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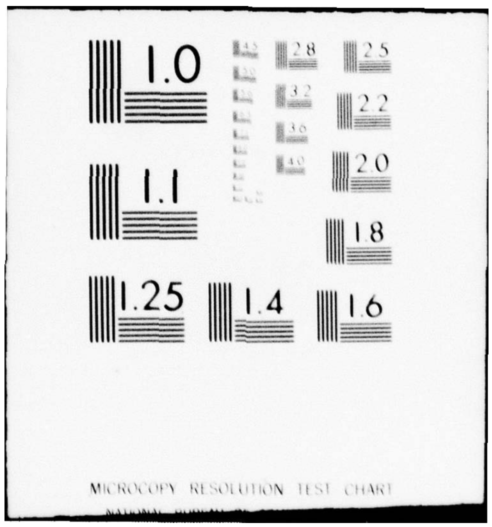
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# An Analysis of the Free Vibration of a Shallow Spherical Membrane Shell

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The free vibration of a shallow spherical membrane shell is analyzed using the thin-shell theory developed by Kraus. The secular equation giving the lowest resonant frequencies of such a shell is derived. The normal and tangential modal displacement functions corresponding to these resonant frequencies are found for the case when the shallow spherical shell has a free edge. In addition, an approximate secular equation is derived for the shallow spherical membrane shell with a free edge by using the Rayleigh-Ritz method. → next page

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7. thin-shell theory
8. resonant frequencies of shells
9. modal displacement functions for shells
10. modal vibrations
11. Rayleigh-Ritz method
12. secular equations
13. Kraus' equations
14. variational integrals
15. spherical sectors

BLOCK 20 (con't): Suitable approximate modal displacement functions, corr[illegible]ng to the resonant frequencies that are determined using this appro[illegible]e secular equation and satisfying the free-edge boundary condition, are found.

CONTENTS

# AN ANALYSIS OF THE FREE VIBRATION OF A SHALLOW SPHERICAL MEMBRANE SHELL

## INTRODUCTION

The ferroelectric element in some transducers is a shell that has the form of a sector of a sphere. In order to design practical underwater acoustic transducers that use spherical-shell-sector elements, it is useful to know the lowest resonant frequencies of such elements and to establish the modal displacement functions that correspond to these resonant frequencies. This report describes the calculation of the lowest resonant frequencies and the corresponding modal displacement functions of a shallow spherical membrane shell that is composed of a homogenous material. The analysis presented here does not provide a complete description of the vibration of a piezoceramic transducer element of this form, in that, it does not explicitly account for the effect of the electrical boundary conditions upon the motion of the shell. Nevertheless, this analysis is necessary before design procedures for transducers, which incorporate piezoelectric shells that are shallow sectors of spheres, can be established. A comprehensive theory that treats a spherical shell composed of a ferroelectric material, and which incorporates the complete set of electromechanical equations, would be the next step in the analytic treatment initiated here.

Note: Manuscript submitted July 17, 1978.

The subsequent material in this report will be presented as follows. First, a description of the spherical shell will be given, and those formulas from differential geometry that will be needed in the later analysis will be written down. Also, the assumptions and limits of shallow-shell theory will be noted. Next, the equations of motion of a shallow spherical membrane shell will be written down in several different forms. The equations will be solved for the normal and tangential displacement functions. Permissible boundary conditions will be discussed. The corresponding secular equations, whose roots yield the eigenfrequencies of the shell's free vibration, will be given. In addition to these secular equations, which result from the solution to the equations of motion that describe the shell, an approximate secular equation will be derived by use of the Rayleigh-Ritz method.

The analysis in this report is, for the most part, a codification of a number of the results found in the excellent book on shell theory by Kraus [1] , and it draws freely and extensively upon the material in this text. It is desirable, however, to collect Kraus' results for shallow spherical membrane shells and to present them in a form that is useful in transducer-design work. Moreover, several errors were found in rederiving Kraus' equations. These have been corrected in the equations reported here. Those interested in further details of the analysis presented in this report should refer to the comprehensive treatment in Kraus' text. The notation used in this report is the same as that used by Kraus, whenever possible.

## GEOMETRY OF A THIN SHALLOW SPHERICAL SHELL

Since a thin shell can be considered as the materialization of a curved surface, results from differential geometry, which is needed to describe such surfaces in space, are incorporated into the theory of thin shells. The motion of a point in a thin shell is, in fact, determined by the motion of the curved reference surface associated with the shell. Consider the geometry of a rotationally symmetric thin shell of general form that is shown in Fig. 1. The reference surface S of this shell may be described in the rectangular coordinate system

