USING PROBABILISTIC INFORMATION IN SOLVING RESOURCE ALLOCATION PROBLEMS FOR A DECENTRALIZED FIRM

BY

JAMES R. FREELAND and GERHARD SCHIEFER

TECHNICAL REPORT NO. 82
SEPTEMBER 1978

PREPARED UNDER CONTRACT
N00014-76-C-0418 (NR-047-061)
FOR THE OFFICE OF NAVAL RESEARCH

DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
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11 Sep 78

Frederick S. Hillier, Project Director

12 26p.

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1. Introduction

The general problem addressed in this paper is how can the headquarters (HQ) of a decentralized firm with k divisions arrive at a resource allocation for the divisions when its initial information about the divisions' opportunities is probabilistic, but it can acquire information by communicating with the divisions. It is assumed that if HQ had complete information, the problem which it would like to solve (and which will be called the firm's optimization problem) can be formulated mathematically as:

$$\text{maximize} \quad \sum_{i=1}^{k} c_i x_i \tag{1}$$

$$\text{subject to} \quad \sum_{i=1}^{k} B_i x_i \leq b$$

$$A_i x_i \leq a_i \qquad (i=1,\ldots,k)$$

$$x_i \geq 0 \qquad (i=1,\ldots,k)$$

where

x_i = vector of decision variables that represent the opportunities of division i;

c_i = vector representing the firm's valuation of one unit of x_i;

b = vector representing the capacity of "common" resources, i.e., resources that can be used by any division i $(i=1,\ldots,k)$;

a_i = vector representing the capacity of resources that can be used by division i only;

B_i = matrix representing the use of common resources per unit of the decision variables x_i;

A_i = matrix representing the use of divisional resources per unit of the decision variables x_i.

In mathematical programming, decomposition methods for solving such problems have been developed which are based on the iterative exchange of information between a coordinating master problem and a number of subproblems. For an overview see Geoffrion [11] or Lasdon [24]. From an economic viewpoint the organization of the information exchange for some of these methods has been extensively discussed with regard to the planning of centralized economies (see Heal [12] and Kornai [22]). From an organizational viewpoint the exchange process has been discussed with regard to the allocation of resources in a decentralized firm (see Atkins [1], Burton and Obel [4], Hurwicz [17], Jennergren [18] and Ruefli [29]) and to the problem of transfer pricing in a decentralized firm (see Enzer[19]).

In the typical decomposition procedure for solving problem (1), the subproblems correspond to divisions while HQ solves the master problem. After a number of information exchanges HQ acquires all information from the divisions which is relevant for the determination of an optimal solution of (1). It is typically assumed that at the beginning of the iterative procedure, HQ has only information about a few or none of the divisions' opportunities but knows b (the vector of available capacity for common resources), and knows the vectors c_i $(i=1,\dots,k)$ of the objective function coefficients or is willing to accept the coefficients used by the divisions. The divisions on the other hand are assumed to be informed about the matrices A_i and B_i and the vectors a_i and c_i $(i=1,\dots,k)$.

As a model for resource allocation in a decentralized system, the existing decomposition schemes are somewhat unrealistic because they may require a large number of iterative exchanges in order to find an optimal solution for the firm's optimization problem (see Christensen and Obel [5], Jennergren [19] and Ljung and Selmer [25]). It is in this context that the importance and effects of different initiation strategies have been discussed, e.g., see Beale, Hughes and Small [2], Burton and Obel [4], Christensen and Obel [5], Kornai [21], Ljung and Selmer [25] and Ruefli [29]. However there have been only a few attempts reported in the literature (see Obel [28] and Weitzman [36]) that show how to deal with situations where HQ has some initial information about divisional opportunities and/or constraints.

It is the main purpose of this paper to show how an iterative information exchange between HQ and divisions that is organized according to the rules of a decomposition procedure can be adapted to situations where HQ possesses stochastic information about the divisional opportunities and constraints. While the discussion is based on the decomposition scheme of Dantzig and Wolfe [9], other procedures such as TenKate [32] or Maier and VanderWeide [26] could be used for the organization of the iterative information exchange as well.

