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EQUIVALENCES BETWEEN MARKOV RENEWAL PROCESSES.(U)

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EQUIVALENCES BETWEEN  
MARKOV RENEWAL PROCESSES

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$\langle Z_n, S_n \rangle$  are equivalent if and only if there is a certain homomorphism between the matrix rings generated by  $Q(t)$ ,  $t \in (0, \infty)$  and  $Y(t)$ ,  $t \in (0, \infty)$ . The equivalence is identical to weak lumpability in the case where  $\langle Z_n, S_n \rangle$  is a renewal process.

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Although the conditions for strong lumpability can be written in an attractive form, they are too restrictive to be of any real interest. Weak lumpability is of more interest since (as will be shown) it occurs in less trivial examples, but the necessary conditions are very complicated. The equivalence defined herein has the advantage of having simple necessary and sufficient conditions.

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## Abstract

We define a form of equivalence between Markov-renewal processes that includes strong and weak lumpability as special cases, and examine its properties.

If  $\{X_n, T_n\}$  is a Markov-renewal process with kernel  $Q(t)$  and  $\{Z_n, S_n\}$  is a Markov-renewal process with kernel  $Y(t)$ , then it is shown that  $\{X_n, T_n\}$  and  $\{Z_n, S_n\}$  are equivalent if and only if there is a certain homomorphism between the matrix rings generated by  $Q(t)$ ,  $t \in [0, \infty]$  and  $Y(t)$ ,  $t \in [0, \infty]$ . The equivalence is identical to weak lumpability in the case where  $\{Z_n, S_n\}$  is a renewal process.

Although the conditions for strong lumpability can be written in an attractive form, they are too restrictive to be of any real interest. Weak lumpability is of more interest since (as will be shown) it occurs in less trivial examples, but the necessary conditions are very complicated. The equivalence defined herein has the advantage of having simple necessary and sufficient conditions.

## CHAPTER 1

1. Introduction. A random process  $(X_n, T_n)$   $n = 1, 2, 3, \dots$  with  $X_n$  taking values in a finite or countable set  $S$  (called the state space), and  $T_n$  taking values in  $[0, \infty]$  is called a Markov renewal process (MRP) if

$$P(X_{n+1} = j, T_{n+1} \leq t | X_0, X_1, \dots, X_n, T_1, T_2, \dots, T_n) = P(X_{n+1} = j, T_{n+1} \leq t | X_n)$$

for all  $n \in \mathbb{Z}^+$ ,  $j \in S$ ,  $t \in [0, \infty]$ . Markov renewal processes arise naturally in queueing systems and since renewal processes and Markov chains are special cases of MRP's, a large class of problems in the study of random processes can be handled with Markov renewal theory.

Consider the departures from an M/G/1/N queue. Let  $T_n$  be the time between the  $(n-1)^{\text{st}}$  and  $n^{\text{th}}$  departure, and let  $X_n$  be the number of customers in line the instant after the  $n^{\text{th}}$  departure. It is well known [3] that  $(X_n, T_n)$  is a MRP on a state space consisting of  $\{0, 1, 2, \dots, N\}$ . Now consider the special case where  $G=M$ . Since the M/M/1 queue is an M/G/1/N queue, the departure process is a MRP with a countable state space. By [1] and [3], though, we know that in steady state the departure process from an M/M/1 queue is a Poisson process, which like any renewal process, is a one state MRP. Thus, in some sense, the infinite state MRP that represents the output from an M/M/1 queue is equivalent to a Poisson process. An enormous amount of work has been done on systems with M/M/1 queues that never would have been possible were it not known that the output from an M/M/1 queue is a Poisson process. Any time it can be shown that a MRP is "equivalent" to a renewal process, the amount of computation necessary to make statements about the process will be drastically reduced. This paper is a first step towards getting such results.

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When two random processes are called equivalent in this paper, it means that certain specific conditions (to be given later) are satisfied by the two processes. The conditions are strong enough to be of interest, and weak enough to assure that there are plenty of examples.

2. Lumpability. Probably the simplest case of equivalence between MRP's is lumpability in Markov chains [ 8 ]. Let  $\{X_n\}$  be a Markov chain on a finite or countable state space  $S$ . Let  $A_1, A_2, \dots, A_n$  be a partition of  $S$ , and let  $F: S \rightarrow \{A_1, A_2, \dots, A_n\}$  be the map that "lumps" the state space  $S$  onto the partition  $\{A_1, A_2, \dots, A_n\}$ . The process  $\{F(X_n)\}$  may or may not be a Markov chain. In general, the probability of going from  $A_i$  to  $A_j$  in  $\{F(X_n)\}$  will depend on precisely which element of  $A_i$  the  $\{X_n\}$  process is in. If for each  $i$  and  $j$ , though, the probability of going from  $A_i$  to  $A_j$  is independent of the state in  $A_i$  that the  $\{X_n\}$  process is in, then the process  $\{F(X_n)\}$  is a Markov chain. When this happens we say  $\{X_n\}$  is strongly lumpable to  $\{F(X_n)\}$ . This is a special case of the equivalence to be defined.

For example, say  $S = \{1, 2, 3\}$  and let  $\{X_n\}$  have transition probability matrix

$$\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

Let  $F(1) = A_1, F(2) = F(3) = A_2$ . The process  $\{F(X_n)\}$  is a Markov chain on  $\{A_1, A_2\}$  with transition probability matrix



$$\begin{pmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} .$$

If  $\{X_n\}$  is strongly lumpable to  $\{F(X_n)\}$  then no matter which state in  $S$  the process starts in,  $\{F(X_n)\}$  will be a Markov chain. In fact, even if the precise state that the process begins in is not known, the ensuing  $\{F(X_n)\}$  process is a Markov chain.

Sometimes, even though  $\{X_n\}$  is not strongly lumpable to  $\{F(X_n)\}$ , the process  $\{F(X_n)\}$  is a Markov chain when  $\{X_n\}$  is in steady state. When this happens we say  $\{X_n\}$  is weakly lumpable to  $\{F(X_n)\}$ .

If  $S$  is a finite set with  $m$  elements and  $F(S)$  has  $n$  elements  $A_1, A_2, \dots, A_n$  ( $n < m$ ), then the following  $m \times n$  matrix,  $U$ , can be constructed. Let

$$U_{ij} = \begin{cases} 0, & \text{if } i \notin A_j, \\ 1, & \text{if } i \in A_j. \end{cases}$$

If  $\{X_n\}$  has a steady state then there is a vector  $\Pi$  that satisfies  $\Pi P = \Pi$  where  $P$  is the transition probability matrix for  $\{X_n\}$ . Let  $\Pi$  be an  $n \times m$  matrix with

$$\Pi_{ij} = \begin{cases} 0, & \text{if } j \notin A_i, \\ \frac{\Pi_j}{\sum_{k \in A_i} \Pi_k}, & \text{if } j \in A_i. \end{cases}$$

The  $i^{\text{th}}$  row of  $\Pi$  is the conditional probability of being in state  $j$  given that the process is in steady state and that the process is in  $A_i$ .

$$\Pi = \left[ \begin{array}{c|ccc|cccc|cccc}
 & \overbrace{\hspace{12em}}^{A_1} & & & \overbrace{\hspace{12em}}^{A_2} & & \dots & & \overbrace{\hspace{12em}}^{A_n} & & & & \\
 & \frac{\Pi_1}{\sum_{k \in A_1} \Pi_k} & \frac{\Pi_2}{\sum_{k \in A_1} \Pi_k} & \dots & \frac{\Pi_{m_1}}{\sum_{k \in A_1} \Pi_k} & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
 & 0 & 0 & \dots & 0 & \frac{\Pi_{m_1+1}}{\sum_{k \in A_2} \Pi_k} & \dots & \frac{\Pi_{m_2}}{\sum_{k \in A_2} \Pi_k} & 0 & 0 & \dots & 0 & \\
 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & \frac{\Pi_{m_{n-1}+1}}{\sum_{k \in A_n} \Pi_k} & \dots & \frac{\Pi_m}{\sum_{k \in A_n} \Pi_k} & \\
 \end{array} \right]$$

$$U = \left[ \begin{array}{c|cccc}
 A_1 & \left\{ \begin{array}{l} 1 \\ 1 \\ \dots \\ \dots \\ 1 \end{array} \right\} & \begin{array}{cccc} 0 & 0 & \dots & \dots \end{array} & \begin{array}{cccc} 0 & \dots & \dots & 0 \end{array} \\
 A_2 & \left\{ \begin{array}{l} 0 \\ 0 \\ \dots \\ \dots \\ 0 \end{array} \right\} & \begin{array}{cccc} 1 & 0 & \dots & \dots \end{array} & \begin{array}{cccc} \dots & \dots & \dots & \dots \end{array} \\
 \vdots & \left\{ \begin{array}{l} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right\} & & & \\
 A_n & \left\{ \begin{array}{l} 0 \\ 0 \\ \dots \\ \dots \\ 0 \end{array} \right\} & \begin{array}{cccc} 0 & 0 & \dots & \dots \end{array} & \begin{array}{cccc} \dots & \dots & \dots & 1 \end{array} \\
 \end{array} \right]$$

Kemeny and Snell [ 8 ], show that  $\{X_n\}$  is strongly lumpable to  $\{F(X_n)\}$  if and only if  $P\bar{U} = \bar{U}(\bar{\Pi}P\bar{U})$ , and that if  $\{F(X_n)\}$  is a Markov chain then its transition probability matrix is  $\bar{\Pi}P\bar{U}$ . They also show that  $\bar{\Pi}P = (\bar{\Pi}P\bar{U})\bar{\Pi}$  or  $P\bar{U} = \bar{U}(\bar{\Pi}P\bar{U})$  is a sufficient condition for  $\{X_n\}$  to be weakly lumpable to  $\{F(X_n)\}$ . For example let  $S = \{1,2,3\}$  and set  $F(1) = A_1$ ,  $F(2) = F(3) = A_2$ . Suppose the transition probability matrix for  $\{X_n\}$  is

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix} .$$

In this case  $\bar{\Pi} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  so

$$\bar{\Pi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} \quad \text{and} \quad \bar{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} .$$

