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REGULARIZATION OF ILL-POSED PROBLEMS

C.W. Groetsch

Analysis and Optimization Branch
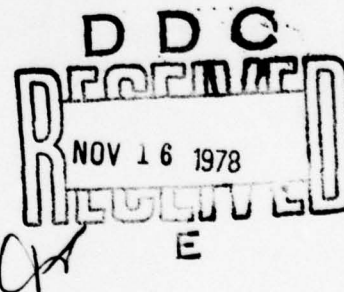Structural Mechanics Division

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Some examples of linear ill-posed problems in engineering are given and a general class of regularization methods for ill-posed linear operator equations is studied. Rates of convergence for the general method are established under various assumptions on the data. Applications are given to a number of iterative and noniterative regularization algorithms.

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FOREWORD

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SECTION I

INTRODUCTION

The concept of a well-posed problem was formulated by Hadamard
early in this century. In broad terms, a problem is well-posed in
the sense of Hadamard if it has a unique solution which depends con-
tinuously on the data of the problem. Specifically, if T is a trans-
formation from a metric space X into a metric space Y, then the prob-
lem

$$Tx = b \tag{1}$$

is said to be well-posed if

(i) for each $b \varepsilon Y$ there is a solution $x \varepsilon X$,

(ii) the solution x is unique, and

(iii) the solution x depends continuously on the "data" b.

A problem which is not well-posed is called "ill-posed." Ill-
posed problems have been intensively studied during the last fifteen
years, especially by Soviet mathematicians (see [12],[23]), because
of their importance in many engineering applications (see [11] and
[14] for specific areas of Air Force interest). In this report we
will be concerned with linear ill-posed problems, that is, we will
study the problem (1) where T is a linear operator on Hilbert space.
A typical problem of this type is the integral equation of the first
kind

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$$\int_a^d k(s,t)x(s)ds = b(t) \tag{2}$$

where the kernel k is a member of $L^2([a,d]x[a,d])$ (the space of

Lebesgue square integrable functions on the rectangle [a,d]x[a,d])

and $b\epsilon L^2[a,d]$ (we allow a or d to be infinite). Such equations are

notoriously ill-posed. For example, if $k(s,t) = t + c$, then (2) can

have a solution only if b is a linear function, violating (i). If

$k(s,t) = \sin(s)$ and $b(t) = 2$, then by the well-known orthogonality

relations,

$$\int_0^\pi k(s,t)(1 + \sin(ms))ds = b(t), \quad m = 2,3,\ldots$$

which violates (ii). Far more serious is the fact that (iii) is

violated for equations of type (2). Indeed, by the Riemann-Lebesgue

lemma, for arbitrary A,

$$\int_o^1 k(s,t) \, A \sin(m\pi s)ds \to 0 \quad \text{as } m \to \infty,$$

and hence solutions do not depend continuously on the data.

Numerical methods for analyzing ill-posed linear problems are

particularly important because a large number of engineering prob-

lems have the form (2). Consider for example the one dimensional

heat equation

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$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} , \quad u(x,0) = h(x).$$

It is well known that the temperature distribution $f(x) = u(x,T)$ at some time $T > 0$ can be expressed in terms of the initial temperature distribution $h(x)$ by

$$f(x) = \frac{1}{2\sqrt{\pi T}} \int_{-\infty}^{\infty} \exp(-(x-\tau)^2/(4T))h(\tau)d\tau.$$

The "inverse" problem of determining the initial temperature distribution $h(x)$, given the distribution $f(x)$ at the later time, is of considerable interest and is an ill-posed problem of type (2).

Another problem of type (2) is the numerical differentiation problem. The nth derivative of a given function $b(t)$ (with $b(0) = b'(0) = \ldots = b^{(n-1)}(0) = 0$) satisfies

$$\int_0^t \frac{1}{(n-1)!} (t-s)^{n-1}x(s)ds = b(t).$$

This problem has been studied extensively within the context of ill-posed problems by Cullum [3], Franklin [5] and others.

Another example is afforded by the work of Lee [13] and Provencher [18] on the determination of the molecular weight distribution of a solute from centrifuge data. In this example the molecular weight distribution $f(m)$ satisfies

$$U(x) = \int_0^{\infty} \frac{\lambda^2 m^2 e^{-\lambda mx}}{1 - e^{-\lambda m}} f(m)dm$$

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where U is a function which is proportional to the measured con-
centration gradient and λ is a constant which is proportional to the
square root of the rotor speed.

