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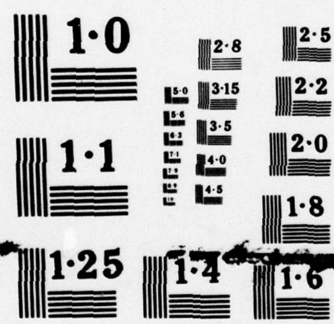
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PROPAGATION OF ONE-DIMENSIONAL WAVES IN INHOMOGENEOUS ELASTIC MEDIA

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Abstract: Formal progressing wave expansions are applied to problems involving one-dimensional wave propagation through inhomogenous elastic media. Expansions for the stress and particle velocity are obtained in addition to the expansion for the particle displacement which is a special case of previous results. Although one-dimensional problems could be solved with the previously reported asymptotic methods, it is more convenient to use the expansion in terms of the stresses to evaluate the expansion coefficients. The procedure is illustrated by solving several problems in layered and nonlayered inhomogeneous media where the compressional wave speed is subject to power and exponential variations with distance.

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Report Date is unavailable per Margaret Putnam (AFWL/SUL); therefore, the latest date appearing in the References is No. 13, 1966.

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1. INTRODUCTION

In recent years, the techniques of geometrical optics have been generalized to treat the electromagnetic wave equations, the acoustic wave equations, the elastic wave equations, and the viscoelastic wave equations, chiefly by workers at the Courant Institute of Mathematical Sciences. The method was originally applied by Luneburg [13] for time periodic waves in the form of an asymptotic expansion in inverse powers of the frequency to show that the leading term of such an expansion for the electromagnetic wave equations is, in fact, the geometrical optics solution. It was shown by several authors [3, 7, 8, 10, 11] that subsequent terms give corrections to the geometrical theory. The method was generalized by Friedlander [3] to treat progressing wave forms. In this form the method is a convenient tool in studying the propagation of discontinuities.

Karal and Keller [5] extended the method to treat general progressing waves in inhomogeneous elastic media. In their analysis, the formulation was in terms of displacements and displacement potentials. For certain problems, it would be more convenient to have the formulation in terms of velocities or stresses. Hence, for the one-dimensional case, these results are presented here, and the method is demonstrated by making use of the stress formulation.

It should be pointed out that, while the procedure is formal, the results for the leading term are consistent with those obtained by the theory of weak solutions and other more fundamental means. For a more complete discussion of these methods see Keller [6] and references contained therein.

After the technique is developed, one-dimensional waves that propagate into an inhomogeneous elastic medium of constant density but varying compressional wave speed are studied. It is expected that these results may be useful for studying a surface generated wave as it propagates downward into the near surface regions of the earth's crust. Related problems, solved by different techniques, have been recently treated by Lindholm and Doshi [12], Whittier [15], Payton [13]. Other related references are contained in these papers.

The method for treating interfaces between different inhomogeneous media is then presented. As an example case, the method is applied to treat an inhomogeneous layer overlying a homogeneous half space, where a step pressure is applied to the free surface.

2. FORMULATION

The one-dimensional equation of motion for a linear elastic continuum, assuming small strains, is

$$\sigma_x = \rho(x)u_{tt} = \rho(x)v_t \tag{1}$$

where σ, ρ, u, and $v \equiv u_t$ are stress (positive in tension), density, particle displacement, and particle velocity; where x and t denote spatial position and time; and where a subscripted variable denotes partial differentiation with respect to that variable. Hooke's law for elastic media is given by

$$\sigma = E(x)\, u_x \tag{2}$$

where E is the modulus of elasticity. The variables $\rho(x)$ and $E(x)$ are assumed to be continuous, differentiable functions of x. Taking the partial derivative of (2) with respect to t, we obtain

$$\sigma_t = E(x)\, v_x \; . \tag{3}$$

Hence, the equations for one-dimensional elastic wave propagation in terms of displacements, velocities, and stresses are given by combining (1), (2), and (3) as

