⑫ (D.S.) **LEVEL** II

⑥ On Extremes of Stationary Processes

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⑨ Technical rept.

Institute of Statistics Mimeo Series #1194

⑪ September 1978

⑫ 25 p.

⑭ MMS-1194

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On Extremes of Stationary Processes

by

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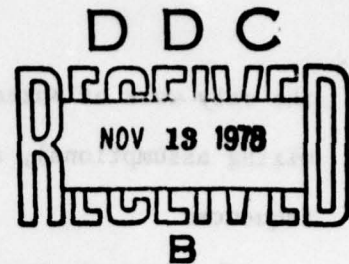
Abstract. Certain aspects of extremal theory for (a) stationary sequences and (b) continuous parameter stationary processes, are discussed in this paper. A slightly modified form of a previously used dependence condition, leads to simple proofs of some key results in extremal theory of stationary sequences. Dependence conditions of a "weak mixing" type are introduced for continuous parameter stationary processes and results of classical extreme value theory extended to that context.

Introduction.

Even though its roots reach much further back into mathematical antiquity, the field of classical extreme value theory may be regarded as half a century old - really beginning from the work of Frechet [8], Fisher and Tippet [7], and somewhat later given an extensive development by Gnedenko [9]. In particular, Gnedenko rigorously proved the central result that the only possible nondegenerate limits of maxima of i.i.d. sequences (under linear normalizations) are the three so called *extreme value distributions* recognized earlier in [8] and [7]. This result - stated explicitly in the next section - will be here referred to as *Gnedenko's Theorem*.

In more recent years there has been a developing interest in extending the results to include *dependent* sequences. Investigations have taken two main lines - the extension of general theory to certain dependent sequences (beginning with

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the only work of Watson [16] under m-dependence, and Loynes [14] under strong

mixing assumptions), and the detailed theory (Berman [1]) for stationary *normal*

sequences.

The general theory for stationary sequences has been further studied by a

number of authors under various dependence restrictions. In particular use has

been made of weak "distributional mixing conditions" suggested in [11].

Under these conditions it may be shown that not only does Gnedenko's Theorem

still hold, but that in many cases the limiting law is the same as would

occur if the terms of the sequence were *independent* with the same marginal

distributions as the original sequence. Further, recent work of R. Davis ([4])

has made possible a yet more complete and satisfying statement of these results.

One of the purposes of the present paper is to show how a slight modification

of the previously used dependence restrictions, leads to very simple proofs for

these existing results. This is described in Section 2.

Our main interest here, however, is to study the behavior of extremes of

continuous parameter stationary processes, which we shall do by relating to the

sequence case. A detailed theory is already in existence for stationary *normal*

processes (involving many studies, begun by the work of Cramér [2],[3]), where

it is known that the so-called "double exponential" extremal limit applies.

Our object here is to determine circumstances under which extremal theory may

be extended to general continuous parameter stationary processes and in parti-

cular, to obtain a form of Gnedenko's Theorem in that context.

These tasks are undertaken in Sections 3-5 where weak mixing conditions-

analogous to those for sequences - are defined and used. As will be seen, the

asymptotic form of the mean number $\mu(x)$ of upcrossings of a high level x per

unit time plays a central role in the discussion and, in particular, in

determining domains of attraction (replacing the tail $1-F(x)$ of the marginal

distribution in the sequence case). We regard this as one of the most

interesting consequences of the theory, and one of potential practical useful-
ness. At the same time this automatically restricts attention to cases for
which such means exist (excluding, for example, the more irregularly behaved
stationary normal processes with infinite second spectral moments). It is
likely that the theory can be extended at the expense of complication (for
example by consideration of so-called "ϵ-upcrossings") but we do not do so here.

Finally we note that we have chosen particular dependence restrictions from
a variety of possible similar conditions. It is certainly possible that differ-
ent forms of these may prove preferable as the theory develops further.

1. Two Results from Classical Extreme Value Theory.

The field of classical extreme value theory deals substantially with
asymptotic distributional questions surrounding $M_n = \max(\xi_1, \xi_2, \ldots, \xi_n)$ as $n \to \infty$,
where $\xi_1, \xi_2, \ldots,$ are i.i.d. random variables (with common d.f. F). That is,
one is first interested in sequences $\{u_n\}$ for which $P\{M_n \leq u_n\}$ converges to a
limit as $n \to \infty$. In this i.i.d. situation it is almost trivial to show that if

$$(1.1) \qquad n(1 - F(u_n)) \to \tau > 0 \text{ as } n \to \infty$$

then

$$(1.2) \qquad P\{M_n \leq u_n\} \to e^{-\tau}$$

and conversely.

The main body of the i.i.d. distributional theory is directed towards the
further study of when such convergence will occur for every x when
$u_n = u_n(x) = x/a_n + b_n$, $a_n(> 0)$ and b_n being constants. Rephrased, we ask
under what circumstances it is true that

$$(1.3) \qquad P\{a_n(M_n - b_n) \leq x\} \to G(x)$$

at each real x (or at least for continuity points of G). Clearly such a G is

non-decreasing with values between zero and one. It is, in fact, quite readily

shown (cf. [5] or [12]) that if G is a non-degenerate d.f., then it must be

max stable in the sense that for each n=1,2,... there exist constants $\alpha_n > 0, \beta_n$

such that $G^n(x) = G(\alpha_n x + \beta_n)$ for all x. Further it is not too difficult to

show (cf. [5], [12] that the "max stable" distributions consist precisely of

the following three general "types" (the so called "extreme value types").