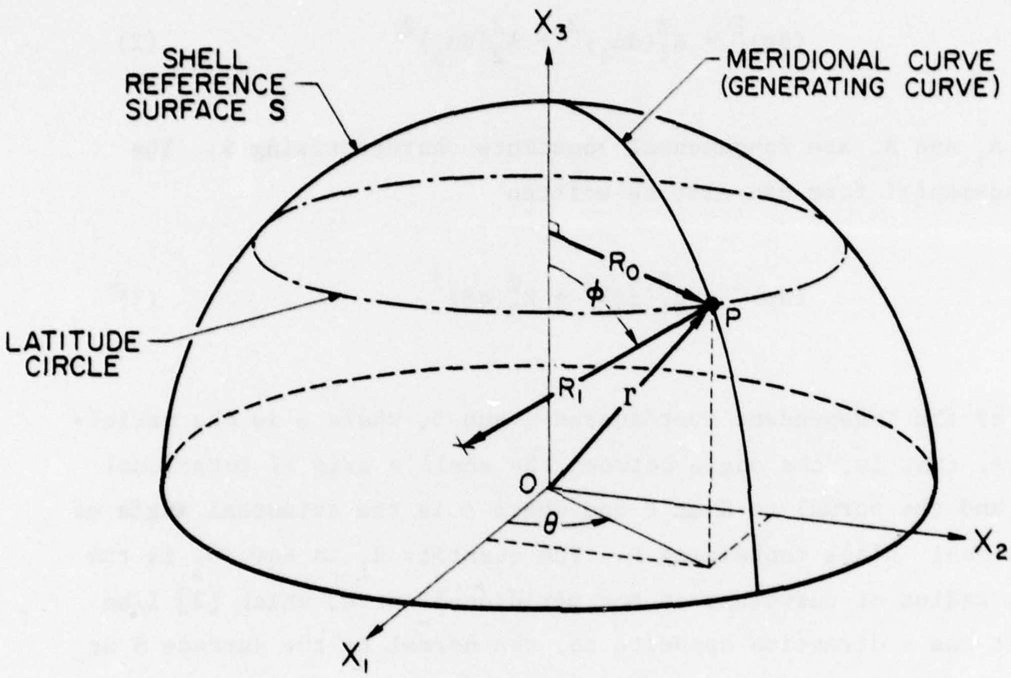


Fig. 1 - Geometry of a rotationally symmetric thin shell

$(x_1, x_2, x_3)$ as a function of two independent curvilinear coordinates of the surface, $\alpha_1$ and $\alpha_2$, by the expressions

$$x_1 = f_1(\alpha_1, \alpha_2),$$

$$x_2 = f_2(\alpha_1, \alpha_2),$$

$$x_3 = f_3(\alpha_1, \alpha_2). \qquad (1)$$

The surface S is symmetric with respect to the $x_3$ axis. The first fundamental form of the surface S is a relation that describes $(dr)^2$, the square of the length of the infinitesimal vector $d\mathbf{r}$ that measures the change in the vector **r** as one moves from point P to an infinitesimally near neighboring point on the surface. If $\alpha_1$ and $\alpha_2$ are orthogonal curvilinear coordinates, then the first fundamental form of S is

$$(ds)^2 = A_1^2(d\alpha_1)^2 + A_2^2(d\alpha_2)^2 \tag{2}$$

in which $A_1$ and $A_2$ are fundamental constants characterizing S. The first fundamental form can also be written

$$(ds)^2 = R_1^2(d\phi)^2 + R_0^2(d\theta)^2 \tag{3}$$

in terms of the independent coordinates $\phi$ and $\theta$, where $\phi$ is the meridional angle, that is, the angle between the shell's axis of rotational symmetry and the normal to S at P and where $\theta$ is the azimuthal angle of the meridional plane containing P. The quantity $R_1$ in Eq. (3) is the principal radius of curvature of the meridional curve, which [2] lies along, but has a direction opposite to, the normal to the surface S at P. The quantity $R_0$ is the radius of the latitude circle that passes through P. Comparing Eqs. (2) and (3), one sees that $\alpha_1$ and $\alpha_2$ may be associated with $\phi$ and $\theta$, respectively, and that

$$A_1 = R_1 \tag{4a}$$

and

$$A_2 = R_0. \tag{4b}$$

Three differential equations, the Gauss-Codazzi conditions, relate the fundamental constants $A_1$ and $A_2$ to the principal radii of curvature of a surface at a point. When applied in the $(\phi, \theta)$ coordinate system, the

Gauss-Codazzi conditions require that

$$\frac{dR_0}{d\phi} = R_1 \cos\phi. \tag{5}$$

The foregoing concepts, which apply to a general, rotationally symmetric shell, can now be specialized to a shallow spherical shell. A shell that is the surface of a segment of a sphere is depicted in Fig. 2. The principal radius of curvature $R_1$ is constant for such a spherical shell. This radius of curvature has been denoted as R, rather than $R_1$, in Fig. 2. It is easily seen that the origin of the $(x_1, x_2, x_3)$ system of Fig. 1 may be placed so that R terminates at the