$$(E\, u_x)_x = \rho u_{tt} \quad , \tag{4}$$

$$(E\, v_x)_x = \rho v_{tt} \quad , \tag{5}$$

$$(\rho^{-1}\sigma_x)_x = E^{-1}\sigma_{tt} \quad . \tag{6}$$

We now assume that the displacement, velocity, and stress solutions may be represented by

$$u \sim \sum_{n=0}^{\infty} U_n(x)\, f_n(t-S(x)) \quad , \tag{7}$$

$$v \sim \sum_{n=0}^{\infty} V_n(x)\, f_n(t-S(x)) \quad , \tag{8}$$

$$\sigma \sim \sum_{n=0}^{\infty} P_n(x)\, f_n(t-S(x)) \quad . \tag{9}$$

The f_n's are related [3] by

$$f_n' = f_{n-1} \tag{10}$$

where the prime denotes total differentiation with respect to the argument. In this manner, all of the f_n's may be related to f_0 (called the waveform) by successive integrations. For example, if f_0 is the Heaviside step function, H(t), then

$$f_n = \frac{(t-S)^n}{n!} H(t-S) \quad . \tag{11}$$

Note that f_0 vanishes for negative argument, i.e., in front of the wavefront whose equation is given by $t = S(x)$, where $S(x)$ is called the phase function. For this case, note that the coefficients U_n, V_n or P_n are the jump conditions for the displacement, velocity, or stress and their derivatives across the wavefront. It is assumed that U_n, V_n, and P_n are identically zero for $n < 0$.

From the relationships in (2), (3), (7), and (9) with the definition $v = u_t$, it may be shown that the coefficients in the displacement, velocity, and stress series are related by

$$U_n = V_{n-1} \quad , \tag{12}$$

$$P_n = E(U'_n - S'U_{n+1}) \quad , \tag{13}$$

$$P_n = E(V'_{n-1} - S'V_n) \quad . \tag{14}$$

3. SOLUTION

Consider the displacement solution first. If (7) is substituted into (4), then

$$\sum_{n=0}^{\infty} \{[(S')^2E - \rho]\, U_n f_{n-2} - [2ES'U_n' + (ES'' + E'S')U_n]\, f_{n-1}$$

$$+ [EU_n'' + E'U_n']\, f_n\} = 0 \ . \qquad (15)$$

Since $U_n = 0$ for $n < 0$, the summations may be changed so that each term is in the form of a coefficient times f_n. Then, because f_n is related to f_0 as illustrated by (11) and because of the arbitrariness of f_0, each coefficient may be equated to zero; hence

$$[(S')^2E - \rho]\, U_{n+1} - 2ES'U_n' - (ES'' + E'S')\, U_n$$

$$+ EU''_{n-1} + E'U'_{n-1} = 0 \ . \qquad (16)$$

Setting $n = -1$, noting that $U_n = 0$ for $n < 0$, and requiring without loss of generality that $U_0 \neq 0$, we obtain

$$(S')^2 = \rho/E = c^{-2} \qquad (17)$$

which may be recognized as the eiconal equation of gemoetrical optics where c(x) is the wave speed. Hence,

$$S(x) = \bar{S} \pm \int_{\bar{x}}^{x} \frac{dx}{c(x)} \tag{18}$$

where $\bar{S} \equiv S(\bar{x})$ is the initial value of the phase function, i.e., its value at $x = \bar{x}$. In (18), the plus sign is taken for waves which propagate in the positive x direction and the minus sign for waves which propagate in the negative x direction. Substituting (18) into (16), we obtain a linear first order ordinary differential equation whose solution is

$$U_n(x) = \bar{U}_n \sqrt{\frac{\bar{\rho}(\bar{x})\bar{c}(\bar{x})}{\rho(x)c(x)}} + \int_{\bar{x}}^{x} A_{n-1}(\tau) \sqrt{\frac{\rho(\tau)c(\tau)}{\rho(x)c(x)}} \, d\tau \tag{19}$$

