ and ω represent the angular orientation of the mirror and the eye, respectively, as indicated in Figure 1. The position of the observed virtual image is located at $p(x_p, y_p)$ such that the following relationships hold: (a) line $Bp \perp Ar$ at $q(x_q, y_q)$ such that line segments $l_{Bq} = l_{qp}$, and (b) line Cp intersects the mirror at point $r(x_r, y_r)$ such that $\phi_1 = \phi_2 = \phi_3$, and line segments $l_{Br} = l_{rp}$.

The problem is to define eye orientation (ω) as a function of mirror orientation (θ). The approach will be to define point $p(x_p, y_p)$ as a function of θ; then ω will be defined as a function of line Cp.

Point $q(x_q, y_q)$ may be determined as a function of θ in the following manner:

line Aq: $y_q = y_A + (x_q - x_A) \tan \theta$ (1)

line Bq: $y_q = y_B - (x_q - x_B) \operatorname{ctn} \theta$ (2)

Subtracting and solving for x_q, we obtain

$$x_q = \frac{y_B - y_A + x_A \tan \theta + x_B \operatorname{ctn} \theta}{\tan \theta + \operatorname{ctn} \theta} . \quad (3)$$

Then,

$$y_q = \frac{x_B - x_A + y_B \tan \theta + y_A \operatorname{ctn} \theta}{\tan \theta + \operatorname{ctn} \theta} . \quad (4)$$

Since line segment Bp is bisected at q,

$$x_p = x_q + (x_q - x_B) \quad (5)$$

$$x_p = \frac{2y_B - 2y_A + 2x_A \tan \theta - x_B \tan \theta + x_B \operatorname{ctn} \theta}{\tan \theta + \operatorname{ctn} \theta} \quad (6)$$

and

$$y_p = y_q + (y_q - y_B) \quad (7)$$

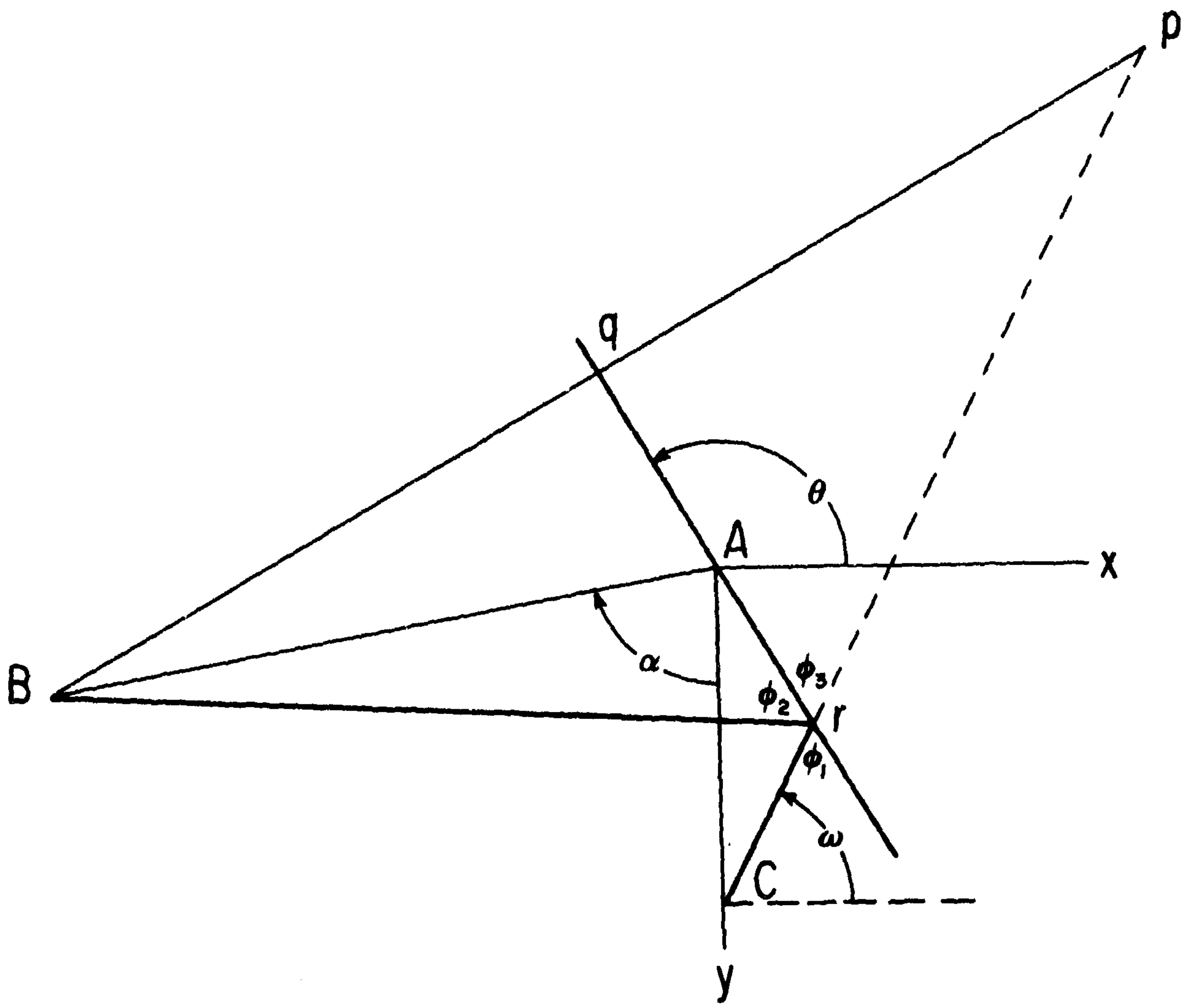

Figure 1. Geometry of the relationship between the target *(B)*, the center of rotation *(A)* of mirror *qAr*, and the center of rotation of the eye *(C)*, when viewing the target image *(p)*. The image direction *(ω)* with respect to the eye is a non-linear function of the mirror angle *(θ)*. Parameters of the relationship are the ratio of distances from center of rotation of the mirror to that of the eye (l_{CA}) and to the target (l_{BA}), and the angle, *BAC*, between them *(α)*.

$$y_D = \frac{2x_B - 2x_A + y_B \tan\theta + 2y_A \operatorname{ctn}\theta - y_B \operatorname{ctn}\theta}{\tan\theta + \operatorname{ctn}\theta} \tag{8}$$

then,

$$\tan\omega = \frac{y_D - y_C}{x_D - x_C} \tag{9}$$

$$\tan\omega = \frac{(x_B - x_A)\sin 2\theta + (y_A - y_B)\cos 2\theta + y_A - y_C}{(y_B - y_A)\sin 2\theta + (x_B - x_A)\cos 2\theta + x_A - x_C} \tag{10}$$

No generality is lost by arbitrarily positioning $A_{(0,\ 0)}$ at the origin, and $C_{(0,\ y_C)}$ on the y-axis. This allows equation (10) to be simplified

$$\tan\omega = \frac{x_B \sin 2\theta - y_B \cos 2\theta - y_C}{y_B \sin 2\theta + x_B \cos 2\theta}. \tag{11}$$

IMAGE VELOCITY: $\frac{d\omega}{d\theta} = f(\theta)$

Since $\frac{d\omega}{dt} = \frac{d\omega}{d\theta} \cdot \frac{d\theta}{dt}$, the angular velocity of the target image with respect to the eye may be obtained as the product of the mirror velocity and the derivative of equation (11).