$\{X_n\}$  is not strongly lumpable to  $\{F(X_n)\}$  since  $P(F(X_n) = A_1 | F(X_{n-1}) = A_2)$  depends on whether  $X_{n-1}$  is equal to 2 or 3. This can be seen formally by noting that  $P\bar{U} \neq \bar{U}(\bar{\Pi}P\bar{U})$ . In steady state, though,  $\{F(X_n)\}$  is a Markov chain since  $\bar{\Pi}P = (\bar{\Pi}P\bar{U})\bar{\Pi}$ . The resulting Markov chain  $\{F(X_n)\}$  has a transition probability matrix

$$\bar{\Pi}P\bar{U} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix} .$$

The necessary conditions for weak lumpability are much less appealing than the necessary and sufficient condition for strong lumpability or the sufficient conditions for weak lumpability. If  $\gamma$  is a probability vector on  $S$  then define

$[\gamma]^i$  to be the vector of conditional probabilities of being in state  $j$  ( $j=1,2, \dots, m$ ), given that the process is in  $A_i$ . For example, the  $i^{\text{th}}$  row of the matrix  $\Pi$  is  $[\Pi]^i$ . Let  $\Gamma_j$  be the set of all finite sequences of states in  $F(S)$  that end with  $A_j$ . If  $A_{i_1}, A_{i_2}, \dots, A_{i_k}, A_j$  and  $A_{j_1}, A_{j_2}, \dots, A_{j_q}, A_j$  are two elements of  $\Gamma_j$  then for  $\{X_n\}$  to be weakly lumpable to  $\{F(X_n)\}$  it must be true that for each  $\alpha \in (1,2, \dots, n)$ ,

$$\sum_{i \in S} \sum_{\beta \in A_\alpha} P_{i\beta} \gamma_i^1 = \sum_{i \in S} \sum_{\beta \in A_\alpha} P_{i\beta} \gamma_i^2$$

where

$$\gamma^1 = [ \dots [ [ [ [ [ [ \Pi ]^1 P ]^2 P ]^3 P ]^4 \dots P ]^k P ]^j ]$$

and

$$\gamma^2 = [ \dots [ [ [ [ [ [ \Pi ]^j P ]^2 P ]^3 P ]^4 \dots P ]^q P ]^j ] .$$

Serfozo [10] showed that strong and weak lumpability can be defined for MRP's in an analogous manner. In fact, the conditions for strong and weak lumpability in MRP's are virtually identical to the conditions for Markov chains. If  $\{X_n, T_n\}$  is a MRP on a finite state space  $S = \{1,2,3, \dots, m\}$ , with kernel  $Q(t)$  (i.e.  $Q_{ij}(t) = P(X_{n+1} = j, T_{n+1} \leq t | X_n = i)$ ) and  $F: S \rightarrow \{A_1, A_2, \dots, A_n\}$  is a partition of the state space then  $\{X_n, T_n\}$  is said to be strongly lumpable to  $\{F(X_n), T_n\}$  if  $\{F(X_n), T_n\}$  is a MRP.

Again, let  $\Pi$  be the steady state vector for the embedded Markov chain (i.e.  $\Pi Q(\infty) = \Pi$ ), and let  $\underline{\Pi}, \underline{U}$  be defined as before. Serfozo shows that  $\{X_n, T_n\}$  is strongly lumpable to  $\{F(X_n), T_n\}$  if and only if  $Q(t)\underline{U} = \underline{U}(\Pi Q(t)\underline{U})$  for all  $t \in [0, \infty]$ . Likewise if for all  $t$ ,  $Q(t)\underline{U} = \underline{U}(\Pi Q(t)\underline{U})$  or  $\underline{\Pi} Q(t) = (\underline{\Pi} Q(t) \underline{U}) \underline{\Pi}$  then  $\{F(X_n), T_n\}$  is a MRP in steady state (i.e. weakly lumpable). Unfortunately, the necessary conditions for weak lumpability are again very complicated.



Definition 1.2. A ring,  $R$  is a collection of objects along with two operations  $+$ ,  $\cdot$  that satisfy the following properties:  $\forall a, b, c \in R$ ,

- (1)  $a + b \in R$
- (2)  $a + b = b + a$
- (3)  $(a + b) + c = a + (b + c)$
- (4)  $\exists 0 \in R$  that satisfies  $a + 0 = a$
- (5)  $\exists -a$  that satisfies  $a + (-a) = 0$
- (6)  $a \cdot b \in R$
- (7)  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- (8)  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- (9)  $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$

The operation  $\cdot$  need not be commutative in a ring. For example, the set of all  $n \times n$  matrices is a ring. In this paper we will be interested in various subsets of the set of all  $n \times n$  matrices that retain all the ring properties.

Definition 1.3. Let  $\{a_i\}$ ,  $i \in I$ , be a collection of elements of a ring,  $R$ . The ring,  $A$ , generated by  $\{a_i\}$  is the smallest subring of  $R$  that contains all of the  $\{a_i\}$ .

For example, consider the ring of integers  $Z$ . The ring generated by  $\{2\}$  is the ring of even integers. For a less trivial example consider the ring,  $N$ , of  $n \times n$  matrices. Let  $M_1, M_2, \dots, M_k$  be elements of  $N$ . A typical element of the ring generated by  $M_1, M_2, \dots, M_k$  might be  $M_1^2 M_4 + M_6^5 M_2^3 - M_k$ .

Definition 1.4. Let  $R_1$  and  $R_2$  be two rings and let  $T: R_1 \rightarrow R_2$  be a map.  $T$  is called a ring homomorphism (homomorphism) if,  $\forall a, b \in R_1$ ,

$$(1) \quad T(a \cdot b) = T(a) \cdot T(b) \quad \text{and}$$

$$(2) \quad T(a + b) = T(a) + T(b).$$

Consider the following example of a ring homomorphism. Let  $\gamma$  be some vector in  $\mathbb{R}^n$  and let  $R_1$  be the set of all  $n \times n$  matrices that have  $\gamma$  as a left eigenvector. If  $A \in R_1$ , define  $T(A) = \alpha$  where  $\alpha$  is the eigenvalue of  $A$  associated with  $\gamma$ . Thus  $T(A + B) = T(A) + T(B)$  and  $T(AB) = T(A)T(B)$  so  $T$  is a homomorphism from  $R_1$  to  $\mathbb{R}$ .

In this paper we will only consider a special class of MRP's defined as follows.

Definition 1.5. An  $m$  state MRP,  $\{X_n, T_n\}$ ,  $1 \leq m \leq \infty$ , with kernel  $Q(t)$  will be called simple if

$$(1) \quad Q_{ij}(t) \text{ is nonnegative, nondecreasing and right continuous, } \forall i, j,$$

$$(2) \quad \sum_{j=1}^m Q_{ij}(\infty) = 1, \quad \forall i,$$

$$(3) \quad Q_{ij}(t) = 0, \quad \forall t \in (-\infty, 0), \quad \forall i, j,$$

$$(4) \quad \exists! \Pi \in \mathbb{R}_+^m \text{ that satisfies}$$

$$(4a) \quad \Pi Q(\infty) = \Pi$$

$$(4b) \quad \Pi U = 1$$

$$(4c) \quad \lim_{n \rightarrow \infty} Q^n(\infty) = U \Pi$$

where  $U = (1, 1, \dots, 1)^T$ .

Conditions (1), (2), and (3) assure that  $Q(t)U$  is a column of nonnegative distribution functions. Condition (4) is equivalent to requiring that the embedded Markov chain  $\{X_n\}$  is irreducible, aperiodic and recurrent non-null. (See [2] for proof of this assertion and other similar results.)

Let  $\{X_k, T_k\}$  be an  $n$  state MRP with kernel  $Q(t)$  and say the initial distribution on the state space is  $\Pi$ . The following quantities are of interest.

- (1)  $P(T_1 \leq t | X_0 = i)$
- (2)  $P(T_1 \leq t)$
- (3)  $P(T_1 \leq t_1, T_2 \leq t_2, \dots, T_m \leq t_m)$

We can write (1) as

$$(4) \quad P(T_1 \leq t | X_0 = i) = \sum_{j=1}^n P(X_1 = j, T_1 \leq t | X_0 = i) = (Q(t)U)_i .$$

Thus the column vector  $Q(t)U$  is a vector of probabilities of a transition before time  $t$  given the initial state. To solve for (2), we weight each initial state by the initial probability distribution, so

$$(5) \quad P(T_1 \leq t) = \Pi Q(t)U.$$

The  $i^{\text{th}}$  element of the vector  $\Pi Q(t)$  is the probability that starting with the initial distribution  $\Pi$ , the process has its first transition before time  $t$ , and the transition is to state  $i$ . Likewise the  $i^{\text{th}}$  element of the vector  $\Pi Q(t_1)Q(t_2)$  is the probability that starting with the initial distribution  $\Pi$ , the process has its first transition before time  $t_1$ , its second transition in less than  $t_2$  time units after the first transition, and ends up in state  $i$ . Inductively, we obtain

$$(6) \quad P(T_1 \leq t_1, T_2 \leq t_2, \dots, T_m \leq t_m) = \Pi Q(t_1)Q(t_2) \cdots Q(t_m)U.$$



In Chapter II we define equivalence between MRP's and investigate its properties. Sections 1-4 deal with the important special case where the equivalence is between a MRP and a renewal process (a one state MRP). Section 5 deals with equivalence between finite state MRP's.

## CHAPTER II

1. Equivalence. A recurrent renewal process  $\{S_n\}$  is a sequence of independent and identically distributed nonnegative random variables with  $S_n < \infty$  with probability one. The sequence  $\{S_n\}$  can be thought of as the times between some fixed event that occurs repeatedly. Associated with each MRP,  $\{X_n, T_n\}$ , is a sequence  $\{T_n\}$ . Suppose  $\{X_n, T_n\}$  is a simple MRP on a finite state space  $S = \{1, 2, \dots, N\}$  with kernel  $Q(t)$ . If  $\gamma$  is the initial distribution on  $S$  then

$$(1) \quad P(T_1 \leq t) = \gamma Q(t)U \quad \text{where } U = (1, 1, \dots, 1)^T, \text{ and}$$

$$(2) \quad P(T_1 \leq t_1, T_2 \leq t_2, \dots, T_n \leq t_n) = \gamma Q(t_1) Q(t_2) \cdots Q(t_n)U.$$

Let  $\Pi$  be the steady state vector associated with the embedded Markov chain  $\{X_n\}$ , and define  $r(t) = \Pi Q(t)U$ .

Lemma 2.1.  $r(t)$  is the cumulative distribution of some nonnegative random variable.