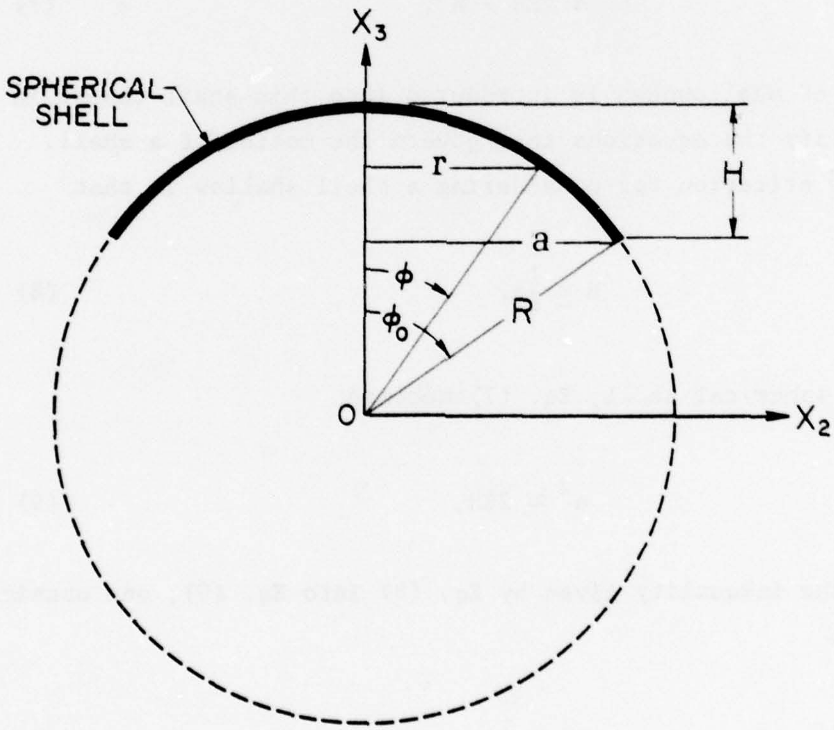


Fig. 2 - Geometry of a spherical shell

center of that sphere from which the shell is derived. Also, for this spherical shell, the meridional radius has been denoted by r instead of by $R_0$. Thus, in the revised notation, Eqs. (4) become

$$A_1 = R_1 = R \tag{6a}$$

and

$$A_2 = R_0 = r. \tag{6b}$$

Figure 2 shows a to be the radius of the spherical shell at its rim, $\phi_0$ to be the meridional angle of points on the rim, and H to be the height of the shell segment. It is also readily shown that, for a general spherical shell,

$$a^2 = 2RH - H^2. \tag{7}$$

The notion of shallowness is introduced into thin-shell theory in order to simplify the equations that govern the motion of a shell. Reissner's [3] criterion for considering a shell shallow is that

$$H \leq \frac{1}{4}a. \tag{8}$$

For a shallow spherical shell, Eq. (7) becomes

$$a^2 \approx 2RH. \tag{9}$$

By inserting the inequality given by Eq. (8) into Eq. (9), one obtains the expression

$$a/R \leq \frac{1}{2}, \tag{10a}$$

which implies that a spherical shell may be considered shallow if

$$\phi_0 \leq 30^{\circ}. \tag{10b}$$

If $\phi$ is small, one has, from Eq. (5), the result that

$$dr \approx R d\phi \tag{11}$$

for a spherical shell. Equation (11) is basic in the theory of shallow shells, even though $\phi$ is not small for all shells that satisfy Reissner's criterion of shallowness; i.e., all those for which Eq. (10b) holds.

FREE VIBRATION OF A SHALLOW SPHERICAL MEMBRANE SHELL

Thin-shell theory is usually considered valid if the thickness h of the shell in question is small in comparison to one of its radii of curvature. A spherical shell is, thus, thin if

$$h/R < \varepsilon, \tag{12}$$

where $\varepsilon$ in engineering studies [4] is usually taken as 1/10 or 1/20. Under conditions where a thin shell is so loaded that all bending moments are zero or are negligibly small, the shell is said to be in a membrane state of stress. The set of equations that governs the motion of a thin shell is considerably simplified if the shell is in a state of membrane stress, in which case it is referred to as a membrane shell. The membrane-shell assumptions will be considered to hold for the spherical shells analyzed subsequently.

The equations of motion of a shallow spherical shell of thickness h, which is not loaded by external forces and which is executing motion that is independent of the circumferential position, i.e., independent of the coordinate $\theta$, are

$$\frac{\partial(rN_r)}{\partial r} - N_\theta = \rho rh \frac{\partial^2 u}{\partial t^2}, \tag{13a}$$

$$\frac{\partial(rQ_r)}{\partial r} - \frac{r}{R}(N_r + N_\theta) = \rho rh \frac{\partial^2 w}{\partial t^2}, \tag{13b}$$

and

$$\frac{\partial(rM_r)}{\partial r} - M_\theta - rQ_r = 0, \tag{13c}$$

in which $\rho$ denotes the density of the shell material, u and w, respectively, denote the tangential and the normal shell displacements, $N_r$ and $N_\theta$ are the stress resultants, $M_r$ and $M_\theta$ are the bending moments, $Q_r$ is the shear-force resultant, t denotes the time, and r and R are as shown in Fig. 2. For a membrane shell,

$$M_r = M_\theta = 0, \tag{14a}$$

and, consequently, by Eq. (13c)

$$Q_r = 0. \tag{14b}$$

As a consequence of Eqs. (14), the three equations of motion reduce to two equations, Thus, if the forces and displacements are assumed to have a harmonic time dependence described by the factor $\exp(j\omega t)$ (which is henceforth dropped), Eqs. (13) become, for a shallow spherical membrane shell in free harmonic vibration, the two coupled differential equations

$$r(\partial N_r/\partial r) + N_r - N_\theta + \rho h \omega^2 u = 0 \tag{15a}$$