$$\frac{d\omega}{d\theta} = \frac{2\,[y_C(y_B \cos 2\theta - x_B \sin 2\theta) + x_B^2 + y_B^2]}{2y_C(y_B \cos 2\theta - x_B \sin 2\theta) + x_B^2 + y_B^2 + y_C^2} \tag{12}$$

$$\frac{d^2\omega}{d\theta^2} = \frac{4y_C(x_B^2 + y_B^2 - y_C^2)\,(y_B \sin 2\theta + x_B \cos 2\theta)}{[2y_C(y_B \cos 2\theta - x_B \sin 2\theta) + x_B^2 + y_B^2 + y_C^2]^2} \tag{13}$$

It is useful to express equations (11), (12), and (13) in terms of the angle BAC (α), and the distances from the center of rotation of the mirror to that of the eye (l_{CA}), and to the target (l_{BA}). From Figure 1, we see that

$$x_B = -l_{BA} \sin\alpha \tag{14}$$

$$y_B = -l_{BA} \cos\alpha \tag{15}$$

$$y_C = -l_{CA} \tag{16}$$

Substituting these in equation (11), and employing the identities,

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \tag{17}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \tag{18}$$

we obtain

$$\tan \omega = \frac{l_{BA} \cos(\alpha + 2\theta) + l_{CA}}{-l_{BA} \sin(\alpha + 2\theta)} \tag{19a}$$

$$= \frac{\cos(\alpha + 2\theta) + \frac{l_{CA}}{l_{BA}}}{-\sin(\alpha + 2\theta)} \tag{19b}$$

$$\frac{d\omega}{d\theta} = \frac{2l_{BA}[l_{BA} + l_{CA} \cos(\alpha + 2\theta)]}{l_{BA}^2 + 2l_{BA}l_{CA} \cos(\alpha + 2\theta) + l_{CA}^2} \tag{20a}$$

$$= \frac{2 + 2\frac{l_{CA}}{l_{BA}} \cos(\alpha + 2\theta)}{1 + 2\frac{l_{CA}}{l_{BA}} \cos(\alpha + 2\theta) + \left(\frac{l_{CA}}{l_{BA}}\right)^2} \tag{20b}$$

$$\frac{d^2\omega}{d\theta^2} = \frac{4l_{BA}l_{CA}(l_{BA}^2 - l_{CA}^2) \sin(\alpha + 2\theta)}{[l_{BA}^2 + 2l_{BA}l_{CA} \cos(\alpha + 2\theta) + l_{CA}^2]^2} \tag{21a}$$

$$= \frac{4\frac{l_{CA}}{l_{BA}}\left[1 - \left(\frac{l_{CA}}{l_{BA}}\right)^2\right] \sin(\alpha + 2\theta)}{\left[1 + 2\frac{l_{CA}}{l_{BA}} \cos(\alpha + 2\theta) + \left(\frac{l_{CA}}{l_{BA}}\right)^2\right]^2} \tag{21b}$$

The physical limitations of the problem place the following constraints upon θ and α (see Figure 1):

$$90^\circ < \theta < 270^\circ \tag{22}$$

$$-90^\circ < \omega < 270^\circ \tag{23}$$

$$-180^\circ < \alpha < 180^\circ \tag{24}$$

It should be noted that the mirror angle (θ) and the tracker or eye angle (ω) are equal to 0^o in the direction of the positive x-axis, and increase with counter-clockwise rotation. The angle BAC (α) is equal to 0^o in the direction of the negative y-axis, and increases in the clockwise direction.

Equation (12) has a minimum (if $l_{CA} < l_{BA}$) or a maximum (if $l_{CA} > l_{BA}$) when

$$y_B \sin 2\theta + x_B \cos 2\theta = 0; \qquad (25a)$$

that is, when

$$\frac{\sin 2\theta}{\cos 2\theta} = -\frac{x_B}{y_B} = \tan 2\theta . \qquad (25b)$$

From Figure 1 we see that

$$\frac{x_B}{y_B} = \tan \alpha . \qquad (26)$$

Thus, $\frac{d\omega}{d\theta}$ has a minimum (or maximum) when

$$\tan 2\theta = -\tan \alpha \qquad (27a)$$

$$= \tan (360^o - \alpha), \qquad (27b)$$

or, when

$$\theta = 180^o - \frac{\alpha}{2} . \qquad (28)$$

From equation (20b), we find the value of $\frac{d\omega}{d\theta}$ at its minimum (or maximum) to be

$$\frac{d\omega}{d\theta}_{\text{min (or max)}} = \frac{2}{1 + \frac{l_{CA}}{l_{BA}}} . \qquad (29)$$

The following limits upon the function $\frac{d\omega}{d\theta} = f(\theta)$ are apparent from equations (20a) and (20b):

$$\lim_{l_{CA} \to 0} \frac{d\omega}{d\theta} = 2 \qquad (30)$$

$$\lim_{l_{BA} \to 0} \frac{d\omega}{d\theta} = 0 \qquad (31)$$

$$\lim_{l_{CA} \to l_{BA}} \frac{d\omega}{d\theta} = 1 \qquad (32)$$

PARAMETERS: $\frac{l_{CA}}{l_{BA}}$, α

We see from equations (19b), (20b), and (21b) that the values of $\omega, \frac{d\omega}{d\theta}, \frac{d^2\omega}{d\theta^2} = f(\theta)$ depend upon the ratio of l_{CA} to l_{BA}, and their included angle, rather than upon the individual values of these distances. Therefore, the parameters of these functions are taken to be α and $\frac{l_{CA}}{l_{BA}}$.

In Figure 2, graphs of $\frac{d\omega}{d\theta} = f(\theta)$ are presented for seven values of $\frac{l_{CA}}{l_{BA}}$, and for one value of α $(\alpha = 0)$. For the purposes of these graphs it is assumed that the dimensions of the target and tracker are negligible, and that the tracker is capable of rotating 360^o.

The solid portions of the curves apply to values of θ over which the target image may be observed through a rotating mirror of maximum length. That is, the center of rotation is on the mirror's midline, and half the mirror's length is equal to l_{CA} or l_{BA}, whichever is smaller. The dotted portions of the curves apply to an extended mirror whose angular movement is limited in one direction by the position of the target, and in the other by the position of the tracker $(90^o < \theta < 270^o - \alpha)$.

When $\alpha = 0$, as in Figure 2, the tracker, target, and center of rotation of the mirror are all aligned. If $\frac{l_{CA}}{l_{BA}} < 1$, counterclockwise rotation of the mirror causes the target image to move from right to left, decelerating across the right side of the mirror through a minimum velocity at $\theta = 180^o - \frac{\alpha}{2}$, and accelerating across the left side of the mirror. If the target is closer to the center of mirror rotation than is the tracker, $\frac{l_{CA}}{l_{BA}} > 1$, the image moves first to the right, accelerating through $\frac{d\omega}{d\theta} = 0$, then through a maximum at $\theta = 180^o - \frac{\alpha}{2}$, and decelerates across the left side of the mirror. If $\frac{l_{CA}}{l_{BA}} = 1$, then $\frac{d\omega}{d\theta} = 1$ over the applicable range of θ's.

