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# A BOUNDARY VALUE PROBLEM ASSOCIATED WITH THE SECOND PAINLEVÉ TRANSCENDENT AND THE KORTEWEG-DE VRIES EQUATION.

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# A BOUNDARY VALUE PROBLEM ASSOCIATED WITH THE SECOND PAINLEVÉ TRANSCENDENT AND THE KORTEWEG-DE VRIES EQUATION

S. P. Hastings and J. B. McLeod

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ABSTRACT

The differential equation considered is $y'' - xy = y|y|^{\alpha}$. For general positive $\alpha$ this equation arises in plasma physics, in work of de Boer and Ludford. For $\alpha = 2$, it yields similarity solutions to the well-known Korteweg-de Vries equation. Solutions are sought which satisfy the boundary conditions

$$(1) \quad y(\infty) = 0$$

$$(2) \quad y(x) \sim \left(-\frac{1}{2}x\right)^{1/\alpha} \text{ as } x \to -\infty .$$

It is shown that there is a unique such solution, and that it is, in a certain sense, the boundary between solutions which exist on the whole real line and solutions which, while tending to zero at plus infinity, blow up at a finite $x$. More precisely, any solution satisfying (1) is asymptotic at plus infinity to some multiple $k\mathrm{Ai}(x)$ of Airy's function. We show that there is a unique $k^*(\alpha)$ such that when $k = k^*(\alpha)$ the condition (2) is also satisfied. If $0 < k < k^*$, the solution exists for all $x$ and tends to zero as $x \to -\infty$, while if $k > k^*$ then the solution blows up at a finite $x$. For the special case $\alpha = 2$ the differential equation is classical, having been studied by Painlevé around the turn of the century. In this case, using an integral equation derived by inverse scattering techniques by Ablowitz and Segur, we are able to show that $k^* = 1$, confirming previous numerical estimates.

AMS(MOS) Subject Classification - 34B15

Key Words - Korteweg-de Vries, Inverse scattering, Connection problems, Painlevé transcendent

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## SIGNIFICANCE AND EXPLANATION

The problem treated in this paper arose originally in the context of plasma physics. Differential equations had been obtained by earlier authors describing the region around a spherical electric probe in a slightly ionized continuum gas. The mathematical problem was to show the existence of a transition solution to these equations by means of which the ion-sheath region near the probe and the quasi-neutral region further away are connected. This problem, originally presented by de Boer and Ludford, is solved in this paper.

Perhaps a more far-reaching application, however, is for a special case when the equations yield particular solutions to the well-known Korteweg-de Vries equation for shallow water waves. In this context the transition, or connection, problem is solved more completely, in that a precise constant is found showing how the behaviour of these solutions at the front of the wave is related to the behaviour at the back.

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A BOUNDARY VALUE PROBLEM ASSOCIATED WITH THE SECOND PAINLEVÉ TRANSCENDENT AND THE KORTEWEG-DE VRIES EQUATION

S. P. Hastings and J. B. McLeod

## 1. Introduction

In [1], in connection with a problem in plasma physics, de Boer and Ludford ask whether there exists a solution to the boundary value problem consisting of the equation

$$\frac{d^2y}{dx^2} - xy = 2y|y|^{\alpha}, \quad -\infty < x < \infty, \tag{1.1}$$

and the boundary conditions

$$y(x) \sim \left(-\frac{1}{2}x\right)^{1/\alpha} \quad \text{as} \quad x \to -\infty, \tag{1.2}$$

$$y(x) \to 0 \quad \text{as} \quad x \to +\infty . \tag{1.3}$$

The quantity $\alpha$ is a strictly positive constant.

As de Boer and Ludford point out, the case $\alpha = 2$ is interesting because (1.1) is then a particular case of what is known as the second Painlevé transcendent. The Painlevé transcendents were first studied by Painlevé himself in a series of papers beginning in 1893 (for a survey of the work see [2] or [3]). These papers dealt with the question of which second order equations have the property that the singularities other than poles of any of the solutions are independent of the particular solution chosen and so dependent only on the equation. Indeed, in the case of the second transcendent, no solution has any singularities at all except for poles and the point at infinity.

The case $\alpha = 2$ of (1.1) is interesting also because of a connection with the Korteweg-de Vries equation, currently the object of considerable attention in many directions. It seems to have been observed first by Whitham (see [4]), building on work of Miura and others, that, if $y(x)$ is a solution of (1.1), and if

$$f = y' - y^2 \quad (' = d/dx),$$

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then

$$u(x,t) = (3t)^{-2/3} \; f(x/(3t)^{1/3})$$

is a similarity solution of the Korteweg-de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

a fact that can be verified by elementary manipulation.