and

$$N_r + N_\theta - \rho R H \omega^2 w = 0. \tag{15b}$$

The two stress resultants $N_r$ and $N_\theta$ are obtained by integrating the actual stress distribution across the thickness of the shell after assuming the form of the displacement functions. These stress resultants are given by the stress-displacement relations

$$N_r = K(\varepsilon_r^0 + \nu\varepsilon_\theta^0) \tag{16a}$$

and

$$N_\theta = K(\varepsilon_\theta^0 + \nu\varepsilon_r^0), \tag{16b}$$

where $\nu$ is the Poisson's ratio of the shell material and

$$K = hE/(1 - \nu^2), \tag{17}$$

in which E is the Young's modulus of the shell material. For a symmetrically vibrating spherical shell, the quantities $\varepsilon_r^0$ and $\varepsilon_\theta^0$ in Eqs. (16) are

$$\varepsilon_r^0 = (\partial u/\partial r) + (w/R) \tag{18a}$$

and

$$\varepsilon_\theta^0 = (u/r) + (w/R). \tag{18b}$$

Figure 3 shows a differential element of the spherical shell in order to illustrate the action of the forces $N_r$ and $N_\theta$ and the

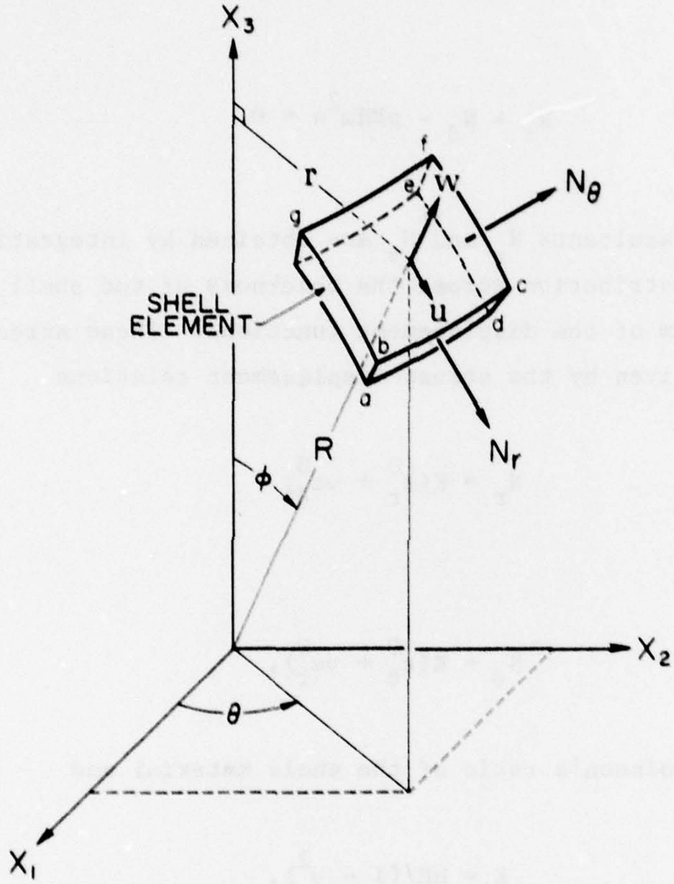


Fig. 3 - Differential element of a shallow spherical membrane shell

directions of the displacement u and w. The shell element shown in Fig. 3 is defined by two infinitesimally separated meridional curves and by two infinitesimally separated latitude circles. The stress resultant $N_r$ acts normal to the face abcd of the element, and the stress resultant $N_\theta$ acts normal to the element face cdef. The direction of the displacement w is normal to the reference surface of the shell, that is, normal to element face bcfg, and the direction of the displacement u is in the meridional direction. Owing to the shell's symmetry, there can, of course, be no displacement in the $\theta$ direction.

If the results expressed by Eqs. (16) through (18) are substituted into Eqs. (15), one obtains, for the two equations of motion, the expressions

$$L(ru) + w' \left[r(1 + \nu)/R\right] + (\rho h\omega^2 ru/K) = 0 \qquad (19)$$

and

$$(ru)' + (2rw/R) - \left\{\left[\rho hR\omega^2 rw\right]/\left[K(1 + \nu)\right]\right\} = 0. \qquad (20)$$

In Eq. (20), the prime denotes differentiation with respect to r

$$(\ldots)' \equiv \frac{d}{dr}(\ldots), \qquad (21a)$$

and the operator L in Eq. (19) is defined

$$L(\ldots) \equiv (\ldots)'' - \frac{1}{r}(\ldots)'$$

$$= \left[r\frac{d^2}{dr^2} + \frac{d}{dr} - \frac{1}{r}\right](\ldots). \qquad (21b)$$