In the procedure to be developed HQ accepts a calculated risk that the plan arrived at will violate constraints in the firm's optimization problem. The proposed procedure is aimed at producing a better solution at the beginning of the communication process between HQ and divisions, and at reducing the number of information exchanges. It will also be shown how the procedure can be adapted to situations where the size of HQ's programming problem is restricted.

2. An Iterative Communication Process

2.1 Headquarter's Initial Programming Problem

Suppose HQ of the firm is informed about the true values of the elements of the vectors $c_i (i=1,\ldots,k)$ and b of the firm's optimization problem, i.e., it is informed about the firm's objective function and the capacity of common resources that are available to the firm. Furthermore, suppose that for the remaining elements of the firm's programming problem, HQ is able either to determine the true value or to formulate (subjective) probability distributions.[1]

On the basis of this information, HQ might initially formulate the firm's optimization problem mathematically as:

$$\begin{aligned} \text{maximize} \quad & Z_H^1 = \sum_{i=1}^{k} c_i \bar{x}_i \\ \text{subject to} \quad & \sum_{i=1}^{k} \bar{B}_i \bar{x}_i \le b \\ & \bar{A}_i \bar{x}_i \le \bar{a}_i \quad (i=1,\ldots,k) \\ & \bar{x}_i \ge 0 \end{aligned} \tag{2}$$

where the elements of the matrices $\bar{A}_i$, $\bar{B}_i$ and the vectors $\bar{a}_i$ $(i=1,\ldots,k)$ are assumed to be either constants or random variables (that might be statistically dependent) which represent either the true values or HQ's estimates of the elements of A_i, B_i and a_i $(i=1,\ldots,k)$ in the firm's problem (1). The vector $\bar{x}_i$ $(i=1,\ldots,k)$ of decision variables corresponds to the divisions' opportunities. They may differ from the actual opportunities which are represented by x_i $(i=1,\ldots,k)$ in problem (1) in their use of common and/or divisional resources because HQ may use probabilistic estimates of them.

The formulation given in (2) is an incomplete representation of HQ's decision problem because it does not reflect HQ's attitude toward the uncertainty in the problem. The formulation may be completed by specifying rules that allow the determination of solutions for the decision situation which accord with HQ's preferences.

2.2 Introduction of Decision Rules

HQ's decision situation is characterized by the fact that the decision variables must be assigned values before the realizations of the random variables, i.e., before the true values formulated in the firm's problem (1), are known. Decision rules for such situations are called zero order decision rules and have been extensively discussed in the literature. (For surveys see Kall [19] and Vajda [34].)

In this paper it is assumed that HQ prefers decisions which restrict the probability that any one of the constraints on the use of common or divisional resources will be violated by the realization of the decisions to specified risk levels. Charnes, Cooper and Symonds [5] have introduced such decision rules into a mathematical programming formulation by means of probability constraints (chance-constrained programming).

If $(1-\alpha)$ and $(1-\beta_i)$, $(i=1,\ldots,k)$, denote vectors that represent risk levels as specified by HQ, then HQ's decision problem can be formulated mathematically as the chance-constrained programming problem:

$$\text{maximize} \qquad z_H^2 = \sum_{i=1}^{k} c_i \bar{x}_i \tag{3a}$$

$$\text{subject to} \qquad \text{Prob}\left(\sum_{i=1}^{k} \bar{B}_i \bar{x}_i \le b\right) \ge \alpha \tag{3b}$$

$$\text{Prob}\left(\bar{A}_i \bar{x}_i \le \bar{a}_i\right) \ge \beta_i \quad (i=1,\ldots,k) \tag{3c}$$

$$\bar{x}_i \ge 0 \quad (i=1,\ldots,k) \tag{3d}$$

While this approach imposes probability constraints on the violation of common and divisional constraints, approaches that require the fulfillment of the constraints (as suggested in linear programming under uncertainty by Beale [2] or Dantzig [7]) or that deal with decision situations where the probability distribution of the objective variable is of interest (see, for example, Markowitz [27], Kataoka [20] or the stochastic programming approach suggested by Tintner [33]) can be considered within the framework of this study as well.

