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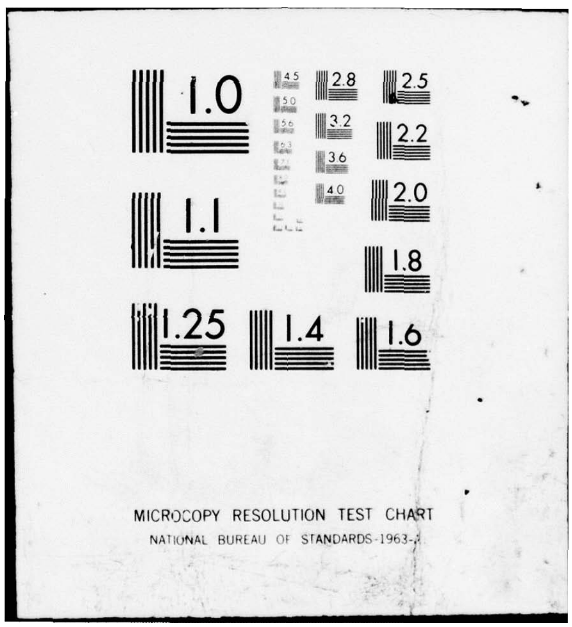
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RELATIONSHIP BETWEEN MEAN RADIATED ENERGY, MEAN SQUARED RADIATED ENERGY, AND SPONTANEOUS POWER SPECTRUM IN A POWER SERIES EXPANSION OF THE EQUATIONS OF MOTION IN A FREE ELECTRON LASER*

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ABSTRACT

To lowest order in the electric field and the inverse electron energy, the classical mean energy radiated by electrons in a free electron laser with a symmetric magnet is equal to one-half the derivative, with respect to energy, of the classical mean-squared radiated energy. The integral for the mean squared energy is also shown to be identical to the integral for the classical spontaneous power spectrum.

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1. Introduction

Light is amplified in a free electron laser in the course of the interaction between the light, a beam of relativistic, free electrons and a spatially periodic transverse magnetic field. To analyze the operation of this device, we need to know the way in which the mean energy radiated by the electrons and the fluctuations in the radiated energy depend on the optical wavelength, the magnet design and the electron energy. In this paper, we analyze the classical equations of motion for an electron in a free electron laser with a static field of arbitrary geometry and show that, to lowest order in the optical electric field and the inverse electron energy, the first moment of the radiated energy per electron is equal to one-half the derivative of the second moment, and that the second moment is proportional to the spontaneous radiation power spectrum. The result defines the range of attainable characteristics for free electron lasers operating at power densities below saturation.

It has been known since the first quantum description of the free electron laser[1] that the gain is proportional to the derivative with respect to energy of the transition rate for spontaneous

radiation into the mode excited by the incident stimulating radiation:

$$\text{available gain} \propto \Gamma_{stim} - \Gamma_{abs}$$

$$\Gamma_{stim} - \Gamma_{abs} = -(n+1)\left(\frac{\hbar\omega}{mc^2}\right)\frac{d\Gamma_{spont}}{d\gamma} \tag{1}$$

where Γ_{stim} is the transition rate for stimulated radiation, Γ_{abs} is the transition rate for absorption and Γ_{spont} is spontaneous transition rate for radiation into the mode in question. The photon energy is $\hbar\omega$ and γmc^2 is the electron mass-energy. The radiation is assumed to propagate parallel to the electrons through the static field. The derivative relationship follows from the difference in the dependence on electron energy of the transition rates for stimulated emission and absorption: the transition rate for absorption is centered at an electron energy which is lower by $\Delta\gamma \approx \frac{\hbar\omega}{mc^2}$ than the energy for stimulated emission. The average radiated energy per electron in the quantum formalism is:

$$\begin{aligned}\langle\gamma_f-\gamma_i\rangle &= \left(\frac{\hbar\omega}{mc^2}\right)(\Gamma_{stim}-\Gamma_{abs})\,T \\ &= (n+1)\left(\frac{\hbar\omega}{mc^2}\right)^2\frac{d\Gamma_{spont}}{d\gamma}\,T\ ,\end{aligned} \tag{2}$$

where γ_i is the initial electron energy, γ_f the final energy, and T the interaction time.

Using the definition of the number of modes $dN = \omega^2 V\, d\omega d\Omega/(2\pi c)^3$ in a volume V, frequency range $d\omega$ and solid angle $d\Omega$, it is possible to express the spontaneous transition rate in terms of the classical spontaneous power spectrum for synchrotron radiation,[2] $(dP(\omega)/d\Omega)$,

$$\Gamma_{spont} = \frac{1}{\hbar\omega} \cdot \frac{1}{dN} \cdot \frac{dP(\omega)}{d\Omega} \cdot d\omega \cdot d\Omega$$
$$= \frac{2\pi}{TV} \frac{e^2 c}{(\hbar\omega)} \left| \int dt\ \underset{\sim}{\beta_\perp}\ e^{i(kz-\omega t)} \right|^2 , \tag{3}$$

where $\beta_\perp$ is the electrons' transverse velocity, and $\underset{\sim}{k}$ is the wave vector for the radiation. We assume a coordinate system in which electrons on the average propagate parallel to $\hat{z}$, the propagation vector $\underset{\sim}{k}$ is parallel to z and the electrons' transverse motion is confined to the x-z plane. Given equation (3), the average radiated energy can be expressed as

$$\langle \gamma_f - \gamma_i \rangle = \frac{2\pi(n+1)(\hbar\omega)}{V} \frac{e^2}{m^2c^2} \cdot \frac{d}{d\gamma} \left| \int_{-T/2}^{T/2} dt\ \underset{\sim}{\beta_\perp}\ e^{i(kz-\omega t)} \right|^2$$
$$\approx \frac{1}{2} \frac{e^2 E^2}{2m^2c^2} \frac{d}{d\gamma} \left| \int_{-T/2}^{T/2} dt\ \underset{\sim}{\beta_\perp}\ e^{i(kz-\omega t)} \right|^2 , \tag{4}$$

where we have set the classical energy density for the field $\frac{E^2}{4\pi} = \frac{(n+1)\hbar\omega}{V}$.

Equation (4) is a key result. It relates the mean radiated energy per electron to the spontaneous power spectrum and the transverse velocity of the electrons in the interaction region. Together with the equations linking the second moment of the radiated energy to $\langle \gamma_f - \gamma_i \rangle$, equation (4) provides an elementary, yet comprehensive means to characterize the operation of the laser. While it is interesting to note that equation (4) can be deduced, as above, from the original quantum analysis of the amplification mechanism,

it is evident that the derivation, relying on both quantum and classical methods, is somewhat deficient in rigor. A self-contained classical derivation of the equation is developed below.

Regarding the fluctuations, although a number of authors have analyzed the fluctuations in the radiated energy,[3] most of the published material has dealt with a special case — the constant period, constant amplitude transverse field. While the derivative relationship between the first and second moments of the radiated energy is implicit in Colson's analysis of the constant period magnet,[4] it remains of considerable interest to determine the way in which the fluctuations change when the magnet design is altered.[5]

Madey and Deacon[6] have pointed out that the commutation relations for the electric and magnetic fields limit the ability to specify the amplitude of the optical field with which the electrons interact. Since the forces acting on the electrons cannot be defined with precision, it is not possible to compute the exact energy lost by the electrons during the interaction. The uncertainty in the field is a quantum effect, and is expected to be the dominant source of fluctuations when the optical power density is low or the wavelength short.

When the optical power density S is large in comparison to $\frac{1}{8\pi^{3/2}} \frac{\hbar\omega_0{}^3}{c\lambda q}$, the relative uncertainty in the field is small, and classical methods can be used to compute the radiated energy with reasonable confidence.[6] In this paper, our interest is in operation at high power and we will assume the validity of the classical approximation. We will also assume that the gain is small so that the magnitude of the optical electric field can be approximated as constant, and that plasma effects can be neglected. The basis of these last approximations is discussed in previous publications.[7]

In the classical problem, the transfer of energy from the electrons to the field is determined by the phase of optical electric field: when the field is parallel to the electron's transverse velocity, the electrons transfer energy to the field. When the field is antiparallel, the field accelerates the electrons. Because the field can remain in phase with the electron's transverse velocity for a considerable fraction of the interaction length, the energy transferred by a given electron to the field depends strongly on the phase of the optical field at the start of the interaction. Since the phase of the field will vary periodically between 0 and 2π as new electrons enter the interaction region, the energy of electrons leaving the interaction region will be periodically modulated.

In the classical approximation, the energy modulation is essential to amplification since it is the means by which the electrons are spatially re-arranged to radiate predominantly in phase with the optical wave. It might therefore be thought that it is impossible to obtain gain without ending up with a spread in the energy of the electrons emerging from the interaction region.

The analysis below indicates that the second of these conclusions is in error. While spatial re-arrangement of the electrons during the interaction remains a pre-condition for amplification, we will see that, for a given gain, the spread in the energy of the electrons emerging from the interaction can, in principle, be

changed substantially by varying the design of the static periodic magnet.

2. Derivation of the Theorem

In discussions of the motion of relativistic electrons it is customary to express the energy in terms of the normalized variable γ and to use $\underset{\sim}{\beta}$ to represent the velocity. If $\underset{\sim}{r}$ is the radius vector to the position of the electron, $\underset{\sim}{p}$, the kinematic momentum and γmc^2 the mass energy, $\underset{\sim}{\beta}$ and γ are defined by the equations:

$$\gamma = \sqrt{p^2+m^2c^2}\ /mc$$

$$\underset{\sim}{\beta} = \dot{\underset{\sim}{r}}/c = \underset{\sim}{p}/\sqrt{p^2+m^2c^2} \quad . \tag{5}$$

With these definitions the magnitude of $\underset{\sim}{\beta}$ is $\sqrt{1-\frac{1}{\gamma^2}}$. As previously noted, we assumed a coordinate system in which the light and the electrons propagate in the +z direction. The static magnetic field is assumed to have only a $\hat{y}$ component so that the electron motion is confined to the x-z plane. Physically, those conditions would apply in a free-electron laser with a linearly polarized static magnet in which the electrons are injected into the interaction region to move on the symmetry plane of the dipoles generating the field. We assume that the interaction region is finite in length and extends between $z = \pm L/2$. We further assume that the periodic magnet is either symmetric or antisymmetric, i.e. $B(z) = \pm B(-z)$.