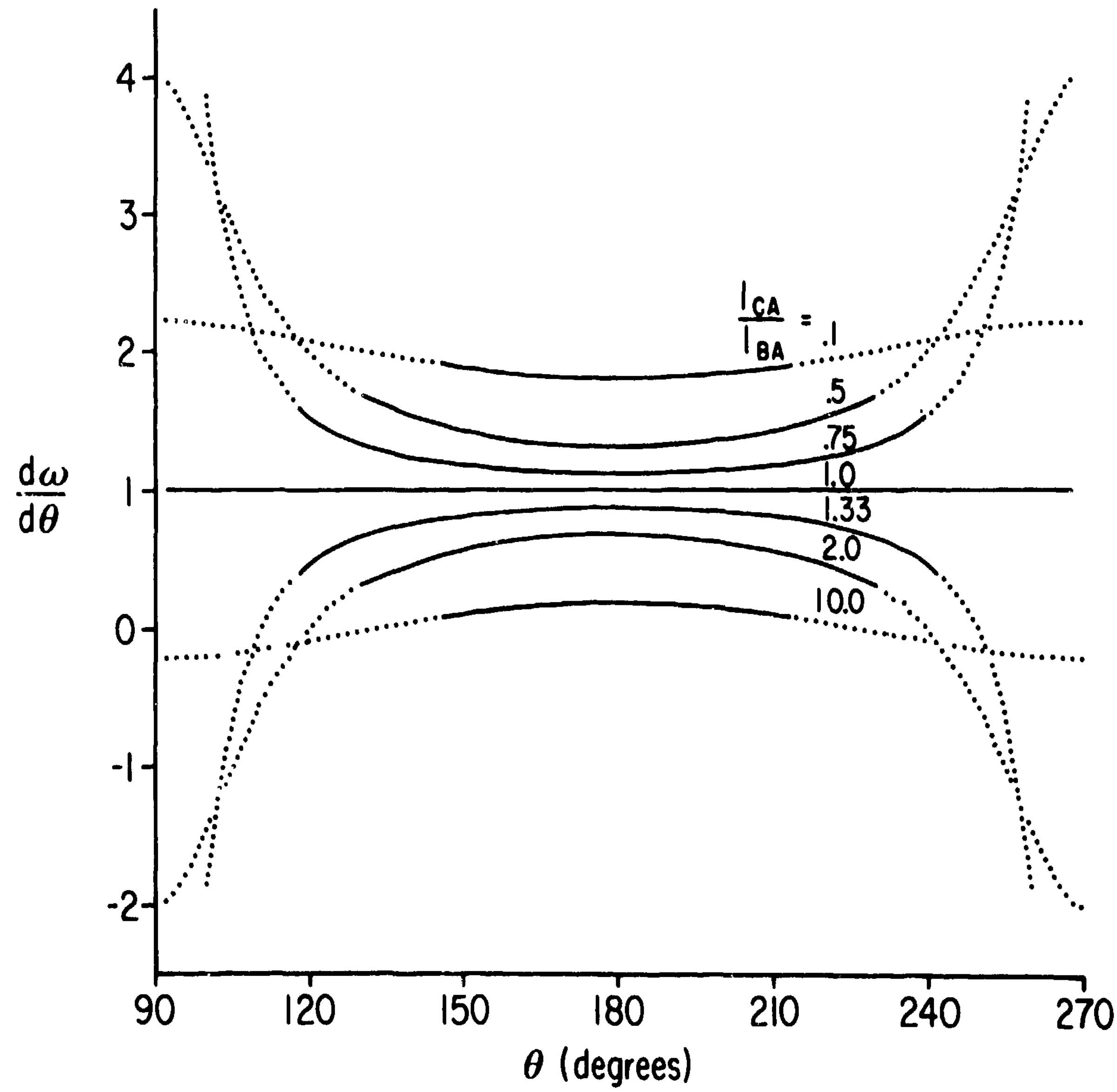

Figure 2. **Rate of change of image direction with respect to mirror angle as a function of mirror position** $\left[\frac{d\omega}{d\theta} = f(\theta)\right]$ **for seven values of** $\frac{l_{CA}}{l_{BA}}$ ***(0.1, 0.5, .75, 1.0, 1.33, 2.0, 10.0; α = 0).***

When $\alpha = 0$, this condition is equivalent to tracking one's own eye in a rotating mirror. In each case the minimum (or maximum) value of $\frac{d\omega}{d\theta}$ is $\frac{2}{1 + \frac{l_{CA}}{l_{BA}}}$ (equation (29)).

Reciprocal values of $\frac{l_{CA}}{l_{BA}}$ produce curves which are symmetrical with respect to $\frac{d\omega}{d\theta} = 1$. This is apparent in Figure 2, and it may be proven by substituting reciprocal values for $\frac{l_{CA}}{l_{BA}}$ in equation (20b):

$$\frac{2 + 2x \cos(\alpha + 2\theta)}{1 + 2x \cos(\alpha + 2\theta) + x^2} - 1 = 1 - \frac{2 + \frac{2}{x}\cos(\alpha + 2\theta)}{1 + \frac{2}{x}\cos(\alpha + 2\theta) + \frac{1}{x^2}} \tag{33}$$

Graphs of equations (20) and (21) are presented in Figure 3 for four values of α and one value of $\frac{l_{CA}}{l_{BA}}$. The values represented by the curve on the right side of Figure 3(b) are the same as those represented by the uppermost curve of Figure 2; $\frac{l_{CA}}{l_{BA}} = 0.1$ and $\alpha = 0$ in both cases. The scale of the ordinate in Figure 3 is increased in order to better illustrate the effects of varying α.

The value of α determines the position of the curve along the abscissa, the minimum occurring at $\theta = 180^o - \frac{\alpha}{2}$ (equation (28)). The value of α does not alter the form of the curve, nor its symmetry with respect to the minimum θ. The requirement for this symmetry may be proven by the following:

At $\theta = 180^o - \frac{\alpha}{2}$,

$$\cos(\alpha + 2\theta) = \cos(\alpha + 360^o - \alpha) = \cos 360^o \tag{34}$$

and,

$$\cos(360^o + \psi_1) = \cos(360^o - \psi_1). \tag{35}$$

Thus, for values of θ which are symmetrical with respect to the minimum, equation (20) predicts that

$$\left.\frac{d\omega}{d\theta}\right|_{+\psi_1} = \left.\frac{d\omega}{d\theta}\right|_{-\psi_1} \tag{36}$$

where, $\psi_1 = (180^o - \frac{\alpha}{2}) - \theta_1$.