2.3 The Iterative Communication Process

It is assumed that HQ is able to communicate with the divisions in order to improve its information about the divisional opportunities, and that the communication process is based on the exchange of information between HQ and divisions as formulated in the decomposition principle of Dantzig and Wolfe [9]. In this scheme, HQ prices common resources whereas the divisions respond with proposals for divisional optimal consumption of common resources and the resulting output in terms of the firm's objective function.

In the proposed procedure, HQ adds the divisional proposals as "certain" information about the divisions' opportunities to its stochastic information, i.e., it incorporates this additional information into the programming problem (3) which, in turn, allows the computation of revised prices for common resources as the optimal values of the dual variables that correspond to the probability constraints, etc.

Denote by v^q a vector that represents the prices for common resources as determined by HQ in iteration q of the iterative communication process. Then division i is assumed to compute its corresponding proposal for the use of common resources by determining the optimal basis solution $\bar{\bar{x}}_i^q$ of the programming problem:

$$\begin{aligned} &\text{maximize} \quad && z_D^1 = (c_i - v^q B_i)x_i \\ &\text{subject to} \quad && A_i x_i \leq a_i \\ & && x_i \geq 0 \end{aligned} \qquad (4)$$

If $\overset{*}{u}{}_i^q$ is defined by the relationship $\overset{*}{u}{}_i^q = c_i \bar{\bar{x}}_i^q$ and u_i^q by $u_i^q = B_i \bar{\bar{x}}_i^q$, then $(\overset{*}{u}{}_i^q, u_i^q)$ represents division i's proposal which it submits to HQ at iteration q of the communication process. After $q = Q$ information exchanges between HQ and the divisions, HQ's programming problem (which corresponds to the master problem in the Dantzig-Wolfe approach) can be formulated mathematically as:

$$\text{maximize} \quad z_H^3 = \sum_{i=1}^{k} c_i \bar{x}_i + \sum_{i=1}^{k} \sum_{q=2}^{Q} \overset{*}{u}{}_i^q \lambda_i^q \qquad (5a)$$

$$\text{subject to:} \quad \text{Prob}\left(\sum_{i=1}^{k} \bar{B}_i \bar{x}_i + \sum_{i=1}^{k} \sum_{q=2}^{Q} u_i^q \lambda_i^q \leq b \right) \geq \alpha \qquad (5b)$$

$$\text{Prob}\left(\bar{A}_i \bar{x}_i - \bar{a}_i \bar{y}_i \leq 0 \right) \geq \beta_i \quad (i=1,\dots,k) \qquad (5c)$$

$$\bar{y}_i - \lambda_i^1 \leq 0 \quad (i=1,\dots,k) \qquad (5d)$$

$$\lambda_i^1 + \sum_{q=2}^{Q} \lambda_i^q = 1 \qquad (i=1,\ldots,k) \qquad (5e)$$

$$\bar{y}_i, \bar{x}_i, \lambda_i^q \geq 0 \quad (i=1,\ldots,k;q=1,\ldots,Q) \quad (5f)$$

In this formulation $y_i (i=1,\ldots,k)$ are transfer variables, and variables λ_i^q $(i=1,\ldots,k;q=1,\ldots,Q)$ are associated with the divisions' proposals and determine the extent of their use in the solution of HQ's programming problem. The proposals that are associated with the variables $\lambda_i^1 (i=1,\ldots,k)$ represent solutions to the divisional programming problems where the divisions' local resources are unused, i.e., where none of the division's decision variables are positive. The formulation of the constraints (5c)-(5e) ensures that HQ is able to use the portion of divisional resources that have not been consumed by the realization of the variables $\lambda_i^q (i=1,\ldots,k;q=2,\ldots,Q)$ for the realization of the decision variables $\bar{x}_i (i=1,\ldots,k)$. Note that HQ must base its calculation on its estimates $\bar{a}_i (i=1,\ldots,k)$ about the capacity of divisional resources.

3. Realization of the Communication Process

For the realization of the iterative communication process one has to convert HQ's planning problem (5) into an equivalent deterministic form which is tractible and which allows finite convergence to a solution which optimizes its objective function and satisfies the formulated constraints in (1) with the specified probability. This section shows how the deterministic form of HQ's problem can be regarded as a master problem in a decomposition procedure based on the Dantzig Wolfe principle.