Type I: $G(x) = \exp(-e^{-x})$, $-\infty < x < \infty$

Type II: $G(x) = \exp(-x^{\alpha})$ $(\alpha > 0)$, $x > 0$ (zero for $x < 0$)

Type III: $G(x) = \exp(-(-x)^{\alpha})$ $(\alpha > 0)$, $x \leq 0$ (1 for $x > 0$)

(In these x may be replaced by $ax + b$ for any $a > 0$, b) .

From these results we see that the only non-degenerate d.f.'s which may

occur as limiting distributions of maxima of i.i.d. sequences as in (1.3), are

the three extreme value types. As noted above, we refer to this as *Gnedenko's*

Theorem.

Further, classical extreme value theory provides necessary and sufficient

conditions on F to ensure that it gives rise to any one G of the three possible

types (i.e. that F "belongs to the domain of attraction of G"). These condi-

tions mainly involve the behavior of the tail 1-F(u) of F as u increases. For

example if 1-F(u) is regularly varying with exponent $-\alpha$ as $u \to \infty$, then F is

attracted to the type II limit law (cf. [5]).

2. Dependent Sequences.

Much of the classical theory may be extended to apply to (stationary)

dependent sequences if the dependence is not too strong. One suitable depen-

dence restriction is that of *strong mixing* under which Gnedenko's Theorem

certainly still holds ([14]). However, strong mixing imposes a uniformity

which is not needed in this context, and weaker forms of mixing will, in fact,

suffice. One convenient such condition is the following which we shall call the "distributional mixing condition $D(u_n)$". Specifically, write $F_{i_1 i_2 \ldots i_n}$ to denote the joint d.f. of $\xi_{i_1} \ldots \xi_{i_n}$, $F_{i_1 \ldots i_n}(u) = F_{i_1 \ldots i_n}(u, u, \ldots u)$, and let $\{u_n\}$ be a real sequence. Then $D(u_n)$ is said to hold if for any choice of n, p, p', $p + p' \leq n$, $i_1 < i_2 \ldots < i_p < j_1 \ldots < j_p$, $j_1 - i_p = \ell$.

$$(2.1) \qquad \left| F_{i_1 \ldots i_p j_1 \ldots j_p}(u_n) - F_{i_1 \ldots i_p}(u_n) F_{j_1 \ldots j_p}(u_n) \right| < \alpha_{n,\ell}$$

where $\alpha_{n,\ell_n} \to 0$ for some sequence $\ell_n = o(n)$.

The following lemma is fundamental in studying the extremes of dependent sequences. This lemma is a modified version of Lemma 2.5 of [11] to which we refer for proof (cf. also [12] Lemma 2.3).

Lemma 2.1. *Let the stationary sequence* $\{\xi_n\}$ *satisfy* $D(u_n)$, *for a given sequence* $\{u_n\}$ *of constants. Then*

$$P\{M_n \leq u_n\} - P^k\{M_{[n/k]} \leq u_n\} \to 0 \quad \text{as} \quad n \to \infty$$

for each integer k ([n/k] *denoting the integer part of* n/k *and*
$M_n = \max(\xi_1, \xi_2, \ldots, \xi_n))$.

The proof of this result - contained in the references cited above - involves the standard technique used under mixing conditions of considering k slightly separated subsets $I_1 \ldots I_k$ of $(1 \ldots n)$ and showing that the maxima on these subsets are approximately independent. □

By means of this result, it is almost trivial to extend Gnedenko's Theorem to include a wide variety of stationary sequences. While this is proved in [11] we sketch a simplified proof, here based on Lemma 2.1.

__Theorem 2.2.__ *Let* $\{\xi_n\}$ *be a stationary sequence,* $M_n = \max(\xi_1, \ldots, \xi_n)$ *and suppose that for some constants* $\{a_n > 0\}$, $\{b_n\}$ *we have*

$$P\{a_n(M_n - b_n) \leq x\} \to G(x)$$

where G *is a non-degenerate d.f. If* $D(u_n)$ *holds with* $u_n = x/a_n + b_n$, *for each* x, *then* G *is one of the three extreme value forms.*

Sketch of proof: Writing $u_n = x/a_n + b_n$ we have $P\{M_n \leq u_n\} \to G(x)$ and hence by Lemma 2.1, $P\{M_{[n/k]} \leq u_n\} \to G^{1/k}(x)$ for each fixed k. This is true with nk replacing n so that $P\{M_n \leq u_{nk}\} \to G^{1/k}(x)$ or, rephrasing,

$$P\{a_{nk}(M_n - b_{nk}) \leq x\} \to G^{1/k}(x) .$$

Thus for a fixed k, M_n has the non-degenerate limit $G^{1/k}$ with normalizing constants a_{nk}, b_{nk} as well as the limit G (with normalizing constants a_n, b_n). But a well known result of Khintchine (cf [6] and [12]) shows that if two such non-degenerate limits exist, they must be the same apart from a linear transformation of the argument, i.e. $G(x) = G^{1/k}(\alpha_k x + \beta_k)$ for some $\alpha_k > 0$, β_k. This shows that G is max stable and hence an extreme value d.f. by the remarks regarding the classical case in Section 1.   ⊔