Proof. From conditions (1) and (3) of Definition 1.5,  $r(t) = 0$  if  $t < 0$  and  $r(t)$  is nondecreasing and right continuous. From condition (2) we have  $r(\infty) = \Pi Q(\infty)U = \Pi U = 1$  so  $r(t)$  is a distribution function of a nonnegative random variable.  $\square$

For the remainder of this chapter all MRP's will be assumed to be simple, and all the renewal processes will be recurrent. The following theorem motivates the definition of equivalence between a MRP and a renewal process.

Theorem 2.2. If  $\{X_n, T_n\}$  is in steady state then  $\{T_n\}$  is a renewal process if and only if  $\forall m, \forall t_1, t_2, \dots, t_m, \prod Q(t_1)Q(t_2) \cdots Q(t_m)U = r(t_1)r(t_2) \cdots r(t_m)$ .

Proof. ( $\implies$ ) if  $\{T_n\}$  is a renewal process then  $P(T_1 \leq t_1, T_2 \leq t_2, \dots, T_m \leq t_m) = P(T_1 \leq t_1) P(T_1 \leq t_2) \cdots P(T_1 \leq t_m)$ . But this says

$$\prod Q(T_1) Q(t_2) \cdots Q(t_m)U = r(t_1) r(t_2) \cdots r(t_m).$$

( $\impliedby$ ) We must show that  $\{T_n\}$  is a sequence of nonnegative independent and identically distributed random variables. Since  $Q(t) \geq 0$  it is clear that  $\{T_n\}$  is nonnegative. Also  $r(t)$  is not a function of  $n$ , so it suffices to show that  $\{T_n\}$  is a sequence of independent random variables. Let  $i_1, i_2, \dots, i_n$  be any  $n$  positive integers. We must show that

$$P(T_{i_1} \leq t_{i_1}, T_{i_2} \leq t_{i_2}, \dots, T_{i_n} \leq t_{i_n}) = P(T_{i_1} \leq t_{i_1}) P(T_{i_2} \leq t_{i_2}) \cdots P(T_{i_n} \leq t_{i_n}).$$

Since

$$P(T_{i_1} \leq t_{i_1}, T_{i_2} \leq t_{i_2}, \dots, T_{i_n} \leq t_{i_n}) = P(T_1 \leq t_1, T_2 \leq t_2, \dots, T_j \leq t_j, \dots, T_{i_n} \leq t_{i_n})$$

where

$$t_j = \begin{cases} t_{i_k}, & \text{if } j = i_k \\ \infty, & \text{if } j \notin (i_1, i_2, \dots, i_n), \end{cases}$$

we have  $P(T_1 \leq t_1, T_2 \leq t_2, \dots, T_{i_n} \leq t_{i_n}) = r(t_1) r(t_2) \cdots r(t_{i_n})$ . But  $r(t_1) r(t_2) \cdots r(t_{i_n}) = r(t_{i_1}) r(t_{i_2}) \cdots r(t_{i_n})$  since  $r(\infty) = 1$ . Thus  $\{T_n\}$  is a renewal process.  $\square$

From now on the following notation will be used. The symbol  $Q$  will denote a simple MRP with kernel  $Q(t)$ . The set of matrices  $\{Q(t)\}$ ,  $t \in [0, \infty]$ , along with an initial probability distribution on the state space, describe the MRP in question, since from them it is possible to determine all transition probabilities. Thus, there is no ambiguity in using the symbol  $Q$  to denote a MRP. Likewise, the symbol  $r$  denotes the renewal process with distribution  $r(t)$  without ambiguity.

Definition 2.1. Let  $Q$  be a simple MRP with steady state vector  $\Pi$ , and let  $r$  be a renewal process. Then  $Q$  is equivalent to  $r$  ( $Q \sim r$ ) if

$$(3) \quad \forall n, t_1, t_2, \dots, t_n, \quad \Pi Q(t_1) Q(t_2) \cdots Q(t_n) U = r(t_1) r(t_2) \cdots r(t_n).$$

By theorem 2.2,  $Q$  is equivalent to a renewal process if and only if  $\{T_n\}$  is a renewal process in steady state. Thus equivalence and weak lumpability are the same thing when the lumping is to one state. From the definition it is clear that  $Q \sim r$  implies that  $r(t) = \Pi Q(t) U$ , which is the steady state distribution of the time between state transitions in  $Q$ . The reason that the steady state vector  $\Pi$  is used in the definition comes from the following theorem.

Theorem 2.3. Let  $Q$  be a simple  $m$  state MRP ( $m \leq \infty$ ). If there exists a probability vector,  $\gamma$ , and a renewal process,  $f$ , such that

$$\forall n, t_1, t_2, \dots, t_n, \quad \gamma Q(t_1) Q(t_2) \cdots Q(t_n) U = f(t_1) f(t_2) \cdots f(t_n),$$

then  $f(t) = r(t)$ ,  $\forall t$  (i.e.  $\gamma Q(t) U = \Pi Q(t) U$ ,  $\forall t$ ).

Proof. We split the proof up into two parts.

(case 1:  $m < \infty$ ). We know that

$$\lim_{n \rightarrow \infty} Q^n(\infty) = U \Pi = \begin{pmatrix} \Pi \\ \Pi \\ \vdots \\ \Pi \end{pmatrix},$$

and so  $\forall \epsilon > 0$ ,  $\exists N$  such that if  $n > N$ ,

$\max_{i,j} |Q_{ij}^{n-1}(\infty) - \pi_j| < \epsilon/m$ . Thus the  $j^{\text{th}}$  column of  $Q^{n-1}(\infty)$  is  $(a_1, a_2, \dots, a_m)^T$

where  $|a_i - \pi_j| < \epsilon/m$  for each  $i$ . Thus

$$\begin{aligned} |(\gamma Q^{n-1}(\infty) - \pi)_j| &= \left| \left( \sum_{k=1}^m \gamma_k a_k \right) - \pi_j \right| = \left| \sum_{k=1}^m \gamma_k (a_k - \pi_j) \right| \\ &\leq \sum_{k=1}^m \gamma_k |a_k - \pi_j| \leq \epsilon/m \sum_{k=1}^m \gamma_k = \epsilon/m. \end{aligned}$$

This says that each element of the vector  $\gamma Q^{n-1}(\infty)$  differs from the corresponding element of  $\pi$  by less than  $\epsilon/m$ . By hypothesis,

$\gamma Q(t_1)Q(t_2)\cdots Q(t_n)U = f(t_1)f(t_2)\cdots f(t_n)$ , so by letting  $t_1 = t_2 = \cdots = t_{n-1} = \infty$  we get  $\gamma Q^{n-1}(\infty)Q(t_n)U = f(t_n)$ . Let  $\gamma Q^{n-1}(\infty) = \pi'$ . To show that  $f(t) = r(t)$  it

suffices to show that  $|\pi'Q(t_n)U - \pi Q(t_n)U|$  is small.

$$|\pi'Q(t_n)U - \pi Q(t_n)U| = |\pi - \pi'|Q(t_n)U \leq \epsilon/m U^T Q(t_n)U \leq \epsilon/m U^T U = \epsilon.$$

Since  $\epsilon$  was arbitrary we have  $r(t) = f(t)$ .

(case 2:  $m = \infty$ ). Fix  $\epsilon$  and choose  $N$  such that  $\sum_{i=N+1}^{\infty} \gamma_i < \epsilon/7$ ,  $\sum_{i=N+1}^{\infty} \pi_i < \epsilon/7$ .

Since  $Q$  is simple we have  $\lim_{n \rightarrow \infty} Q_{ij}^n(\infty) = \pi_j \quad \forall i$ , so we can choose  $M$  such that  $n > M$

implies  $|Q_{ij}^n(\infty) - \pi_j| < \epsilon/7N$  for  $i, j \leq N$ .

We must show  $|f(t) - r(t)|$  is small,  $\forall t$ . Since  $f(t) = \gamma Q^n(\infty)Q(t)U$  for any value of  $n$ , it suffices to show that for  $n > M$ ,  $|\gamma Q^n(\infty)Q(t)U - r(t)|$  is small,  $\forall t$ . Since  $r(t) = \pi Q(t)U$ ,

$$\begin{aligned} |\gamma Q^n(\infty)Q(t)U - r(t)| &\leq |\gamma Q^n(\infty) - \pi|Q(t)U \leq \sum_{j=1}^{\infty} |\gamma Q^n(\infty) - \pi|_j \\ &= \sum_{j=1}^N |\gamma Q^n(\infty) - \pi|_j + \sum_{j=N+1}^{\infty} |\gamma Q^n(\infty) - \pi|_j. \end{aligned}$$

We have,

$$\sum_{j=1}^N |\gamma Q^n(\infty) - \pi|_j \leq \sum_{j=1}^N \sum_{i=1}^{\infty} \gamma_i |Q_{ij}^n(\infty) - \pi_j| = \sum_{i=1}^{\infty} \gamma_i \sum_{j=1}^N |Q_{ij}^n(\infty) - \pi_j|$$

$$\begin{aligned}
&= \sum_{i=1}^N \gamma_i \sum_{j=1}^N |Q_{ij}^n(\infty) - \pi_j| + \sum_{i=N+1}^{\infty} \gamma_i \sum_{j=1}^N |Q_{ij}^n(\infty) - \pi_j| \\
&\leq \sum_{i=1}^N \gamma_i \sum_{j=1}^N \varepsilon/7N + \sum_{i=N+1}^{\infty} \gamma_i \left( \sum_{j=1}^N Q_{ij}^n(\infty) + \sum_{j=1}^N \pi_j \right) \leq \varepsilon/7 + 2\varepsilon/7 = 3\varepsilon/7.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{j=N+1}^{\infty} |\gamma Q^n(\infty) - \pi|_j &\leq \sum_{j=N+1}^{\infty} (\gamma Q^n(\infty))_j + \sum_{j=N+1}^{\infty} \pi_j = 1 - \sum_{j=1}^N (\gamma Q^n(\infty))_j + \sum_{j=N+1}^{\infty} \pi_j \\
&\leq 1 + \varepsilon/7 - \sum_{j=1}^N \sum_{i=1}^{\infty} \gamma_i Q_{ij}^n(\infty) \leq 1 + \varepsilon/7 - \sum_{j=1}^N \sum_{i=1}^N \gamma_i Q_{ij}^n(\infty) \\
&\leq 1 + \varepsilon/7 - \sum_{j=1}^N \sum_{i=1}^N \gamma_i (\pi_j - \varepsilon/7N) = 1 + \varepsilon/7 - \sum_{i=1}^N \gamma_i \sum_{j=1}^N (\pi_j - \varepsilon/7N) \\
&\leq 1 + \varepsilon/7 - (1 - \varepsilon/7)(1 - \varepsilon/7 - \varepsilon/7) = 1 + \varepsilon/7 - 1 + 3\varepsilon/7 - 2\varepsilon^2/49 \leq 4\varepsilon/7.
\end{aligned}$$

$$\text{Thus } |f(t) - r(t)| \leq \sum_{j=1}^{\infty} |\gamma Q^n(\infty) - \pi|_j \leq 3\varepsilon/7 + 4\varepsilon/7 = \varepsilon. \quad \square$$

2. Conditions for Equivalence. We are now ready to find the conditions for  $Q$ -r.

Theorem 2.4. Let  $Q$  be a MRP with steady state vector  $\pi$ , and let  $r$  be a renewal process. If  $\forall t, \pi Q(t) = r(t)\pi$  then  $Q$ -r.