3.1 A Deterministic Equivalent for HQ's Programming Problem

Consider a single probability constraint j of (5b) and denote it by

$$\text{Prob}\left\{\sum_{i=1}^{k}\bar{b}_i^j\bar{x}_i + \sum_{i=1}^{k}\sum_{q=2}^{Q}u_i^{jq}\lambda_i^q \le b^j\right\} \le \alpha^j$$

If some elements of the vectors $\bar{b}_i^j(i=1,\ldots,k)$ are random variables, then for any possible values of $\bar{x}_i(i=1,\ldots,k)$ and $\lambda_i^q(i=1,\ldots,k;q=2,\ldots,Q)$ the variable

$$w = \left(\sum_{i=1}^{k}\bar{b}_i^j\bar{x}_i + \sum_{q=2}^{Q}u_i^{jq}\lambda_i^q - b^j\right)$$

is a random variable as well. Denote the expected value and variance of w by $E(w)$ and $V(w)$, respectively.

Furthermore, assume that the functional form of the probability distribution for w is known, and that the fractiles of this distribution are completely determined by its mean and variance.[2] Let $F(u)$ denote the cumulative distribution function of the standardized variable $u = (w-E(w)/\sqrt{V(w)})$,[3] and define u_{α_j} by the relationship $F(u_{\alpha_j}) = \alpha^j$ where α^j corresponds to a specified risk level $(1-\alpha^j)$. The deterministic equivalent form of the probability constraint can then be formulated as:

$$E(w) + u_{\alpha_j}\sqrt{V(w)} \le 0$$

which can be specified as

$$\left[E\left(\sum_{i=1}^{k}\bar{b}_i^j\bar{x}_i\right) + \sum_{i=1}^{k}\sum_{q=2}^{Q}u_i^{jq}\lambda_i^q\right] + u_{\alpha_j}\sqrt{V\left(\sum_{i=1}^{k}\bar{b}_i^j\bar{x}_i\right)} \le b^j$$

It has been shown elsewhere (see Hillier [13] and Kataoka [20]) that for continuous decision variables and $u_{\alpha_j} \ge 0$ the deterministic equivalent form of the probability constraint is convex. Furthermore, convex separable and linear approximations have been developed for dealing with the problem (see Hillier [13] and Schiefer [31]).

A similar transformation can be applied to all probability constraints of HQ's problem (5). Consequently, the deterministic equivalent form of the problem is a convex programming problem which can be solved by means of convex programming algorithms[4] or, by converting it to a linear approximation and using the simplex method.

3.2 Properties of the Iterative Solution Process

In the context of the Dantzig-Wolfe decomposition approach the divisional programming problems (4) and the deterministic equivalent of (5) can be regarded as the subproblems and the master problem derived from the following problem:

$$\text{maximize} \quad z_B = \sum_{i=1}^{k} c_i \bar{x}_i + \sum_{i=1}^{k} c_i x_i \tag{6a}$$

$$\text{subject to:} \quad \text{Prob}\left(\sum_{i=1}^{k} \bar{B}_i \bar{x}_i + \sum_{i=1}^{k} B_i x_i \le b \right) \ge \alpha \tag{6b}$$

$$\text{Prob}\left(\tilde{A}_i \bar{x}_i - \tilde{a}_i \bar{y}_i \le 0 \right) \ge \beta_i \quad (i=1,\ldots,k) \tag{6c}$$

$$\bar{y}_i - y_i \le 0 \quad (i=1,\ldots,k) \tag{6d}$$

$$a_i y_i + A_i x_i \le a_i \quad (i=1,\ldots,k) \tag{6e}$$

$$\bar{y}_i, y_i, \bar{x}_i, x_i \ge 0 \quad (i=1,\ldots,k) \tag{6f}$$

By proceeding as described earlier the problem in (6) can be transformed to a deterministic form where the probability constraints are replaced by convex non-linear constraints or their linear approximations. This form has a block angular structure in the matrix and is linear in common constraints which relate to the divisional problems. Thus, one could solve the problem in (6) by applying the decomposition principle for linear problems to its deterministic equivalent form.[5]

It is well known that such a procedure will converge finitely and that the objective function will increase monotonically at each iteration. Furthermore, the speed of convergence has been examined in computer experiments (see [3], [6], [24], [25], and [30]). However while the communication process will lead to solution for problem (6), HQ is interested in finding a solution for the problem given in (1).