The other result of the classical theory quoted - the equivalence of (1.1) and (1.2), also extends to this dependent context, but requires a further restriction. One such convenient condition - which we refer to as $D'(u_n)$ - is the following

$$(2.2) \quad (D'(u_n)) \qquad \limsup_{n \to \infty} n \sum_{j=2}^{[n/k]} P\{\xi_1 > u_n, \xi_j > u_n\} \to 0 \quad \text{as} \quad k \to \infty$$

(in which [ ] denotes the integer part). The first part of the following result has been known for some years ([11]). A version of the converse part was obtained recently by R. Davis [4] under a slightly different D'-condition. Here we give a very simple proof using the present form of (2.2).

Theorem 2.3. *Suppose that* $D(u_n)$, $D'(u_n)$ *hold for the stationary sequence* $\{\xi_n\}$, *where* $\{u_n\}$ *is a given sequence of constants. If* (1.1) *holds* (i.e. $n P\{\xi_1 > u_n\} \to \tau$) *then* (1.2) *holds* (i.e. $P\{M_n \le u_n\} \to e^{-\tau}$) *and conversely.*

Proof: Fix k and write, for each n, $n' = \lfloor n/k \rfloor$. Using the fact that $\{M_{n'} > u_n\} = \bigcup_{j=1}^{n'} \{\xi_j > u_n\}$, standard inequalities for the probability of a union, and stationarity, it is simply shown that

$$(2.3) \qquad 1 - n'(1 - F(u_n)) \le P\{M_{n'} \le u_n\} \le 1 - n'(1 - F(u_n)) + S_n$$

where $S_n = n' \sum_{j=2}^{n'} P\{\xi_1 > u_n, \xi_j > u_n\}$ and $\limsup_{n \to \infty} S_n = o(k^{-1})$ as $k \to \infty$ by $D'(u_n)$.

If (1.1) holds, $1 - F(u_n) \sim \tau/(n'k)$, so that $n \to \infty$ in (2.3) gives

$$1 - \tau/k \le \liminf_{n \to \infty} P\{M_{n'} \le u_n\} \le \limsup_{n \to \infty} P\{M_{n'} \le u_n\} \le 1 - \tau/k + o(k^{-1}) .$$

By taking kth powers throughout, using Lemma 2.1, and letting $k \to \infty$ we see that (1.2) holds.

Conversely suppose that (1.2) holds. Then (2.3) gives

$$1 - P\{M_{n'} \le u_n\} \le n'(1 - F(u_n)) \le 1 - P\{M_{n'} \le u_n\} + S_n .$$

where $P\{M_{n'} \le u_n\} \to e^{-\tau/k}$ by (1.2) and Lemma 2.1 so that we obtain

$$1 - e^{-\tau/k} \le k^{-1} \liminf_{n \to \infty} n(1 - F(u_n)) \le k^{-1} \limsup_{n \to \infty} n(1 - F(u_n))$$

$$\le 1 - e^{-\tau/k} + o(1/k)$$

from which by multiplying by k and letting $k \to \infty$, (1.1) follows. □

The importance of this result may be seen from the following corollary, which shows in particular that under D, D' conditions the same limit (if any) occurs in the dependent case as would occur if the sequence were i.i.d. That is the maximum for the stationary sequence then has the same asymptotic distribution as it would if the individual terms were independent with the same marginal d.f. *Hence the tail of this d.f. may still be used in criteria to determine domains of attraction.* In this corollary $\hat{M}_n$ will be used to denote the maximum of n independent random variables, each having the same marginal d.f. F as the ξ_i.

__Corollary__: *Suppose that $D(u_n)$, $D'(u_n)$ hold for the stationary sequence $\{\xi_n\}$. Then $P\{M_n \leq u_n\} \to \rho > 0$ if and only if $P\{\hat{M}_n \leq u_n\} \to \rho$.*

In particular suppose that for some constants $a_n > 0$, b_n, $D(u_n)$, $D'(u_n)$ hold with $u_n = x/a_n + b_n$ for each x, and G is a non-degenerate d.f. Then

$$P\{a_n(M_n - b_n) \leq x\} \to G(x)$$

if and only if

$$P\{a_n(\hat{M}_n - b_n) \leq x\} \to G(x)$$

and G is, of course, necessarily then one of the extreme value types.

__PROOF__. The first part follows at once from the equivalence of (1.1) and (1.2) writing ($\rho = e^{-\tau}$) in both independent and dependent cases. The second part follows by identifying ρ and G(x) where G(x) > 0 (and using continuity where G = 0 - each extreme value d.f. being continuous). []

Finally in this section we note that the conditions $D(u_n)$, $D'(u_n)$ are satisfied when $\{\xi_n\}$ is a stationary *normal* sequence under appropriate restrictions on the covariance function. The simplest of these is that the covariance

function $\{r_n\}$ should satisfy $r_n \log n \to 0$ as $n \to \infty$, but even weaker conditions yet are known cf. [13]).