where

$$A_{n-1} = \frac{\pm 1}{2\rho c} \frac{d}{dx} (EU'_{n-1}) \cdot \tag{20}$$

In a similar manner, the solution for the coefficients of (8) and (9) are

$$V_n(x) = \bar{V}_n \sqrt{\frac{\bar{\rho}(\bar{x})\bar{c}(\bar{x})}{\rho(x)c(x)}} + \int_{\bar{x}}^{x} B_{n-1}(\tau) \sqrt{\frac{\rho(\tau)c(\tau)}{\rho(x)c(x)}}\, d\tau \tag{21}$$

where

$$B_{n-1} = \pm \frac{1}{2\rho c} \frac{d}{dx} (EV'_{n-1}) , \tag{22}$$

and

$$P_n(x) = \bar{P}_n \sqrt{\frac{\rho(x)c(x)}{\bar{\rho}(\bar{x})\bar{c}(\bar{x})}} + \int_{\bar{x}}^{x} C_{n-1}(\tau) \sqrt{\frac{\rho(x)c(x)}{\rho(\tau)c(\tau)}}\, d\tau \tag{23}$$

where

$$C_{n-1} = \pm \frac{\rho c}{2} \frac{d}{dx} \left[\frac{P'_{n-1}}{\rho}\right] . \tag{24}$$

In the above expressions, the $\pm$ signs are associated with waves propagating in the $\pm x$ directions.

4. EXAMPLE PROBLEMS FOR INHOMOGENEOUS HALF-SPACES

Consider the half-space $x \geq 1$, defined by the properties $E = E(x)$ and $\rho = \bar{\rho}$ = constant, initially at rest. For $t > 0$, a pressure is applied to the surface $x = 1$ so that $\sigma(1,t)$ is known. The solution will first be obtained for a step pressure input and Duhamel's theorem will be applied to obtain the solution for a general input pressure function. Thus, from (9)

$$\sigma(1,t) \sim \sum_{n=0}^{\infty} \bar{P}_n(1)\, f_n(t-\bar{S}) = -H(t) \tag{25}$$

where bars indicate that the variables are evaluated at $x = \bar{x} = 1$ and $H(t)$ is the Heaviside step function. Thus,

$$\begin{aligned} \bar{P}_n &= -1 \quad, && n = 0 \quad, \\ &= 0 \quad, && n > 0 \text{ and } n < 0 \quad, \\ \bar{S} &= 0 \quad, && \\ f_0 &= H(t) \quad. && \end{aligned} \tag{26}$$

The phase function S is then determined from (18) and the general f_n is obtained from (11). Since the density will be constant in the example cases, C_{n-1} from (24) reduces to

$$C_{n-1} = \frac{1}{2} c(x) P''_{n-1} \quad . \tag{27}$$

Case I. $c(x) = \bar{c}\, x^{2\alpha}$; $\rho = \bar{\rho}$

In this case, $E = \bar{\rho}\, \bar{c}^{-2} x^{4\alpha}$ and $C_{n-1} = \frac{\bar{c}x^{2\alpha}}{2} P''_{n-1}$. From (18), it may be shown that

$$S(x) = \frac{x^{-2\alpha + 1} - 1}{\bar{c}(1 - 2\alpha)} \quad , \qquad \alpha \neq \frac{1}{2} \quad ,$$

$$= \frac{1}{\bar{c}} \log x \quad , \qquad \alpha = \frac{1}{2} \quad . \tag{28}$$

From (23), (25), and (27), it may be shown by induction that if $\alpha \neq \frac{1}{2}$

$$P_n = x^{\alpha} \sum_{j=0}^{n} A_{jn}\, x^{(2\alpha-1)j} \quad , \qquad \alpha \neq \frac{1}{2} \quad . \tag{29}$$