As a final example, we give the two dimensional integral equa-
tion

$$\int_\Omega \frac{p(x´,y´)\ dx´dy´}{[(x-x´)^2 + (y-y´)^2]^{\frac{1}{2}}} = \delta - f_1 (x,y) - f_2(x,y)$$

which was studied by Singh and Paul [21] and concerns the pressure
distribution in the contact of nonconforming elastic bodies.

Integral equations of the first kind also arise in the deter-
mination of the shape of conducting bodies from backscattered elec-
tromagnetic radiation ([16],[17]), seismic prospecting [2], antenna
theory [4], remote probing of the atmosphere ([22],[24]), medical
tomography [6] and system identification ([1],[15]).

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SECTION II
GENERALIZED INVERSES

We will henceforth assume that H_1 and H_2 are Hilbert spaces and that $T:H_1 \to H_2$ is a bounded linear operator. The inner product and norm in each space will be denoted by $(\cdot,\cdot)$ and $||\cdot||$, respectively. The range and nullspace of T will be denoted by R(T) and N(T), respectively. Our task is to solve the ill-posed problem (1) for $x \epsilon H_1$ given $b \epsilon H_2$. Of course, if $b \notin R(T)$ then (i) is violated and there is no solution. In such a case we might reasonably adopt the more flexible attitude of replacing b in the right hand side of (1) by the point in R(T) which is nearest to b. However, if R(T) is not closed, such a closest point may not exist. We are then led to accept as a generalized solution any vector $u \epsilon H_1$ which satisfies

$$Tu = Pb \qquad (3)$$

where P is the projection of H_2 onto $\overline{R(T)}$, the closure of R(T). Any vector u satisfying (3) is called a least squares solution of equation (1). We note that a least squares solution will exist for any vector b whose projection onto $\overline{R(T)}$ lies in R(T), i.e., for all vectors b in the dense subspace $R(T) \oplus R(T)^{\perp}$ of H_2. It is not difficult to see that least squares solutions may also be characterized as vectors $u \epsilon H_1$ which satisfy either of the conditions

$$||Tu - b|| \leq ||Tx - b||, \text{ for all } x \epsilon H_1, \qquad (4)$$

or

$$T^*Tu = T^*b \qquad (5)$$

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where T^* is the adjoint of T (see [7] for a proof of this and other simple facts pertaining to this section).

We have seen that if we consider least squares solutions instead of traditional solutions, then difficulty (i) is to a certain extent obviated. The problem of nonuniqueness, however, remains at this point. Indeed, if $N(T) \neq \{0\}$ then there may be infinitely many least squares solutions, for if u is a least squares solution, then so is u+v for any $v \epsilon N(T)$. Fortunately, there is a natural way of selecting a least squares solution which is unique in a certain sense. We see from (5) that the set of all least squares solutions is a closed convex set. This set therefore contains a unique vector of smallest norm and it is this vector which we will accept as the unique generalized solution of equation (1). Let $\mathcal{D}(T^+) = R(T) \oplus R(T)^{\perp}$. The operator

$$T^+ : \mathcal{D}(T^+) \rightarrow H_1$$

which associates with each $b \ \epsilon \ \mathcal{D}(T^+)$ the unique least squares solution of equation (1) with minimal norm is called the generalized inverse of T. It is not difficult to show that T^+ is a closed linear operator (see [7]). If T^+ were continuous then problems (i), (ii), (iii) would be solved, at least for $b \epsilon \mathcal{D}(T^+)$. But alas this is not the case. It is not difficult to show that T^+ is continuous if and only if R(T) is closed. Unfortunately the range of an integral operator is closed if and only if its kernel is degenerate (see [7]). We are therefore led to seek approximations to T^+ by bounded linear

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operators. Such approximations, when applied to b, are called reg-

ularizors of equation (1).