3. Continuous Parameter Processes - General Framework.

We turn attention now to a stationary process $\{\xi(t): t \geq 0\}$ in continuous "time". It will be assumed, without further comment, that $\xi(t)$ has (a.s.) continuous sample functions, and continuous one-dimensional distributions. If E is any set of real numbers we write $M(E) = \sup\{\xi(t): t \in E\}$, and $N_u(E)$ for the number of upcrossings of a level u by $\xi(t)$ in the set E. It will be convenient to write $M(t)$, $N_u(t)$ when $E = [0,t]$.

Our interest lies in asymptotic distributional properties of $M(T)$ when T becomes large. Since for an integer n

(3.1) $M(n) = \max(\zeta_1, \ldots, \zeta_n)$

where

(3.2) $\zeta_i = \max\{\xi(t): i - 1 \leq t \leq i\}$

it is reasonable to expect that the properties of $M(T)$ may be approached via the theory for sequences. Of course the members of the sequence are *not* now the original r.v.'s of the process, but their *maxima over fixed intervals*. Hence our basic distribution $P\{\xi_1 \leq x\}$ of the sequence case must be replaced by the distribution of ζ_1 i.e. $P\{\zeta_1 \leq x\} = P\{\max(\xi(t): 0 \leq t \leq 1) \leq x\}$. Since it is the tail of this distribution which plays a crucial role (e.g. in (1.1)), one of our first tasks must be to discuss the asymptotic behavior of $P\{M(1) > u\}$, i.e. the tail probability for the maximum over a *fixed* interval. The other task - to obtain convenient distributional mixing conditions on the $\xi(t)$ to ensure that appropriate $D(u_n)$, $D'(u_n)$ conditions hold for the ζ_i sequence - will be taken up in Section 5.

Before embarking on these tasks it is convenient to give the following form of Gnedenko's Theorem assuming that the "partial maxima" ζ_i satisfy appropriate $D(u_n)$ - conditions. A more complete statement will be given in Section 5 where, as noted, conditions on the *original* process $\xi(t)$ will be obtained.

Theorem 3.1. *Suppose that for some families of constants* $\{a_T > 0\}$, $\{b_T\}$ *we have*

$$(3.3) \qquad P\{a_T(M(T) - b_T) \leq x\} \to G(x), \text{ as } T \to \infty$$

(for a non-degenerate d.f. G), and that the $\{\zeta_i\}$ *sequence defined by (3.2) satisfies* $D(u_n)$ *whenever* $u_n = x/a_n + b_n$ *(for each fixed real x). Then G is one of the three extreme value types.*

Proof. (3.3) must hold, in particular, as $T \to \infty$ through integral values. Since the ζ_n-sequence is clearly stationary, the result follows by replacing ξ_n by ζ_n in Theorem 2.2, and using (3.1). []

Corollary: *The result holds in particular if the* $D(u_n)$ *conditions are replaced by the assumption that* $\{\zeta(t)\}$ *is strongly mixing. For then the sequence* $\{\zeta_i\}$ *is strongly mixing and thus satisfies the* $D(u_n)$*-condition.* []

4. Maxima over Fixed Intervals.

As noted above, the trail of the distribution of the maximum of $\xi(t)$ in a fixed interval (e.g. [0,1]), plays a crucial role with respect to the asymptotic distributional properties of M(T) as $T \to \infty$. Accordingly we consider here the behavior of $P\{M(h) > u\}$ as $u \to \infty$ for a fixed h. We shall relate this simply to the mean number of upcrossings of the level u, using the notation $N_u(E)$, $N_u(t)$ as defined above. Here we assume the finiteness of first and second moments of such quantities as $N_u(h)$. These would usually be checked from known formulae (cf. [10]) giving these moments in terms of distributions of ξ and its

derivative (cf. Eqn. 4.4 below). In cases where these moments are not finite, it appears possible to couch the discussion in terms of the so-called "ϵ-upcrossings" ([15]), but we do not attempt to do so here, for the reasons expressed in the Introduction.

For $t > 0$, write

$$(4.1) \qquad I_t(u) = P\{\xi(0) < u < \xi(t)\}/t.$$

It is easily checked that if (for fixed u) a sequence $q_n \to 0$ may be found such that $\liminf\limits_{n\to\infty} I_{q_n}(u) < \infty$, then $\lim\limits_{q\to 0} I_q(u)$ exists, finite, and

$$(4.2) \qquad \mu(= \mu(u)) = E\, N_u(0,1) = \lim_{q\to 0} I_q(u) .$$

We remark in passing that $I_t(u)$ may be simply written in terms of the distribution of $\zeta(0)$, $\xi(t)$ and that if this is absolutely continuous (4.2) may be transformed to give

$$(4.3) \qquad I_t(u) = \int_0^\infty dz \int_0^z p_t(u - tx, z)dx$$

where $p_t(x,z)$ is the joint density for $\xi(0)$ and $(\xi(t) - \xi(0))/t$. This leads easily - under appropriate conditions - to the well known expression

$$(4.4) \qquad \mu = \int_0^\infty z\, p(u,z)dz$$

where p is the joint density for $\xi(0)$ and its derivative $\xi'(0)$ (cf. [10]). For example this may be evaluated when ξ is a standard (zero mean, unit variance) stationary normal process with covariance function $r(\tau)$, to give
$$\mu = [(-r''(0))^{\frac{1}{2}}/2\,)]e^{-u^2/2}.$$

In our situation here, u is not usually fixed, and we shall want to let u converge to the right hand endpoint (finite or infinite) of the distribution F of $\xi(t)$. We shall use "$u \to \infty$" to denote this, with the understanding

that this is to be replaced by "u → b" if F has a finite right hand endpoint b.
We shall sometimes (though not in this section) need the further assumption
that q = q(u) may be chosen depending on u so that

$$(4.5) \qquad\qquad I_q(u) \sim \mu \quad \text{as } u \to \infty$$

(i.e. $\lim_{q' \to 0} I_{q-}(u) \sim I_{q(u)}(u)$ as $u \to \infty$). This assumption may be verified from
(4.3), for example in the case of stationary normal processes, under appropriate
conditions.