Formally, the energy of an electron moving through the interaction region is given by:

$$\gamma(z) = \gamma_i + \frac{e}{mc} \int_0^t dt\, \underset{\sim}{\beta}(z,t,\phi) \cdot \underset{\sim}{E}(z,t,\phi)$$

$$= \gamma_i + \frac{e}{mc^2} \int_{-L/2}^{z} \frac{dz}{\beta_\parallel}\, \underset{\sim}{\beta_\perp} \cdot \underset{\sim}{E} \tag{6}$$

where γ_i is the energy at the start of the interaction, and $\underset{\sim}{E}$ is the electric field at the point z, evaluated at the time t(z) that the electron arrives at z. The value of $\underset{\sim}{B}$ and the value of the electric field at (z,t) is also affected by the phase ϕ of the field at the start of the interaction. This dependence is noted by making $\underset{\sim}{\beta}$ and $\underset{\sim}{E}$ functions of ϕ in addition to z and t. We have defined $\beta_\parallel$, the longitudinal velocity, as $\hat{z}\cdot\underset{\sim}{\beta}$ and $\underset{\sim}{\beta_\perp}$, the transverse velocity, as $\underset{\sim}{\beta} - \beta_\parallel \hat{z}$.

If we define γ_f as the energy of the electrons at the end of the interaction region, $\gamma_f = \gamma(L/2)$, the energy radiated by the electron during the interaction is:

$$\gamma_f - \gamma_i = \frac{e}{mc^2} \int_{-L/2}^{L/2} \frac{dz}{\beta_\parallel}\, \underset{\sim}{\beta_\perp} \cdot \underset{\sim}{E} \ . \tag{7}$$

To obtain the average of the radiated energy, we need to average over the phase of the electric field at the start of the interaction:

$$\langle(\gamma_f - \gamma_i)\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \ \frac{e}{mc^2} \int_{-L/2}^{L/2} \frac{dz}{\beta_\parallel}\, \underset{\sim}{\beta_\perp} \cdot \underset{\sim}{E} \ . \tag{8}$$

Equation (8) also defines the first moment of the radiated energy. The second moment can similarly be defined as:

$$\langle(\gamma_f - \gamma_i)^2\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \left[\frac{e}{mc^2} \int_{-L/2}^{L/2} \frac{dz}{\beta_\parallel}\, \underset{\sim}{\beta_\perp} \cdot \underset{\sim}{E}\right]^2 \ . \tag{9}$$

To compute the first and second moments we need to know how $\underset{\sim}{\beta}$ and t vary with position and electron energy as the electrons move through the interaction region. Formally t(z) is given by:

$$t(z) = \frac{1}{c} \int_{-L/2}^{z} \frac{dz}{\beta_\parallel} \tag{10}$$

so, to compute t(z) we need to know $\underset{\sim}{\beta}$. The computation of $\underset{\sim}{\beta}$ can be substantially simplified if we assume the static magnetic and optical fields are independent of the x coordinate. With this assumption, the x component of canonical momentum is conserved and we may write:

$$p_x = -\frac{eA_x}{c} + \text{constant} \ . \tag{11}$$

Given that the transverse momentum of the electrons has to be minimized to limit the radial spread of the beam during the interaction, the constant in equation (11) may be taken to be zero in situations of practical interest. The transverse velocity $\beta_\perp$ is then:

$$\underset{\sim}{\beta_\perp} = \frac{p_x}{\gamma mc}\, \hat{x} \approx -\frac{eA_x}{\gamma mc^2}\, \hat{x} \ . \tag{12}$$

Given $\underset{\sim}{\beta_\perp}$, $\beta_\parallel$ is:

$$\beta_\parallel = \sqrt{1 - \frac{1}{\gamma^2} - \beta_\perp^2}$$

(cont'd)

$$= \sqrt{1 - \frac{1}{\gamma^2}\,[1 + (eA_x/mc^2)^2]} \quad . \tag{13}$$

Although $\underset{\sim}{\beta}$ depends on t(z) through the vector potential of the optical field, this dependence can be neglected in situations of practical interest. Given a static field of the form $\underset{\sim}{B}_o = B_o \sin(2\pi z/\lambda_q)\hat{y}$ and an optical field of the form $\underset{\sim}{B}_{opt} = B_{opt} \sin(kz-\omega t+\phi)\hat{y}$, the vector potential has the form:

$$\underset{\sim}{A} = -\left[\frac{\lambda_q}{2\pi} B_o \cos(2\pi z/\lambda_q) + \frac{1}{k}\cos(kz-\omega t+\phi)\right]\hat{x} \quad . \tag{14}$$

Noting that the optical field strength B_{opt} typically will be small in comparison with the static field B_o and that the optical wavelength is smaller by a factor $\sim \gamma^2$ than the magnet period λ_q, the vector potential can be approximated by the vector potential A_{ox} of the static field, which is independent of time.

If the optical power density is not too large, it is reasonable to expand the energy $\gamma(z)$ in a power series in the amplitude E of the optical electric field.[8] If $\gamma(z)$ is set equal to:

$$\gamma(z) = \gamma_i + \gamma_1 + \gamma_2 + \ldots \tag{15}$$

where $\gamma_1 \propto E$, $\gamma_2 \propto E^2$ and $\gamma_n \propto E^n$, and $\underset{\sim}{\beta}^o$ and t^o are defined by:

$$\underset{\sim}{\beta}_\perp{}^o = \frac{1}{\gamma_i}\left(\frac{e}{mc^2}\right) A_{ox}(z)\,\hat{x}$$

$$\beta_\parallel{}^o = \sqrt{1 - (1/\gamma_i{}^2) - (\underset{\sim}{\beta}_\perp{}^o)^2} \tag{16}$$

$$t^o(z) = \frac{1}{c}\int_{-L/2}^{z} dz \left(\frac{1}{\beta_\parallel{}^o}\right) \quad .$$

The leading terms in the expansion for $\gamma(z)$ are found to be:

$$\gamma_1(z) = \left(\frac{e}{mc^2}\right)\int_{-L/2}^{z} \frac{dz}{\beta_\parallel{}^o}\; \underset{\sim}{\beta}_\perp{}^o \cdot \underset{\sim}{E}(z,t^o,\phi)$$

$$\begin{aligned}\gamma_2(z) = \left(\frac{e}{mc^2}\right)\int_{-L/2}^{z} dz \Bigg\{ &\Bigg[\frac{d}{d\gamma_i}\,\frac{\beta_\perp{}^o(z)}{\beta_\parallel{}^o(z)}\cdot \underset{\sim}{E}(z,t^o,\phi) \\ &\left(\frac{e}{mc^2}\right)\int_{-L/2}^{z}\frac{dz'}{\beta_\parallel{}^o(z')}\,\underset{\sim}{\beta}_\parallel{}^o(z')\cdot\underset{\sim}{E}(z',t^o,\phi)\Bigg] \\ &+\left[\frac{\beta_\perp{}^o(z)}{\beta_\parallel{}^o(z)}\cdot\frac{\partial}{\partial t}\underset{\sim}{E}(z,t^o,\phi)\cdot\int_{-L/2}^{z} dz'\,\frac{1}{c}\left(\frac{d}{d\gamma_i}\,\frac{1}{\beta_\parallel{}^o(z')}\right) \\ &\left(\frac{e}{mc^2}\right)\int_{-L/2}^{z'}\frac{dz''}{\beta_\parallel{}^o(z'')}\,\underset{\sim}{\beta}_\perp{}^o(z'')\cdot\underset{\sim}{E}(z'',t^o,\phi)\right]\Bigg\} \quad .\end{aligned} \tag{17}$$

Although the derivative of $\underset{\sim}{\beta}$ appears in the second order term in the form $\frac{d}{d\gamma_i}(\beta_\perp{}^o/\beta_\parallel{}^o)$, the velocity of relativistic electrons is comparatively insensitive to the energy. The dominant term in the integrand for the second order term is associated with the change in the optical electric field due to the change in the time of arrival of the electron $\Delta t = \frac{1}{c}\int dz\left[\frac{1}{\beta_\parallel(z)} - \frac{1}{\beta_\parallel{}^o(z)}\right]$. Terms associated with the change in the electric field due to the phase advance of the electrons also dominate the higher order terms γ_3, γ_4, ... In general, the integrand of the nth order term will be, approximately, proportional to $\frac{\partial^n}{\partial t^n}E(z,t^o,\phi)(\Delta t)^n \approx (\omega\Delta t)^n\,E(Z,t^o,\phi)$.

The number of terms required to analyze operation of the laser depends on the objective of the analysis and the optical power density. If the optical power density is small in comparison to the power density for saturation, knowledge of the first order term γ_1 is sufficient to estimate the second moment of the radiated energy while γ_2 must be kept to compute the first moment. Terms of 3rd order and higher, neglected in this analysis, would be required to compute saturation effects.