Proof. If  $\pi Q(t) = r(t)\pi$  then  $\forall n, t_1, t_2, \dots, t_n,$

$$\begin{aligned}
\pi Q(t_1) Q(t_2) \cdots Q(t_n) U &= r(t_1) \pi Q(t_2) \cdots Q(t_n) U \\
&\vdots \\
&= r(t_1) r(t_2) \cdots r(t_n) \pi U = r(t_1) r(t_2) \cdots r(t_n). \quad \square
\end{aligned}$$

Theorem 2.5. If  $\forall t, Q(t)U = Ur(t)$  then  $Q-r$ .

Proof. If  $Q(t)U = Ur(t)$  then  $\forall n, t_1, t_2, \dots, t_n,$

$$\begin{aligned} \Pi Q(t_1) Q(t_2) \cdots Q(t_n) U &= \Pi Q(t_1) Q(t_2) \cdots Q(t_{n-1}) U r(t_n) \\ &\quad \vdots \\ &= \Pi U r(t_1) r(t_2) \cdots r(t_n) = r(t_1) r(t_2) \cdots r(t_n). \quad \square \end{aligned}$$

Theorems 2.4 and 2.5 are special cases of the sufficient conditions for weak lumpability that Serfozo gives in [10]. Theorem 2.4 says that if the steady state vector,  $\Pi$ , is a left eigenvector of  $Q(t)$  for every  $t$  then  $Q-r$ , where  $r(t)$  is the eigenvalue of  $Q(t)$  corresponding to the eigenvector  $\Pi$ . Notice that in theorem 2.5 it was not important that the starting vector was  $\Pi$ . Any vector that satisfied  $\gamma U = 1$  would have worked. This is because  $Q(t)U = Ur(t)$  is a necessary and sufficient condition for strong lumpability of  $Q$  to  $r$ . Theorem 2.5 says that if the row sums of the matrix  $Q(t)$  are the same for all  $t$  then  $Q-r$  where  $r(t)$  is the common value of the row sums of  $Q(t)$ . If the row sums of  $Q(t)$  are the same, then no matter which state the process is in, the time until the next transition has the same distribution. Thus, knowing the state that the process is in gives no extra information about the time until the next transition. It is clear that in such cases the times between state transitions is a renewal process. The intuitive justification for theorem 2.4 is less obvious, but most interesting cases of equivalence seem to be of that type. We will see later that Burke's theorem is a simple corollary of theorem 2.4.

Let  $Q$  be a MRP with  $n$  states ( $1 < n \leq \infty$ ), and steady state vector  $\Pi$ . We define the following subsets of  $R^n$ .

Definitions 2.2.

- (A) Let  $S = \{v \in \mathbb{R}^n: vQ(t)U = (vU)r(t), \forall t\}$  where  $r(t) = \Pi Q(t)U$ .
- (B) Let  $V$  be the largest subset of  $S$  that is invariant under multiplication by  $Q(t)$  (i.e.  $v \in V \implies \forall t, vQ(t) \in V$ ).
- (C) Let  $P = \{v \in \mathbb{R}^n: v \geq 0, vU = 1\}$ .  $P$  is the set of probability vectors.
- (D) Let  $K = V \cap P$ .

Lemma 2.6.  $S$  and  $V$  are subspaces of  $\mathbb{R}^n$ .  $K$  is a compact and convex set if  $n < \infty$ .

Proof. Clearly  $\{\Pi\} \in S$  and  $\{0\} \in S$ , so  $S$  has at least two elements. Say  $\gamma_1, \gamma_2 \in S$ . Then  $\gamma_1 Q(t)U = r(t)(\gamma_1 U)$  and  $\gamma_2 Q(t)U = r(t)(\gamma_2 U)$ . Thus  $(a\gamma_1 + b\gamma_2)Q(t)U = r(t)(a\gamma_1 + b\gamma_2)U$  so  $a\gamma_1 + b\gamma_2 \in S$ . Let  $W$  be any invariant subset of  $S$  and let  $W^*$  be the subspace generated by  $W$ . Say  $w \in W^*$ .

$$w = \sum_{i=1}^k a_i w_i \text{ where } w_i \in W \text{ } i=1, 2, \dots, k. \text{ But } wQ(t) = \sum_{i=1}^k a_i w_i Q(t) = \sum_{i=1}^k a_i w'_i$$

where  $w'_i \in W$  since  $W$  is invariant. Thus  $wQ(t) \in W^*$ , so  $W^*$  is invariant. Since  $V$  is defined to be the largest invariant set, it must be a subspace.

The set  $P$  of probability vectors in  $\mathbb{R}^n$  is closed convex and bounded. So is  $P \cap V$  since  $V$  is a subspace, thus  $K$  is compact and convex.  $\square$

Consider the column vector  $Q(t)U$ . If  $v \in S$  then  $\frac{v}{(vU)} Q(t)U = \Pi Q(t)U, \forall t$ .

This says that  $(\frac{v}{(vU)} - \Pi) Q(t)U = 0, \forall t$ , which says that the vector  $\frac{v}{(vU)} - \Pi$  is orthogonal to the vector  $Q(t)U, \forall t$ . Thus we have the following important lemma.

Lemma 2.7. If  $Q$  is an  $n$  state MRP and there exists times  $t_1, t_2, \dots, t_n$  such that  $\{Q(t_1)U, Q(t_2)U, \dots, Q(t_n)U\}$  is a linearly independent set then  $S$  is the subspace of  $\mathbb{R}^n$  generated by  $\{\Pi\}$ .

Proof. If  $v \in S$  then  $(\frac{v}{(vU)} - \Pi)$  must be orthogonal to each  $Q(t)U$ . But only the zero vector can be orthogonal to  $n$  independent vectors in  $\mathbb{R}^n$ . Thus the only elements of  $S$  are the points on the line through  $\Pi$  and the origin.  $\square$

We now show the importance of the sets  $V$  and  $K$ .

Theorem 2.8.  $Q\text{-}r \iff \Pi \in V$ .

Proof. ( $\implies$ ) If  $Q\text{-}r$  then  $\forall n, t_1, t_2, \dots, t_n, \Pi Q(t_1) Q(t_2) \cdots Q(t_n)U = r(t_1) r(t_2) \cdots r(t_n)$ . Say  $\Pi Q(t_1) Q(t_2) \cdots Q(t_n) \notin S$ . Then  $\exists t$  such that  $\Pi Q(t_1) Q(t_2) \cdots Q(t_n) Q(t)U \neq (\Pi Q(t_1) \cdots Q(t_n)U) r(t)$ . But this says that  $Q\text{-}r$ . Thus  $\forall n, t_1, t_2, \dots, t_n$  we have  $\Pi Q(t_1) Q(t_2) \cdots Q(t_n) \in S$ . Let  $W = \{w: w = \Pi Q(t_1) Q(t_2) \cdots Q(t_n) \text{ for some } n, t_1, t_2, \dots, t_n\}$ .  $W$  must be a subset of  $S$ , and clearly  $W$  is invariant under multiplication by  $Q(t)$ . Thus  $W \subseteq V$ . But  $\Pi \in W$  since  $\Pi Q(\infty) = \Pi$ , so  $\Pi \in V$ .

( $\impliedby$ ) If  $\Pi \in V$  then  $\Pi \in K$ . Let  $v \in K$ . Since each element of  $Q(t)$  is nonnegative,  $v Q(t) \geq 0$ . Also  $v Q(t) = v'$  where  $v' \in V$ . Finally,  $v Q(t)U = r(t)$  since  $v \in K$  so  $v'U = r(t)$ . This implies that  $v(\frac{Q(t)}{r(t)}) \in K$ , so  $K$  is invariant under multiplication by  $Q(t)/r(t)$ . Thus  $\Pi Q(t_1) Q(t_2) \cdots Q(t_n)U = r(t_1) r(t_2) \cdots r(t_n)$ .  $\square$

Corollary 2.8.1.  $Q\text{-}r \iff \Pi \in K$ .

Proof. If  $\Pi \in K$  then  $\Pi \in V$  so  $Q\text{-}r$ . If  $Q\text{-}r$  then  $\Pi \in V$ . But  $\Pi U = 1$  so  $\Pi \in K$ .  $\square$

Theorems 2.4 and 2.5 were sufficient conditions for  $Q\text{-}r$  which are relatively easy to use in practice. Theorem 2.8 gives a necessary and



sufficient condition but there is not yet any simple way of determining what  $V$  is. The following theorem is a necessary condition for  $Q-r$  which is also useful in practice.

Theorem 2.9. If  $Q$  is a finite state MRP then  $Q-r \Rightarrow r(t)$  is an eigenvalue of  $Q(t)$ ,  $\forall t$ .

Proof. If  $Q-r$  then  $\Pi \in K$ , so  $K$  is not empty. We have shown that if  $Q$  is finite dimensional then  $K$  is compact and convex and invariant under multiplication by  $Q(t)/r(t)$ . Thus by the Brouwer Fixed Point Theorem [9],  $\forall t, \exists \gamma_t$  such that  $\gamma_t \in K$  and  $\gamma_t(Q(t)/r(t)) = \gamma_t$ . This says  $\gamma_t Q(t) = r(t)\gamma_t$  so  $r(t)$  is an eigenvalue of  $Q(t)$ .  $\square$

Corollary 2.9.1.  $Q-r \Rightarrow \det(Q(t) - r(t)I) = 0, \forall t$ .