We also at times require the following condition

$$(4.6) \qquad\qquad E\, N_u(h)[N_u(h) - 1] = o(\mu) \quad \text{as } u \to \infty,$$

for a given fixed h. This is an intuitively appealing "regularity" condition
which asserts that $E\, N_u^2(h)$ is appropriately close to $E\, N_u(h)$ which will be the
case if $N_u(h)$ is zero or one with high probability, i.e. if $P\{N_u(h) > 1\}$ is
appropriately small for large u. (4.6) would be verified in practice by using
(4.4) and the corresponding result giving $E\, N_u(h)(N_u(h) - 1)$ in terms of appro-
priate joint densities ([10]).

The following result gives the desired tail distribution for the maxima of
$\xi(t)$ over fixed intervals.

Theorem 4.1. *Let h be fixed. Suppose that (4.6) holds for the stationary
process $\{\xi(t)\}$, and that $P\{\xi(0) > u\} = o(\mu)$ as $u \to \infty$. $(\mu = \mu(u) = E\, N_u(0,1))$.
Then*

$$(4.7) \qquad\qquad P\{M(h) > u\} \sim \mu h \quad \text{as } u \to \infty.$$

Proof. **Clearly**

$$(4.8) \qquad P\{N_u(h) \geq 1\} \leq P\{M(h) > u\} \leq P\{\xi(0) > u\} + P\{N_u(h) \geq 1\}$$

$$\leq \mu h + o(\mu)$$

by assumption, and since $P\{N_u(h) \geq 1\} \leq E\, N_u(h) = \mu\, h$. Now if $p_j = P\{N_u(h) = j\}$, we have

$$E\, N_u(h)[N_u(h) - 1] = \sum_{j=2}^{\infty} j(j-1)p_j \geq \sum_{j=2}^{\infty} jp_j = \mu\, h - P\{N_u(h) = 1\}$$

so that

$$P\{N_u(h) \geq 1\} \geq P\{N_u(h) = 1\} \geq \mu\, h - E\, N_u(h)[N_u(h) - 1]$$
$$= \mu\, h - o(\mu)$$

by (4.6). Hence from (4.8),

$$\mu\, h - o(\mu) \leq P\{M(h) > u\} \leq \mu\, h + o(\mu)$$

from which the result follows.   []

From this result with $h = 1$ we see at once that $P\{\zeta_1 > u\} \sim \mu$ (with ζ_1 defined by (3.2)). Hence if assumptions on $\xi(t)$ can be found under which the ζ_i satisfy appropriate $D(u_n)$, $D'(u_n)$ conditions, *then the sequence theory applies to M(n) given by (3.1) provided the function* $\mu(u)$ *is used in lieu of the tail of the distribution* (e.g. in (1.1), or in classical criteria for the domains of attraction). As with Gnedenko's Theorem (Theorem 3.1) it will be convenient to state these results explicitly here under D (and D') assumptions for the ζ_i's, completing these results in Section 5 where the dependence conditions will be recast in terms of the original process $\{\xi(t)\}$.

<u>Theorem 4.2</u>. *Suppose that the conditions of Theorem 4.1 are satisfied by the stationary process* $\{\xi(t)\}$ *with* $h = 1$, *let* $\{u_n\}$ *be a sequence of constants, and write* $u_T = u_{[T]}$ *for each* $T > 0$. *Suppose that the sequence* $\{\zeta_n\}$ *(defined by (3.2)) satisfies* $D(u_n)$, $D'(u_n)$. *Then, for a given* $\tau > 0$,

(4.9) $$P\{M(T) \leq u_T\} \to e^{-\tau} \quad \text{as } T \to \infty$$

if and only if

(4.10) $$\mu_T = \mu(u_T) \sim \tau/T \quad \text{as } T \to \infty$$

or equivalently if and only if

(4.10)' $$\mu_n = \mu(u_n) \sim \tau/n \quad \text{as } n \to \infty.$$

<u>Proof</u>. It is clear that (4.10) and (4.10)' are equivalent from the definition of u_T as $u_{[T]}$. If (4.9) holds, it holds as $T \to \infty$ through integral values. It then follows from Theorem 2.3 that $nP\{\zeta_1 > u_n\} \to \tau$ and hence from Theorem 4.1 that (4.10)' holds, as required.