Our objective in this analysis is to demonstrate that $\langle(\gamma_f-\gamma_i)\rangle \approx \frac{1}{2}\frac{d}{d\gamma_i}\langle(\gamma_f-\gamma_i)^2\rangle$ to lowest order in E and $(1/\gamma)$ and to establish the similarity of the functional dependence on energy of the second moment and the spectral power density for spontaneous radiation. We will first compute $\langle(\gamma_f-\gamma_i)^2\rangle$. To order E^2, the second moment of the radiated energy is

$$\langle(\gamma_f-\gamma_i)^2\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, [\gamma_1(z = \tfrac{L}{2})]^2$$
$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi\, [\left(\frac{e}{mc^2}\right)\int_{-L/2}^{L/2} \frac{dz}{\beta_\parallel^o}\, \underline{\beta_\perp}^o\cdot \underset{\sim}{E}(z,t^o,\phi)]^2 . \qquad (18)$$

if we express the electric field as:

$$\underset{\sim}{E}(z,t,\phi) = \mathrm{Re}\, \hat{x}\, E_o e^{i(kz-\omega t+\phi)} , \qquad (19)$$

$\gamma_1(z)$ has the form:

$$\gamma_1(z) = \mathrm{Re}\left(\frac{eE_o}{mc^2}\right)\int_{-L/2}^{z} \frac{dz'}{\beta_\parallel^o(z')}\, \beta_x^o(z')\, e^{i(kz'-t^o+\phi)}$$
$$=\left(\frac{eE_o}{mc^2}\right)\{[\int_{-L/2}^{z} \frac{dz'}{\beta_\parallel^o(z')}\, \beta_x^o(z')\cos(kz'-\omega t^o)]^2 , \qquad (20)$$
$$+ [\int_{-L/2}^{z} \frac{dz'}{\beta_\parallel^o(z')}\, \beta_x^o(z')\sin(kz'-\omega t^o)]^2\}^{\frac{1}{2}}\cos(\phi+\psi)$$

where ψ is given by:

$$\psi = \cos^{-1}\left\{\frac{\int_{-L/2}^{z}\frac{dz'}{\beta_\parallel^o(z')}\beta_x^o(z')\cos(kz'-\omega t^o)}{\{[\int_{-L/2}^{-z}\frac{dz'}{\beta_\parallel^o(z')}\beta_x^o(z')\cos(kz-\omega t^o)]^2 + [\int_{-L/2}^{z}\frac{dz'}{\beta_\parallel^o(z')}\beta_x^o(z')\sin(kz'-\omega t^o)]^2\}^{1/2}}\right. \qquad (21)$$

We note that $\gamma_1(z)$ varies as the cosine of the phase of the optical field at the start of the interaction. The second moment of the radiated energy is, therefore:

$$\langle(\gamma_f-\gamma_i)^2\rangle = \frac{1}{2}\left(\frac{eE_o}{mc^2}\right)^2 \{[\int_{-L/2}^{L/2}\frac{dz'}{\beta_\parallel^o(z')}\,\beta_x^o(z')\cos(kz'-\omega t^o)]^2$$
$$+ [\int_{-L/2}^{L/2}\frac{dz'}{\beta_\parallel^o(z')}\,\beta_x^o(z')\sin(kz'-\omega t^o)]^2\} . \qquad (22)$$

Given equation (22), it is straightforward to establish the connection to the spectral power density for spontaneous radiation. The power spectrum has the form:[2]

$$\frac{dP(\omega)}{d\Omega} = \frac{1}{T}\,\frac{e^2\omega^2}{4\pi^2 c}\,|\int_{-\infty}^{+\infty} dt\, \hat{n}\times(\hat{n}\times\underset{\sim}{\beta})\exp[+i\omega(t-\frac{\hat{n}\cdot \underset{\sim}{r}}{c})]|^2 \qquad (23)$$

where $dP(\omega)/d\Omega$ is the energy radiated per unit time per unit frequency per unit solid angle, and $\hat{n}$ is a unit vector from the origin to the point of observation. For radiation in the forward direction $\hat{n} = \hat{z}$, and $\hat{n}\times(\hat{n}\times\underset{\sim}{\beta}) = -\underline{\beta_\perp} = -\hat{x}\,\beta_x$ for the problem in question. The velocity in equation (23) is the velocity in the absence of the optical field and is, therefore, equal to $\underset{\sim}{\beta}^{o}$. The integral over time can be converted to an integral over z by replacing dt with $(dz/\beta_{\parallel}^{o})$ and t by $t^{o}(z)$. Clearly the limits in the integral over z are extended only over the region in which $\underset{\sim}{\beta}_{\perp}$ is finite, $z = \pm(L/2)$, by the previous definition of the interaction region. The spontaneous power spectrum can therefore be written as:

$$\frac{dP(\omega)}{d\Omega} = \frac{1}{T} \frac{e^2\omega^2}{4\pi^2 c} \left| \int_{-L/2}^{L/2} \frac{dz}{\beta_{\parallel}^{o}(z)} \beta_x(z)\, e^{i(\omega t^{o}-kz)} \right|^2$$

$$= \frac{1}{2\pi^2} \frac{1}{T} \frac{m^2c^3\omega^2}{E_o^2} \langle (\gamma_f-\gamma_i)^2 \rangle \quad . \tag{24}$$

The spontaneous power spectrum and the second moment of the radiated energy are related by a factor which depends only on the optical frequency, the amplitude of the optical electric field, and the interaction time.

The proof of the equality $\langle (\gamma_f-\gamma_i) \rangle = \frac{1}{2} \frac{d}{d\gamma_i} \langle (\gamma_f-\gamma_i)^2 \rangle$ is somewhat more involved. As previously noted, and as is apparent from equation (20) it is necessary to include the second order term $\gamma_2(z)$ to compute the first moment of the radiated energy. To lowest order in the electric field, the first moment is:

$$\langle \gamma_f-\gamma_i \rangle = \langle \gamma_2(z = L/2) \rangle$$

(cont'd)

$$= \frac{1}{2\pi} \int_0^{2\pi} d\phi \left(\frac{eE_o}{mc^2}\right) \mathrm{Re} \int_{-L/2}^{L/2} dz \left\{ \left[e^{i(kz-\omega t^{o}+\phi)} \left(\frac{d}{d\gamma_i} \frac{\beta_x^{o}(z)}{\beta_{\parallel}^{o}(z)} \right) \right.\right.$$

$$\cdot \left(\frac{eE_o}{mc^2}\right) \mathrm{Re} \int_{-L/2}^{z} \frac{dz'}{\beta_{\parallel}^{o}(z')} \beta_x^{o}(z')\, e^{i(kz'-\omega t^{o}+\phi)}]$$

$$- i\left(\frac{\omega}{c}\right) \left[\frac{\beta_x^{o}(z)}{\beta_{\parallel}^{o}(z)} e^{i(kz-\omega t^{o}+\phi)} \int_{-L/2}^{z} dz' \left(\frac{d}{d\gamma_i} \frac{1}{\beta_{\parallel}^{o}(z')} \right) \right.$$

$$\left.\left. \cdot \left(\frac{eE_o}{mc^2}\right) \mathrm{Re} \int_{-L/2}^{z'} \frac{dz''}{\beta_{\parallel}^{o}(z'')} \beta_x^{o}(z'')\, e^{i(kz''-\omega t^{o}+\phi)} \right] \right\} \tag{25}$$

where we have used the expontential notation for $\underset{\sim}{E}$ defined in equation (19). We are trying to demonstrate that this integral is equal to $\frac{1}{2} \frac{d}{d\gamma_i} \langle (\gamma_f-\gamma_i)^2 \rangle$ to lowest order in E. From equation (18)

$$\frac{1}{2} \frac{d}{d\gamma_i} \langle (\gamma_f-\gamma_i)^2 \rangle \approx \frac{1}{2} \frac{d}{d\gamma_i} \left\{ \frac{1}{2\pi} \int_0^{2\pi} d\phi \right.$$

$$\left. \times \left[\left(\frac{eE_o}{mc^2}\right) \mathrm{Re} \int_{-L/2}^{L/2} \frac{dz}{\beta_{\parallel}^{o}(z)} \beta_x^{o}(z) e^{i(kz-\omega t^{o}+\phi)} \right]^2 \right\}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\phi \left[\left(\frac{eE_o}{mc^2}\right) \mathrm{Re} \int_{-L/2}^{L/2} \frac{dz''}{\beta_{\parallel}^{o}(z'')} \beta_x^{o}(z'')\, e^{i(kz''-\omega t^{o}+\phi)} \right]$$

$$\cdot \left\{ \left(\frac{eE_o}{mc^2}\right) \mathrm{Re} \int_{-L/2}^{L/2} dz \left[e^{i(kz-\omega t^{o}+\phi)} \left(\frac{d}{d\gamma} \frac{\beta_x^{o}(z)}{\beta_{\parallel}^{o}(x)} \right) \right.\right.$$

$$\left.\left. - i\left(\frac{\omega}{c}\right) \frac{\beta_x^{o}(x)}{\beta_{\parallel}^{o}(z)} e^{i(kz-\omega t^{o}+\phi)} \int_{-L/2}^{z} dz' \left(\frac{d}{d\gamma_i} \frac{1}{\beta_{\parallel}^{o}(z')} \right) \right] \right\} . \tag{26}$$

We note that, while both equations (25) and (26) contain terms proportional to $\frac{d}{d\gamma} [\beta_x^{o}(z)/\beta_{\parallel}^{o}(z)]$ and $\frac{d}{d\gamma} [1/\beta_{\parallel}^{o}(z')]$, the limits to and order of the integrals in the two expressions is not the same.