4. Solution for the Firm's Optimization Problem

4.1 Properties of the Solution Found by Headquarters

A solution, x_i^* $(i=1,\ldots,k)$, for the firm's optimization problem can be derived from the solutions of problem (6) by the relationship

$$x_i^+ = \bar{x}_i + x_i$$

where corresponding elements of the decision vectors $\bar{x}_i$ and x_i $(i=1,\ldots,k)$ represent identical decision alternatives. The two parts of the solution differ in so far as they are based either on "certain" information about divisional opportunities that has been supplied by the divisions during the communication process or on HQ's estimates.

It should be obvious that the problem given in (6) is identical with the firm's problem in (1) if HQ is able to determine the true values of the elements of B_i, A_i and a_i $(i=1,\ldots,k)$, i.e., $\bar{B}_i=B_i$, $\bar{A}_i=A_i$ and $\bar{a}_i=a_i$ $(i=1,\ldots,k)$. In this case feasible solutions of (6) are feasible solutions for (1) and vice versa, and the optimal solutions for (6) and (1) are identical. However, if some of HQ's estimates are stochastic, a solution of (6) with some elements of $\bar{x}_i$ $(i=1,\ldots,k)$ positive might violate constraints in problem (1).

The introduction of HQ's attitude towards a possible violation of constraints by formulating chance-constraints implies that HQ is willing to accept as a plan an optimal solution for the problem given in (6). Although this plan may not be strictly feasible relative to (1), it may avoid the collection of all relevant information about the division's opportunities. HQ's willingness to accept possible infeasibilities is expressed by the values of α and $\beta_i (i=1,\ldots,k)$, or equivalently by the values of u_α and $u_{\beta i}$ $(i=1,\ldots,k)$.

In order to state general properties regarding the optimal solution to (6) relative to α and β_i $(i=1,\ldots,k)$ suppose that F(u) corresponds to a random variable with a normal distribution. For $\alpha^j \to 1$ and $\beta_i^j \to 1$ $(i=1,\ldots,k)$ for all j, u_α^j and $u_{\beta_i}^j$ $(i=1,\ldots,k)$ for all j will become very large. In this case, the optimal basis solution of (6) will include only variables x_i $(i=1,\ldots,k)$, i.e., the optimal basis solution will be completely based on certain information about the divisional opportunities. Furthermore, the basis solution will be the same as if the decomposition principle were applied to the firm's problem (1). This situation refers to an HQ with an extreme aversion against risk.

On the other hand, for $\alpha^j \to 0$ and $\beta_i^j \to 0$ $(i=1,\dots,k)$ for all j, u_α^j and $u_{\beta_i}^j$ $(i=1,\dots,k)$ for all j will become very large negative numbers and HQ's plan derived from (6) will include only variables $\overline{x}_i$ $(i=1,\dots,k)$. In such a situation HQ will never find it advantageous to iteratively exchange information with the divisions because the optimal basis solution of HQ's initial optimization problem (2) represents the optimal basis solution for (6).

In general if HQ has any aversion toward risk then values for α^j and β_i^j $(i=1,\dots,k)$ will be chosen which result in positive values for u_α^j and $u_{\beta_i}^j$ $(i=1,\dots,k)$, e.g., in the case of normality α^j and β_i^j $(i=1,\dots,k)$ would be selected from within the ranges:

$$.5 < \alpha^j \le 1$$

$$.5 < \beta_i^j \le 1 \quad (i=1,\dots,k).$$

However, in this case the optimal basis solution of (6) might include both, variables $\overline{x}_i$ and x_i $(i=1,\dots,k)$, i.e. HQ might compute the solution by using divisional proposals that have been reported by the divisions and its initial estimates about the divisions' opportunities . The iterative communication process may be terminated before HQ has collected all certain information about the divisions' opportunities that would be relevant for the solution of (1).

The same is true in situations where HQ terminates the communication process before it reaches the optimal solution for (6). This may be the case if the maximum possible improvement in the objective function value for (6) that could be reached by continuing the information exchange is considered insufficient. (For a computation of lower and upper bounds on the optimal objective function value, see Lasdon [24].)