The differential equations for the normal and tangential displacements of the shell, w(r) and u(r), are coupled when expressed in the form of Eqs. (19) and (20). Uncoupled differential equations for w and u may, however, be found in the following way. Suppose one solves Eq. (20) for w, and obtains the result

$$w = \left[u' + (u/r)\right]/\beta, \qquad (22)$$

where

$$\beta = (\rho h\omega^2 R)/[K(1 + \nu)] - (2/R). \tag{23}$$

Then, if Eq. (22) is differentiated with respect to r, the expression

$$w' = L(ru)/(\beta r) \tag{24}$$

is produced. Substituting the value of w', given by Eq. (24), into Eq. (19) allows one to obtain the uncoupled differential equation for the tangential displacement of the shell, viz

$$L(ru) + (\mu/a)^2(ru) = 0, \tag{25}$$

where

$$\mu^2 = a^2\Omega^2(1+\nu)[2 - \Omega^2(1 - \nu)]/[R^2(1 - \Omega^2)], \tag{26}$$

in which

$$\Omega^2 = \rho\omega^2 R^2/E. \tag{27}$$

Now Eq. (19) can be solved for u, with the result that

$$u = -[K/(\rho rh\omega^2)]\{[r(1 + \nu)/R]\, w' + L(ru)\}. \tag{28}$$

Upon substituting the value of L(ru) that is obtained from Eq. (24) into Eq. (28), one obtains the result that

$$u = [R(1 - \Omega^2)]/[\Omega^2(1 + \nu)]w', \tag{29}$$

which yields, upon differentiation, the equation

$$u' = [R(1 - \Omega^2)]/[\Omega^2(1 + \nu)]\, w''. \tag{30}$$

The expressions for u and u', given by Eqs. (29) and (30), when substituted into Eq. (22), results in the uncoupled differential equation for the normal displacement of the shell, viz

$$w'' + w'/r + (\mu/a)^2 w = 0. \tag{31}$$

When one differentiates Eq. (31) with respect to r and multiplies both sides by r, he obtains the equation

$$L(rw') + (\mu/a)^2(rw') = 0. \tag{32}$$

Since Eqs. (25) and (32) have the same form, their respective solutions u(r) and w'(r) must be proportional. This can easily be shown by substituting the result

$$L(ru) = -(\mu/a)^2 ru, \tag{33}$$

obtained from Eq. (25), into Eq. (19). When this is done, one obtains the result that

$$u/w' = R(1 + \nu)/[(\mu R/a)^2 - (1 - \nu^2)\Omega^2]. \tag{34}$$

One notes that if the change of variable $s = (\mu/a)r$ is made in Eq. (31), the result is just Bessel's equation of order zero. The general solution to Eq. (31) therefore is

$$w = -A(a/\mu)J_0(\mu r/a), \qquad (35)$$

where $J_0(\ldots)$ is the Bessel function of the first kind of order zero and where A is an arbitrary constant. The zero-order Bessel function of the second kind $Y_0(\ldots)$ does not enter the general solution to Eq. (31), since w(r) cannot be singular at r = 0. From Eqs. (34) and (35), one also obtains the result

$$u = \{R(1 + \nu)/[(\mu R/a)^2 - (1 - \nu^2)\Omega^2]\} AJ_1(\mu r/a), \qquad (36)$$

in which $J_1(\ldots)$ is the Bessel function of the first kind of order one.

Either of two possible boundary conditions may be imposed at the edge r = a of a shallow spherical membrane shell. At r = a, one can have either

$$u(a) = 0 \qquad (37)$$

or else

$$N_r(a) = 0. \qquad (38)$$

The boundary condition that is expressed by Eq. (37) corresponds to a tangentially clamped shell rim, while that expressed by Eq. (38) corresponds to a free edge. Note that neither of the two permissible boundary conditions is imposed upon the normal displacement w.

Equation (36) requires that $\mu$ must be a root of the secular equation

$$J_1(\mu) = 0 \tag{39}$$

for the tangentially-clamped boundary condition that is expressed by Eq. (37). For the boundary condition expressed by Eq. (38), which characterizes a shell with a free edge, one finds that $\mu$ must be a root of the secular equation

$$J_0(\mu) - \{(\mu R^2/a^2)/[(\mu R/a)^2 - \Omega^2(1 - \nu^2)]\}$$
$$\times [(\nu - 1)J_1(\mu) + \mu J_0(\mu)] = 0. \tag{40}$$

Equation (40) is obtained as follows. First, one calculates the various terms appearing in the stress resultant $N_r$, which is given by Eqs. (16a) and (18), using values for u, u' and w obtained from Eqs. (35) and (36). One then sets the result obtained equal to zero.

The roots of Eq. (40) determine the resonant frequencies of the spherical shell with a free edge. Before Eq. (40) can be solved, however, it is necessary to express the quantity $\Omega^2$ therein in terms of $\mu$. To do this, one rewrites Eq. (26) in the form

$$\Omega^4 - \{[2(1 + \nu) + (\mu R/a)^2]/(1 - \nu^2)\}\ \Omega^2 + (\mu R/a)^2/(1 - \nu^2) = 0 \tag{41}$$