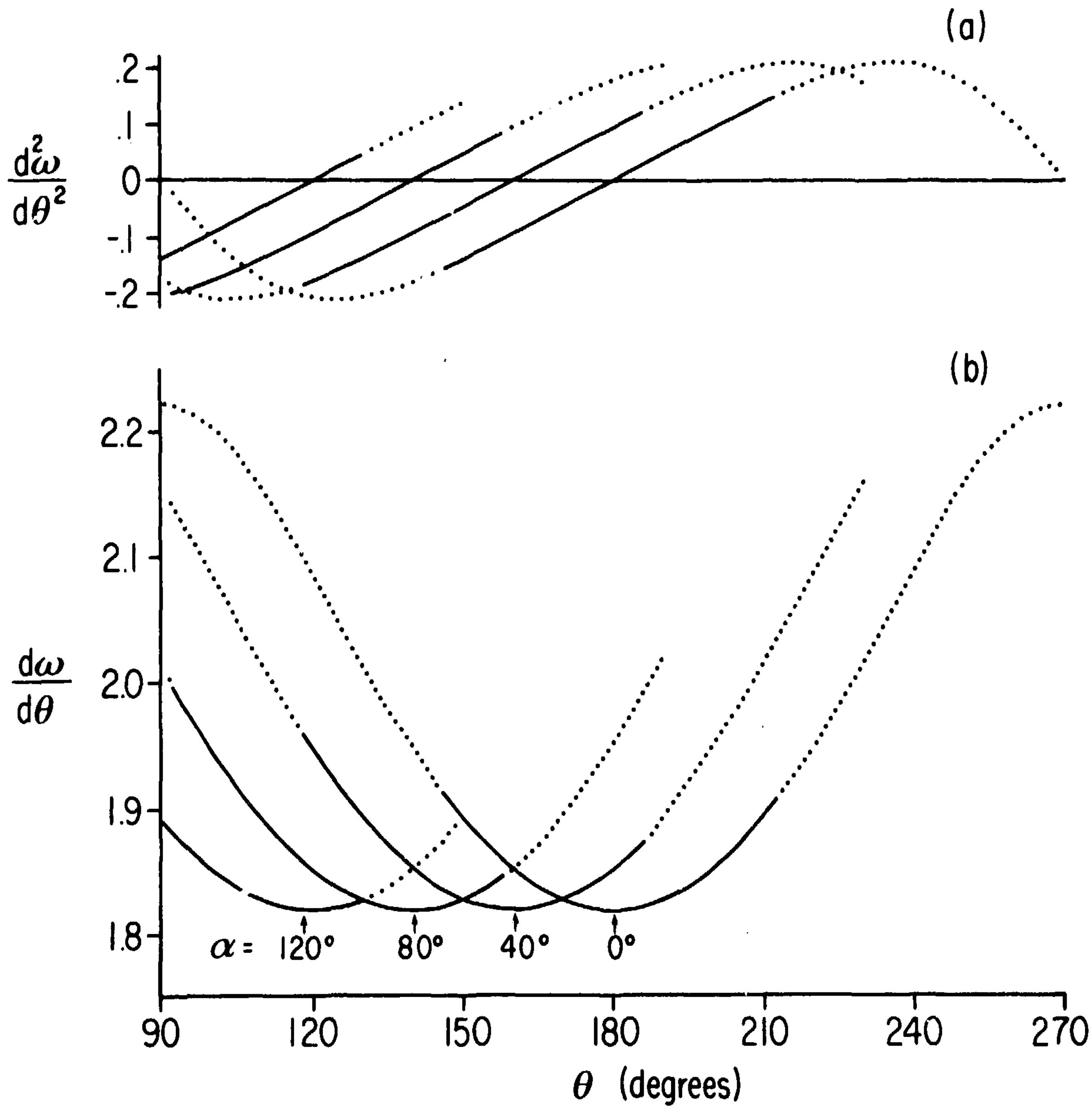

Figure 3. (a) $\frac{d^2\omega}{d\theta^2} = f(\theta)$. (b) $\frac{d\omega}{d\theta} = f(\theta)$. **Rates of change of image direction with respect to mirror angle as functions of mirror position for four values of** α *(0°, 40°, 80°, 120°;* $\frac{l_{CA}}{l_{BA}} = 0.1$*)*.

Thus, the function $\frac{d\omega}{d\theta} = f(\theta)$ is symmetrical with respect to $\theta = 180^o - \frac{\alpha}{2}$, and this symmetry is not influenced by the value of α. However, the values of θ for which the target image may be observed through a rotating mirror are determined, in part, by α. The solid portions of the curves in Figure 3 apply to the values of θ over which the target is observable through a rotating mirror of maximum length. The "observation window" for a given aperture shifts along the abscissa as a function of α. When the midpoint of the aperture is on the axis of rotation, the window is symmetrical with respect to $\theta = 180^o - \frac{\alpha}{2}$ only when $\alpha = 0$. For other values of α the observation window is no longer centered on $\theta = 180^o - \frac{\alpha}{2}$, but shifts in the direction of α with respect to this point. The magnitude of $\frac{d\omega}{d\theta}$ can easily exceed 2 for an extended mirror. However, $\frac{d\omega}{d\theta} \leq 2$ for all values of θ which lie within the observation window of a rotating mirror $(l_{Ar} < l_{CA})$. The extent of the observation window depends upon the mirror aperture, which may be calculated once the mirror intercept, $r_{(x_r,\, y_r)}$, has been defined.

MIRROR INTERCEPT: $r_{(x_r,\, y_r)}$.

The coordinates of the point at which the principle incident ray is reflected on the mirror may be determined by the intercept of lines Ar and Cr:

$$\text{line } Ar: \quad y_r = y_A + (x_r - x_A) \tan \theta \tag{37}$$

$$\text{line } Cr: \quad y_r = y_C + (x_r - x_C) \tan \omega \tag{38}$$

Setting point $A_{(0,0)}$ at the origin and point $C_{(0,y_C)}$ on the y-axis, and solving for the intercept, we obtain

$$x_r = \frac{y_C}{\tan \theta - \tan \omega}, \tag{39}$$

$$y_r = \frac{y_C \tan \theta}{\tan \theta - \tan \omega}. \tag{40}$$

The length of line segment l_{Ar} is

$$l_{Ar} = \sqrt{x_r^2 + y_r^2} \quad . \tag{41}$$

Substituting equations (39) and (40),

$$l_{Ar} = \sqrt{\frac{y_C^2\ (1 + \tan^2\theta)}{(\tan\theta - \tan\omega)^2}} \quad . \tag{42}$$

Employing the identity,

$$\sec^2\theta = 1 + \tan^2\theta \ , \tag{43}$$

we obtain the magnitude of l_{Ar} :

$$l_{Ar} = \frac{y_C \sec\theta}{\tan\theta - \tan\omega} \quad . \tag{44}$$

From Figure 1, it is clear that the mirror intercept is coincident with the center of rotation of the mirror $(l_{Ar} = 0)$ when $\omega = 90^o$. From equation (19), we see that when $\omega = 90^o$,

$$\sin\ (\alpha + 2\theta) = 0 \ . \tag{45}$$

Within the physical limitations expressed by equations (22), (23), and (24), this relationship requires that $\theta = 180^o - \frac{\alpha}{2}$, which is the value of θ at $\frac{d\omega}{d\theta}_{\text{min (max)}}$.

Within the boundaries of the observation window, the aperture limits for any desired range of θ may be calculated by substituting equation (19) in equation (44). When $\frac{d\theta}{dt}$ is known, the resulting equation may be used to define the aperture limits which will provide a desired exposure time as is required, for example, in dynamic visual acuity experiments.

IMAGE DISTANCE: l_{CP}

The optical distance from the eye to the target image (l_{CP}) is equal to the sum of the distance from the eye to the point of reflection on the mirror (l_{Cr}) and the distance from the mirror intercept to the target (l_{Br}). These distances vary as the target is tracked across the rotating mirror.

From Figure 1, we see that

$$l_{Cr} = \sqrt{x_r^2 + (y_r - y_C)^2} \tag{46a}$$

$$= \frac{y_C \sec \omega}{\tan \theta - \tan \omega} \tag{46b}$$

$$l_{Br} = \sqrt{(x_r - x_B)^2 + (y_r - y_B)^2} \tag{47}$$

$$l_{CP} = l_{Cr} + l_{Br} \, . \tag{48}$$

This distance is maximum when the target image is in alignment with the axis of rotation of the mirror $(\omega = 90^o, \ \theta = 180^o - \frac{\alpha}{2})$.

IMAGE DIMENSIONS: $\Delta \tau$, $\Delta \omega$

The angular size of the target image also undergoes change as a function of θ. Both the vertical (parallel to axis of mirror rotation) and horizontal angular dimensions vary because of the changing image distance. The horizontal dimension undergoes further variation due to the nonlinearity of the function $\omega = f(\theta)$ reflected in equation (19). The angular dimensions of the target image may be calculated for a given value of θ in the following manner.