4.2 Implementation of a Plan

Whenever the communication process is terminated, HQ faces the problem of what plan to implement. If none of the variables $\overline{x}_i$ $(i=1,\dots,k)$ are positive in the solution of HQ's programming problem, then the plan $x_i^+ = x_i^0$ $(i=1,\ldots,k)$ that has been derived according to the rules developed in the decomposition principle is feasible. This might even be true for solutions with positive values of the $\bar{x}_i$ $(i=1,\ldots,k)$ if neither the common nor the divisional constraints are very "tight."

In other situations the firm might have to "pay" for HQ's acceptance of estimates about the divisions' opportunities. If a computed solution should result in an infeasible plan, the firm has either to purchase additional common and/or divisional resources[7] to remove the infeasibility or to accept a plan which uses only the available capacity of resources but yields an objective function value which is lower than computed by HQ. In these cases, depending on the specific situation, a variety of alternative possibilities for the determination of a feasible plan could be formulated. With regard to the availability of common and divisional resources from sources outside the firm one might realize the following procedures:

1. HQ instructs the divisions to implement the plan x_i^0 $(i=1,\ldots,k)$. In this case, the firm may have to purchase additional common and/or divisional resources.
2. If only common resources are available for purchase, HQ may achieve feasibility by instructing the divisions to realize a plan which is "as close as possible" to its computed plan x_i^0 $(i=1,\ldots,k)$ but does not violate divisional constraints, i.e., a plan which could be determined by each division i by solving the problem:

$$\begin{aligned} &\text{minimize} && z_D^2 = |x_i - x_i^0| \\ &\text{subject to} && A_i x_i \leq a_i \\ & && x_i \geq 0 \end{aligned} \tag{7}$$

3. In situations where the purchase of resources is not possible, HQ cannot assure feasibility by merely informing the divisions about x_i^0 $(i=1,\ldots,k)$.

A feasible plan may be achieved (at least for situations where none of the divisions is a net producer of common resources) by reporting to the divisions an allocation d_i $(i=1,\ldots,k)$ of common resources[8] which satisfies $\sum_{i=1}^{k} d_i \leq b$ and instructing each division to realize a plan which it could determine by solving the problem:

$$\begin{array}{ll} \text{maximize} & z_D^3 = c_i x_i \\ \text{subject to} & B_i x_i \leq d_i \\ & A_i x_i \leq a_i \\ & x_i \geq 0 \end{array} \tag{8}$$

5. Reduction of the Size of HQ's Programming Problem

The procedure described in the previous sections increases the number of constraints and variables in HQ's programming problem over what would be required if the stochastic information was ignored. However, there are sometimes restrictions on the size of HQ's programming problem. In fact, the development of decomposition methods for mathematical programming was primarily aimed at reducing the size of programming problems that had to be solved by HQ.

In cases where restrictions on the size of HQ's programming problem exist but are not too strict, some of the initial stochastic information could be preserved by using it in aggregated form, i.e., replacing divisional constraints and/or opportunities by aggregates that are based on linear combinations. If the elements in the disaggregated matrices $\bar{B}_i$, $\bar{A}_i$ and vectors $\bar{a}_i$ $(i=1,\ldots,k)$ of HQ's problem (5) are random elements, the elements in the aggregated matrices and vectors are also. Their probability distributions can be estimated directly or derived from the probability distributions of the disaggregated random elements.[9] Then HQ's programming problem on the basis of aggregated information can be formulated as (5) and the iterative communication process can be organized as outlined in preceding sections for situations where HQ's programming problem is based on disaggregated information[10] if the following assumptions hold:

1. Only divisional constraints and/or opportunities are aggregated.

2. The probability distributions of the aggregated elements agree with the requirements stated in preceding sections for the probability distributions of disaggregated random elements.