SECTION III

A GENERAL METHOD

We will denote the operator T^*T by $\tilde{T}$ and the operator TT^* by
$\hat{T}$. Note that $\tilde{T}$ and $\hat{T}$ are self adjoint linear operators whose spectra lie in the interval $[0, ||T||^2]$. If $0 \notin \sigma(\tilde{T})$ (the spectrum of
$\tilde{T}$), then by (5) we have $T^\dagger = \tilde{T}^{-1}T^*$. In general, however, $0 \in \sigma(\tilde{T})$,
but this last equation nevertheless leads us to seek approximations
to $T^\dagger$ by operators of the form $U(\tilde{T})T^*$ where U is a continuous function on $[0, ||T||^2]$ which approximates the function $f(t) = t^{-1}$ in
some sense. Specifically, we will consider a family (net) of real
valued functions $\{U_\beta(t): \beta \in S\}$, indexed by a subset S of the positive
real numbers with $\infty \in \bar{S}$, where each U_β is continuous on $[0, ||T||^2]$
and such that

$$|tU_\beta(t)| \leq M \qquad \text{for all } t \text{ and } \beta \qquad (6)$$

and

$$U_\beta(t) \to t^{-1} \qquad \text{as } \beta \to \infty \text{ for each } t \neq 0. \qquad (7)$$

The following is proved in [7].

<u>Theorem 1</u>. Suppose $b \in \mathcal{N}(T^\dagger)$ and let $x_\beta = U_\beta(\tilde{T})T^*b$. Then $x_\beta \to T^\dagger b$
as $\beta \to \infty$.

To this we now add,

<u>Theorem 2</u>. If $b \notin \mathcal{N}(T^\dagger)$, then $\{x_\beta\}$ has no weakly convergent subnet.

Proof.   Suppose $\{x_{\beta'}\}$ is a subnet of $\{x_\beta\}$ which converges weakly
to $z \in H_1$, denoted $x_{\beta'} \xrightarrow{W} z$. By the weak continuity of bounded linear
operators we then have $Tx_{\beta'} \xrightarrow{W} Tz$.

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Now,

$$Pb - Tx_\beta = Pb - TU_\beta(\tilde{T})T^*b$$

$$= Pb - \hat{T}U_\beta(\hat{T})Pb.$$

However, by (6) and (7), the operator $\hat{T}U_\beta(\hat{T})$ converges pointwise to the projection of H_2 onto $N(\hat{T})^\perp = N(T^*)^\perp = \overline{R(T)}$. Therefore $Pb - Tx_\beta \to 0$. It then follows that $Pb = Tz$, a contradiction. #

In the proof above we have used the fact that $U_\beta(\tilde{T})T^* = T^*U_\beta(\hat{T})$. This is easy to see if U_β is a polynomial. In the general case the identity follows from the Weierstrass approximation theorem. Using the fact that bounded sets in Hilbert space are weakly compact, we have:

<u>Corollary 3</u>. If $b \notin \mathcal{D}(T^+)$, then $||x_\beta|| \to \infty$ as $\beta \to \infty$.

Theorem 1 and Corollary 3 demonstrate dramatically the unequivocal nature of the approximations $\{x_\beta\}$.

Several authors have established rates of convergence for various approximations to T^+b under the stronger assumption that $Pb \in R(\hat{T})$ (see [20],[9],[10]). We see from Corollary 3 that the very least we must require to get convergence at all is that $Pb \in R(T)$. In order to strengthen this condition only slightly and thereby obtain a rate of convergence we note that

$$R(T) = R(TP_{N(T)^\perp})$$

and, in the pointwise sense,

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$$P_{N(T)^{\perp}} = \lim_{\nu \to 0^+} \tilde{T}^{\nu}.$$

It therefore seems reasonable to replace the hypothesis $b \in \mathcal{D}(T^{\dagger})$, i.e., $Pb \in R(T)$, by the hypothesis $Pb \in R(\tilde{T T}^{\nu})$ for some $\nu > 0$. In order to gauge the rate of convergence we will replace (7) by the stronger condition

$$t^{\nu}|1 - tU_{\beta}(t)| \leq \omega(\beta,\nu) \quad \text{for } \nu > 0 \qquad (8)$$

where $\omega(\beta,\nu) \to 0$ as $\beta \to \infty$ for each $\nu > 0$ (the case $\nu = 1$ was considered in [8]).

<u>Lemma 4.</u>   If $\nu > 0$, then $R(\tilde{T}^{\nu}) \subseteq N(T)^{\perp}$.