Let the target extend in two dimensions to a height, Δz, normal to the plane of incidence, and a width, Δw, normal to l_{BA} along the plane of incidence. The vertical angle subtended by the target image at the eye is given by

$$\Delta \tau = \tan^{-1} \frac{\Delta z}{l_{CP}} \, . \tag{49}$$

The horizontal angular subtense of a target of width Δw may be calculated with respect to the center of rotation of the mirror to be

$$\Delta \alpha = \tan^{-1} \frac{\Delta w}{l_{BA}} \, . \tag{50}$$

Then, equation (19) may be employed to calculate the angular width of the target image with respect to the eye ($\Delta\,\omega$).

$$\Delta\,\omega = \omega_{\alpha} - \omega_{\alpha + \Delta\,\alpha} \tag{51}$$

where

$$\omega_{\alpha + \Delta\,\alpha} = \tan^{-1} \frac{\cos(\alpha + \Delta\,\alpha + 2\theta) + \frac{l_{CA}}{l_{BA}}}{-\sin(\alpha + \Delta\,\alpha + 2\theta)} \,. \tag{52}$$

In Figure 4 are presented graphs representing angular subtense of the target image as a function of θ, using two values of α and two values of $\frac{l_{CA}}{l_{BA}}$. The solid lines represent the vertical dimension ($\Delta\,\tau$), and the dotted lines represent the horizontal dimension ($\Delta\,\omega$). The ranges of θ for these plots represent the observation windows of a rotating mirror whose length is $2l_{CA}$, and for which $l_{Ax\,(max)} = l_{CA}$. In Figure 4(a) and (b), $\frac{l_{CA}}{l_{BA}} = 0.5$, and $\alpha = 0^{o}, 80^{o}$, respectively. In Figure 4(c), both values of α are represented for $\frac{l_{CA}}{l_{BA}} = 0.1$. Linear target sizes have been adjusted so that the minimum visual angle subtended is 0.1^{o} for both values of $\frac{l_{CA}}{l_{BA}}$. Both the angular height and width of the target vary as a function of mirror orientation. These variations are symmetrical with respect to $\theta = 180^{o} - \frac{\alpha}{2}$. The effects of changes in image distance with mirror rotation are seen in the plots of image height, the distance decreasing as θ deviates from $\theta = 180^{o} - \frac{\alpha}{2}$. However, the change in aspect angle with mirror rotation serves to retard the rate of change of the image width. Further, since image distance varies as a function of ω, the target's angular height will not be uniform across its width, and this distortion in image height will vary as a function of θ.

DISCUSSION

The dynamic characteristics of an image viewed through a rotating mirror are governed by two parameters: the ratio of distances from the center of rotation of the mirror to that of the tracker, and to the target $\left(\frac{l_{CA}}{l_{BA}}\right)$, and the angular displacement of the target from the tracker with respect to the mirror's center

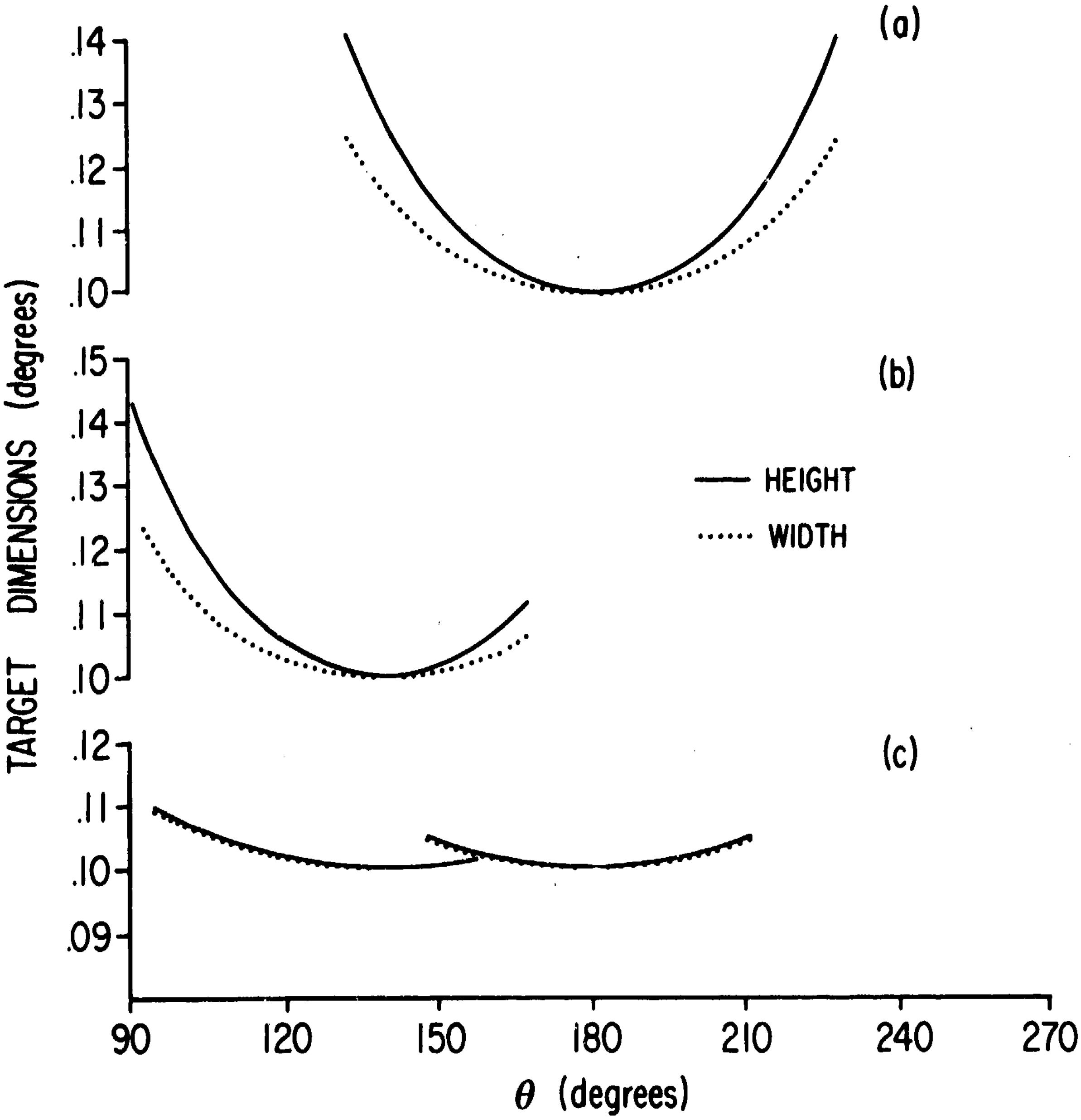

Figure 4. **Angular dimensions of the target image as a function of mirror position. (a)** $\frac{l_{CA}}{l_{BA}} = 0.5,\ \alpha = 0$. **(b)** $\frac{l_{CA}}{l_{BA}} = 0.5,\ \alpha = 80^o$. **(c)** $\frac{l_{CA}}{l_{BA}} = 0.1,\ \alpha = 80^o,\ 0^o$. **The minimum angle subtended by the target image in each case is** 0.1^o.

of rotation (α). The direction (ω) of the target image with respect to the tracker varies as a nonlinear function of mirror orientation (θ), as do the image distance (l_{CP}) and angular dimensions ($\Delta\ \tau$, $\Delta\ \omega$).