Conversely if (4.10)' holds it follows similarly from Theorems 2.2 and 4.1 that $P\{M(n) \leq u_n\} \to e^{-\tau}$. Now if n denotes the integer part [T] of T, we have

$$P\{M(T) \leq u_T\} = P\{M(T) \leq u_n\}$$
$$= P\{M(n) \leq u_n\} - P\{M(n) \leq u_n < M(T)\}.$$

As noted the first term converges to $e^{-\tau}$. But the second term is dominated by $P\{\zeta_{n+1} > u_n\} = P\{\zeta_1 > u_n\} \sim \mu_n$ by Theorem 4.1, and $\mu_n \to 0$ by (4.10)', so that (4.9) follows. □

<u>Corollary 1</u>. *Suppose that the conditions of Theorems 3.1 and 4.1 hold and that the $\{\zeta_n\}$ sequence also satisfies $D'(u_n)$, for all u_n of the form $x/a_n + b_n$. Then G(x) is precisely the extreme value d.f. which would be obtained for the maxima $\hat{M}_n = \max(\hat{\zeta}_1 \ldots \hat{\zeta}_n)$ of an i.i.d. sequence $\hat{\zeta}_1, \hat{\zeta}_2 \ldots$ whose tail distribution satisfies*

(4.11) $$P\{\hat{\zeta}_i > x\} \sim \mu(x) .$$

Thus the classical criteria for domain of attraction may be applied with $\mu(x)$ replacing the tail distribtuion of the sequence terms.

Proof: By the theorem we have that $\mu_n \sim \tau/n$ where $\tau = -\log G(x)$, whenever $G(x) > 0$. Since $P\{\zeta_i > x\} \sim \mu(x)$ by Theorem 4.1, there exist i.i.d. random variables $\hat{\zeta}_i$ (having the same marginal distributions as ζ_i) satisfying (4.11). Hence $P\{\hat{\zeta}_1 > u_n\} \sim \mu(u_n) = \mu_n \sim \tau/n$ so that $P\{\hat{M}_n \leq u_n\} \to e^{-\tau}$. Rephrased, this states

$$P\{a_n(\hat{M}_n - b_n) \leq x\} \to G(x),$$

as required.

Conversely the following result is similarly proved.

Corollary 2. *Suppose that the conditions of Theorem 4.1 hold and that there are i.i.d. random variables $\hat{\zeta}_1\hat{\zeta}_2\ldots$ such that $P\{a_n(\hat{M}_n - b_n) \leq x\} \to G(x)$ for some non-degenerate G, constants $a_n > 0, b_n$, $\hat{M}_n = \max(\hat{\zeta}_1, \hat{\zeta}_2\ldots\hat{\zeta}_n)$, and such that $P\{\hat{\zeta}_1 > x\} \sim \mu(x)$ as $x \to \infty$. If $D(u_n)$, $D'(u_n)$ are satisfied by the sequence $\{\zeta_n\}$ for all u_n of the form $x/a_n + b_n$ then $P\{a_T(M(T) - b_T) \leq x\} \to G(x)$ (where $a_T = a_{[T]}$, $b_T = b_{[T]}$).* []

It may seem curious that it is the tail $1-F(u)$ of the distribution of one of the random variables which is central in the sequence case, whereas the mean number of upcrossings of a level per unit time plays the same role in the continuous context. However, it is worth noting that for an i.i.d. sequence, the probability of an upcrossing of a level u in unit time is $P\{\xi_{i-1} < u < \xi_i\} = F(u)[1 - F(u)] \sim 1 - F(u)$ for large u. Since only zero or one such upcrossing may occur in unit time, this is also the mean number of such upcrossings per unit time. Thus the tail $1 - F(u)$ of the distribution is, in the discrete case, also the asymptotic form of the mean number of upcrossings of u per unit time, and so naturally corresponds to $\mu(u)$ in the continuous case.

5. Dependence Restrictions for $\xi(t)$.

In the sequence case we used the conditions $D(u_n)$, $D'(u_n)$ to restrict dependence. We now define continuous analogs $(D_c(u_n), D'_c(u_n))$ of these conditions which will apply to the process $\xi(t)$, and may replace the D, D' requirements for the ζ_n-sequence in the previous theorems.

The condition $D_c(u_n)$ is the following analog of $D(u_n)$, where again $\{u_n\}$ is any real sequence. In this $I_q(u)$ is defined by (4.1) and if $F_{t_1 \ldots t_n}(x_1 \ldots x_n)$ denotes a finite dimensional d.f. of $\xi(t)$, $F_{t_1 \ldots t_n}(u)$ is written for $F_{t_1 \ldots t_n}(u \ldots u)$.

Specifically $\{\xi(t)\}$ will be said to satisfy $D_c(u_n)$ if there is a sequence $\{q_n\}$, $q_n \to 0$ as $n \to \infty$, such that $I_{q_n}(u_n) \sim \mu_n = \mu(u_n)$ as $n \to \infty$ and, for any integers n, $i_1 < i_2 \ldots < i_p < j_1 \ldots < j_p$, $\leq n/q_n$, $j_1 - i_p \geq \ell/q_n$ we have (writing $q_n = q$)

(5.1) $\left| F_{i_1 q \ldots i_p q, j_1 q \ldots j_p, q}(u_n) - F_{i_1 q \ldots i_p q}(u_n) F_{j_1 q \ldots j_p, q}(u_n) \right| < \alpha_{n,\ell}$

where $\alpha_{n,\ell} \to 0$ for some sequence $\ell_n = o(n)$.