and solves this quadratic equation in $\Omega^2$ for its two roots $\Omega_1^2(\mu)$ and $\Omega_2^2(\mu)$. To each of these two values of $\Omega^2$ there is a set of roots $\mu_m$ of Eq. (40). When inserted into Eq. (40), the root $\Omega_1^2(\mu)$ generates a sequence of eigenvalues $\mu_{1m}$, with $\{m = 1, 2, \ldots\}$, when that equation is solved. Another sequence of eigenvalues $\mu_{2m}$, again with $\{m = 1, 2, \ldots\}$, is obtained when this procedure is repeated with $\Omega_2^2(\mu)$. There are, thus, two families of solutions $\mu_{km}$, with $k = 1$ or $k = 2$, to Eq. (40). To obtain the resonant frequencies of the shallow spherical membrane shell, each root $\mu_{km}$ is inserted into Eq. (41). Each of the two values of $\mu_{km}$ yields in turn two values of $\Omega^2$, which can be designated $\Omega_{1km}^2$ and $\Omega_{2km}^2$. There thus appear to be four families of resonant frequencies of the shallow spherical membrane shell with a free edge. Recalling Eq. (27), one notes that the resonant angular frequencies are

$$\omega_{ikm} = (\Omega_{ikm}/R)\ (E/\rho)^{\frac{1}{2}}, \tag{42}$$

with $i = 1$ or $i = 2$ and $k = 1$ or $k = 2$. Of course, only those roots $\mu_{km}$ which yield real values of the resonant frequencies $\omega_{ikm}$ can be considered to be of physical significance. (One notes in passing that there are only two families of resonances for a spherical shell with a tangentially clamped edge. Since Eq. (39) does not involve $\Omega^2$, it yields directly a sequence of eigenvalues $\mu_m$. Each of these $\mu_m$ when inserted into Eq. (41) gives rise to two values of $\Omega^2$, which can be denoted $\Omega_{m1}^2$ and $\Omega_{m2}^2$. One resonant frequency $\omega_{mk}$ of the shell corresponds to each such $\Omega_{mk}^2$.)

The modal displacement functions corresponding to the roots of Eq. (40) can now be expressed. First, note from Eq. (35) that the expression for the normal displacement w does not depend explicitly upon $\Omega$. Thus, there are just two families of modal normal displacement functions corresponding to the two sequences of eigenvalues $\mu_{1m}$ and $\mu_{2m}$. These modal normal displacement functions are

$$w_{km} = -(a/\mu_{km})J_0(\mu_{km}r/a), \tag{43}$$

with k = 1 or k = 2. On the other hand, since by Eq. (36) the tangential displacement depends upon $\Omega$, there appear to be four families of modal tangential displacement functions, namely

$$u_{ikm} = \{R(1+\nu)/[(\mu_{km}R/a)^2 - (1-\nu^2)\Omega_{ikm}^2]\}J_1(\mu_{km}r/a), \tag{44}$$

with i = 1 or i = 2 and k = 1 or k = 2. The modal functions of the gravest modes of the shallow spherical membrane shell are those corresponding to setting m equal to one in Eqs. (43) and (44).

## MODAL VIBRATIONS OF A SHALLOW SPHERICAL MEMBRANE SHELL WITH A FREE EDGE, CALCULATED USING THE RAYLEIGH-RITZ METHOD

In order to calculate the resonant frequencies of free vibration of a shallow spherical membrane shell that has a free edge, one is required to find the roots of Eq. (40). Moreover, as was explained previously, before Eq. (40) can be solved, it is necessary first to solve Eq. (41) for $\Omega^2$ in terms of $\mu$ and to insert each of the two values of $\Omega^2$ thus obtained into Eq. (40) before proceeding to find the roots of this latter equation. While such a procedure for calculating the resonant frequencies of a spherical shell with a free edge is clearly possible, it is sufficiently complicated to make simpler approximate methods of calculating shell's resonant frequencies attractive.

One such approximate method of calculating the resonant frequencies of a shell's vibration is the Rayleigh-Ritz method. The formulation of the Rayleigh-Ritz analysis that is appropriate in thin-shell theory is based upon the variational equation

$$\int_{\alpha_1}\int_{\alpha_2} \{[\mathscr{L}_1(u,v,w) + \rho h\omega^2 u]\delta u$$

$$+ [\mathscr{L}_2(u,v,w) + \rho h\omega^2 v]\delta v$$

$$+ [\mathscr{L}_3(u,v,w) + \rho h\omega^2 w]\delta w\} A_1 A_2 \, d\alpha_1 d\alpha_2 = 0. \tag{45}$$

in which the $\mathscr{L}_i(\ldots)$ are the differential operators that appear in the equations of motion for free vibration of a thin shell, once the shearing forces, the stress resultants, and the bending moments have been eliminated. In terms of these operators, the equations of motion appear in the form

$$\mathscr{L}_1(u,v,w) = \rho h \frac{\partial^2 u}{\partial t^2}, \tag{46a}$$

$$\mathscr{L}_2(u,v,w) = \rho h \frac{\partial^2 v}{\partial t^2}, \tag{46b}$$

and

$$\mathscr{L}_3(u,v,w) = \rho h \frac{\partial^2 w}{\partial t^2}. \tag{46c}$$

For symmetric free vibration of the kind being considered, there is no tangential displacement of the shell in the direction of $\theta$. Therefore, for symmetric motion, the displacement $v$ does not appear in Eq. (45), and one has a variational equation of the form

$$\int_{\alpha_1}\int_{\alpha_2} \{[\mathcal{L}_1(u,w) + \rho h\omega^2 u]\delta u$$

$$+ [\mathcal{L}_2(u,w) + \rho h\omega^2 w]\delta w\}A_1 A_2 \, d\alpha_1 d\alpha_2 = 0. \qquad (47)$$

Likewise for symmetric motion, there is no appearance of v in Eqs. (46).