Proof. This is just a restatement of the theorem.  $\square$

Although it is unrealistic to check to see whether  $r(t)$  is an eigenvalue of each  $Q(t)$ , theorem 2.9 says that one can show that  $Q-r$  by merely finding a value of  $t$  where  $r(t)$  is not an eigenvalue. The following theorem is useful in the same way.

Theorem 2.10. If  $Q$  is an  $n$  state MRP ( $n < \infty$ ) and  $\exists t_1, t_2, \dots, t_n$  such that  $\{Q(t_1)U, Q(t_2)U, \dots, Q(t_n)U\}$  is a linearly independent set then

$$Q-r \iff \Pi Q(t) = r(t)\Pi, \forall t.$$

Proof. By lemma 2.7,  $S$  is a one dimensional subspace, so either  $V = S$  or  $V = \{0\}$ . By theorem 2.8,  $Q-r \implies \Pi \in V$  so  $K$  must consist of the single vector  $\{\Pi\}$ . Since  $K$  is invariant under multiplication by  $Q(t)/r(t)$  we have  $\Pi Q(t) = r(t)\Pi$ . The converse is a restatement of theorem 2.4.  $\square$

Using the results so far, a rough algorithm for determining whether a MRP is equivalent (or weakly lumpable) to a renewal process can be formulated. First of all, if a MRP,  $Q$ , is equivalent to a renewal process, the renewal process it is equivalent to must be  $r$  where  $r(t) = \Pi Q(t)U$ , and  $\Pi$  is the steady state vector of the embedded Markov chain  $Q(\infty)$ . The first step should be to try to show that  $Q \neq r$  since in general that will be the case. If  $Q$  is an  $n$  state MRP choose  $t_1, t_2, \dots, t_n$  randomly or otherwise and compute  $\{Q(t_1)U, \dots, Q(t_n)U\}$ . If  $\{Q(t_1)U, \dots, Q(t_n)U\}$  is a linearly independent set (as it will be in general), then  $Q \neq r$  if and only if  $\Pi$  is a left eigenvector of each  $Q(t)$ . Choose some  $t$  and perform the multiplication  $\Pi Q(t)$ . If this product is not  $r(t)\Pi$  then  $Q \neq r$ . If  $\Pi Q(t) = r(t)\Pi$  the chances are it was not a coincidence. See if  $\Pi$  is indeed a left eigenvector of each  $Q(t)$  by writing out explicitly  $\Pi Q(t)$  as a function of  $t$ . If  $\Pi$  is a left eigenvector of each  $Q(t)$  then  $Q = r$ .

If  $\{Q(t_1)U, \dots, Q(t_n)U\}$  was not a linearly independent set, it becomes more complicated. If  $\dim\{Q(t_1)U, \dots, Q(t_n)U\} = 1$  check to see if  $U$  is a right eigenvector of each  $Q(t)$  by summing the elements of the rows of  $Q(t)$ . If the row sums of  $Q(t)$  are the same for all  $t$  then  $Q$  is strongly lumpable to  $r$  which implies equivalence.

Another way of trying to quickly show that  $Q \neq r$  is as follows. Choose  $t_1$  and  $t_2$  and perform the multiplication  $(\Pi - \frac{\Pi Q(t_1)}{r(t_1)})Q(t_2)U$ . If  $\Pi \in K$  then so is  $\Pi Q(t_1)/r(t_1)$ . But for any  $v_1$  and  $v_2$  in  $K$ ,  $(v_1 - v_2)Q(t)U = 0 \quad \forall t$ . Thus if the result of the multiplication is not zero then  $Q \neq r$ .

If no conclusion has been reached yet then either try different values of time or attempt to find the subspaces  $S$  and  $V$ . Finding  $S$  is easy, finding  $V$  is much more difficult. The next section gives a more abstract description of  $V$  and  $K$ .

3. Equivalence as a Homomorphism. Let  $Q$  be an  $n$  state MRP ( $1 < n \leq \infty$ ) with steady state vector  $\Pi$ . Associated with each  $t \in [0, \infty]$  is a matrix  $Q(t)$ . Let  $\mathcal{Q}$  be the ring of matrices generated by  $\{Q(t)\}, t \in [0, \infty]$ . For each probability vector  $\gamma$  in  $\mathbb{R}^n$  we have a map  $F_\gamma: \mathcal{Q} \rightarrow \mathbb{R}$  where

$$(4) \quad F_\gamma(A) = \gamma AU.$$

A necessary and sufficient condition for equivalence can now be written in a very simple form.

Theorem 2.11.  $Q\text{-}r \iff F_\Pi$  is a homomorphism.

Proof. ( $\implies$ ) If  $Q\text{-}r$  then  $\forall n, t_1, t_2, \dots, t_n, \Pi Q(t_1) Q(t_2) \cdots Q(t_n) U = (\Pi Q(t_1) U) (\Pi Q(t_2) U) \cdots (\Pi Q(t_n) U)$ . Also if  $A_1, A_2 \in \mathcal{Q}$  then  $\Pi(A_1 + A_2)U = \Pi A_1 U + \Pi A_2 U$

so  $F_\Pi$  is a homomorphism.

( $\impliedby$ ) If  $F_\Pi$  is a homomorphism then  $\forall n, t_1, t_2, \dots, t_n,$

$$\Pi Q(t_1) Q(t_2) \cdots Q(t_n) U = (\Pi Q(t_1) U) (\Pi Q(t_2) U) \cdots (\Pi Q(t_n) U) = r(t_1) r(t_2) \cdots r(t_n),$$

so  $Q\text{-}r$ .  $\square$

Theorem 2.11 says that the question of whether or not a MRP is equivalent to a renewal process is identical to the question of whether a certain map from a matrix ring to the real line is a homomorphism. If there were a good way of deciding when there are homomorphisms between matrix rings and the reals the analysis of equivalence between simple MRP's and renewal processes would be complete. This is certainly a deep question that will demand more research.

The maps  $\{F_\gamma\}$  also give an alternate characterization of the sets  $K$  and  $V$ .

Theorem 2.12.  $\gamma \in K \iff F_\gamma$  is a homomorphism.

Proof. ( $\implies$ ) If  $\gamma \in K$  then  $\Pi \in K$ , so  $Q$ -r. Thus  $F_\Pi$  is a homomorphism. But  $K \subset S$ , and from the definition of  $S$ ,  $F_\Pi = F_\gamma$ .

( $\impliedby$ ) Let  $T = \{v: vQ(t)U = (vU)\gamma Q(t)U\}$ , and let

$W = \{w: w = \gamma Q(t_1)Q(t_2)\cdots Q(t_n) \text{ for some } n, t_1, t_2, \dots, t_n\}$ .

$W$  is a subset of  $T$ ; for if not, there would be some  $t_1, t_2, \dots, t_n$  such that  $\gamma Q(t_1)Q(t_2)\cdots Q(t_n) \notin T$ . But this would imply that for some  $t$ ,  $\gamma Q(t_1)Q(t_2)\cdots Q(t_n)Q(t)U \neq (\gamma Q(t_1)\cdots Q(t_n)U)(\gamma Q(t)U)$  which says that  $F_\gamma$  is not a homomorphism. The steady state vector must be in the closure of  $W$  since  $\Pi = \lim_{n \rightarrow \infty} \gamma Q^n(\infty)$ , thus  $\forall \epsilon > 0, \exists w \in W$  that satisfies  $|w - \Pi|U < \epsilon$ .

Also  $w$  can be chosen so that  $wU = 1$  since  $\gamma Q^n(\infty)U = 1, \forall n$ . Thus

$$|\Pi Q(t)U - \gamma Q(t)U| = |\Pi Q(t)U - wQ(t)U| = |\Pi - w|Q(t)U \leq |\Pi - w|U < \epsilon.$$

Since  $\epsilon$  was arbitrary,  $\Pi Q(t)U = \gamma Q(t)U$  so  $\Pi \in T$ . But if  $\Pi \in T$  then  $T = S$ , so  $\gamma \in K$ .  $\square$

If we allow  $F_\gamma$  to be defined for any vector  $\gamma$  in  $\mathbb{R}^n$ , then by a similar argument it can be shown that  $V$  is the set of all vectors,  $v$ , that make  $F_v$  a homomorphism.

#### 4. Examples.

EXAMPLE 1. Disney, Farrell and DeMoraes [ 3 ] show that the output of an M/D/1/1 queue is a renewal process. The results obtained thus far allow for a quick verification of this fact.

The output process from an M/D/1/1 queue is a two state MRP with kernel

$$Q(t) = \begin{pmatrix} Q_{00}(t) & Q_{01}(t) \\ Q_{10}(t) & Q_{11}(t) \end{pmatrix},$$

where  $Q_{ij}(t)$  is the probability that

- (A) a customer departs at time zero leaving  $i$  customers in the queue, and
- (B) the next departure is before time  $t$ , and when that customer leaves there are  $j$  customers left in the queue.

If arrivals are Poisson with rate  $\lambda$ , and the service times are deterministic with rate  $d$ , then

$$Q_{00}(t) = \begin{cases} 0, & \text{if } t < d, \\ \int_0^{t-d} \lambda e^{-\lambda s} e^{-\lambda d} ds = e^{-\lambda d} - e^{-\lambda t}, & \text{if } t \geq d. \\ 0 \end{cases}$$

$$Q_{01}(t) = \begin{cases} 0, & \text{if } t < d, \\ \int_0^{t-d} \lambda e^{-\lambda s} (1 - e^{-\lambda d}) ds = (1 - e^{-\lambda d})(1 - e^{-\lambda(t-d)}), & \text{if } t \geq d. \\ 0 \end{cases}$$

$$Q_{10}(t) = \begin{cases} 0, & \text{if } t < d, \\ e^{-\lambda d} & \text{if } t \geq d. \end{cases}$$

$$Q_{11}(t) = \begin{cases} 0, & \text{if } t < d, \\ 1 - e^{-\lambda d}, & \text{if } t \geq d. \end{cases}$$

Thus,

$$Q(t) = \begin{pmatrix} e^{-\lambda d} (1 - e^{-\lambda(t-d)}) & (1 - e^{-\lambda d})(1 - e^{-\lambda(t-d)}) \\ e^{-\lambda d} & 1 - e^{-\lambda d} \end{pmatrix} \mathbf{1}_{(d, \infty)}(t)$$

where

$$\mathbf{1}_{(d, \infty)}(t) = \begin{cases} 0, & \text{if } t < d, \\ 1, & \text{if } t \geq d. \end{cases}$$

The embedded Markov chain is

$$Q(\infty) = \begin{pmatrix} e^{-\lambda d} & 1 - e^{-\lambda d} \\ e^{-\lambda d} & 1 - e^{-\lambda d} \end{pmatrix},$$

so the steady state vector,  $\Pi = (e^{-\lambda d}, 1 - e^{-\lambda d})$ . Performing the multiplication  $\Pi Q(t)$ , we get

$$\Pi Q(t) = ((1 - e^{-\lambda t}) e^{-\lambda d}, (1 - e^{-\lambda t})(1 - e^{-\lambda d})) \mathbf{1}_{(d, \infty)}(t) = (1 - e^{-\lambda t}) \mathbf{1}_{(d, \infty)}(t) \Pi.$$

By theorem 2.4 we know that  $Q$ -r where  $r(t) = (1 - e^{-\lambda t}) \mathbf{1}_{(d, \infty)}(t)$ .