Substituting (29) into (23) and simplifying, we obtain

$$A_{jn} = \frac{\bar{c}}{2} A_{j-1,\, n-1} \frac{[\alpha + 2\alpha-1)(j-1)][-\alpha + j(2\alpha-1)]}{j(2\alpha-1)} , \quad 1 \le j \le n ,$$

$$= -\frac{\bar{c}}{2} \sum_{j=1}^{n} A_{j-1,n-1} \frac{[\alpha + (2\alpha-1)(j-1)][-\alpha + j(2\alpha-1)]}{j(2\alpha-1)} , \quad j = 0,\ n > 0,$$

$$= -1 , \qquad j = n = 0 ,$$

$$= 0 , \qquad j < 0 \text{ or } j > n ,$$

where $A_{j,n} \equiv A_{jn}$. If $\alpha = \frac{1}{2}$, it may again be shown by induction that

$$P_n = \sqrt{x} \sum_{j=0}^{n} A_{jn} (\log x)^j , \qquad \alpha = \frac{1}{2} , \tag{31}$$

where

$$A_{jn} = \frac{\bar{c}}{2} [(j+1) A_{j+1,\, n-1} - \frac{1}{4j} A_{j-1,\, n-1}] , \quad 1 \le j \le n ,$$

$$= 0 , \quad j \le 0,\ n > 0 \text{ or } j > n ,$$

$$= -1, \quad j = n = 0 . \tag{32}$$

All possible values of α have been included in the above results. In particular, the case $\alpha = 1$ is of interest because it may be shown from

(30) that $A_{00} = -1$ and $A_{jn} = 0$ for all other values of j and n. Hence, the series terminates after a single term,

$$\sigma = -x\, H\,[t - (x-1)/\bar{c}x]\ ,\ \alpha = 1, \tag{33}$$

which may be shown to be the exact solution for this case. Note, in this case, that the stress is unbounded for increasing x, and that, for $\bar{c}t > 1$, $\sigma = -x$ since $(x-1)/x < 1$ for all positive x. That the stress should grow without bound is non-physical. However, we note that the assumption $c = \bar{c}x^2$ is also non-physical for unbounded x. Hence, if the model $c = \bar{c}x^2$ is realistic for $x \leq L < \infty$, it is expected that the solution (30) is applicable for $x \leq L < \infty$.

From the above results, it is seen that the stress amplitude may either increase or decrease with increasing x depending on the value of α. In the case $\alpha = 0$, the above results reduce to the homogeneous elastic case where the stress level is independent of the distance traveled. For $\alpha > 0$, the stress level increases and for $\alpha < 0$, the stress level decreases with increasing x.

Case II. $c = \bar{c}\exp\{2\beta(x-1)\}$, $\rho = \bar{\rho}$

In this case, $E = \bar{\rho}\,\bar{c}^2 \exp\{4\beta(x-1)\}$ and $C_{n-1} = \frac{\bar{c}}{2}\exp\{2\beta(x-1)\}$ P''_{n-1}. From (18), it may be shown that

$$S(x) = \frac{1 - \exp\{-2\beta(x-1)\}}{2\beta\bar{c}} \quad . \tag{34}$$

From (23), (25), and (27), it may be shown by induction that

$$P_n = \exp\{\beta(x-1)\} \sum_{j=0}^{n} A_{jn} \exp\{2\beta j(x-1)\} \ . \tag{35}$$

Substituting (35) into (23) and simplifying, we obtain

$$\begin{aligned} A_{jn} &= \frac{(2j-1)^2 \bar{c}\beta}{4j} A_{j-1,\, n-1} \ , & 1 \le j \le n \ , \\ &= \frac{-\bar{c}}{4} \sum_{j=1}^{n} \frac{(2j-1)^2 \beta}{j} A_{j-1,\, n-1} \ , & j = 0,\ n > 0 \ , \\ &= -1 \ , & j = n = 0 \ , \\ &= 0 \ , & j < 0 \text{ or } j > n \ . \end{aligned} \tag{36}$$

Again, the above solution reduces to the limiting homogeneous elastic case for $\beta = 0$. As x increases, the stress grows if $\beta > 0$ and attenuates if $\beta < 0$.