We have two objectives in the present paper. The first is to answer the de Boer-Ludford question in the affirmative, and indeed to prove even more, that the boundary value problem (1.1-3) has one and only one solution. In order to state the result fully, we recall first the definition of the Airy function $Ai(x)$. This is defined to be the solution of the equation

$$Ai'' - xAi = 0 \tag{1.4}$$

for which

$$Ai(x) \sim \pi^{-1/2}|x|^{-1/4}\cos\left(\frac{2}{3}|x|^{3/2} - \frac{1}{4}\pi\right) \quad \text{as} \quad x \to -\infty \tag{1.5}$$

and

$$Ai(x) \sim \frac{1}{2}\pi^{-1/2}x^{-1/4}\exp\left(-\frac{2}{3}x^{3/2}\right) \quad \text{as} \quad x \to +\infty . \tag{1.6}$$

Since the Airy function can be expressed in terms of Bessel functions of order $\frac{1}{3}$, the asymptotic expansions are merely a reflection of the well known ones for Bessel functions. Indeed, in the standard notation for Bessel functions [5],

$$Ai(x) = 3^{-1/2}\pi^{-1}x^{1/2}K_{1/3}\left(\frac{2}{3}x^{3/2}\right), \tag{1.7}$$

a result that is perhaps best proved by verifying that both $Ai(x)$ and $x^{1/2}K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$ satisfy the equation (1.4) and then comparing their asymptotic expansions to confirm that they are in the ratio given by (1.7).

Our existence and uniqueness theorem is then as follows.

Theorem 1. For each $\alpha > 0$ the problem (1.1-3) has a unique solution, and this solution has the following properties:

(i) $y > 0$, $y' < 0$;

(ii) if $\alpha \le 1$, then $y'' > 0$, while if $\alpha > 1$, then $y''$ has precisely one zero, with $y''(x) > 0$ for large positive $x$ and $y''(x) < 0$ for large negative $x$;

(iii) as $x \to \infty$, $y(x)$ is asymptotic to some multiple $k^*(\alpha)Ai(x)$ of the Airy function defined in (1.4-6).

Furthermore, any solution of (1.1) satisfying (1.3) is asymptotic to $kAi(x)$ for some $k$, and, conversely, for any $k$, there is a unique solution of (1.1) asymptotic to $kAi(x)$. If $|k| < k^*(\alpha)$, then the solution asymptotic to $kAi(x)$ exists for all $x$ and as $x \to -\infty$ is asymptotic to

$$(1.8) \qquad d|x|^{-1/4}\sin\left\{\frac{2}{3}|x|^{3/2} - \frac{c_1|x|^{\frac{1}{2}-\frac{1}{4}\alpha}}{\frac{1}{2}-\frac{1}{4}\alpha} - c_2\right\} \text{ if } \alpha < 2,$$

for some constants $d, c_1, c_2$, where

$$(1.9) \qquad c_1 = 2\pi^{-1}|d|^{\alpha}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha + \frac{3}{2}\right)/\Gamma\left(\frac{1}{2}\alpha + 2\right),$$

to

$$(1.10) \qquad d|x|^{-1/4}\sin\left\{\frac{2}{3}|x|^{3/2} - \frac{3}{4}d^2\log|x| - c_2\right\} \text{ if } \alpha = 2,$$

and to

$$(1.11) \qquad d|x|^{-1/4}\sin\left\{\frac{2}{3}|x|^{3/2} - c_2\right\} \text{ if } \alpha > 2\,.$$

If $|k| > k^*(\alpha)$, the solution becomes infinite at a finite value of $x$.

Since (1.1) is left unchanged by the transformation $y \to -y$, we can, and shall, take $k > 0$ in the rest of the paper.

C. Conley, in unpublished notes, has proved the existence, but not the uniqueness, of the solution of (1.1-3). His existence proof, like ours, is based on a "shooting" technique, but his proof requires a distinction between the cases $0 < \alpha \le 1$ and $\alpha > 1$ which ours does not, and it is based on Wazewski's principle and separation theorems in two dimensions while ours uses the connectedness of the real line.

Our second objective, and our second theorem, are concerned with the case $\alpha = 2$, and the remainder of this introduction is confined to that.

In this situation, Rosales [6] observed numerically that $k^*(2) = 1 + O(10^{-13})$, which raises the obvious conjecture that in fact $k^*(2) = 1$, and this we prove.

Theorem 2. $$k^*(2) = 1 .$$

Theorem 2 is an example of a nonlinear connection problem, since we are relating the asymptotic behaviour of the solution of (1.1-3) as $x \to +\infty$ to the asymptotic behaviour as $x \to -\infty$. Linear connection problems have been one of the main areas in ordinary differential equations for over a hundred years, but nonlinear connection problems are very rare. One reason at least for this is that the method which is perhaps the most useful one for linear problems is not in general applicable. This is to consider $x$ as a complex variable and pass from $x = -\infty$ to $x = +\infty$ along a large semicircle in the x-plane. Provided that the coefficients in the equation have a reasonably simple asymptotic behaviour as $|x| \to \infty$, it may be possible to construct an asymptotic expansion for the solution at all points on the large semicircle, and so relate a specific behaviour as $x \to +\infty$ to a specific behaviour as $x \to -\infty$.