EXAMPLE 2. Burke's Theorem [1] implies that the output from a steady state M/M/1 queue is a Poisson process. The output process is a MRP with kernel  $Q(t)$  where  $Q_{ij}(t)$  is defined exactly as in the first example, except that in this case  $i$  and  $j$  range over all the nonnegative integers. If the service rate is  $\mu$ , and the arrival rate is  $\lambda$  then  $Q(t)$  has the form

$$\begin{pmatrix}
 q_0(t) & q_1(t) & q_2(t) & q_3(t) & q_4(t) & \cdots \\
 f_0(t) & f_1(t) & f_2(t) & f_3(t) & f_4(t) & \cdots \\
 0 & f_0(t) & f_1(t) & f_2(t) & f_3(t) & \cdots \\
 0 & 0 & f_0(t) & f_1(t) & f_2(t) & \cdots \\
 0 & 0 & 0 & f_0(t) & f_1(t) & \cdots \\
 0 & 0 & 0 & 0 & f_0(t) & \cdots \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
 \end{pmatrix}$$

where

$$f_j(t) = \int_0^t \frac{(\lambda s)^j}{j!} e^{-\lambda s} \mu e^{-\mu s} ds = \frac{\lambda^j \mu}{(\lambda + \mu)^{j+1}} - \mu \lambda^j e^{-(\lambda + \mu)t} \sum_{k=0}^j \frac{t^{j-k}}{(j-k)! (\lambda + \mu)^{k+1}},$$

$$q_j(t) = \int_0^t \lambda e^{-\lambda s} f_j(t-s) ds$$

$$= \frac{\lambda^j \mu}{(\lambda + \mu)^{j+1}} (1 - e^{-\lambda t}) - \lambda^{j+1} e^{-\lambda t} \sum_{k=0}^j \frac{1}{(\lambda + \mu)^{k+1} \mu^{j-k}}$$

$$+ \lambda^{j+1} \sum_{k=0}^j \sum_{p=0}^{j-k} \frac{e^{-(\lambda + \mu)t} t^{j-k-p}}{(j-k-p)! (\lambda + \mu)^{k+1} \mu^p}.$$

Because of the special structure of  $Q(\infty)$ , it can be shown that the steady state vector,  $\Pi = (1 - \rho)^{-1} (1, \rho, \rho^2, \rho^3, \dots)$  where  $\rho = \lambda/\mu$ . Performing the multiplication  $\Pi Q(t)$  we get

$$\begin{aligned}
 (\Pi Q(t))_j &= (1-\rho)^{-1} (q_j(t) + \sum_{k=0}^j f_{j-k}(t) \rho^{k+1}) \\
 &= (1-e^{-\lambda t})(1-\rho^{-1})\rho^j = (1-e^{-\lambda t})\Pi_j.
 \end{aligned}$$

This says that the steady state output is a renewal process with distribution  $(1-e^{-\lambda t})$ .

EXAMPLE 3. Now consider the M/M/1/N queue. The output from this queue is an N+1 state MRP with kernel  $Q(t) =$

$$\begin{pmatrix}
 q_0(t) & q_1(t) & q_2(t) & \cdots & q_{N-1} & \sum_{j=N}^{\infty} q_j(t) \\
 f_0(t) & f_1(t) & f_2(t) & \cdots & f_{N-1} & \sum_{j=N}^{\infty} f_j(t) \\
 0 & f_0(t) & f_1(t) & \cdots & f_{N-2} & \sum_{j=N-1}^{\infty} f_j(t) \\
 0 & 0 & f_0(t) & \cdots & f_{N-3} & \sum_{j=N-2}^{\infty} f_j(t) \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & f_0(t) & \sum_{j=1}^{\infty} f_j(t)
 \end{pmatrix},$$

where  $q_j(t)$  and  $f_j(t)$  are as in the last example. The steady state vector

$$\Pi^N = \frac{1-\rho}{1-\rho^{N+1}} (1, \rho, \rho^2, \rho^3, \dots, \rho^{N-1}, k), \text{ where } k = \rho^N + \frac{e^{-\mu t} \rho^{N+1}}{1-\rho}.$$

If the steady state output from the M/M/1/N queue is a renewal process then the renewal process would have to have distribution

$$r_N(t) = \Pi^N Q_N(t) U = 1 - \frac{e^{-\lambda t}}{1-\rho^{N+1}} + \frac{\rho^{N+1} e^{-\mu t}}{1-\rho^{N+1}}.$$



Assume  $\Pi^N \in V$ . Then  $\forall x, \Pi^N Q_N(x) \in V$ . Since  $V$  satisfies

$$v \in V \implies \left( \frac{v}{vU} - \Pi^N \right) \perp Q_N(t)U, \forall t, \text{ we must have } \left( \frac{\Pi^N Q_N(x)}{r_N(x)} - \Pi^N \right) Q_N(t)U = 0, \forall t.$$

$Q_N(t)U$  has the form  $(a, b, b, b, \dots, b)$  where  $a$  and  $b$  are positive, thus

$$\begin{aligned} \left( \frac{\Pi^N Q_N(x)}{r_N(x)} - \Pi^N \right) Q_N(t)U &= \frac{1-\rho}{1-\rho^{N+1}} \left( \frac{1-e^{-\lambda x}}{r_N(x)} - 1 \right) a + \frac{1-\rho}{1-\rho^{N+1}} \left( \frac{1-e^{-\lambda x}}{r_N(x)} - 1 \right) b \sum_{i=1}^{N-1} \rho^i \\ &+ \frac{1-\rho}{1-\rho^{N+1}} \left( \frac{1-e^{-\lambda x}}{r_N(x)} (\rho^N + e^{-\mu x} \frac{\rho^{N+1}}{1-\rho}) - \rho^N \right) b. \end{aligned}$$

Since  $r_N(x) < 1 - e^{-\lambda x}$ , each term in the expression for

$$\left( \frac{\Pi^N Q_N(x)}{r_N(x)} - \Pi^N \right) Q_N(t)U \text{ is strictly positive, so } \Pi^N \notin V. \text{ Thus by theorem 2.8,}$$

the output of an M/M/1/N queue is not equivalent to any renewal process. (See [3].)

5. Equivalence Between Finite State MRP's. Let  $Q$  be a  $k$  state MRP with steady state vector  $\Pi$ , let  $Y$  be an  $m$  state MRP ( $m < k$ ), and let  $\{A_1, A_2, \dots, A_m\}$  be a partition of the states of  $Q$ . Let  $\tilde{\Pi}$  be an  $m \times k$  matrix defined by

$$\tilde{\Pi}_{ij} = \begin{cases} 0, & \text{if } j \notin A_i, \\ \frac{\Pi_j}{\sum_{a \in A_i} \Pi_a}, & \text{if } j \in A_i, \end{cases}$$

and let  $U$  be a  $k \times m$  matrix defined by

$$U_{ij} = \begin{cases} 0, & \text{if } i \notin A_j, \\ 1, & \text{if } i \in A_j. \end{cases}$$

$\Pi Q(t)U$  is an  $m \times m$  matrix whose  $(i,j)$  element is  $P_{\Pi}(X_1 \in A_j, T_1 \leq t | X_0 \in A_i)$ . Say  $F$  is the function that maps the state space of  $Q$  to  $\{A_1, A_2, \dots, A_m\}$ . Serfozo [10] shows that if  $\{F(X_n), T_n\}$  is a MRP, its kernel is  $\Pi Q(t)U$ .

The definition of equivalence between a MRP and a renewal process has a natural generalization to equivalence between two finite state MRP's.

Definition 2.3. Let  $Q$  be a MRP with state space  $\{1, 2, \dots, k\}$  and steady state vector  $\Pi$ . Let  $Y$  be an  $m$  state MRP ( $m < k$ ). Let  $F: \{1, 2, \dots, k\} \rightarrow \{A_1, \dots, A_m\}$  be a partition of the states of  $Q$ , and define  $\Pi$  and  $U$  as above. We say  $Q$  is equivalent to  $Y$  via the partition  $F$ ,  $(Q \stackrel{F}{\sim} Y)$  if  $\forall n, t_1, \dots, t_n$ ,

$$\Pi Q(t_1) Q(t_2) \cdots Q(t_n) U = Y(t_1) Y(t_2) \cdots Y(t_n).$$

Before attempting to produce conditions for this type of equivalence we should know what this equivalence means. In the case where  $Y$  has one state (a renewal process) we showed that equivalence is the same as weak lumpability. If  $Q$  is  $\{X_n, T_n\}$  and  $Y$  is  $\{Z_n, S_n\}$  then definition 2.3 says that for each  $i, j \in \{1, 2, \dots, m\}$ , and  $\forall n, t_1, \dots, t_n$ ,

$$(1) \quad P_{\Pi}(X_n \in A_j, T_n \leq t_n, \dots, T_1 \leq t_1 | X_0 \in A_i) = P(Z_n = j, S_n \leq t_n, \dots, S_1 \leq t_1 | Z_0 = i).$$

For weak lumpability between  $Q$  and  $Y$  we would need that for each  $i, j \in \{1, 2, \dots, m\}$  and  $\forall n, t_1, t_2, \dots, t_n$ ,  $i_1, i_2, \dots, i_{n-1} \in \{1, 2, \dots, m\}$ ,

$$(2) \quad P_{\Pi}(X_n \in A_j, X_{n-1} \in A_{i_{n-1}}, \dots, X_1 \in A_{i_1}, T_n \leq t_n, \dots, T_1 \leq t_1 | X_0 \in A_i) \\ = P(Z_n = j, Z_{n-1} = i_{n-1}, \dots, Z_1 = i, S_n \leq t_n, \dots, S_1 \leq t_1 | Z_0 = i).$$

It seems inconceivable that every  $Q$  and  $Y$  that satisfy (1) would also satisfy (2) but all attempts to find a counter example have failed so far. We can show that weak lumpability implies equivalence, though.