The discussion about the formulation of probability distributions for aggregated information about divisional constraints and/or opportunities is based on the assumption that at least some of the elements of the matrices B_i, A_i or the vectors a_i $(i=1,\dots,k)$ in the firm's optimization problem (1) are unknown to HQ but that HQ is able to formulate the probability distributions of estimates. However, a similar approach can be applied to situations where HQ is in fact able to determine the true values of B_i, A_i or a_i $(i=1,\dots,k)$ but where restrictions on the size of HQ's programming problem allow only the formulation of a reduced number of divisional constraints.[11] In such a case a communication process between HQ and divisions that is organized according to the rules of the decomposition principle as outlined by Dantzig and Wolfe [9] would be based on a programming problem at HQ's level where all information about divisional constraints that is not obtained during the iterative communication process is neglected. However, the concept of considering aggregated divisional constraints in HQ's programming problem and the formulation of probability distributions for the elements in the corresponding matrices and vectors allows the introduction of at least parts of HQ's initial information into its programming problem.

Suppose restrictions on the size of HQ's programming problem allow the consideration of one divisional constraint for each of the k divisions in the firm's optimization problem (1). Now, consider the divisional constraints $A_i x_i \leq a_i$ that correspond to division i in problem (1). Let A_i be a $(m \times n)$ matrix and denote the elements of a column j $(j=1,\ldots,n)$ by a_i^j $(i=1,\ldots,m)$. Regard the a_i^j $(i=1,\ldots,m)$ as possible realizations of a random variable a^j $(j=1,\ldots,n)$. Define $\bar{A}_i$ by

$$\bar{A}_i = [a^1, a^2, \ldots, a^n].$$

Furthermore, denote by $\bar{a}_i$ a random variable with the elements of the vector a_i as possible realizations.

Now, apply a similar transformation to all matrices A_i $(i=1,\ldots,k)$ and vectors a_i $(i=1,\ldots,k)$ in (1). Then by imposing probability constraints on the violation of divisional constraints by a solution of the firm's optimization problem (1) one could formulate a decomposition procedure for the solution of (1). In such a procedure the iterative exchange of information between HQ and divisions is organized as described in preceding sections with divisional problems (4) and a master problem to be solved by HQ which can be formulated mathematically as:

$$\begin{array}{lll}
\text{maximize} & z = \sum_{i=1}^{k} c_i x_i + \sum_{i=1}^{k} \sum_{q=2}^{Q} \overset{*}{u}{}_i^q \lambda_i^q & (9) \\
\text{subject to} & \sum_{i=1}^{k} B_i x_i + \sum_{i=1}^{k} \sum_{q=2}^{Q} u_i^q \lambda_i^q \leq b & \\
& \text{Prob}\left\{ \bar{A}_i x_i - \bar{a}_i \bar{y}_i \leq 0 \right\} \geq \beta_i \quad (i=1,\ldots,k) & \\
& \bar{y}_i - \lambda_i^1 \leq 0 \quad (i=1,\ldots,k) & \\
& \lambda_i^1 + \sum_{q=2}^{Q} \lambda_i^q = 1 \quad (i=1,\ldots,k) & \\
& x_i, \bar{y}_i, \lambda_i^q \geq 0 &
\end{array}$$

6. Conclusion

This paper has shown an iterative information exchange derived from the Dantzig-Wolfe decomposition principle can be adapted to situations where HQ possesses probabilistic information about the divisions. A chance constraint formulation was used to express HQ's attitude toward accepting solutions which violate constraints. Under extreme risk aversion the same solution is found as when the stochastic information is ignored; however, in general the solution will depend on both the initial stochastic information and on the "certain" information gathered from the divisions.

By accepting a calculated risk of developing a plan that might violate constraints of the firm's optimization problem, the use of stochastic information enables HQ to terminate the procedure at an earlier stage of the communication process. Thus avoiding collecting additional information from the divisions.

FOOTNOTES

1. In situations where c_i ($i=1,\ldots,k$) expresses the valuation of output by the divisions, c_i might be unknown to HQ as well. It can be shown that such a situation could be handled within the framework of this study if HQ is able to formulate probability distributions for c_i.

2. See Hillier [14] for a discussion of relations between the probability distributions of the random variables in $\bar{b}_i^j$ ($i=1,\ldots,k$) and the probability distribution of their linear combination and the conditions under which the probability distribution of w can be regarded as normal or at least (by some version of the Central Limit Theorem) approximately normal.

3. See Hillier and Lieberman [15]. If the functional form of the probability distribution of w is not known, Tchebychev's inequality can be used to formulate an upper bound for u_{α_j}. See Hillier [14] for a discussion.