Proof.   Suppose $\{f_n\}$ is a sequence of continuous real valued functions on $[0,||T||^2]$ such that $f_n(t) \to t^{\nu-1}$ for $t \neq 0$ and $tf_n(t)$ is uniformly bounded (for example, we may take $f_n(t) = t^{\nu-1}$ for $t \geq 1/n$ and $f_n(t) = n^{2-\nu}t$ for $0 \leq t \leq 1/n$). Let $\{E_t\}$ be the resolution of the identity generated by the self-adjoint operator $\tilde{T}$. By the bounded convergence theorem we then have

$$\tilde{T}^{\nu}y = \int_0^{||T||^2} t^2 dE_t y = \int_0^{||T||^2} \lim_n tf_n(t)dE_t y$$

$$= \lim_n \int_0^{||T||^2} tf_n(t)dE_t y = \lim_n \tilde{T}f_n(\tilde{T})y \in N(T)^{\perp}. \#$$

We now state a rate of convergence result. The vector $T^{\dagger}b$ will be denoted by x and the error $x - x_{\beta}$ by e_{β}.

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Theorem 5.   If $Pb = T\tilde{T}^\nu w$, where $\nu > 0$, then $\|e_\beta\| \leq \omega(\beta,\nu)\|w\|$.

Proof.   Since $Tx = Pb = T\tilde{T}^\nu w$ and since $x - \tilde{T}^\nu w \in N(T)^\perp$, we see

that $x = \tilde{T}^\nu w$. Now,

$$x_\beta = U_\beta(\tilde{T})T^* b = U_\beta(\tilde{T})T^* Pb$$

$$= U_\beta(\tilde{T})\tilde{T}x = U_\beta(\tilde{T})\tilde{T}^{\nu+1}w.$$

Therefore $e_\beta = x - x_\beta = \tilde{T}^\nu(I - U_\beta(\tilde{T})\tilde{T})w$.

By the Spectral Mapping Theorem and Radius Formula, we then have

$$\|e_\beta\| \leq \omega(\beta,\nu)\|w\|. \#$$

        In our next result we become more cavalier in our

assumptions on the data.

Lemma 6.   If $Pb = \hat{T}^\nu w$ where $\nu \geq 1$, then $\|e_\beta\|^2 \leq \omega(\beta,\nu-1)\|Te_\beta\| \|w\|$.

Proof.   As in the previous proof we find that $x = T^*\hat{T}^{\nu-1}w$. Also,

$$x_\beta = U_\beta(\tilde{T})T^* Pb = U_\beta(\tilde{T})T^*\hat{T}^\nu w$$

$$= T^* U_\beta(\hat{T})\hat{T}^\nu w.$$

Therefore $e_\beta = x - x_\beta = T^*(I - U_\beta(\hat{T})\hat{T})\hat{T}^{\nu-1}w$, and

$$\|e_\beta\|^2 = (e_\beta, T^*(I - U_\beta(\hat{T})\hat{T})\hat{T}^{\nu-1}w)$$

$$= (Te_\beta, (I - U_\beta(\hat{T})\hat{T})\hat{T}^{\nu-1}w) \leq \omega(\beta,\nu-1)\|w\| \|Te_\beta\|. \#$$

Theorem 7.   If $Pb = \hat{T}^\nu w$ where $\nu \geq 1$, then $\|e_\beta\|^2 \leq \omega(\beta,\nu)\omega(\beta,\nu-1)\|w\|$.

Proof.   In Lemma 6 we saw that

$$e_\beta = T^*(I - U_\beta(\hat{T})\hat{T})\hat{T}^{\nu-1}w,$$

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therefore

$$\tilde{T}e_\beta = T^* T^\nu (I - U_\beta(\hat{T})\hat{T})w.$$

We then have

$$||Te_\beta||^2 = (\tilde{T}e_\beta, e_\beta) = (T^\nu(I - U_\beta(\hat{T})\hat{T}w, Te_\beta)$$

$$\leq \omega(\beta, \nu)||Te_\beta||, \text{ i.e., } ||Te_\beta|| \leq \omega(\beta, \nu).$$

Substituting into the result of Lemma 6 completes the proof.#

In the next section we will give a number of examples of
specific computational techniques to which the above results apply.