In general, when the plane of incidence is normal to the axis of mirror rotation, the function $\frac{d\omega}{d\theta} = f(\theta)$ is nonlinear and is symmetrical with respect to $\frac{l_{CA}}{l_{BA}} = 1$ for reciprocal values of $\frac{l_{CA}}{l_{BA}}$. Further, $\frac{d\omega}{d\theta} = f(\theta)$ has a minimum at $\theta = 180^o - \frac{\alpha}{2}\left(\frac{d\omega}{d\theta}_{min} = \frac{2}{1 + \frac{l_{CA}}{l_{BA}}}\right)$, and is symmetrical with respect to that value of θ for all values of $\frac{l_{CA}}{l_{BA}}$ and α. This symmetry applies, as well, to variations in distance (l_{CP}) and angular dimensions ($\Delta\ \tau$, $\Delta\ \omega$) of the target image as functions of mirror orientation (θ). Calculated values of these variables as functions of θ are represented in Appendix A. The range of mirror orientations (θ) over which the target image may be observed depends upon the aperture of the mirror. Thus, apertures may be designed to control that segment of the above functions which is to be observed.

It is frequently assumed that a target image viewed through a rotating mirror moves with respect to the observer at twice the angular rate of the mirror rotation. Equations (20) and (21) indicate this to be true only in the trivial case where $\frac{l_{CA}}{l_{BA}} = 0$. The only practical case in which $\frac{d\omega}{d\theta} = f(\theta)$ is linear occurs when $l_{CA} = l_{BA}$; then $\frac{d\omega}{d\theta} = 1$, but the angular height of the target image changes dramatically.

The significance of these dynamic characteristics of the target image depends, of course, upon the application. In many cases, the small, dynamic changes encountered are acceptable. Then, it is required only that adjustments be made in estimates of mean values of image variables. Often, it is desired to maximize the size of the observation window, and to minimize accelerations and changes in image size. This need is served by minimizing $\frac{l_{CA}}{l_{BA}}$ and α. Maximizing l_{BA} is directly effective and free of complications. Problems in minimizing l_{CA} and α derive from the dimensions of the tracker and its ability to obscure the target. One temptation may be to displace the target vertically. Resulting variations in vertical angular displacement, τ, as a function of mirror orientation may be calculated in the manner of equation (49).

Since $\frac{d\omega}{dt} = \frac{d\omega}{d\theta} \cdot \frac{d\theta}{dt}$, the function $\frac{d\omega}{dt}$ may be linearized exactly by adjusting the mirror speed, $\frac{d\theta}{dt}$, in accordance with equation (20).

SUMMARY

1. $$\tan \omega = \frac{\cos(\alpha + 2\theta) + \frac{l_{CA}}{l_{BA}}}{-\sin(\alpha + 2\theta)}$$

2. for rotating mirror ($l_{Az} < l_{CA}, l_{BA}$): $0 < \frac{d\omega}{d\theta} < 2$

3. a. $$\lim_{l_{CA} \to 0} \frac{d\omega}{d\theta} = 2, \quad \frac{d^2\omega}{d\theta^2} = 0$$

   b. $$\lim_{l_{BA} \to 0} \frac{d\omega}{d\theta} = 0, \quad \frac{d^2\omega}{d\theta^2} = 0$$

   c. $$\lim_{l_{CA} \to l_{BA}} \frac{d\omega}{d\theta} = 1, \quad \frac{d^2\omega}{d\theta^2} = 0$$

4. for $\theta = 180^o - \frac{\alpha}{2}$, $\frac{l_{CA}}{l_{BA}} < 1$

$$\frac{d\omega}{d\theta}_{min} = \frac{2}{1 + \frac{l_{CA}}{l_{BA}}}$$

$$l_{Az} = 0$$

$$\omega = 90^o$$

$$l_{CP_{max}} = l_{CA} + l_{BA}$$

$$\Delta \tau_{min}, \Delta \omega_{min} = \frac{\Delta z}{l_{CA} + l_{BA}}$$

5. Symmetry: $\frac{d\omega}{d\theta} - 1 \ (for \ \frac{l_{CA}}{l_{BA}} = x) = 1 - \frac{d\omega}{d\theta} \ (for \ \frac{l_{CA}}{l_{BA}} = \frac{1}{x})$

$$\frac{d\omega}{d\theta}_{(\theta_{min} - \theta_i)} = \frac{d\omega}{d\theta}_{(\theta_{min} + \theta_i)}$$

REFERENCES

1. Polásek, J. C., Matrix analysis of gimbaled mirror and prism systems. J. Opt. Soc. Amer., 57: 1193-1201, 1967.

2. Marquis, D. C., Optical tracking; a brief survey of the field. Appl. Opt., 5: 481-7, 1966.

3. Ludvigh, E. and Miller, J. W., Study of visual acuity during the ocular pursuit of moving test objects. I. Introduction. J. Opt. Soc. Amer., 48: 799-802, 1958.

APPENDIX A

Calculated Values of ω, $\frac{d\omega}{d\theta}$, $\frac{d^2\omega}{d\theta^2}$, l_{Ar}, l_{Cp}, $\Delta\,\tau$, $\Delta\,\omega$ as Functions θ

TABLE I

Calculated Values of Variables related to the Observation of a Target through a Rotating Mirror. $l_{CA} = 1, l_{BA} = 10$

θ				ω	$\frac{d\omega}{d\theta}$	$\frac{d^2\omega}{d\theta^2}$	l_{AE}	l_{CD}	$\Delta\tau$	$\Delta\omega$
$\alpha = 0^o$	$\alpha = 40^o$	$\alpha = 80^o$	$\alpha = 120^o$							
100				-67.83815	2.20429	-0.20042	1.79056	9.06676	0.12132	0.24247
110				-46.01799	2.15547	-0.34675	1.70853	9.25630	0.11884	0.23710
120	100			-24.79128	2.08791	-0.41414	1.57459	9.53939	0.11531	0.22967
130	110			-4.27686	2.01510	-0.41001	1.39281	9.87558	0.11139	0.22166
140	120	100		15.52900	1.94761	-0.35730	1.16870	10.22120	0.10762	0.21424
150	130	110		34.71500	1.89189	-0.27834	0.90909	10.53565	0.10441	0.20811
160	140	120	100	53.41679	1.85109	-0.18813	0.62185	10.78522	0.10199	0.20362
170	150	130	110	71.79072	1.82642	-0.09438	0.31572	10.94504	0.10050	0.20091
180 *	160 *	140 *	120 *	90.00000	1.81818	0.00000	0.00000	11.00000	0.10000	0.20000
190	170	150	130	108.20928	1.82642	0.09438	0.31572	10.94504	0.10050	0.20091
200	180	160	140	126.58321	1.85109	0.18813	0.62185	10.78522	0.10199	0.20362
210	190	170	150	145.28500	1.89189	0.27834	0.90909	10.53565	0.10441	0.20811
220	200	180		164.47100	1.94761	0.35730	1.16870	10.22120	0.10762	0.21424
230	210	190		184.27686	2.01510	0.41001	1.39281	9.87558	0.11139	0.22166
240	220			204.79128	2.08791	0.41414	1.57459	9.53939	0.11531	0.22967
250	230			226.01799	2.15547	0.34675	1.70853	9.25630	0.11884	0.23710
260				247.83815	2.20429	0.20042	1.79056	9.06676	0.12132	0.24247