In the above examples, the response was determined for a step input compressive stress at $x = 1$. The stress solution has the form

$$\sigma_H \sim \sum_{n=0}^{\infty} P_n(x) \frac{[t - S(x)]^n}{n!} H[t - S(x)] \tag{37}$$

where $P_n(x)$ is given by (29), (31), or (35) and $S(x)$ is given by (28) or (34). In general, the solution for an arbitrary boundary stress $\sigma(1,t)$ may be determined from Duhamel's theorem

$$\sigma = \int_0^t \frac{\partial \sigma_H(\tau)}{\partial \tau} \sigma(1, t - \tau)\, d\tau. \tag{38}$$

5. REFLECTION AND TRANSMISSION OF WAVES AT AN INTERFACE

Consider a stress wave, traveling in the direction of increasing x in a medium "a," incident at an interface, $x = \bar{x}$, between medium "a" and medium "b." Let the compressional wave speed and density of the two media be denoted by c_a, c_b, ρ_a, and ρ_b. It is assumed that a wave is transmitted and a wave is reflected. Due to the linearity of the equations, the stress field can be obtained by superposition. The boundary conditions at the interface require that $\bar{\sigma}_a = \bar{\sigma}_b$ and $\bar{v}_a = \bar{v}_b$. Assume that

$$\sigma_a = \sum_{n=0}^{\infty} P_{1n} f_n (t - S_1) + \sum_{n=0}^{\infty} P_{2n} f_n (t - S_2),$$

$$\sigma_b = \sum_{n=0}^{\infty} P_{3n} f_n (t - S_3), \tag{39}$$

$$v_a = \sum_{n=0}^{\infty} V_{1n} f_n (t - S_1) + \sum_{n=0}^{\infty} V_{2n} f_n (t - S_2) \quad ,$$

$$v_b = \sum_{n=0}^{\infty} V_{3n} f_n (t - S_3) \quad . \tag{40}$$

Where the subscripts, $i = 1, 2, 3$, denote the incident, reflected, and transmitted waves respectively. Since the form for all of the waves is that represented in the previous sections, the solution is known providing the initial values, $\bar{S}_i$, $\bar{P}_{in}$, $\bar{V}_{in}$, are known. It is assumed that the incoming wave has been defined by the methods of the previous section. Hence, $\bar{S}$, and $\bar{P}_{1n}$ are known.

From (14) and (17) it can be deduced that

$$\rho c V_{jn} = (-1)^j (P_{jn} - F_{j,n-1}) \quad j = 1,2,3 \; , \tag{41}$$

where $$F_{j,n-1} \equiv \rho c^2 V'_{j,n-1} \tag{42}$$

and $S' = c^{-1}$ for the outgoing waves and $S' = -c^{-1}$ for incoming wave. Note that $F_{j,n-1}$ is a function of the coefficients P_k where $k \leq n-1$. In particular, (41) yields the well known result from the conservation of momentum across a steadily moving discontinuity, $P_{j0} = -\rho c|V_{j0}|$.

When equations (39)-(42) are combined with the boundary conditions, $\bar{\sigma}_a = \bar{\sigma}_b$ and $\bar{v}_a = \bar{v}_b$, we find that

$$\bar{S}_1 = \bar{S}_2 = \bar{S}_3 \ , \tag{43}$$

and

$$\begin{aligned} \bar{P}_{2n} - \bar{P}_{3n} &= -\bar{P}_{1n} \ , \\ \bar{P}_{2n} + \bar{r}\bar{P}_{3n} &= \bar{P}_{1n} - \bar{F}_{1,n-1} + \bar{F}_{2,n-1} + \bar{r}\bar{F}_{3,n-1}, \end{aligned} \tag{44}$$

where

$$\bar{r} = \bar{\rho}_a \ \bar{c}_a / \bar{\rho}_b \ \bar{c}_b \ . \tag{45}$$

Hence:

$$\begin{aligned} \bar{P}_{2n} &= [(1-\bar{r}) \ \bar{P}_{1n} - \bar{F}_{1,n-1} + \bar{F}_{2,n-1} + \bar{r} \ \bar{F}_{3,n-1}] \ (1 + \bar{r})^{-1} \ , \\ \bar{P}_{3n} &= [2\bar{P}_{1n} - \bar{F}1,n-1 + \bar{F}_{2,n-1} + \bar{r} \ \bar{F}_{3,n-1}] \ (1 + \bar{r})^{-1} \end{aligned} \tag{46}$$

Thus, since $\bar{S}_1$ and $\bar{P}_{1n}$ are known, the initial values at the interface $(x = \bar{x})$ are determined and the procedures of the preceding section may be applied to obtain the formal solution. One should be careful to choose the proper sign in (18), (20), (22), and (24) so that the direction of propagation is accounted for. It will be noted that the above results reduce to the limiting homogeneous elastic results in the proper way.