We have avoided for long enough the problem of polluted
data. We now take up this question. Suppose that the data b is the
result of measurements so that instead of b we have in our possession
a corrupted version b^ϵ satisfying $||b - b^\epsilon|| \leq \epsilon$. We operate on the
vector b^ϵ to obtain the approximations x_β^ϵ given by

$$x_\beta^\epsilon = U_\beta(\tilde{T})T^* b^\epsilon.$$

Let $\phi(\beta) = \sup\{|tU_\beta(t)|:t \in [0,||T||^2]\}$, and recall that $\phi(\beta)$ is
bounded (by (6)).

<u>Lemma 8.</u> $||Tx_\beta - Tx_\beta^\epsilon|| \leq \epsilon \phi(\beta).$

Proof. $\tilde{T}(x_\beta - x_\beta^\epsilon) = \tilde{T}U_\beta(\tilde{T})T^*(b - b^\epsilon),$ therefore

$$||Tx_\beta - Tx_\beta^\epsilon||^2 = (\tilde{T}(x_\beta - x_\beta^\epsilon), x_\beta - x_\beta^\epsilon)$$

$$= (\tilde{T}U_\beta(\tilde{T})T^*(b - b^\epsilon), x_\beta - x_\beta^\epsilon)$$

$$= (\hat{T}U_\beta(\hat{T})(b - b^\epsilon), T(x_\beta - x_\beta^\epsilon))$$

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$$\leq \phi(\beta)||b - b^\epsilon|| \; ||T(x_\beta - x_\beta^\epsilon)||$$

$$\leq \epsilon\phi(\beta)||Tx_\beta - Tx_\beta^\epsilon||.\#$$

Suppose now that $g(\beta) = \sup\{|U_\beta(t)|: t \in [0, ||T||^2]\}$. We note that

$$g(\beta) \to \infty \quad \text{as} \quad \beta \to \infty. \tag{9}$$

Indeed, if this were not the case, then there would be a constant L such that $|U_\beta(t)| < L$ for all t and β. But then $|tU_\beta(t)| \leq Lt \to 0$ as $t \to 0$, contradicting (7).

Lemma 9. $||x_\beta - x_\beta^\epsilon|| \leq \epsilon\sqrt{g(\beta)\phi(\beta)}$.

Proof. Since $x_\beta - x_\beta^\epsilon = T^*U_\beta(\hat{T})(b - b^\epsilon)$, we have, by use of Lemma 8,

$$||x_\beta - x_\beta^\epsilon||^2 = (x_\beta - x_\beta^\epsilon, T^*U_\beta(\hat{T})(b - b^\epsilon))$$

$$= (T(x_\beta - x_\beta^\epsilon), U_\beta(\hat{T})(b - b^\epsilon))$$

$$\leq \epsilon^2\phi(\beta)g(\beta).\#$$

Suppose now that $Pb = \hat{T}w$ (we could also use the other hypotheses considered above, but we choose to consider this simple case to illustrate the ideas). By the triangle inequality we have

$$||x - x_\beta^\epsilon|| \leq ||x - x_\beta|| + ||x_\beta - x_\beta^\epsilon||.$$

Lemma 9 and Theorem 7, then give

Theorem 10. If $Pb = \hat{T}w$, then

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$$||x - x_\beta^\epsilon|| \leq (||w||\omega(\beta,1)\omega(\beta,0))^{\frac{1}{2}} + \epsilon(g(\beta)\phi(\beta))^{\frac{1}{2}}.$$

The first term on the right hand side of this inequality goes to zero as $\beta \to \infty$. However, by (9) and (7), the second term becomes infinitely large as $\beta \to \infty$. This illustrates the classic dilemma in the numerical treatment of ill-posed problems. Even if computations are performed exactly, small errors in the data may eventually grow and overpower the approximations.

In view of Theorem 10, the question naturally arises as to whether it is ever possible to obtain convergent approximations even if the data can be measured as precisely as desired. Specifically, is there an effective way of choosing a "stopping parameter" $\beta(\epsilon)$ such that $e_{\beta(\epsilon)} \to 0$ as $\epsilon \to 0$? This problem of choice of regularization parameters is of great import and still has not been satisfactorily answered. For the wide class of methods considered here the question is particularly difficult, for as we shall see in the next section, the parameter may take on discrete or continuous values depending upon the specific method under consideration.

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SECTION IV

SPECIFIC METHODS

In this section we will consider some specific choices for the functions $\{U_\beta(t)\}$ and we will find functions $\omega(\beta,\nu)$ which determine rates of convergence. The index set S in all examples below will be either the set of nonnegative reals or nonnegative integers. In the discrete case, the parameter β will be denoted by n.