Theorem 2.13. If  $Q$  is weakly lumpable to  $Y$  via  $F$  then  $Q \stackrel{F}{\sim} Y$ .

Proof. Let  $Q$  be  $\{X_n, T_n\}$  and  $Y$  be  $\{Z_n, S_n\}$ .

$$\begin{aligned}
 (\prod Q(t_1) \cdots Q(t_n) U)_{ij} &= P_{\Pi}(F(X_n) = j, T_n \leq t_n, \dots, T_1 \leq t_1 | F(X_0) = i) \\
 &= \sum_{i_1=1}^k \sum_{i_2=1}^k \cdots \sum_{i_{n-1}=1}^k P(F(X_n) = j, F(X_{n-1}) = i_{n-1}, \dots, F(X_1) = i_1, \\
 &\quad T_n \leq t_n, \dots, T_1 \leq t_1 | F(X_0) = i) \\
 &= \sum_{i_1=1}^k \cdots \sum_{i_{n-1}=1}^k P_{\Pi}(T_n \leq t_n, \dots, T_1 \leq t_1 | F(X_n) = j, \dots, F(X_1) = i, F(X_0) = i) \\
 &\quad \cdot P(F(X_n) = j, \dots, F(X_1) = i_1 | F(X_0) = i).
 \end{aligned}$$

Since  $Q$  is weakly lumpable to  $Y$  we know that in steady state,  $F(X_n)$  is a MRP with kernel  $Y(t)$  so

$$\begin{aligned}
 P_{\Pi}(T_n \leq t_n, \dots, T_1 \leq t_1 | F(X_n) = j, \dots, F(X_0) = i) \\
 = \frac{Y_{i, i_1}(t_1)}{Y_{i, i_1}(\infty)} \frac{Y_{i_1, i_2}(t_2)}{Y_{i_1, i_2}(\infty)} \cdots \frac{Y_{i_{n-1}, j}(t_n)}{Y_{i_{n-1}, j}(\infty)},
 \end{aligned}$$

and

$$P_{\Pi}(F(X_n) = j, \dots, F(X_1) = i_1 | F(X_0) = i) = Y_{i, i_1}(\infty) Y_{i_1, i_2}(\infty) \cdots Y_{i_{n-1}, j}(\infty).$$

Thus

$$\begin{aligned}
 (\prod Q(t_1) \cdots Q(t_n) U)_{ij} &= \sum_{i_1=1}^k \cdots \sum_{i_{n-1}=1}^k \frac{Y_{i, i_1}(t_1)}{Y_{i, i_1}(\infty)} \cdots \frac{Y_{i_{n-1}, j}(t_n)}{Y_{i_{n-1}, j}(\infty)} \\
 &= \sum_{i_1=1}^k \cdots \sum_{i_{n-1}=1}^k Y_{i, i_1}(t_1) \cdots Y_{i_{n-1}, j}(t_n) = (Y(t_1) Y(t_2) \cdots Y(t_n))_{ij}. \quad \square
 \end{aligned}$$

Since the definition of equivalence between finite state MRP's is analogous to the definition of equivalence between a MRP and a renewal process, one might assume that the conditions for equivalence would be similar. First of all, by theorem 2.13 any sufficient condition for weak lumpability will also be a sufficient condition for equivalence, thus we have

Theorem 2.14. If  $\forall t, \Pi Q(t) = Y(t)\Pi$ , then  $Q \stackrel{F}{\sim} Y$ .

Theorem 2.15. If  $\forall t, Q(t)U = UY(t)$ , then  $Q \stackrel{F}{\sim} Y$ .

Theorems 2.14 and 2.15 can be proved the same way theorems 2.4 and 2.5 were proved. They can also be found in Serfozo [11].

Assume that for each  $t$ ,  $Y(t)$  is an invertible  $m \times m$  matrix ( $m < \infty$ ), and  $Q(t)$  is a  $k \times k$  matrix ( $m < k < \infty$ ).  $Q(t)$  need not be invertible. Let  $M$  be the set of all  $m \times k$  matrices,  $M$ , with  $M_{ij} \geq 0$  and  $MU = \alpha I$  where  $\alpha$  is a scalar. In other words  $M$  must have the form

$$M = \begin{array}{|c|c|c|c|} \hline \boxed{X \cdots X} & & & \\ \hline & \boxed{X \cdots X} & \circ & \\ \hline & & \cdot & \\ \hline \circ & & \cdot & \\ \hline & & & \boxed{X \cdots X} \\ \hline \end{array}$$

and each row sum must be the same.

If  $Y(t)^{-1} \Pi Q(t) \notin M$  then it must be true that  $Y(t)^{-1} \Pi Q(t) U \neq I$  which says that  $\Pi Q(t) U \neq Y(t)$ . Thus a simple necessary condition for equivalence is that  $Y(t)^{-1} \Pi Q(t) \in M$ .

Let  $S = \{M \in M : MQ(t)U = (MU)Y(t), \forall t\}$  and let  $V$  be the largest subset of  $S$  that is invariant under multiplication by  $Y(t)^{-1}$  on the left and  $Q(t)$  on the right (i.e.  $M \in V \implies \forall t, Y(t)^{-1}MQ(t) \in V$ ). Let  $K = \{M \in V : MU = I\}$ .

These definitions are analogous to the definitions in Section 2 and not too surprisingly we have

Theorem 2.16.  $Q \stackrel{F}{\sim} Y \iff \Pi \in K.$

Proof. ( $\implies$ ) Assume  $\Pi \notin K.$  Then  $\exists t_1, t_2, \dots, t_n$  such that

$Y(t_n)^{-1} Y(t_{n-1})^{-1} \dots Y(t_1)^{-1} \Pi Q(t_1) Q(t_2) \dots Q(t_n) \notin M.$  But that says  $\exists t$  so that  $Y(t_n)^{-1} \dots Y(t_1)^{-1} \Pi Q(t_1) \dots Q(t_n) Q(t) U \neq Y(t_n)^{-1} \dots Y(t_1)^{-1} \Pi Q(t_1) \dots Q(t_n)$

But if  $Q \stackrel{F}{\sim} Y$  then  $Y(t_n)^{-1} \dots Y(t_1)^{-1} \Pi Q(t_1) \dots Q(t_n) U = I$  so

$\Pi Q(t_1) \dots Q(t_n) Q(t) U \neq Y(t_1) \dots Y(t_n) Y(t)$  which is a contradiction.

( $\impliedby$ ) If  $\Pi \in K$  then  $\forall n, t_1, t_2, \dots, t_n$   $Y(t_n)^{-1} \dots Y(t_1)^{-1} \Pi Q(t_1) \dots Q(t_n) U = I$ , so

$\Pi Q(t_1) \dots Q(t_n) U = Y(t_1) \dots Y(t_n).$   $\square$

In general  $Y(t)$  will not be invertable for all  $t.$  This poses a serious problem. In the renewal case this problem did not exist since  $r(t)$  was a scalar, not a matrix. For equivalence we need  $\Pi Q(t) = Y(t) \Pi'$  where in some sense  $\Pi'$  acts just like  $\Pi.$  Unfortunately, if  $Y(t)$  is singular we cannot solve for  $\Pi'$  uniquely, and we therefore have trouble defining the sets  $V$  and  $K.$

Another result that carries over from the renewal case rather easily is that equivalence is identical to a certain ring homomorphism.

Let  $Q$  be the ring generated by  $\{Q(t)\}, t \in [0, \infty],$  and let  $Y$  be the ring generated by  $\{Y(t)\}, t \in [0, \infty].$  Define  $\phi_{\Pi, F}: Q \rightarrow Y$  to be  $\phi_{\Pi, F}(A) = \Pi A U.$

Theorem 2.17.  $Q \stackrel{F}{\sim} Y \iff \phi_{\Pi, F}$  is a homomorphism.

The proof here is identical to the proof in the renewal case.

One result that has no counterpart in the renewal case is

Theorem 2.18. Let  $Q$ ,  $Y$  and  $D$  be  $k$ ,  $m$  and  $l$  state simple MRP's ( $k > m > l$ ).  
If  $Q \stackrel{F}{\sim} Y$  and  $Y \stackrel{G}{\sim} D$  then  $Q \stackrel{G \circ F}{\sim} D$ .

To prove this theorem we need the following lemma.

Lemma 2.19. If  $Q \stackrel{F}{\sim} Y$  and  $\Pi$  is the steady state vector for  $Q$  then the steady state vector for  $Y$  is  $\gamma$  where  $\gamma_i = \sum_{p \in F^{-1}(i)} \Pi_p$  (i.e.  $\gamma = \Pi U$ ).

Proof. Since the steady state vector is unique in the class of simple MRP's, it suffices to show that  $\gamma$  satisfies  $\gamma Y(\infty) = \gamma$ .