For the shallow spherical shell, one has, from the previous discussion of shell geometry and from Eqs. (6) and (11), the result that

$$A_1 A_2 \, d\alpha_1 d\alpha_2 = r \, dr d\theta. \qquad (48)$$

Since the shell's motion is independent of $\theta$, the integral over $\alpha_2 = \theta$ can be carried out directly after one substitutes the result of Eq. (48) into Eq. (47). The resulting expression is a variational integral over r:

$$\int_0^a \{[\mathcal{L}_1(u,w) + \rho h\omega^2 u]\delta u$$

$$+ [\mathcal{L}_2(u,w) + \rho h\omega^2 w]\delta w\}r \, dr = 0. \qquad (49)$$

Note that, in the case of harmonic free vibration of a shallow spherical membrane shell, Eqs. (19) and (20) can be put in the form of the general equations of shell motion that are expressed symbolically by Eq. (46). When this is done, one can obtain from Eq. (49), for the shell in question, the expression

$$\int_0^a \Big\{ [L(ru) + w'\{r(1 + \nu)/R\} + (\Omega/R)^2(1 - \nu^2)ru]\delta u$$

$$- [\{(1 + \nu)/R\}\{(ru)' + (2rw/R)\}$$

$$- (\Omega/R)^2(1 - \nu^2)rw]\delta w \Big\} \, dr = 0. \qquad (50)$$

In obtaining Eq. (50), the results expressed by Eqs. (17) and (27) have been used. Equation (50) is the form of the Rayleigh-Ritz variational equation that is needed for finding the resonant frequencies of a shallow spherical membrane shell.

The next step in the Rayleigh-Ritz method is to find suitable approximate displacement functions, that is, suitable approximate expressions for u and w. In order to use Eq. (50), however, these approximate displacement functions must be such that the boundary conditions appropriate to the problem in question are satisfied. For the vibrating spherical shell with a free edge, this boundary condition is given by Eq. (38), which, when written in terms of the displacement functions u and w, has the form

$$u' + [(1 + \nu)/R]w + (\nu/r)u = 0. \qquad (51)$$

Equation (51) holds at $r = a$.

Approximate forms of the displacement functions are found in the following way. First, note that the exact forms of the displacement functions are available. In particular, one has from Eq. (36) that

$$u = BJ_1(\mu r/a), \tag{52}$$

where B is a constant. Consider, on the one hand, the form of the series expansion of a first-order Bessel function:

$$J_1(x) = \frac{x}{2}\left( 1 - \frac{x^2}{8} + \frac{x^4}{192} - \dots \right) . \tag{53a}$$

Now suppose, on the other hand, that one makes the approximation

$$J_1(x) \approx \sin(bx) = bx\left( 1 - \frac{b^2x^2}{6} + \frac{b^4x^4}{120} - \dots \right), \tag{53b}$$

in which b is a constant chosen so as to make the first zero of $J_1(x)$ coincide with the first axis-crossing of the sine function. The approximation expressed by Eq. (53b) can be expected to be reasonably good for values of x somewhat beyond the first zero of $J_1(x)$. Upon examining Eqs. (52) and (53), it is not too difficult to see that a good approximation of the displacement function u would be

$$u = A(r/a)[1 - \gamma(r/a)^2], \tag{54}$$

in which A is an arbitrary constant and in which $\gamma$ is a constant chosen in such a way that the boundary condition in question is satisfied. For example, in the case of the tangentially clamped shell, for which the boundary condition is expressed by Eq. (37), one would take $\gamma = 1$.

The approximate displacement function expressed by Eq. (54) can be expected to be a good approximation of the tangential modal functions corresponding to the lowest resonant frequencies,

Since the boundary condition for a spherical shell with a free edge is expressed by Eq. (51), an approximation for the displacement function w, corresponding to the approximation for u that is expressed by Eq. (54), is also needed. This may be readily found by inserting the value of u expressed by Eq. (54) into Eq. (22). The result is that

$$w = A[2/(a\beta)][1 - 2\gamma(r/a)^2], \tag{55}$$

in which, after combining Eqs. (17), (23), and (27), one expresses $\beta$ by the equation

$$\beta = [(1 - \nu)\Omega^2 - 2]/R. \tag{56}$$

Note that, since Eq. (22) is just a rearrangement of Eq. (20), the approximate displacement functions, given by Eqs. (54) and (55), will satisfy Eq. (20).

The approximate displacement functions u and w, which are given by Eqs. (54) and (55), are now substituted into Eq. (51), which expresses the boundary condition for the shell with a free edge. The resulting equation, when solved for $\gamma$, yields the expression

$$\gamma = [(1 + \nu)\Omega^2]/[(3 + \nu)\Omega^2 - 2]. \tag{57}$$

Thus, if $\gamma$ is given the value expressed by Eq. (57), the approximate displacement functions will satisfy the proper boundary condition. Note from the presence of $\Omega^2$ in Eq. (57) that both of the approximate displacement functions u and w must depend upon the frequency of vibration of the shell, if the rim of the vibrating shell is to be stress free.