As a first example we consider Showalter's integral formula [19]:

$$T^\dagger b = \int_0^\infty \exp(-u\tilde{T})T^* b\, du.$$

The functions U_β for this example have the form

$$U_\beta(t) = \int_0^\beta \exp(-ut)\, du$$

and may be motivated in terms of Borel summability [7]. It is not difficult to see that a function $\omega(\beta,\nu)$ satisfying (8) is given by

$$\omega(\beta,\nu) = \beta^{-\nu} \qquad (\nu > 0).$$

The choice $U_\beta(t) = (t + \beta^{-1})^{-1}$ ($\beta > 0$) leads to Tychonov's regularization of order zero [23]. Here one can readily verify that

$$\omega(\beta,\nu) = \beta^{-\nu} \qquad \text{for } 0 < \nu \le 1.$$

In order to obtain approximations with this rate for $\nu > 1$ we may use extrapolated regularization [9]. That is, for a given $\beta > 0$ we set

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$$U_\beta^{(0)}(t) = (t + \beta^{-1})^{-1}$$

and define Richardson extrapolants by

$$U_\beta^{(j)}(t) = (2^j U_{2\beta}^{(j-1)}(t) - U_\beta^{(j-1)}(t))/(2^j - 1),$$
$$j = 1, 2, \ldots .$$

It is not difficult to show (see [9, lemma 2.1]) that for $k = 0, 1, 2, \ldots$

$$t^{k+1}|1 - tU_\beta^{(k)}(t)| = \prod_{i=0}^{k} (\frac{t}{2^i \beta t + 1})$$

$$\leq \beta^{-k-1}$$

Therefore, for the kth extrapolant we may apply Theorem 7 with $\omega(\beta,k) = \beta^{-k-1}$, $k = 1, 2, \ldots$, to obtain the rate $\beta^{-k+\frac{1}{2}}$ (see [9, Theorem 3.2]).

We now consider some iterative regularization methods. Below, α will be a parameter satisfying $0 < \alpha < 2 \, ||T||^{-2}$.

If the functions $U_n(t)$, $n = 0, 1, 2, \ldots$ are defined by

$$U_n(t) = \alpha \sum_{k=0}^{n} (1 - \alpha t)^k$$

then (6) and (7) are satisfied and one can show that

$$n^\nu t^\nu |1 - tU_n(t)| = n^\nu t^\nu |1 - \alpha t|^{n+1}$$

is uniformly bounded. From this we find that the rate of convergence of the iterative process

$$x_o = \alpha T^* b, \quad x_{n+1} = (I - \alpha \tilde{T})x_n + \alpha T^* b$$

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is determined by the function $\omega(n,\nu) = n^{-\nu}$.

Newton's method for approximating t^{-1} leads to the sequence of functions defined by

$$U_o(t) = \alpha, \quad U_{n+1}(t) = U_n(t)(2 - tU_n(t)).$$

For this sequence of functions it is not difficult to see that

$$t^{\nu}|1 - tU_n(t)| = 0(2^{-\nu n}) \quad \text{for } \nu > 0.$$

Therefore the rate of convergence of the corresponding iterative method is determined by the function $\omega(n,\nu) = 2^{-\nu n}$.

Showalter and Ben-Israel [20] have extrapolated on the previous method to obtain methods with a higher rate of convergence. For a positive integer $p \geq 2$ they define the hyperpower methods in terms of the sequence

$$U_o(t) = \alpha, \quad U_{n+1}(t) = U_n(t) \sum_{k=0}^{p-1} (1 - tU_n(t))^k.$$

For these methods the results above may be used to obtain the convergence rate $0(p^{-\nu n})$.

In [11] Lardy considered the approximations

$$x_o = 0, \quad \tilde{T}x_n + x_n = x_{n-1} + T^*b, \quad n = 1,2,\ldots$$

to $T^{\dagger}b$, where T is an unbounded operator. We may apply the results above in the case of a bounded operator if we define the functions U_n by

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$$U_n(t) = \sum_{k=1}^{n} (t + 1)^{-k}.$$

One can verify, as in the first iterative example above, that the function $\omega(n,\nu) = n^{-\nu}$ determines a rate of convergence.