In discussing $D(u_n)$ for the ζ_n-sequence, it will be convenient to first obtain the following lemma, which enables us to approximate the maximum of ξ in a fixed interval by the maximum of discrete values $\zeta(jq)$ for points jq ($j=0,1,2\ldots$) in that interval. In this lemma, and subsequently, $N_u^{(q)}(E)$ will denote the number of upcrossings of the level u by the sequence $\xi(jq)$ in a set E, i.e. the number of integers j such that $jq \in E$ for which $\xi(jq) < u < \xi[(j+1)q]$. We also write $N_u^{(q)}(t)$ when $E = [0,t]$.

Lemma 5.1. *Suppose that (4.5) holds. Then, as $u \to \infty$, if I is an interval of fixed length h,*

(i) $\quad E N_u^{(q)}(I) = \mu h + o(\mu)$

(ii) If also $P\{\xi(0) > u\} = o(\mu)$, *then we have*

$$0 \leq P\{ \bigcup_{jq \in I} (\xi(jq) \leq u)\} - P\{M(1) \leq u\} = o(\mu)$$

as $u \to \infty$ *(uniformly in all such fixed length intervals).*

<u>Proof</u>. (i) The number, $m = m_u$, say, of points $jq \in I$, clearly satisfies $m \sim h/q$ as $u \to \infty$ so that, by stationarity,

$$E\, N_u^{(q)}(I) = m\, P\{\xi(0) < u < \xi(q)\} \sim h\, I_q(u) \sim \mu h$$

by (4.5), so that (i) follows.

(ii) The difference in probabilities is clearly non-negative and dominated by

(5.2) $$P\{\xi(a) > u\} + P\{N_u(I) > 0,\ N_u^{(q)}(I) = 0\}$$

if a denotes the left hand endpoint of I. But the first term is equal to $P\{\xi(0) > u\} = o(\mu)$ and the second is dominated by $P\{N_u(I) - N_u^{(q)}(I) > 0\} \leq E\{N_u(I) - N_u^{(q)}(I)\}$ ($N_u - N_u^{(q)}$ being non-negative) so that (by (i)), (5.2) does not exceed

$$\mu h - [\mu h + o(\mu)] + o(\mu) = o(\mu)$$

as required.   □

By using this lemma we may now obtain conditions under which $D(u_n)$ holds for the sequence $\{\zeta_n\}$. We write $\mu_n = \mu(u_n)$ for the mean number of upcrossings of u_n by $\xi(t)$ per unit time.

<u>Lemma 5.2</u>. *Let* $\{u_n\}$ *be a sequence of constants and suppose that* $D_c(u_n)$ *holds for the process* $\xi(t)$. *Suppose also that* $n\mu_n$ *is bounded, and* $P\{\xi(0) > u_n\} = o(\mu_n)$. *Then the sequence* $\{\zeta_n\}$ *of maxima of* ξ *over the intervals* $[n-1, n]$, *satisfies* $D(u_n)$.

Proof. Fix n, $i_1 < i_2 \ldots < i_p < j_1 \ldots < j_{p'} \leq n$, $j_1 - j_p \geq \ell$. Let $q = q_n$ be chosen so that (5.1) holds. Then, writing $I_r = [i_r - 1, i_r]$, $J_s = [j_s - 1, j_s]$, we have, by Lemma 5.1,

$$0 \leq P\{\bigwedge_{r=1}^{p} \{\xi(jq) < u_n, jq \in I_r\} \cap \bigwedge_{s=1}^{p'} \{\xi(jq) \leq u_n, jq \in J_s\}\}$$

$$- P\{\bigwedge_{r=1}^{p} \{\zeta_{i_r} \leq u_n\} \cap \bigwedge_{s=1}^{p'} \{\zeta_{j_s} \leq u_n\}\}$$

$$\leq (p + p') \, o(\mu_n) = o(n\mu_n)$$

which tends to zero as $n \to \infty$ since $n\mu_n$ is bounded. By applying this as stated and also to the groups of I_r and J_s intervals separately, we see at once that the difference, R_n, say, between

$$(5.3) \quad |P\{\bigwedge_{r=1}^{p} (\zeta_{i_r} \leq u_n) \cap \bigwedge_{s=1}^{p'} (\zeta_{j_s} \leq u_n)\} - P\{\bigwedge_{r=1}^{p} (\zeta_{i_r} \leq u_n)\} P\{\bigwedge_{s=1}^{p'} (\zeta_{j_s} \leq u_n)\}|$$

and

$$(5.4) \quad |P\{\bigwedge_{r=1}^{p} (\xi(jq) \leq u_n, jq \in I_r) \cap \bigwedge_{s=1}^{p'} (\xi(jq) \leq u_n, jq \in J_s)\}$$

$$- P\{\bigwedge_{r=1}^{p} (\xi(jq) \leq u_n, jq \in I_r)\} P\{\bigwedge_{s=1}^{p'} (\xi(jq) \leq u_n, jq \in J_s)\}|$$

tends to zero as $n \to \infty$. But since the smallest jq in any J_s is at least $j_1 - 1$, and the largest in any I_r is at most i_p, (5.1) shows that (5.4) does not exceed $\alpha_{n, \ell - 1}$ where $\alpha_{n, \ell_n} \to 0$ for some sequence $\ell_n = o(n)$. Hence (5.3) does not exceed $\alpha^*_{n, \ell} = \alpha_{n, \ell - 1} + R_n$ and $\alpha^*_{n, \ell^*_n} \to 0$ for $\ell^*_n = \ell_n + 1$, so that the sequence $\{\zeta_n\}$ satisfies $D(u_n)$ as asserted. $\square$

The continuous version of Gendenko's Theorem (Theorem 3.1) may now be restated in terms of conditions on the process $\xi(t)$.