Equation (50) can now be used to derive an approximate secular equation for the resonant frequencies of a shallow spherical membrane shell. First, note that from Eq. (54) one has, for the variation in u, the expression

$$\delta u = (r/a)[1 - \gamma(r/a)^2]\delta A. \tag{58}$$

Next, the displacement functions given by Eqs. (54) and (55) and the result expressed by Eq. (58) are substituted into Eq. (50). Because the displacement functions u and w satisfy Eq. (20), the bracketed term in the integral that multiplies $\delta w$ will vanish identically. Therefore, one need not deal further with that term. The substitution in question, which is somewhat simplified if one recalls Eq. (24), yields the equation

$$A\delta A \int_0^a \left\{ \left[\frac{8(\gamma/a^4)(1 - \Omega^2)}{(1 - \nu)\Omega^2 - 2}\right] r^3 \left[1 - \gamma(\frac{r}{a})^2\right] + \left[\frac{\Omega^2(1 + \nu)}{a^2R^2}\right] r^3 \left[1 - \gamma(\frac{r}{a})^2\right]^2 \right\} dr = 0. \tag{59}$$

When the integration in Eq. (59) is performed, the equation

$$A\delta A\{\gamma(1-\Omega^2)(3-2\gamma)/[(1-\nu)\Omega^2-2]$$

$$+ [\Omega^2 a^2(1+\nu)(3\gamma^2-8\gamma+6)/(4R)^2]\} = 0 \qquad (60)$$

is obtained. Since the variation $\delta A$ is arbitrary and since the constant A must be non-zero for non-trivial displacement functions, Eq. (60) is satisfied only if the term within braces vanishes. If the value of $\gamma$ given by Eq. (57) is used, Eq. (60) with the term in braces set equal to zero leads to the following secular equation:

$$C_1\Omega^6 + C_2\Omega^4 + C_3\Omega^2 + C_4 = 0, \qquad (61)$$

in which the constants $C_i$ are

$$C_1 = (1-\nu)(33+10\nu+\nu^2), \qquad (62a)$$

$$C_2 = -[2(61-14\nu-3\nu^2) + (4R/a)^2(7+\nu)], \qquad (62b)$$

$$C_3 = 8(17-\nu) + (4R/a)^2(13+\nu), \qquad (62c)$$

and

$$C_4 = -48[2(R/a)^2 + 1]. \qquad (62d)$$

Equation (61) is cubic in $\Omega^2$. Suppose $\Omega_0^2$ is the smallest real root of Eq. (61). The resonant angular frequency of the gravest mode of a shallow spherical membrane shell with a free edge is, therefore,

$$\omega_0 = (\Omega_0/R)(E/\rho)^{\frac{1}{2}}. \qquad (63)$$

It is known that the Rayleigh-Ritz method always yields a resonant frequency that is somewhat higher than the exact result (i.e., the resonant frequency determined from Eq. (40). Therefore, the approximate resonant frequency of a shallow spherical membrane shell given by Eq. (63) will be somewhat greater than that found using the exact theory that was outlined in the previous section. Once the smallest real root $\Omega_0^2$ of Eq. (61) has been calculated, the modal displacement functions $u_0$ and $w_0$ of the gravest mode can be found. These modal displacements are

$$u_0 = (r/a)\Big[1 - [(1+\nu)\Omega_0^2]/[(3+\nu)\Omega_0^2 - 2](r/a)^2\Big], \tag{64a}$$

and

$$w_0 = (2R/a)[(1-\nu)\Omega_0^2 - 2]^{-1}$$

$$\times \Big\{1 - 2[(1+\nu)\Omega_0^2]/[(3+\nu)\Omega_0^2 - 2](r/a)^2\Big\}. \tag{64b}$$

At a nodal circle, the normal displacement function $w_0$, if it exists vanishes. From Eq. (55), it is seen that a nodal circle exists at

$$r = a\Big\{[(3+\nu)\Omega_0^2 - 2]/[2(1+\nu)\Omega_0^2]\Big\}^{\frac{1}{2}} \tag{65}$$

provided that

$$\Omega_0^2 < 2/(1-\nu) \tag{66}$$

or provided that the lowest resonant angular frequency $\omega_0$ is such that

$$\omega_0 < R^{-1}\Big\{2E/[(1-\nu)\rho]\Big\}^{\frac{1}{2}}. \tag{67}$$

Equations (66) and (67) follow from Eq. (65) owing to the fact that the quantity $(r/a)$ cannot exceed unity.

SUMMARY

In this report, the secular equation giving the resonant frequencies of a freely vibrating shallow spherical membrane shell has been calculated by two means. The exact form of this secular equation, under the assumptions of the theory, is given by Eq. (40). An approximate form of the secular equation resulting from the Rayleigh-Ritz analysis is given by Eq. (61). The modal displacement functions for normal and tangential motions that are associated with the roots of the exact secular equation are respectively given by Eqs. (43) and (44). Corresponding to the roots of the approximate secular equation, is the expression for the normal modal displacement functions, given by Eq. (64b), and the expression for the tangential modal displacement functions, given by Eq. (64a).

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