o establish the equivalence of equations (25) and (26), we make se of the symmetry of the magnet. In computing $\langle(\gamma_f-\gamma_i)\rangle$, we ssumed that the electrons started at $z = -L/2$ and traveled in the z direction. We can equivalently start the electrons at $z = +L/2$ ith the velocity equal to $-\beta_\parallel^{\,o}(z = -L/2)$. Given the symmetry of he magnet, the first moment of the radiated energy for this eversed geometry would be the same as for the normal configuration. f the transverse and longitudinal velocities in the reversed eometry are defined as $\bar\beta_x^{\,o}(z)$ and $\bar\beta_\parallel^{\,o}(z)$, and the time as $\bar t^{\,o}(z)$, e can write:

$$\begin{aligned}\langle\gamma_f-\gamma_i\rangle \approx \frac{1}{2\pi}\int_0^{2\pi} d\phi\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{L/2}^{-L/2} dz\{[e^{i(kz+\omega\bar t^o+\phi)}\left(\frac{d}{d\gamma}\frac{\bar\beta_x^{\,o}(z)}{\bar\beta_\parallel^{\,o}(z)}\right)\\ \cdot\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{L/2}^{z}\frac{dz'}{\bar\beta_\parallel^{\,o}(z')}\bar\beta_x^{o}(z')\,e^{i(kz'+\omega\bar t^o+\phi)}]\\ +i\left(\frac{\omega}{c}\right)[\frac{\bar\beta_x^{\,o}(z)}{\bar\beta_\parallel^{\,o}(z)}e^{i(kz+\omega\bar t^o+\phi)}\int_{L/2}^{z}dz'\left(\frac{d}{d\gamma}\frac{1}{\bar\beta_\parallel^{\,o}(z')}\right)\\ \cdot\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{L/2}^{z'}\frac{dz''}{\bar\beta_\parallel^{\,o}(z'')}\bar\beta_x^{\,o}(z'')\,e^{i(kz''+\omega\bar t^o+\phi)}]\}\ .\end{aligned} \tag{27}$$

ue to the symmetry of the magnet, $\bar\beta_x^{\,o}(z) = \pm\,\beta_x^{\,o}(z)$ and therefore:

$$\begin{aligned}\bar\beta_\parallel^{\,o}(z) &= -\,\beta_\parallel^{\,o}(z)\\ \bar t^{\,o}(z) &= \frac{1}{c}\int_{L/2}^{z}\frac{dz}{\bar\beta_\parallel^{\,o}(z)} = \frac{1}{c}\int_{z}^{L/2}\frac{dz}{\beta_\parallel^{\,o}(z)}\\ &= t^o\,(z = L/2) - t^o(z)\ .\end{aligned} \tag{28}$$

Using these identities, we can reverse the limits on equation (27) to obtain:

$$\begin{aligned}\langle\gamma_f-\gamma_i\rangle \approx \frac{1}{2\pi}\int_0^{2\pi} d\phi'\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{L/2} dz\{[e^{i(kz-\omega t^o+\phi')}\left(\frac{d}{d\gamma}\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\right)\\ \cdot\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{z}^{L/2}\frac{dz'}{\beta_\parallel^{\,o}(z')}\beta_x^{o}(z')\,e^{i(kz'-\omega t^o+\phi')}]\\ +i\left(\frac{\omega}{c}\right)\left[\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}e^{i(kz-\omega t^o+\phi')}\int_{z}^{L/2}dz'\left(\frac{d}{d\gamma}\frac{1}{\beta_\parallel^{\,o}(z')}\right)\right.\\ \left.\cdot\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{z'}^{L/2}\frac{dz''}{\beta_\parallel^{\,o}(z'')}\beta_x^{o}(z'')\,e^{i(kz''-\omega t^o+\phi')}\right]\}\end{aligned} \tag{29}$$

where $\phi' = \phi + \left(\frac{\omega}{c}\right)\int_{-L/2}^{L/2}\frac{dz}{\beta_\parallel^{\,o}(z)}$. Expressing the integrals over z' and z'' as the difference of the integrals between the limits $z,z' = \pm\,L/2$ and $(z,L/2)$, $(z',L/2)$, equation (29) can be rewritten:

$$\begin{aligned}\approx \frac{1}{2\pi}\int_0^{2\pi} d\phi'\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{L/2} dz\{[e^{i(kz-\omega t^o+\phi')}\left(\frac{d}{d\gamma}\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\right)\\ \cdot\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{L/2}\frac{dz'}{\beta_\parallel^{\,o}(z')}\beta_x^{\,o}(z')\,e^{i(kz'-\omega t^o+\phi')}]\\ -[e^{i(kz-\omega t^o+\phi')}\left(\frac{d}{d\gamma}\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\right)\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{z}\frac{dz'}{\beta_\parallel^{\,o}(z')}\beta_x^{o}(z')\,e^{i(kz'-\omega t^o+\phi')}]\\ +i\left(\frac{\omega}{c}\right)\left[\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}e^{i(kz-\omega t^o+\phi')}\int_{-L/2}^{L/2}dz'\left(\frac{d}{d\gamma}\frac{1}{\beta_\parallel^{\,o}(z')}\right)\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{L/2}\frac{dz''}{\beta_\parallel^{\,o}(z'')}\beta_x^{o}(z'')\,e^{i(kz''-\omega t^o+\phi')}\right]\\ -i\left(\frac{\omega}{c}\right)\left[\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}e^{i(kz-\omega t^o+\phi')}\int_{-L/2}^{L/2}dz'\left(\frac{d}{d\gamma}\frac{1}{\beta_\parallel^{\,o}(z')}\right)\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{z'}\frac{dz''}{\beta_\parallel^{\,o}(z'')}\beta_x^{o}(z'')\,e^{i(kz''-\omega t^o+\phi')}\right]\\ -i\left(\frac{\omega}{c}\right)\left[\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}e^{i(kz-\omega t^o+\phi')}\int_{-L/2}^{z}dz'\left(\frac{d}{d\gamma}\frac{1}{\beta_\parallel^{\,o}(z')}\right)\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{L/2}\frac{dz''}{\beta_\parallel^{\,o}(z'')}\beta_x^{o}(z'')\,e^{i(kz''-\omega t^o+\phi')}\right]\\ +i\left(\frac{\omega}{c}\right)\left[\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}e^{i(kz-\omega t^o+\phi')}\int_{-L/2}^{z}dz'\left(\frac{d}{d\gamma}\frac{1}{\beta_\parallel^{\,o}(z')}\right)\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{z'}\frac{dz''}{\beta_\parallel^{\,o}(z'')}\beta_x^{o}(z'')\,e^{i(kz''-\omega t^o+\phi')}\right]\}\end{aligned} \tag{30}$$

One of the terms on the right hand side of equation (30) is identically equal to zero:

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi' \left(\frac{eE_o}{mc^2}\right) \mathrm{Re}\int_{-L/2}^{L/2} dz\, i \left(\frac{\omega}{c}\right)\left[\frac{\beta_x{}^o(z)}{\beta_\parallel{}^o(z)}\; e^{i(kz-\omega t^o+\phi')}\right.$$

$$\cdot \int_{-L/2}^{L/2} dz' \;\frac{d}{d\gamma}\; \frac{1}{\beta_\parallel{}^o(z')} \;\left(\frac{eE_o}{mc^2}\right)$$

$$\left.\mathrm{Re}\int_{-L/2}^{L/2} \frac{dz''}{\beta_\parallel{}^o(z'')}\, \beta_x{}^o(z'')\; e^{i(kz''-\omega t^o+\phi')}\right]$$

$$\propto \int_0^{2\pi} d\phi\, \sin(\phi'+\psi)\, \cos(\phi'+\psi) = 0 \quad . \qquad (31)$$

For this term, the integral over z' depends only on the electron energy and magnet design and is independent of ϕ'. While the integral over z'' is, from equation (20), proportional to $\cos(\phi'+\psi)$, the i in the integral over z shifts the phase of the real part of the integral by 90°:

$$\mathrm{Re}\int_{-L/2}^{L/2} dz\, i \left(\frac{\omega}{c}\right) \frac{\beta_x{}^o}{\beta_\parallel{}^o}\; e^{i(kz-\omega t^o+\phi')} \approx \sin(\phi'+\psi) \quad . \qquad (32) \quad .$$

As noted in eq. (31), the integral of $\sin(\phi'+\psi)\cos(\phi'+\psi)$ is zero. Two of the remaining terms on the right hand side of equation (30) are equal to a term in equation (26) for $\frac{1}{2}\;\frac{d}{d\gamma_i}\;\langle(\gamma_f-\gamma_i)^2\rangle$. The term

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi' \left(\frac{eE_o}{mc^2}\right) \mathrm{Re}\int_{-L/2}^{L/2} dz\, (-i) \left(\frac{\omega}{c}\right)\left[\frac{\beta_x{}^o(z)}{\beta_\parallel{}^o(z)}\; e^{i(kz-\omega t^o+\phi')}\right.$$

(cont'd)

(cont'd)

$$\cdot \int_{-L/2}^{z} dz' \left(\frac{d}{d\gamma}\; \frac{1}{\beta_\parallel{}^o(z'')}\right) \left(\frac{eE_o}{mc^2}\right)$$

$$\left.\mathrm{Re}\int_{-L/2}^{L/2} \frac{dz''}{\beta_\parallel{}^o(z'')}\;\; \beta_x{}^o(z'')\; e^{i(kz''-\omega t^o+\phi')}\right] \qquad (33)$$

in equation (30) can be identified immediately with the identical term in equation (26). The term

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi' \left(\frac{eE_o}{mc^2}\right) \mathrm{Re}\int_{-L/2}^{L/2} dz(-i) \left(\frac{\omega}{c}\right)\left[\frac{\beta_x{}^o(z)}{\beta_\parallel{}^o(z)}\;\; e^{i(kz-\omega t^o+\phi')}\right.$$

$$\int_{-L/2}^{L/2} dz' \left(\frac{d}{d\gamma}\; \frac{1}{\beta_\parallel{}^o(z')}\right) \left(\frac{eE_o}{mc^2}\right) \qquad (34)$$

$$\left.\mathrm{Re}\int_{-L/2}^{z'} \frac{dz''}{\beta_\parallel{}^o(z'')}\;\; \beta_x{}^o(z'')\; e^{i(kz''-\omega t^o+\phi')}\right]$$

is equal to the term in equation (33). To demonstrate this, let $\phi'' = \phi'+\frac{\pi}{2}$, exchange the dummy variables z and z'', and rewrite eq. (34) as:

$$-\frac{1}{2\pi}\int_0^{2\pi} d\phi'' \left(\frac{eE_o}{mc^2}\right) \mathrm{Re}\int_{-L/2}^{z'} dz(-i) \left(\frac{\omega}{c}\right)\left[\frac{\beta_x{}^o(z)}{\beta_\parallel{}^o(z)}\; e^{i(kz-\omega t^o+\phi'')}\right.$$

$$\int_{-L/2}^{L/2} dz' \left(\frac{d}{d\gamma}\; \frac{1}{\beta_\parallel{}^o(z')}\right) \left(\frac{eE_o}{mc^2}\right) \qquad (35)$$

$$\left.\mathrm{Re}\int_{-L/2}^{L/2} \frac{dz''}{\beta_\parallel{}^o(z'')}\, \beta_x{}^o(z'')\; e^{i(kz''-\omega t^o+\phi')}\right].$$

This term has the same integrand as equation (33), but differs from that equation in the limits in the integrals over z and z' and in the overall sign. The limits of the integrals in the two expressions define the domains of integration as shown in Figure 1.