* $\theta = 180^o - \frac{\alpha}{2}$

TABLE II

Calculated Values of Variables related to the Observation of a Target through a Rotating Mirror. $l_{CA} = 1$, $l_{BA} = 2$

θ				ω	$\frac{d\omega}{d\theta}$	$\frac{d^2\omega}{d\theta^2}$	l_{Ar}	l_{Co}	Δr	$\Delta\omega$
$\alpha = 0^o$	$\alpha = 40^o$	$\alpha = 80^o$	$\alpha = 120^o$							
100				-52.12201	3.41696	-5.32793	1.31308	1.11410	0.26927	0.51254
110				-22.48426	2.54973	-4.11669	1.25292	1.39134	0.21562	0.38246
120	100			-0.00000	2.00000	-2.30940	1.15470	1.73205	0.17320	0.30000
130	110			18.33449	1.69680	-1.27507	1.02139	2.07495	0.14458	0.25452
140	120	100		34.37370	1.52682	-0.72885	0.85705	2.38633	0.12572	0.22902
150	130	110		49.10661	1.42857	-0.42418	0.66667	2.64575	0.11339	0.21429
160	140	120	100	63.08249	1.37202	-0.23722	0.45603	2.83975	0.10564	0.20580
170	150	130	110	76.63627	1.34251	-0.10700	0.23153	2.95952	0.10137	0.20138
180*	160*	140*	120*	90.00000	1.33333	0.00000	0.00000	3.00000	0.10000	0.20000
190	170	150	130	103.36373	1.34251	0.10700	0.23153	2.95952	0.10137	0.20138
200	180	160	140	116.91751	1.37202	0.23722	0.45603	2.83975	0.10564	0.20580
210	190	170	150	130.89339	1.42857	0.42418	0.66667	2.64575	0.11339	0.21429
220	200	180		145.62630	1.52682	0.72885	0.85705	2.38633	0.12572	0.22902
230	210	190		161.66551	1.69680	1.27507	1.02139	2.07495	0.14458	0.25452
240	220			180.00000	2.00000	2.30940	1.15470	1.73205	0.17320	0.30000
250	230			202.48426	2.54973	4.11669	1.25292	1.39134	0.21562	0.38246
260				232.12201	3.41696	5.32793	1.31308	1.11410	0.26927	0.51254

* $\theta = 180^o - \frac{\alpha}{2}$

TABLE III

Calculated Values of Variables related to the Observation of a Target through a Rotating Mirror. $l_{CA} = 1$, $l_{BA} = 1.3333$

θ $\alpha = 0^\circ$	θ $\alpha = 40^\circ$	θ $\alpha = 80^\circ$	θ $\alpha = 120^\circ$	ω	$\frac{d\omega}{d\theta}$	$\frac{d^2\omega}{d\theta^2}$	l_{AT}	l_{CP}	$\Delta\tau$	$\Delta\omega$
100				-29.01141	3.86015	-19.18724	1.12548	0.52144	0.44746	0.67554
110				-1.42818	2.05815	-4.93560	1.07392	0.85730	0.27217	0.36018
120	100			16.10326	1.53842	-1.72169	0.98973	1.20183	0.19415	0.26923
130	110			30.33882	1.33599	-0.76239	0.87547	1.52139	0.15337	0.23380
140	120	100		43.16507	1.23997	-0.38892	0.73461	1.80021	0.12961	0.21700
150	130	110		55.28540	1.18917	-0.21254	0.57142	2.02756	0.11508	0.20811
160	140	120	100	67.02380	1.16133	-0.11473	0.39088	2.19555	0.10627	0.20323
170	150	130	110	78.55717	1.14719	-0.05082	0.19845	2.29858	0.10151	0.20076
180*	160*	140*	120*	90.00000	1.14284	0.00000	0.00000	2.33330	0.10000	0.20000
190	170	150	130	101.44283	1.14719	0.05082	0.19845	2.29858	0.10151	0.20076
200	180	160	140	112.97620	1.16133	0.11473	0.39088	2.19555	0.10627	0.20323
210	190	170	150	124.71460	1.18917	0.21254	0.57142	2.02756	0.11508	0.20811
220	200	180		136.83493	1.23997	0.38892	0.73461	1.80021	0.12961	0.21700
230	210	190		149.66118	1.33599	0.76239	0.87547	1.52139	0.15337	0.23380
240	220			163.89674	1.53842	1.72169	0.98973	1.20183	0.19415	0.26923
250	230			181.42818	2.05815	4.93560	1.07392	0.85730	0.27217	0.36018
260				209.01141	3.86015	19.18724	1.12548	0.52144	0.44746	0.67554

* $\theta = 180^\circ - \frac{\alpha}{2}$

TABLE IV

Calculated Values of Variables related to the Observation of a Target through a Rotating Mirror. $l_{CA} = 1$, $l_{BA} = 1$

θ				ω	$\frac{d\omega}{d\theta}$	$\frac{d^2\omega}{d\theta^2}$	l_{Ar}	l_{CP}	Δr	Δω
α = 0°	α = 40°	α = 80°	α = 120°							
100				10.00000	1.00000	0.00000	0.98481	0.34730	0.57586	0.20000
110				20.00000	1.00000	0.00000	0.93969	0.68404	0.29238	0.20000
120	100			30.00000	1.00000	0.00000	0.86603	1.00000	0.20000	0.20000
130	110			40.00000	1.00000	0.00000	0.76604	1.28558	0.15557	0.20000
140	120	100		50.00000	1.00000	0.00000	0.64279	1.53209	0.13054	0.20000
150	130	110		60.00000	1.00000	0.00000	0.50000	1.73205	0.11547	0.20000
160	140	120	100	70.00000	1.00000	0.00000	0.34202	1.87939	0.10642	0.20000
170	150	130	110	80.00000	1.00000	0.00000	0.17365	1.96962	0.10154	0.20000
180*	160*	140*	120*	90.00000	1.00000	0.00000	0.00000	2.00000	0.10000	0.20000
190	170	150	130	100.00000	1.00000	0.00000	0.17365	1.96962	0.10154	0.20000
200	180	160	140	110.00000	1.00000	0.00000	0.34202	1.87939	0.10642	0.20000
210	190	170	150	120.00000	1.00000	0.00000	0.50000	1.73205	0.11547	0.20000
220	200	180		130.00000	1.00000	0.00000	0.64279	1.53209	0.13054	0.20000
230	210	190		140.00000	1.00000	0.00000	0.76604	1.28558	0.15557	0.20000
240	220			150.00000	1.00000	0.00000	0.86603	1.00000	0.20000	0.20000
250	230			160.00000	1.00000	0.00000	0.93969	0.68404	0.29238	0.20000
260				170.00000	1.00000	0.00000	0.98481	0.34730	0.57586	0.20000

* $\theta = 180^\circ - \frac{\alpha}{2}$

TABLE V

Calculated Values of Variables related to the Observation of a Target through a Rotating Mirror. $l_{CA} = 1,\ l_{BA} = .75$