The iterative method

$$x_o = T^*b, \quad x_{n+1} = x_n + (T^*b - \tilde{T}x_n)/(n + 2),$$

was investigated in [10]. The appropriate functions U_n are given by

$$U_n(t) = \sum_{k=0}^{n} (k + 1)^{-1} \prod_{j=0}^{k-1} (1 - t/(1 + j)).$$

This leads to the iterative method

$$x_o = T^*b, \quad x_{n+1} = x_n + (T^*b - \tilde{T}x_n)/(n+2).$$

Following the analysis given in [10] one can show that the rate of convergence of this method is governed by the function $\omega(n,\nu) = (\log n)^{-\nu}$.

REFERENCES

1. D.R. Audley and D.A. Lee, Considerations related to ill-posed and well-posed problems in system identification, ARL TR 73-D196, December, 1973.

2. E.C. Bullard and R.I.B. Cooper, The determination of masses necessary to produce a given gravitational field, Proc. Royal Soc. A 194(1968), 332-347.

3. J. Cullum, Numerical differentiation and regularization, SIAM J. Numer. Anal. 8(1971), 254-265.

4. T.S. Fong, On the problem of optimal antenna aperture distribution, J. Franklin Inst. 283(1967), 235-249.

5. J.N. Franklin, On Tikhonov's method for ill-posed problems, Math. Comp. 28(1974), 889-907.

6. R. Gorenflo (ed.), Inkorrekt Gestellte Probleme I, Preprint 58/77, Fachbereich Mathematik, Freie Universität Berlin, Berlin, 1977.

7. C.W. Groetsch, Generalized Inverses of Linear Operators: Representation and Approximation, Dekker, New York, 1977.

8. C.W. Groetsch, On rates of convergence for approximations to the generalized inverse, J. Numer. Func. Anal. Opt., to appear.

9. C.W. Groetsch and J.T. King, Extrapolation and the method of regularization for generalized inverses, J. Approx. Th., to appear.

10. C.W. Groetsch and B.J. Jacobs, Iterative methods for generalized inverses based on functional interpolation, in "Recent Applications of Generalized Inverses" (M.Z. Nashed, ed.), Pitman, London, to appear.

11. L.J. Lardy, A series representation for the generalized inverse of a closed linear operator, Tech. Rep. 74-18, Dept. of Math., Univ. of Maryland, April, 1974.

12. M.M. Lavrentiev, Some Improperly Posed Problems of Mathematical Physics, Springer-Verlag, Berlin, 1967.

13. D.A. Lee, On the determination of molecular weight distributions from sedimentation-diffusion equilibrium data at a single rotor speed, J. Polymer Sci. 8(1970), 1039-1056.

14. D.A. Lee et al., Some practical aspects of the treatment of ill-posed problems by regularization, ARL TR 75-0022, February, 1975.

15. M.Z. Nashed (ed.), Generalized Inverses and Applications, Academic Press, New York, 1976.

16. W.L. Perry, On the Bojarski-Lewis inverse scattering method, IEEE Trans. Antennas Prop. 6(1974), 826-829.

17. W.L. Perry, Approximate solution of inverse problems with piecewise continuous solutions, Radio Sci. 12(1977), 637-642.

18. S.W. Provencher, Numerical solution of linear integral equations of the first kind. Calculation of molecular weight distributions from sedimentation equilibrium data, J. Chem. Phys. 46(1967), 3229-3236.

19. D. Showalter, Representation and computation of the pseudoinverse, Proc. Amer. Math. Soc. 18(1967), 584-586.

20. D. Showalter and A. Ben-Israel, Representation and computation of the generalized inverse of a bounded linear operator between two Hilbert spaces, Accad. Naz. dei Lincei 48(1970), 184-194.

21. K. Singh and B. Paul, A method of solving ill-posed integral equations of the first kind, Comp. Meth. Appl. Mech. Eng. 2(1973), 339-348.

22. O.N. Strand and E.R. Westwater, Statistical estimation of the numerical solution of a Fredholm integral equation of the first kind, J.A.C.M. 15(1968), 100-114.

23. A.N. Tikhonov and V.Y. Arsenin, Solutions of Ill-Posed Problems, Wiley, New York, 1977.

24. V.F. Turchin et al., The use of mathematical-statistics methods in the solution of incorrectly posed problems, Sov. Phys. Uspekhi 13(1971), 681-703.