In the z,z' plane, the domains of integration for the two terms consist of the two halves of the square $z = \pm L/2$, $z' = \pm L/2$, where the cut is along the diagonal $z = z'$. From equation (31), if the limits of integration for either integral extended over the full square, the integral would be zero. Since the integrals in equations (33) and (35) extend over complementary areas within the square, they must have equal magnitude and opposite signs. Noting that the integral in equation (35) is multiplied by (-1), the terms in equations (33) and (35) are identical.

If the expression for $\langle\gamma_f-\gamma_i\rangle$ in equation (25) is set equal to the expression in equation (30), and use is made of the identities derived above in equations (31)-(35), we obtain the result:

$$\frac{2}{2\pi}\int_0^{2\pi} d\phi \left(\frac{eE_o}{mc^2}\right) \mathrm{Re} \int_{-L/2}^{L/2} dz\{[e^{i(kz-\omega t^o+\phi)} \left(\frac{d}{d\gamma}\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\right)\left(\frac{eE_o}{mc^2}\right)$$

$$\mathrm{Re}\int_{-L/2}^{z} \frac{dz'}{\beta_\parallel^{\,o}(z')}\,\beta_x^{\,o}(z')\,e^{i(kz'-\omega t^o+\phi)}]$$

$$- i\left(\frac{\omega}{c}\right)\left[\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\,e^{i(kz-\omega t^o+\phi)}\int_{-L/2}^{z} dz'\left(\frac{d}{d\gamma_i}\frac{1}{\beta_\parallel^{\,o}(z')}\right)\left(\frac{eE_o}{mc^2}\right)\right.$$

$$\mathrm{Re}\int_{-L/2}^{z'} \frac{dz''}{\beta_\parallel^{\,o}(z'')}\,\beta_x^{\,o}(z'')\,e^{i(kz''-\omega t^o+\phi)}]\}$$

$$= \frac{2}{2\pi}\int_0^{2\pi} d\phi\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{L/2} dz\{[\tfrac{1}{2}e^{i(kz-\omega t^o+\phi)}\left(\frac{d}{d\gamma}\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\right)\left(\frac{eE_o}{mc^2}\right)$$

$$\mathrm{Re}\int_{-L/2}^{L/2} \frac{dz'}{\beta_\parallel^{\,o}(z')}\,\beta_x^{\,o}(z')\,e^{i(kz'-\omega t^o+\phi')}$$

$$- i\left(\frac{\omega}{c}\right)\left[\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\,e^{i(kz-\omega t^o+\phi)}\int_{-L/2}^{L/2} dz'\left(\frac{d}{d\gamma}\frac{1}{\beta_\parallel^{\,o}(z')}\right)\left(\frac{eE_o}{mc^2}\right)\right.$$

$$\mathrm{Re}\int_{-L/2}^{z'} \frac{dz''}{\beta_\parallel^{\,o}(z'')}\,\beta_x^{\,o}(z'')\,e^{i(kz''-\omega t^o+\phi)}]\}\,. \qquad (36)$$

Recalling the definitions for $\langle\gamma_f-\gamma_i\rangle$ and $\frac{d}{d\gamma_i}\langle(\gamma_f-\gamma_i)^2\rangle$ given in equations (25) and (26), equation (36) indicates that to lowest order in E,

$$\langle\gamma_f-\gamma_i\rangle = \frac{1}{2}\,\frac{d}{d\gamma_i}\langle(\gamma_f-\gamma_i)^2\rangle$$

$$-\frac{1}{4\pi}\int_0^{2\pi} d\phi\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{L/2} dz\,[e^{i(kz-\omega t^o+\phi)}\left(\frac{d}{d\gamma}\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\right)$$

$$\cdot\left(\frac{eE_o}{mc^2}\right)\mathrm{Re}\int_{-L/2}^{L/2}\frac{dz'}{\beta_\parallel^{\,o}(z')}\,\beta_x^{\,o}(z')\,e^{i(kz'-\omega t^o+\phi)}]\,. \qquad (37)$$

Noting that

$$\frac{d}{d\gamma}\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)} = \frac{1}{\beta_\parallel^{\,o}(z)}\frac{d\beta_x^{\,o}}{d\gamma} + \frac{\beta_x^{\,o}}{(\beta_\parallel^{\,o})^3}\left(\frac{1}{\gamma_i^3} + \beta_x^{\,o}\frac{d\beta_x^{\,o}}{d\gamma}\right) \qquad (38)$$

and that from equation (12), $\beta_x \approx \frac{eA_x}{\gamma mc^2}$, we find for the case under consideration that:

$$\frac{d}{d\gamma}\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)} = -\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\left[\frac{1}{\gamma_i} + O\left(\frac{1}{\gamma_i^3}\right)\right]$$

$$\approx -\frac{1}{\gamma_i}\frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\,. \qquad (39)$$

The second term on the right hand side of equation (27) can therefore be approximated as:

$$\frac{1}{4\pi}\int_0^{2\pi} d\phi \left(\frac{eE_o}{mc^2}\right) \mathrm{Re} \int_{-L/2}^{L/2} dz \Big[e^{i(kz-\omega t^o+\phi)} \left(\frac{d}{d\gamma}\, \frac{\beta_x^{\,o}(z)}{\beta_\parallel^{\,o}(z)}\right) \left(\frac{eE_o}{mc^2}\right)$$

$$\mathrm{Re} \int_{-L/2}^{L/2} \frac{dz'}{\beta_\parallel^{\,o}(z')}\, \beta_x^{\,o}(z')\, e^{i(kz'-\omega t^o+\phi)} \Big]$$

$$\approx -\frac{1}{2\gamma_i} \langle (\gamma_f-\gamma_i)^2 \rangle \ . \tag{40}$$

This term will be small in comparison to $\frac{d}{d\gamma}\langle(\gamma_f-\gamma_i)^2\rangle$ for cases of practical interest in which the range in energy for which $\langle(\gamma_f-\gamma_i)^2\rangle$ differs from zero is likely to be small in comparison to γ_i. We can therefore write:

$$\langle(\gamma_f-\gamma_i)\rangle \approx \frac{1}{2}\, \frac{d}{d\gamma_i}\, \langle(\gamma_f-\gamma_i)^2\rangle \ , \tag{41}$$

the result to be proven. We note that, if equation (41) is combined with equation (24), we obtain:

$$\langle(\gamma_f-\gamma_i)\rangle = \frac{1}{2}\, \frac{e^2 E_o^{\,2}}{2m^2c^4}\, \frac{d}{d\gamma_i} \left| \int_{-L/2}^{L/2} \frac{dz}{\beta_\parallel^o(z)}\, \beta_x^o(z)\, e^{i(\omega t^o - kz)} \right|^2 \tag{42}$$

which, with the substitution $dt = dz/\beta_\parallel^o c$, is identical to equation (4).

3. Limits of Applicability

We have established the relationships:

$$\begin{aligned} \langle(\gamma_f-\gamma_i)^2\rangle &\approx 2\pi^2\, \frac{E_o^{\,2} T}{m^2 c^3 \omega^2}\, \frac{dP(\omega)}{d\Omega} \\ \langle(\gamma_f-\gamma_i)\rangle &\approx \frac{1}{2}\, \frac{d}{d\gamma_i}\, \langle(\gamma_f-\gamma_i)^2\rangle \end{aligned} \tag{43}$$

assuming: (1) a(n) (anti)symmetric periodic magnet with no horizontal or vertical gradients, (2) an electron beam with a transverse canonical momentum equal to zero, and (3) a "weak", linearly polarized radiation field. Some of these restrictions are fundamental while others were assumed merely to simplify the algebra.

Although the quantum formulation of the gain as the derivative of the spontaneous lineshape is suggestive of a more general result, the restriction to symmetric, or anti-symmetric, magnet geometries appears to be fundamental to the establishment of the relation between $\langle\gamma_f-\gamma_i\rangle$ and $\langle(\gamma_f-\gamma_i)^2\rangle$. The proof outlined above is based on the observation [equation (27)] that the radiated energy in a symmetric magnet is independent of the end of the magnet at which the interaction is started. We cannot invoke this equivalence in an asymmetric magnet.

While the assumptions that the transverse canonical momentum is zero and that the static field has no horizontal or vertical gradients are not fundamental, their use implies the existence of limits on the emittance of the electron beam at the start of the interaction. The field in a static, periodic magnet must, of necessity, have a transverse gradient to satisfy Maxwell's equations. There may be restricted regions within the field in which the gradient is zero — as noted earlier, there will be no gradients if motion is confined to the symmetry plane between the pole faces of a linearly polarized magnet array — but a real electron beam will have a finite thickness and some portion of the electron beam will pass through a region of the field in which the gradient is non-zero.