Theorem 5.3. *Suppose that for some families of constants $\{a_T > 0\}$, $\{b_T\}$ we have*

$$P\{a_T (M(T) - b_T) \leq x\} \to G(x)$$

for a non-degenerate d.f. G and that, for each sequence $u_n = \{x/a_n + b_n\}$ *(all real x),* $\{\xi(t)\}$ *satisfies* $D_c(u_n)$ *and* $P\{\xi(0) > u\} = o(\mu)$, $\limsup_n n\,\mu_n < \infty$, *where* $\mu = \mu(u)$, $\mu_n = \mu(u_n)$. *Then G is one of the three extreme value types.*

This follows at once from Theorem 3.1 and Lemma 5.2. □

The condition $D_c(u_n)$ (and the other conditions of this Theorem) are not too difficult to verify when $\xi(t)$ is a normal process. The appropriate D' condition for the continuous case - which we shall call $D'_c(u_n)$ - is simpler to state and more interesting than $D_c(u_n)$, though evidently also more difficult to verify. Like $D_c(u_n)$, $D'_c(u_n)$ is a natural analog of the corresponding sequence condition $(D'(u_n))$.

Specifically, we say that $\{\xi(t)\}$ satisfies $D'_c(u_n)$, for a given sequence $\{u_n\}$, if

(5.5) $\limsup_{n \to \infty} E\, N_{u_n}([n/k])(N_{u_n}([n/k]) - 1) = o(k^{-1})$ as $k \to \infty$.

We note that while this seems formally quite different from the stated $D'(u_n)$ condition for sequences, the difference is somewhat illusory, since the sequence condition may also be recast in terms of a second factorial moment (of numbers of exceedances). Theorem 4.2 may now be restated in terms of the assumptions D_c, D'_c as follows.

Theorem 5.4. *Let* $\{u_n\}$ *be a sequence of constants,* $u_T = u_{[T]}$. *Suppose that the stationary process* $\{\xi(t)\}$ *satisfies* $D_c(u_n)$, $D'_c(u_n)$, *that (4.6) holds and that* $P\{\xi(0) > u\} = o(\mu)$ *as* $u \to \infty$. *Then (4.9) holds* $(P\{M(T) \leq u_T\} \to e^{-\tau})$ *if and only if (4.10) holds* $(\mu_T = \mu(u_T) \sim \tau/T)$.

Proof: For fixed k, writing n' = [n/k] and arguing as in Theorem 4.1 we obtain

(5.6) $n'\mu_n - E\,N_{u_n}(n')(N_{u_n}(n') - 1) \leq P\{M(n') > u_n\} \leq P(\xi(0) > u_n) + n'\mu_n$.

If (4.9) holds since the central term of (5.6) does not exceed $P\{M(n) > u_n\}$

which converges to $1-e^{-\tau}$, it follows from the left hand inequality in (5.6) that $n' \mu_n$ (and hence $n \mu_n$) is bounded when k is chosen so that the left hand side of (5.5) is finite. If (4.10) holds it is trivially true that $n \mu_n$ is bounded. Hence under (4.9) or (4.10), by Lemma 5.2 the sequence $\{\zeta_n\}$ of maxima of $\xi(t)$ over the intervals $[n-1,n]$, satisfy $D(u_n)$. Hence also by Lemma 2.1 (with ζ_i for ξ_i),

$$(5.7) \qquad P\{M(n) \le u_n\} - P^k\{M(n') \le u_n\} \to 0 \quad \text{as} \quad n \to \infty$$

for each k. The rest of the proof now follows the same arguments as in Theorem 2.3. For example if (4.10) holds we may let $n \to \infty$ in (5.6) and use (5.5) to obtain

$$1 - \tau/k \le \liminf_{n \to \infty} P\{M(n') \le u_n\} \le \limsup_{n \to \infty} P\{M_n(n') \le u_n\}$$

$$\le 1 - \tau/k + o(1/k)$$

from which (4.9) follows by taking the kth power, using (5.7) and letting $k \to \infty$. Similarly (4.9) implies (4.10). $\qquad\qquad []$

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On Extremes of Stationary Processes		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
		6. PERFORMING ORG. REPORT NUMBER Mimeo Series No. 1194
7. AUTHOR(s) M.R. Leadbetter		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0809
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics University of North Carolina Chapel Hill, North Carolina 27514		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research, Statistics and Probability Arlington, VA 22217		12. REPORT DATE September 1978
		13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release: Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Extreme values, stochastic processes, maxima

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Certain aspects of extremal theory for (a) stationary sequences and (b) continuous parameter stationary processes, are discussed in this paper. A slightly modified form of a previously used dependence condition, leads to simple proofs of some key results in extremal theory of stationary sequences. Dependence conditions of a "weak mixing" type are introduced for continuous parameter stationary processes and results of classical extreme value theory extended to that context.

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