Since

$$\Pi_{ij} = \begin{cases} 0, & \text{if } j \notin F^{-1}(i), \\ \frac{\Pi_j}{\sum_{p \in F^{-1}(i)} \Pi_p}, & \text{if } j \in F^{-1}(i), \end{cases}$$

we have

$$\gamma Y(\infty) = \gamma \Pi Q(\infty) U = \Pi Q(\infty) U = \Pi U = \gamma. \quad \square$$

Proof of Theorem. Let  $\Pi$  be the steady state vector for  $Q$ ,  $\gamma$  the steady state vector for  $Y$  and let  $\Pi_F$  and  $\gamma_G$  be the matrices induced by the partitions  $F$  and  $G$ . We first show that  $\gamma_G \Pi_F = \Pi_{G \circ F}$  where  $\Pi_{G \circ F}$  is the  $l \times k$  matrix induced on  $\Pi$  by the partition  $G \circ F$ . First of all, if  $G \circ F(j) \neq i$  then  $(\gamma_G \Pi_F)_{ij} = 0$ . If  $G \circ F(j) = i$  then since each column of  $\gamma_G$  and  $\Pi_F$  has only one nonzero element, there is some  $b$  such that  $(\gamma_G \Pi_F)_{ij} = (\gamma_G)_{ib} (\Pi_F)_{bj}$ . From the lemma,

$$(\gamma_G)_{ib} = \frac{\gamma_k}{\sum_{p \in G^{-1}(i)} \gamma_p} = \frac{\sum_{a \in F^{-1}(b)} \Pi_a}{\sum_{p \in G^{-1}(i)} \sum_{q \in F^{-1}(p)} \Pi_q} = \frac{\sum_{a \in F^{-1}(b)} \Pi_a}{\sum_{q \in [G \circ F]^{-1}(i)} \Pi_q}$$

Thus,

$$(\gamma_{G \circ F}^{\Pi})_{ij} = \left( \frac{\sum_{a \in F^{-1}(b)} \Pi_a}{\sum_{q \in [G \circ F]^{-1}(i)} \Pi_q} \right) \left( \frac{\Pi_j}{\sum_{a \in F^{-1}(b)} \Pi_a} \right) = \frac{\Pi_j}{\sum_{q \in [G \circ F]^{-1}(i)} \Pi_q} .$$

Thus,

$$(\gamma_{G \circ F}^{\Pi})_{ij} = \begin{cases} 0, & \text{if } G \circ F(j) \neq i, \\ \frac{\Pi_j}{\sum_{q \in [G \circ F]^{-1}(i)} \Pi_q}, & \text{if } G \circ F(j) = i, \end{cases}$$

so

$$\gamma_{G \circ F}^{\Pi} = \Pi_{G \circ F} \text{ as desired.}$$

If we let  $U_F$  be the  $k \times m$  summing matrix associated with  $F$ , and  $U_G$  be the  $m \times l$  summing matrix associated with  $G$ , it is easy to show that  $U_F U_G = U_{G \circ F}$  where  $U_{G \circ F}$  is the  $k \times l$  summing matrix associated with  $G \circ F$ .

Since  $Q \stackrel{F}{\sim} Y, \forall n, t_1, t_2 \dots t_n, \Pi_F Q(t_1) \dots Q(t_n) U_F = Y(t_1) \dots Y(t_n)$ .

Also, since  $Y \stackrel{G}{\sim} D, \gamma_G Y(t_1) \dots Y(t_n) U_G = D(t_1) D(t_2) \dots D(t_n)$ . Thus

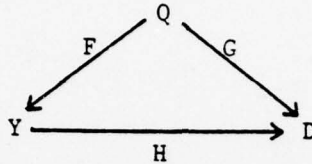
$\gamma_{G \circ F}^{\Pi} Q(t_1) \dots Q(t_n) U_F U_G = D(t_1) \dots D(t_n)$ . But this says

$\Pi_{G \circ F} Q(t_1) \dots Q(t_n) U_{G \circ F} = D(t_1) \dots D(t_n)$ , so  $Q \stackrel{G \circ F}{\sim} D$ .  $\square$

There are several open questions on the topic of equivalence between finite state MRP's. Among them are finding an example of two finite state MRP's,  $Q$  and  $Y$ , that are equivalent but not weakly lumpable; and finding conditions for equivalence when the matrices  $Y(t)$  are allowed to be singular.

If  $Q \stackrel{F}{\sim} Y$  and  $Y \stackrel{G}{\sim} D$ , then we have shown that  $Q \stackrel{G \circ F}{\sim} D$ . Now suppose  $Q \stackrel{F}{\sim} Y$  and  $Q \stackrel{G}{\sim} D$ . Under what conditions is there a partition,  $H$ , such that  $Y \stackrel{H}{\sim} D$ ?

In other words, when is there an  $H$  that makes the following diagram commute?



6. Equivalence on a Subset of  $[0, \infty]$ . We return now to equivalence between a MRP and a renewal process. Say  $Q$  is an  $n$  state MRP with steady state vector  $\Pi$ ,  $r$  is a renewal process and  $Q \sim r$ . Let  $Q'$  be another  $n$  state MRP that has the property that  $t \leq T$  implies that  $Q(t) = Q'(t)$ . If  $t_1 \leq T$   $i=1,2,\dots,k$  then  $\Pi Q(t_1) \cdots Q(t_k) U = r(t_1) \cdots r(t_k)$ . But since  $Q(t) = Q'(t)$  for  $t \leq T$  we also have  $\Pi Q'(t_1) \cdots Q'(t_k) U = r(t_1) \cdots r(t_k)$ . Although it is possible that  $Q' \sim r$  (in fact  $\Pi$  might not even be the steady state vector for  $Q'$ ) we do have a sort of equivalence between  $Q'$  and  $r$  on  $[0, T]$ .

The motivation behind this section is that in general a MRP,  $Q$ , is not equivalent to any renewal process and therefore computing probabilities of successive interdeparture times involves matrix multiplication. If there is some  $B \subset \mathbb{R}$  such that if  $t_1, t_2, \dots, t_k \in B$  then  $P(T_1 \leq t_1, \dots, T_k \leq t_k) = P(T_1 \leq t_1) P(T_1 \leq t_2) \cdots P(T_1 \leq t_k)$ , computing probabilities would only involve scalar multiplication (so long as all the times are chosen from  $B$ ). Also, it seems interesting to consider the concept of a MRP acting like different renewal processes on different subsets of  $[0, \infty]$ .



Definition 2.4. Let  $Q$  be an  $n$  state MRP and let  $r$  be a renewal process. We say  $Q \sim r$  on  $A$  if  $\exists \gamma$  such that  $\gamma \geq 0$  and  $\forall k$ , and  $\forall t_1, \dots, t_k \in A$ ,  
 $\gamma Q(t_1) \cdots Q(t_k) U = r(t_1) \cdots r(t_k)$ .

Notice that if  $Q \sim r$  on  $A$  then  $Q \sim r'$  on  $A$  for any  $r'$  that satisfies  $r'(t) = r(t), \forall t \in A$ . For an arbitrary MRP,  $Q$ , one can always construct a set  $B \subset \mathbb{R}$  such that  $Q \sim r$  on  $B$  for some  $r$  (although sometimes  $B$  will consist of only one point). Choose some  $t \in [0, \infty]$ . By the Perron-Frobenius theorem, there is a largest positive eigenvalue,  $\alpha(t) \leq 1$ , of  $Q(t)$  and an associated eigenvector  $v(t)$  that satisfies  $v(t)U = 1$ . Let  $B = \{s: v(t)Q(s) = \alpha(s)v(t)\}$ . Clearly  $Q \sim \alpha$  on  $B$ .

Definition 2.5. If  $\{A_\beta\}, \beta \in J$ , is a cover of  $[0, \infty]$  (i.e.  $\bigcup_\beta A_\beta = [0, \infty]$ ), and for each  $\beta \in J$  there is a renewal process  $r_\beta$  such that  $Q \sim r_\beta$  on  $A_\beta$  then call  $\{A_\beta\}$  a renewal-cover of  $Q$ .

Such a cover exists for any  $Q$  since if necessary there can be a different renewal process for each  $t \in [0, \infty]$ . The interesting question is whether or not one can find a finite or countable renewal-cover. If  $Q \sim r$  then there is a renewal-cover with one element (i.e.  $Q \sim r$  on  $[0, \infty]$ ).

Theorem 2.20. Say  $Q$  is a finite state MRP such that  $Q_{ij}(t)$  is a continuous function of  $t$  for each  $i$  and  $j$ . Let  $\alpha(t)$  be the largest positive eigenvalue of  $Q(t)$  and let  $v(t)$  be its associated nonnegative, normalized eigenvector. If the set  $\{v(t)\}, t \in [0, \infty]$ , is finite or countable then so is the smallest renewal-cover, and there exists an open interval  $A \subset [0, \infty]$  such that  $Q \sim \alpha$  on  $A$ .

Proof. Let  $\{v_1, v_2, \dots\}$  be a list of the elements of  $\{v(t)\}$  and let  $A_n = \{s: v_n Q(s) = \alpha(s)v_n\}$ .  $\{A_n\}_{n=1,2,\dots}$  is the desired renewal-cover. Say

$t_1, t_2, \dots \in A_n$  and  $t_i \rightarrow t$ . Since  $Q(t)$  is continuous  $\lim_{i \rightarrow \infty} v_n Q(t_i) = v_n Q(t)$ ,  
and  $\lim_{i \rightarrow \infty} v_n Q(t_i) = \lim_{i \rightarrow \infty} \alpha(t_i) v_n = \alpha(t) v_n$ . Thus  $v_n Q(t) = \alpha(t) v_n$  so  $A_n$  is  
closed. By the Baire Category Theorem one of the  $A_n$  must contain an open  
interval.  $\square$

## BIBLIOGRAPHY

- [1] P. J. Burke, (1956), "The Output of a Queueing System", Op. Res., 4, 699-714.
- [2] E. Çinlar, (1975), Introduction to Stochastic Processes, Prentice-Hall Englewood Cliffs, New Jersey.
- [3] R. L. Disney, R. L. Farrell and P. R. de Morais, (1972), "A Characterization of M/G/1/N Queues with Renewal Departure Processes", Mgmt. Sci., 19, 1222-1228.
- [4] J. L. Doob, (1953), Stochastic Processes, John Wiley & Sons, New York.
- [5] P. D. Finch, (1959), "The Output Process of the Queueing System M/G/1", J. R. Statis. Soc., B21, 375-380.
- [6] T. Harris, (1955), "The Existence of Stationary Measures for Certain Markov Processes", Proceedings of the third Berkeley Symposium on Mathematical Statistics and Probability, University of California Press.
- [7] I. N. Herstein, (1964), Topics in Algebra, Xerox College Publishing.
- [8] J. Kemeny and L. Snell, (1960), Finite Markov Chains, Van Nostrand, Princeton, New Jersey.
- [9] H. L. Royden, (1968), Real Analysis (second ed.), Collier-Macmillan Limited, London.
- [10] R. Serfozo, (1969), "Time and Space Transformations of a Semi-Markov Process", Doctoral Thesis in Applied Math., Northwestern Univ., Evanston, Illinois.
- [11] R. Serfozo, (1971), "Functions of Semi-Markov Processes", Siam. J. Appl. Math., 20, 530-535.
- [12] T. L. Vlach and R. L. Disney, (1969), "The Departure Process from the GI/G/1 Queue", J. Appl. Prob., 6, 704-707.