The effects of the electrons' initial transverse momentum and the transverse gradient are similar in that both result in motion at an angle to the axis. A gradient imposes a systematic long-period motion on the motion of electrons which enter the magnet off axis. The long period motion can either be oscillatory, or exponentially diverging,[9] depending on the character and magnitude of the gradient but, in all cases, an electron injected off axis will eventually move at an angle to the axis, even if its initial transverse momentum was zero. With respect to the laser interaction, the most important effect of motion at an angle to the axis is to modify the expression for the longitudinal velocity:

$$\beta_{\parallel}^{o} \rightarrow \sqrt{1 - \frac{1}{\gamma^2} - (\beta_x^{o\prime})^2}$$

$$\beta_x^{o\prime} \approx \beta_x^{o} + \theta(z) \qquad (44)$$

$$(\theta \ll 1)$$

where $\theta(z)$ is the angle between the tangent to the trajectory executing the long-period motion and the tangent to the trajectory of an electron moving through the field along the axis. If β_x^{o} is assumed to average to zero over a period short in comparison to the time scale for the long period motion caused by the gradient,

$$\langle \beta_{\parallel}^{o} \rangle \approx \sqrt{1 - \frac{1}{\gamma^2} - (\beta_x^{o})^2 - \theta^2} \quad . \qquad (45)$$

The long period motion can be neglected as long as θ^2 is small in

comparison to $(1/\gamma)^2$ and $(\beta_x^{o})^2$. The numerical limits to θ^2 will be determined by the error which can be tolerated in $t(z) = \frac{1}{c}\int_{-L/2}^{z} (dz/\beta_{\parallel}^{o})$.

The magnitude of θ will be determined by the position and transverse momentum at the start of the interaction. Clearly, the electrons must be injected into the interaction region with a transverse momentum, and consequent angular divergence, which is within the tolerable limit to θ. But the existence of a transverse gradient also establishes a limit on the transverse dimensions of the electron beam through the dependence of θ on the initial position. We are justified in neglecting the transverse gradient and transverse momentum only so long as the angular divergence and radius of the electron beam at the start of the interaction fall within these limits.

With regard to the characteristics of the optical field, a linearly polarized field was assumed for simplicity. The same theorem can be proven for circularly polarized radiation and a circularly polarized magnet for which $|\beta_x| \sim |\beta_y|$. The restriction to weak optical electric fields is a condition for the use of the power series expansion for $\gamma(z)$. Some insight into the range of optical power densities for which the approximation $\gamma(z) \approx \gamma_i + \gamma_1(z) + \gamma_2(z)$ is applicable can be gathered from an inspection of equation (17). The first and second order terms are calculated under the assumption that the phase advance $\omega\Delta t$ of the electrons due to acceleration by the optical electric field remains small throughout the interaction. In the approximation of the second .

moment $\langle(\gamma_f-\gamma_i)^2\rangle$ by $\langle\gamma_1{}^2\rangle$, this is necessary to insure that t(z), the time of arrival of an electron at z, can be approximated by $t^o(z)$, the arrival time in the absence of the field. The electric field changes in time at the rate $\frac{\partial}{\partial t}\underset{\sim}{E} \approx \omega\underset{\sim}{E}$ indicating that the phase advance $\omega(t-t_o) \equiv \omega\Delta t$ must be small compared to one if we wish to approximate E(t) by $E(t^o)$. The same condition, $\omega\Delta t \ll 1$, applies in the approximation of $\langle(\gamma_f-\gamma_i)\rangle \approx \langle\gamma_1(z)\rangle$. For $\langle(\gamma_f-\gamma_i)\rangle$, the condition is required to justify the approximation of $\underset{\sim}{E}(t)$ by $\underset{\sim}{E}(t^o) + \Delta t \cdot \frac{\partial}{\partial t}\underset{\sim}{E}(t^o)$.

Physically, we would expect that the condition $\omega\Delta t \ll 1$ would be satisfied so long as the phase advance did not exceed, say 0.1 radians, while at a phase advance $\omega\Delta t \approx \pi/2$, an account would obviously have to be taken of the higher order terms in the power series expansion. As shown by Colson[10], the phase advance of the electrons can approach $\omega\Delta t \approx 2\pi$ when a free electron laser is operated at optical power densities near saturation. The condition $\omega\Delta t \ll 1$ will therefore be satisfied only when the power density is small in comparison to the power density for saturation.

The range of applicability of the approximation is illustrated in Figures 2 and 3. Figure 2 indicates the dependence of the first moment on the optical power density for the Stanford Free Electron Laser operated at 3 microns. The first moment increases linearly with the optical power density to a power density of 10^7 watts/cm^2, where the radiated energy begins to saturate. Figure 3 shows the dependence of the first and second moments of radiated energy on electron energy as the radiated energy is raised from 10^4 to 10^7 watts/cm^2 and compares the result for $\langle\gamma_f-\gamma_i\rangle$ obtained by

integrating the full equations of motion with the result implied by the derivative relationship between the first and second moments. As can be seen from Figure 3, the derivative relationship remains valid to a power density of 10^5 watts/cm^2, approximately one hundredth of the power density for saturation.

As described in the introduction, the power density must also be sufficiently large to justify the use of the classical approximation. Considering, as an example, the Stanford FEL with a 3.2 cm magnet period, use of the classical approximation at a wavelength of 1 μ is justified only when the power density is much larger than 0.1 ω/cm^2. A power density in excess of 100 watts/cm^2 would be required at λ = 0.1 μ.

4. Discussion

The operation of a free electron laser can be characterized by first and second moments of the electrons' radiated energy. In the foregoing analysis we have shown that the first moment is approximately equal to the derivative, with respect to the initial electron energy, of the second moment. The second moment has also been related to the magnet design through the identification of the second moment with the power spectrum for spontaneous radiation.

Given these results, the prototype constant period, constant amplitude helical FEL magnet is seen to constitute a special case. The average and mean squared radiated energy for electrons moving through this structure have a magnitude and a functional dependence which is unique to the (sin x/x)^2 lineshape for spontaneous radiation emitted by electrons moving through this structure. Significantly different characteristics would be obtained if, for example, the magnet was modified to generate a gaussian or lorentzian spontaneous lineshape.

In principle, the static magnet can be designed so that the first and second moments of the radiated energy have an arbitrary functional dependence on the initial electron energy, subject only to the constraint $\langle \gamma_f-\gamma_i \rangle \approx \frac{1}{2} \frac{d}{d\gamma_i} \langle (\gamma_f-\gamma_i)^2 \rangle$. Typically, it would be desirable to get the highest gain possible out of a structure while minimizing the spread in energy of the electrons leaving the laser. In principle, the spread could be reduced nearly to zero by using a magnet in which the spontaneous power spectrum started from zero with a discontinuous slope as in Figure 4. But would such a lineshape be realizable in practice?

Several observations can be made concerning this problem. Equation (24), relating the spontaneous power spectrum to the transverse velocity, can be inverted to obtain $\beta_x(z)$ and the magnetic field once the functional dependence of $dP(\omega)/d\Omega$ on ω is known. We will typically start with the desired dependence of $dP(\omega)/d\Omega$ on γ, as in the figure. To obtain the dependence on ω, we note that the functional form of $dP(\omega)/d\Omega$ is fixed by the convolution of $[\beta_\perp{}^o(z)/\beta_\parallel{}^o(z)]$ and $\exp[i(kz-\omega t^o)]$. The phase of the exponential varies with z as:

$$\frac{\partial}{\partial z} (kz-\omega t^o) = k - \left(\frac{\omega}{c}\right) \frac{1}{\beta_\parallel{}^o}$$

$$= \frac{1}{2\gamma^2} \left(\frac{\omega}{c}\right) \left(1 + \frac{eA_{ox}}{mc^2}\right)^2 + O\left(\frac{1}{\gamma^4}\right). \qquad (46)$$

To order $(1/\gamma^4)$, the phase depends only on the ratio (ω/γ^2).

This relationship permits the variation of the phase factor with ω to be expressed in terms of its variation with γ. Using this relationship, and noting the dependence of $(\beta_x{}^o/\beta_\parallel{}^o)$ on γ derived in equation (39), the value of $dP/d\Omega$ at the frequency ω and the fixed energy γ can be expressed in terms of the value of $dP/d\Omega$ at the fixed frequency ω_o:

$$dP(\omega,\gamma)/d\Omega \approx \frac{dP\left(\omega_o, (\omega/\omega_o)^{1/2}\gamma\right)}{(\omega_o/\omega)^{1/2}} . \qquad (47)$$

If the linewidth is assumed small, the functional dependence of $dP/d\Omega$ on frequency will be essentially identical to its dependence on energy.

Given dPd/Ω and the assumption of a(n) (anti)symmetric magnet, the integral over z in equation (24) is given to within a phase factor $e^{i\psi}$ by:

$$\int_{-L/2}^{L/2} \frac{dz}{\beta_\parallel{}^o(z)} \beta_x{}^o(z)\, e^{i(kz-\omega t^o)} = e^{i\psi} \left|\frac{2\pi}{e\omega}\right| \left(cT \frac{dP(\omega)}{d\Omega}\right)^{1/2} . \qquad (48)$$

for a symmetric magnet, and by:

$$\int_{-L/2}^{L/2} \frac{dz}{\beta_\parallel{}^o(z)} \beta_x{}^o(z)\, e^{i(\omega t^o-kz)} = e^{i\psi} \left(\frac{2\pi}{e\omega}\right) \left(cT \frac{dP(\omega)}{d\Omega}\right)^{1/2} . \qquad (49)$$

for an antisymmetric magnet. Given that the static magnetic field is weak, so that $(\beta_x{}^o)^2 \ll \frac{1}{\gamma^2}$, the time $t^o(z)$ can be approximated as:

$$t^o = \frac{1}{c}\int_{-L/2}^{z} \frac{dz}{\beta_\parallel{}^o(z)}$$

$$\approx \frac{1}{c}\int_{-L/2}^{z} \frac{dz}{\sqrt{1-(\frac{1}{\gamma_i})^2}}$$

$$= \frac{(z+L/2)}{c\beta} \quad . \tag{50}$$

The left hand side of equation (47) [(48)] is then the Fourier transform of $\beta_x{}^o(z)$:

$$\frac{1}{c\beta}\int_{-L/2}^{L/2} dz\ \beta_x{}^o(z)\ e^{+ik(1-\frac{1}{\beta})z} \approx e^{i\psi}\left\{\begin{matrix}\left|\frac{2\pi}{e\omega}\right| \\ \left(\frac{2\pi}{e\omega}\right)\end{matrix}\right\}(cT\ \frac{dP(\omega)}{d\Omega})^{1/2} \tag{51}$$

and the equation can be inverted to obtain $\beta_x{}^o$:

$$\beta_x{}^o(z) \approx e^{i\psi}\ c\beta(1-\beta)\int_{\infty}^{\infty} dk \left\{\begin{matrix}\left|\frac{1}{e\omega}\right| \\ \left(\frac{1}{e\omega}\right)\end{matrix}\right\}(cT\ \frac{dP(\omega)}{d\Omega})^{1/2}\ e^{ik(1-\beta)z} \ . \tag{52}$$

$$(\omega = ck) \ .$$

Given the transverse velocity, the static magnetic field can be computed from equation (12).