One interesting result from (46) is that, in the limiting case of $\bar{r} = 1$, there is no reflected wave only if $\bar{F}_{1,n-1} = \bar{F}_{2,n-1} = \bar{F}_{3,n-1}$ for all $n \geq 1$. Thus, even though the impedance is continuous at $\bar{x}$ it is still possible that a wave will be reflected if the impedance of medium b varies in a different manner than that of medium a. Note that, in the case $\bar{\rho}_a \bar{c}_a = \bar{\rho}_b \bar{c}_b$, the reflected wave has no discontinuity since $\bar{P}_{20} = 0$. Rather, its amplitude increases from zero at the front in a manner which depends on the functions representing the impedance in media a and b. This general result was pointed out for a particular case by Payton [13].

As an example case, consider that for $1 \leq x \leq x_1$, $\rho = \bar{\rho}$ and $c = \bar{c}x^2$ where $\bar{c} = c$ at $x = 1$. For $x \geq x_1$ let $c = cx_1^2$ and $\rho = \bar{\rho}$. Thus, the interface is between an inhomogeneous layer overlying a homogeneous elastic half space. Further, the impedance is continuous at the interface. Consider the result when a step pressure is applied at the surface $x = 1$. The incident stress wave is given by (33). The stress field is given by the first of (39) in the layer and by the second of (39) in the half space. This solution will apply until the time that the reflected wave reaches the surface $x = 1$. At $x = x_1$, the phase functions are

$$\bar{S}_1 = \bar{S}_2 = \bar{S}_3 = \frac{x_1 - 1}{\bar{c}x_1} \tag{47}$$

and from (18)

$$S_2 = \frac{x_1(1 + x) - 2x}{\bar{c}\, x_1\, x} \quad ; \quad S_3 = \frac{x_1^2 - 2x_1 + x}{\bar{c}\, x_1^2} \quad . \tag{48}$$

It may be shown from (41) and (46) that, since $P_{10} = -x$ and $P_{1n} = 0$ for $n > 0$, $F_{3n} = 0$ for $n \geq 0$; $F_{1n} = 0$ for $n \geq 1$; $\bar{P}_{2n} = \bar{P}_{3n} = \bar{F}_{2n-1}/2$ for $n \geq 1$; and

$$\begin{aligned} P_{2n} &= -(-\bar{c}x_1/2)^n x \quad , & n \geq 1 \\ &= 0 \quad , & n \leq 0 \end{aligned}$$

$$\begin{aligned} P_{3n} &= -(-\bar{c}x_1/2)^n x_1 \quad , & n \geq 1 \\ &= -x_1 \quad , & n = 0 \\ &= 0 \quad , & n < 0 \end{aligned} \tag{49}$$

Thus, the solution is determined by substitution of (48) and (49) into (39). As was pointed out previously this solution is applicable only until the time that the reflected wave strikes the surface x = 1. At this point the procedure may be repeated for that boundary and so on.

6. SUMMARY

A formal procedure for obtaining solutions for one-dimensional wave propagation in inhomogeneous elastic media has been presented and applied to two general cases. There is no proof that the resulting expansions are convergent to the exact solution; however, in all cases where exact solutions are known and where the technique has been applied, it has been found that the correct asymptotic solutions have been given. In one of the example cases treated here, the series terminated and a closed-form exact solution was obtained.

The method of treating a single interface between different inhomogeneous media was presented in detail. Multiple interfaces may be treated in exactly the same way. One interesting result was that reflected waves may be generated in the case of an incident wave at the point where two different inhomogeneous media are joined even though the impedances at the interface are matched.

The method, while tedious, is direct and involves only ordinary differentiation, integration, and algebra. The terms of the series are evaluated recursively once the initial values of the coefficients and the phase function (the barred variables) are known. These initial values are determined from application of the boundary conditions in a straight forward manner.

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