4. See Dantzig [8] or Wagner [35] for a comprehensive list of references.

5. Whinston [37] has discussed the application of the decomposition principle in situations where the common constraints are nonlinear and, in addition, not even separable. For its application to concave programs see, for example, Holloway [16].

6. It is assumed throughout the paper that the true values of the elements of $\bar{B}_i$, $\bar{A}_i$ and $\bar{a}_i$ ($i=1,\ldots,k$) for which HQ has formulated estimates are within the range of the probability distributions for the random variables.

7. The price per unit of additional resources could be considered in the deterministic equivalent form of HQ's problem (5) by an approach similar to the one used in stochastic programming with simple recourse. See Ziemba [38] or, in a chance-constrained programming context, Schiefer [31].

8. The determination of d_i ($i=1,\ldots,k$) which correspond to the computed solution of HQ's programming problem is no problem if the common constraints are separable in such a way that for each constraint j it is possible to compute $\alpha_i^j = E(\bar{b}_i^j\bar{x}_i^0) + \sum_{q=2}^{Q} u_i^{jq}\lambda_i^q + u_{\alpha_j}\sqrt{V(\bar{b}_i^j\bar{x}_i^0)}$. Otherwise, one might compute an approximation as

$$d_i^j = E(\bar{b}_i^j\bar{x}_i^0) + \sum_{q=2}^{Q} u_i^{jq}\lambda_i^q + (1/k)\ u_{\alpha_j}\sqrt{V(\sum_{i=1}^{k} \bar{b}_i^j\bar{x}_i^0)}$$

9. See Hillier [14] for a discussion of the functional form of the probability distribution of a linear combination of random variables.

10. In this case, $\bar{A}_i$, $\bar{B}_i$ and $\bar{a}_i$ $(i=1,\ldots,k)$ in (5) are supposed to represent the aggregated information.

11. Only the restrictions on the number of divisional constraints and not the number of opportunities is discussed here. This is done because the number of constraints is usually the most limiting factor in the size of programming problems, and the aggregation of divisional opportunities does not affect the feasibility of computed solutions for the firm's optimization problem.

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2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): USING PROBABILISTIC INFORMATION IN SOLVING RESOURCE ALLOCATION PROBLEMS FOR A DECENTRALIZED FIRM
5. TYPE OF REPORT & PERIOD COVERED: TECHNICAL REPORT
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): JAMES R. FREELAND & GERHARD SCHIEFER
8. CONTRACT OR GRANT NUMBER(s): N00014-76-C-0418
9. PERFORMING ORGANIZATION NAME AND ADDRESS: DEPARTMENT OF OPERATIONS RESEARCH, STANFORD UNIVERSITY, STANFORD, CALIF.
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR-047-061
11. CONTROLLING OFFICE NAME AND ADDRESS: OPERATIONS RESEARCH PROGRAM CODE 434, OFFICE OF NAVAL RESEARCH, ARLINGTON, VIRGINIA 22217
12. REPORT DATE: SEPTEMBER 1978
13. NUMBER OF PAGES:
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): APPROVED FOR PUBLIC RELEASE, DISTRIBUTION IS UNLIMITED.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number): CAPITAL BUDGETING, LINEAR PROGRAMMING, RESOURCE ALLOCATION, DECOMPOSITION, CHANCE-CONSTRAINED PROGRAMMING
20. ABSTRACT (Continue on reverse side if necessary and identify by block number): SEE REVERSE SIDE

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Abstract

This paper formulates a general linear programming problem for a multidivision firm where headquarters possesses probabilistic information regarding each division's opportunities. It is assumed that headquarters is willing to risk implementing a plan which may not be optimal in order to avoid collecting detailed information from the divisions. Headquarters willingness to take a risk is modelled via the use of chance constraints. An iterative procedure which is derived from the Dantzig-Wolfe decomposition principle is presented which allows headquarters to combine deterministic information from the divisions with its stochastic information to arrive at a resource allocation plan. Characteristics of the resulting plan are discussed relative to headquarter's risk attitude and its probabilistic information. The procedure is adapted to situations where the size of headquarters programming problem has to be reduced.

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