The length of the required magnet will probably be the principal criteria in the determination as to whether a given lineshape can successfully be realized. There are several reasons for this. Clearly, the magnet length must be phystically compatible with the space available for the interaction. More important, the length enters in the computation of the allowable

electron beam energy spread and emittance, and the power density for saturation. In general, the restrictions on the electron beam quality become more stringent, and the saturation power density falls, as the interaction length increases. The lineshape and interaction length must be compatible with the electron beam generated for use with the laser and the anticipated power density of the stimulating radiation field.

Finally, we note that the relation $\langle(\gamma_f-\gamma_i)\rangle \approx \frac{1}{2}\ \frac{d}{d\gamma}\langle(\gamma_f-\gamma_i)^2\rangle$ and the result in equation (20) for the dependence of $(\gamma_f-\gamma_i)$ on the phase of the optical field can be used to advantage in simulations of laser operation below saturation in the storage ring and other applications. These simple analytic approximations very substantially reduce the time required to compute the effect of laser operation on the distribution of electrons moving through the laser, particularly for magnet designs other than the standard constant period helix.

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8. For a detailed description of the power series expansion of the equations of motion in a constant period helix, see W. B. Colson, Op. cit.
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Symbols:

Physical Constants

c = velocity of light = 2.998×10^{10} cm sec^{-1}

m = electron mass = 9.11×10^{-28} grams

e = electron charge (esu) = 4.8×10^{-10} stat-coulombs

$\hbar$ = Planck's constant = 1.055×10^{-27} erg-sec

Electron Coordinates:

$\underset{\sim}{r}$ = electron spatial position

$\underset{\sim}{v} = \dot{\underset{\sim}{r}}$ = electron velocity

$\underset{\sim}{\beta} = \dot{\underset{\sim}{r}}/c$ = normalized electron velocity

$\beta_{\parallel} = \underset{\sim}{\beta}\cdot\hat{z}$ = normalized longitudinal electron velocity

$\underset{\sim}{\beta}_{\perp} = \underset{\sim}{\beta} - \beta_{\parallel}\hat{z}$ = normalized transverse electron velocity

$\beta_{\parallel}^{o}(z)$ = normalized longitudinal electron velocity in absence of optical field

$\beta_{\perp}^{o}(z)$ = normalized transverse electron velocity in absence of optical field

$\underset{\sim}{p}$ = electron momentum (gm-cm-sec^{-1})

$E = \sqrt{p^2c^2+m^2c^4}$ = electron mass-energy (ergs)

$\gamma = E/mc^2$ = normalized electron mass-energy

γ_i = normalized electron energy at start of interaction

γ_f = normalized electron energy at end of interaction

$\langle(\gamma_f-\gamma_i)^n\rangle$ = nth moment of the radiated energy. The first moment is the average, or mean radiated energy. The second moment is the mean-squared radiated energy.

$\gamma_o, \gamma_1, \ldots \gamma_k, \ldots$ = the subscript k indicates the term proportional to E^k in the power series expansion of $\gamma(z)$.

$t(z)$ = time of arrival of the electron at z (sec)

$\Delta t(z)$ = change in time of arrival of the electron at z due to the optical field.

Radiation Field Coordinates

ω = optical frequency (sec^{-1})

$k = \omega/c$ = optical wave number (cm^{-1}), $\underset{\sim}{k}$ is assumed parallel to z.

$\underset{\sim}{E}$ = optical electric field (stat volts cm^{-1})

$\underset{\sim}{B}_{opt}$ = optical magnetic field (gauss)

$\underset{\sim}{A}_{opt}$ = vector potential of optical field

n = number of photons in mode interacting with the electrons

$dN = [\omega^2 d\omega d\Omega/(2\pi c)^3]\cdot V$ = number of optical modes in a volume V in a frequency range $d\omega$ and solid angle $d\Omega$.

Magnet Geometry

λ_q = magnet period (cm)

L = magnet length (magnet defined to extend between $z = \pm L/2$) (cm)

B_o = amplitude of static magnetic field (gauss)

$\underset{\sim}{A}_o = A_{ox}\hat{x}$ = vector potential of static magnetic field

Miscellaneous

$\Gamma_{spont}, \Gamma_{stim}, \Gamma_{abs}$ = quantum transition rates for spontaneous radiation, stimulated radiation, and absorption by the electron into or from the mode excited by the incident radiation.

$dP(\omega)/d\Omega$ = classical power spectrum for synchrotron radiation by an electron moving through the static magnetic field in the absence of a stimulating optical radiation field.

FIGURE CAPTIONS

1. The figure indicates the domains of integration for the double integrals over z and z' in equations (33) and (35). Note that the domain of integration for the integral over z and z' in equation (31) is the union of the domains for the integrals in equation (33) and (35).

2. The figure shows the dependence of the first moment of the radiated energy on the optical power density for electrons moving through the constant period magnet in the Stanford Free Electron Laser (FEL). The radiated energy was computed by numerical integrations of the complete equations of motion. The magnet period for the Stanford FEL is 3.2 cm and the interaction length 520 cm. A magnetic field of 2.3 kilogauss and an optical wavelength of 3.2 μ were assumed for the calculation. The dashed line indicates the mean radiated energy computed using only the second order term $\gamma_2(z)$ in the power series expansion for γ.

3. The figure shows the dependence of the first and second moments of the radiated energy on electron energy for the Stanford FEL at optical power densities of 10^4, 10^5, 10^6, and 10^7 watts/cm^2 at 3.2 microns. The solid curves were computed by integration of the full equations of motion. The dotted curves on the graphs for the first moments were computed by numerical differentiation of the matching curves for $\langle(\gamma_f-\gamma_i)^2\rangle$ according to equation (41).

4. The figure shows an idealized spontaneous lineshape for a free electron laser in which electrons entering the magnet at the energy γ^* would lose energy during the interaction without accumulating an energy spread. The second moment of the radiated energy has the same dependence on energy as the spontaneous power spectrum while the first moment, shown below the spontaneous lineshape, is proportional to the derivative of the second moment.

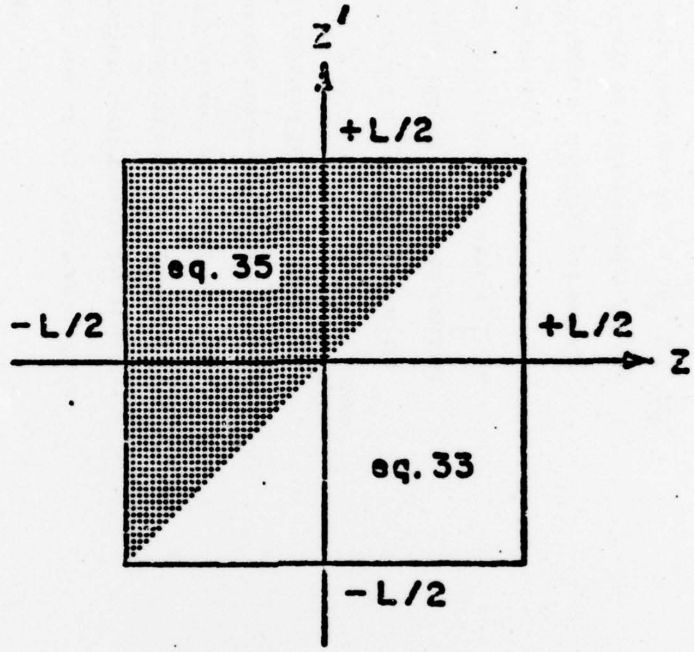
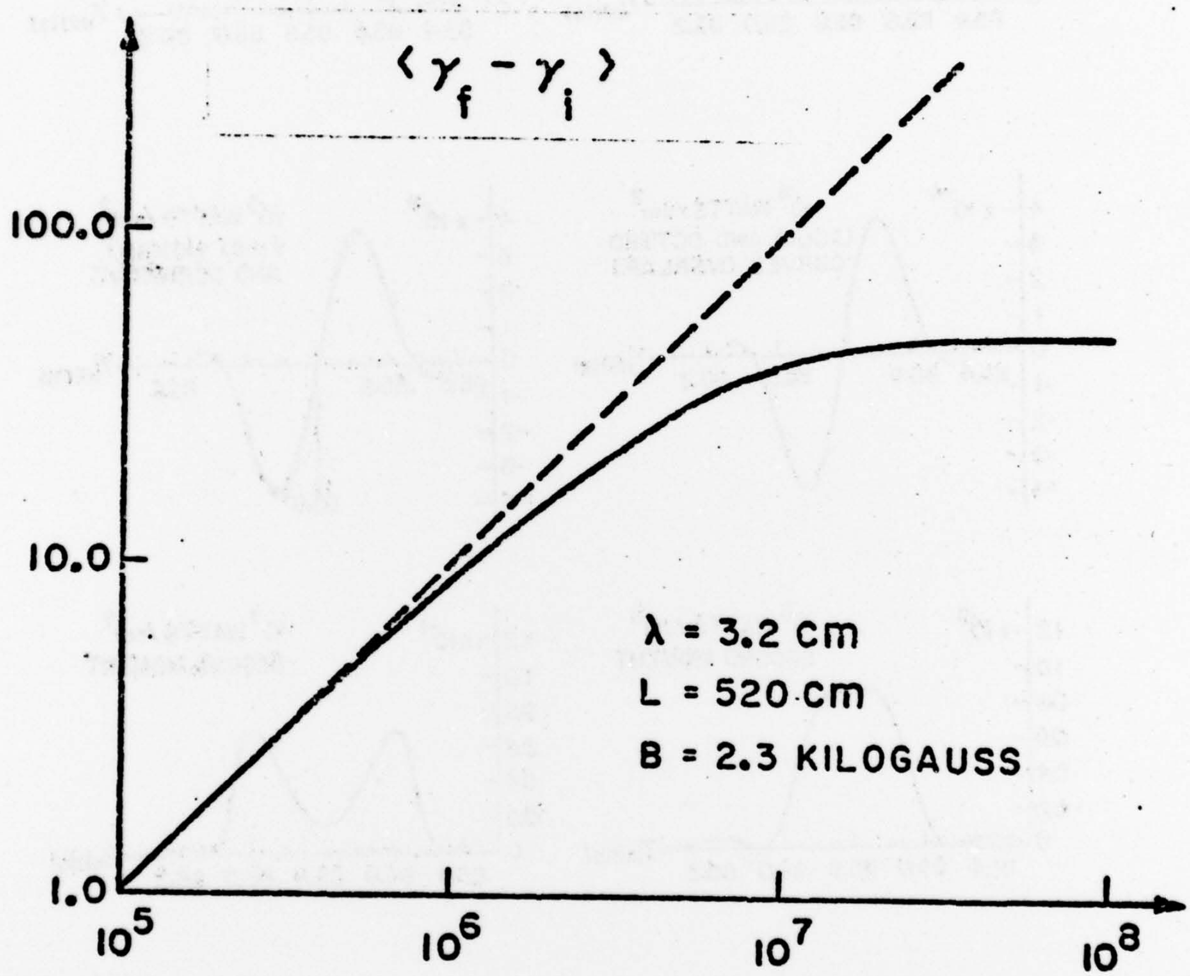