θ, α = 0°	θ, α = 40°	θ, α = 80°	θ, α = 120°	ω	$\frac{d\omega}{d\theta}$	$\frac{d^2\omega}{d\theta^2}$	l_{AT}	l_{CP}	Δτ	Δω
100				49.01381	-1.86020	19.18621	0.84412	0.39110	0.44744	-0.43407
110				41.42985	-0.05821	4.93578	0.80545	0.64299	0.27217	-0.01359
120	100			43.89789	0.46154	1.72180	0.74231	0.90139	0.19414	0.10769
130	110			49.66199	0.66399	0.76245	0.65661	1.14106	0.15337	0.15493
140	120	100		56.83551	0.76001	0.38895	0.55096	1.35017	0.12961	0.17733
150	130	110		64.71500	0.81081	0.21255	0.42857	1.52069	0.11508	0.18919
160	140	120	100	72.97646	0.83865	0.11474	0.29316	1.64668	0.10627	0.19569
170	150	130	110	81.44295	0.85279	0.05082	0.14884	1.72396	0.10151	0.19898
180*	160*	140*	120*	90.00000	0.85714	0.00000	0.00000	1.75000	0.10000	0.20000
190	170	150	130	98.55705	0.85279	-0.05082	0.14884	1.72396	0.10151	0.19898
200	180	160	140	107.02354	0.83865	-0.11474	0.29316	1.64668	0.10627	0.19569
210	190	170	150	115.28500	0.81081	-0.21255	0.42857	1.52069	0.11508	0.18919
220	200	180		123.16449	0.76001	-0.38895	0.55096	1.35017	0.12961	0.17733
230	210	190		130.33801	0.66399	-0.76245	0.65661	1.14106	0.15337	0.15493
240	220			136.10211	0.46154	-1.72180	0.74231	0.90139	0.19414	0.10769
250	230			138.57015	-0.05821	-4.93578	0.80545	0.64299	0.27217	-0.01359
260				136.98619	-1.86020	-19.18621	0.84412	0.39110	0.44744	-0.43407

* $\theta = 180^\circ - \frac{\alpha}{2}$

TABLE VI

Calculated Values of Variables related to the Observation of a Target through a Rotating Mirror. $l_{CA} = 1,\ l_{BA} = .75$

θ: α = 0°	θ: α = 40°	θ: α = 80°	θ: α = 120°	ω	$\frac{d\omega}{d\theta}$	$\frac{d^2\omega}{d\theta^2}$	l_{AX}	l_{CP}	Δ τ	Δ ω
100				72.12201	-1.41696	5.32793	0.65654	0.55705	0.26927	-0.42508
110				62.48426	-0.54973	4.11669	0.62646	0.69567	0.21562	-0.16492
120	100			60.00000	-0.00000	2.30940	0.57735	0.86603	0.17320	-0.00000
130	110			61.66551	0.30320	1.27507	0.51070	1.03747	0.14458	0.09096
140	120	100		65.62630	0.47318	0.72885	0.42853	1.19317	0.12572	0.14195
150	130	110		70.89339	0.57143	0.42418	0.33333	1.32288	0.11339	0.17143
160	140	120	100	76.91751	0.62798	0.23722	0.22801	1.41987	0.10564	0.18839
170	150	130	110	83.36373	0.65749	0.10700	0.11577	1.47976	0.10137	0.19724
180*	160*	140*	120*	90.00000	0.66667	0.00000	0.00000	1.50000	0.10000	0.20000
190	170	150	130	96.63627	0.65749	-0.10700	0.11577	1.47976	0.10137	0.19724
200	180	160	140	103.08249	0.62798	-0.23722	0.22801	1.41987	0.10564	0.18839
210	190	170	150	109.10661	0.57143	-0.42418	0.33333	1.32288	0.11339	0.17143
220	200	180		114.37370	0.47318	-0.72885	0.42853	1.19317	0.12572	0.14195
230	210	190		118.33449	0.30320	-1.27507	0.51070	1.03747	0.14458	0.09096
240	220			120.00000	-0.00000	-2.30940	0.57735	0.86603	0.17320	-0.00000
250	230			117.51574	-0.54973	-4.11669	0.62646	0.69567	0.21562	-0.16492
260				107.87799	-1.41696	-5.32793	0.65654	0.55705	0.26927	-0.42508

* $\theta = 180^{\circ} - \frac{\alpha}{2}$

TABLE VII

Calculated Values of Variables related to the Observation of a Target through a Rotating Mirror. $l_{CA} = 1$, $l_{BA} = .1$

θ				ω	$\frac{d\omega}{d\theta}$	$\frac{d^2\omega}{d\theta^2}$	l_{AB}	l_{CD}	$\Delta\tau$	$\Delta\omega$
$\alpha = 0^0$	$\alpha = 40^0$	$\alpha = 80^0$	$\alpha = 120^0$							
100				87.83815	-0.20429	0.20042	0.17906	0.90668	0.12132	-0.22467
110				86.01799	-0.15547	0.34675	0.17085	0.92563	0.11884	-0.17099
120	100			84.79128	-0.08791	0.41414	0.15746	0.95394	0.11531	-0.09669
130	110			84.27686	-0.01510	0.41001	0.13928	0.98756	0.11139	-0.01661
140	120	100		84.47100	0.05239	0.35730	0.11687	1.02212	0.10762	0.05761
150	130	110		85.28500	0.10811	0.27834	0.09091	1.05357	0.10441	0.11890
160	140	120	100	86.58321	0.14891	0.18813	0.06219	1.07852	0.10199	0.16377
170	150	130	110	88.20928	0.17358	0.09433	0.03157	1.09450	0.10050	0.19091
180*	160*	140*	120*	90.00000	0.18182	0.00000	0.00000	1.10000	0.10000	0.19997
190	170	150	130	91.79072	0.17358	-0.09438	0.03157	1.09450	0.10050	0.19091
200	180	160	140	93.41679	0.14891	-0.18813	0.06219	1.07852	0.10199	0.16377
210	190	170	150	94.71500	0.10811	-0.27834	0.09091	1.05357	0.10441	0.11890
220	200	180		95.52900	0.05239	-0.35730	0.11687	1.02212	0.10762	0.05761
230	210	190		95.72314	-0.01510	-0.41001	0.13928	0.98756	0.11139	-0.01661
240	220			95.20872	-0.08791	-0.41414	0.15746	0.95394	0.11531	-0.09669
250	230			93.98201	-0.15547	-0.34675	0.17085	0.92563	0.11884	-0.17099
260				92.16185	-0.20429	-0.20042	0.17906	0.90668	0.12132	-0.22467

* $\theta = 180^0 - \frac{\alpha}{2}$

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1. REPORT NUMBER NAMRL-1252	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Dynamics of an Image Viewed through a Rotating Mirror		5. TYPE OF REPORT & PERIOD COVERED Interim
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) James E. Goodson, CDR MSC USN		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Aerospace Medical Research Laboratory Naval Air Station Pensacola, Florida 32508		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS BUMED MR 041.01.03.0154 ONR RR 041.01.02.NR 201.0?8
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Medical Research and Development Command National Naval Medical Center Bethesda, Maryland 20014		12. REPORT DATE August 1978
		13. NUMBER OF PAGES 27
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
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18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Dynamic Visual Acuity
Open Loop Tracking
Rotating Mirror
Image Velocity
Target Velocity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is frequently assumed that the virtual image of a target viewed through a rotating mirror moves with respect to the observer at twice the angular rate of mirror rotation. This assumption is false, and leads in imprecise treatment of open loop tracking systems. Of particular interest is a class of Dynamic Visual Acuity experiments in which acuity targets are viewed through a rotating mirror, where control of image velocity, exposure time, and image dimensions are of critical importance. → next page

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20.

Expressions are derived which describe the direction of the target image with respect to the observer as a function of mirror position. This relationship is nonlinear, and depends upon the rotation of the mirror (A) to the observer (C), and to the target (B), and upon the included angle ($\angle BAC$). Expressions are further derived for image velocity, acceleration, mirror intercept, and image dimensions as functions of mirror position.