Figure 1



$\langle \gamma_f - \gamma_i \rangle$
100.0
10.0
1.0
10^5
10^6
10^7
10^8
λ = 3.2 cm
L = 520 cm
B = 2.3 KILOGAUSS

Figure 2

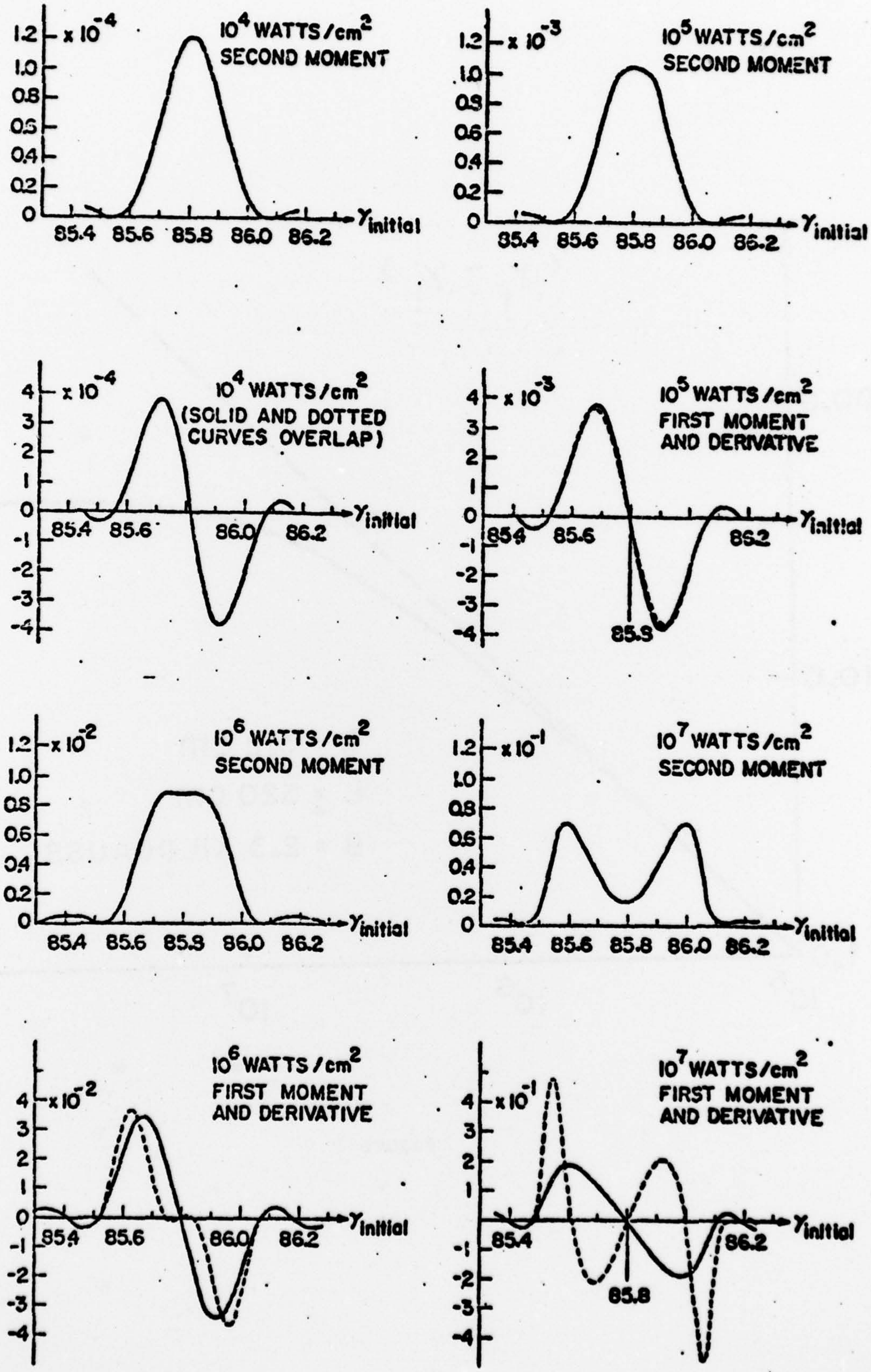
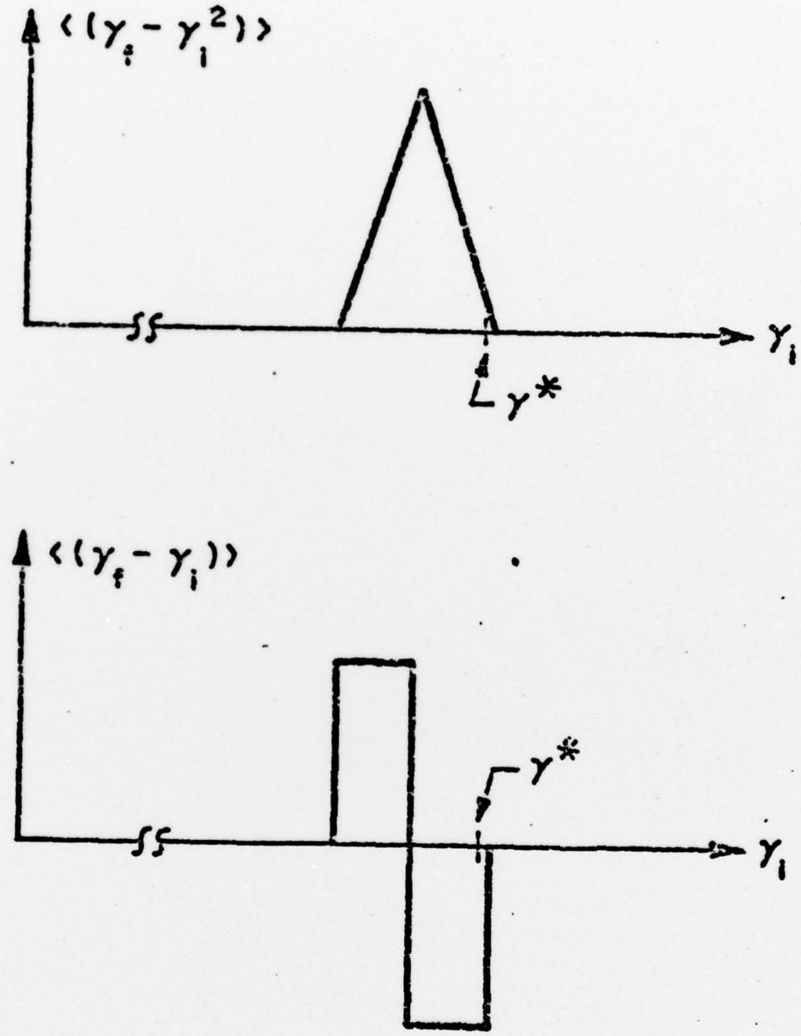

10^4 WATTS/cm^2
SECOND MOMENT
x 10^{-4}
10^5 WATTS/cm^2
SECOND MOMENT
x 10^{-3}
10^4 WATTS/cm^2
(SOLID AND DOTTED CURVES OVERLAP)
x 10^{-4}
10^5 WATTS/cm^2
FIRST MOMENT AND DERIVATIVE
x 10^{-3}
85.8
10^6 WATTS/cm^2
SECOND MOMENT
x 10^{-2}
10^7 WATTS/cm^2
SECOND MOMENT
x 10^{-1}
10^6 WATTS/cm^2
FIRST MOMENT AND DERIVATIVE
x 10^{-2}
10^7 WATTS/cm^2
FIRST MOMENT AND DERIVATIVE
x 10^{-1}
85.8
$\gamma_{initial}$
85.4
85.6
85.8
86.0
86.2

Figure 3



$\langle(\gamma_f - \gamma_i^2)\rangle$
γ_i
γ^*
$\langle(\gamma_f - \gamma_i)\rangle$
γ^*
γ_i

Figure 4