In nonlinear problems, in general, this method fails because, even if the coefficients in the equations are very reasonable, the solutions may not continue to exist as $|x| \to \infty$. For one important class of equations, however, something can be saved, and these are the Painlevé transcendents, and in particular those transcendents such as the second for which all solutions have no singularities other than poles in the finite part of the plane. Indeed, Boutroux, in two long memoirs [7], [8] (see also [3]), has studied the asymptotics of solutions of the first Painlevé transcendent in considerable detail, and, as he remarks, the ideas extend to the second transcendent also. The essential result is that the solutions behave asymptotically, at least locally, like elliptic functions, and although Boutroux does not specifically consider any connection problems, the solution of these is a matter of piecing together different elliptic functions in different sectors on the large semicircle in the complex plane.

Even if this programme is feasible, it certainly involves formidable technical difficulties, and it turns out that we can in any case avoid it by solving our connection problem by using the relation already mentioned between the second Painlevé transcendent and the Korteweg-de Vries equation. Ablowitz and Segur [4] have pointed out that the fact that the Korteweg-de Vries equation can be solved by the inverse scattering technique implies that the solution of (1.1) which is asymptotic to $kAi(x)$ as $x \to \infty$ can be regarded as the solution of a linear integral equation, and we use this fact to establish Theorem 2.

This does however raise the question whether there is a deeper connection between the Painlevé transcendents, for which, exceptionally amongst nonlinear ordinary differential equations, there is a routine for solving nonlinear connection problems, and nonlinear evolution equations such as the Korteweg-de Vries equation, for which, again exceptionally, there exists an inverse scattering technique relating behaviour for large negative time to behaviour for large positive time. Ablowitz and Segur have already pointed out that, just [illegible] [illegible]ond Painlevé transcendent is associated with the Korteweg-de Vries equation, s[illegible] first transcendent is associated with the Boussinesq equation and the third wit[illegible] sine-Gordon equation. It would certainly seem a reasonable conjecture that any similarity solution of a nonlinear evolution equation for which an inverse scattering technique applies should necessarily satisfy an ordinary differential equation whose solutions possess no singularities other than poles, and this would in turn lead to a test for the availability of an inverse scattering technique for any given nonlinear evolution equation, which is one of the open problems mentioned by Miura in [9]; but we do not pursue these questions further here.

Theorem 2 solves the connection problem for the particular solution of (1.1) (with $\alpha = 2$) which is asymptotic to $k^*Ai(x)$ as $x \to \infty$. If the solution is asymptotic to $kAi(x)$ with $0 < k < k^*$, then Theorem 1 asserts that the solution exists for all $x$, and Ablowitz and Segur find on heuristic grounds that it has the asymptotic form as $x \to -\infty$ given by (1.10), where $k$ and $d$ are related by the formula

$$d^2 = -\pi^{-1}\log(1 - k^2) \quad (1.12)$$

This would certainly imply Theorem 2, and it seems likely that a more detailed application of the asymptotic methods used in this paper would in fact also prove (1.12), but again we do not pursue this further here.

The arrangement of the paper is that the existence part of Theorem 1 is proved in §§2-5, along with the qualitative properties of the solution, the uniqueness part in §6, and the asymptotic behaviour in §7. Theorem 2 is proved in §§8-9.

We would like to thank Professors C. Conley, Y. Sibuya, W. Wasow and H. Weinberger for helpful conversations.

## 2. Existence of a solution

The proof depends on a series of lemmas, some of which are almost immediate. We give the main proof, leaving the verification of those lemmas which require extended arguments to later sections.

**Lemma 1.** There exists a unique solution of (1.1) which is asymptotic to $kAi(x)$ as $x \to \infty$, $k$ being any given positive number. This solution may not exist for all $x$ as $x$ decreases to $-\infty$, but at each $x$ for which it continues to exist the solution and its derivatives are continuous functions of $k$. We denote this solution by $y_k(x)$.

This lemma requires little proof. Perhaps the simplest technique is to recognize that $y_k$ must satisfy the integral equation

$$(2.1) \qquad y_k(x) = kAi(x) + 2\int_x^\infty \{Ai(x)Bi(t) - Bi(x)Ai(t)\}y_k(t)|y_k(t)|^\alpha dt,$$

where $Bi(x)$ is a solution of the equation

$$w'' - xw = 0$$

which is linearly independent of $Ai(x)$ and which we can take to have the asymptotic behaviour

$$Bi(x) \sim \pi^{1/2}x^{-1/4}\exp\left(\tfrac{2}{3}x^{3/2}\right) \quad \text{as} \quad x \to \infty .$$

The equation (2.1) can then be solved (uniquely) by iteration, and this gives both $y_k$ and its continuous dependence on $k$.

**Lemma 2.** *The set of $k(>0)$ for which $y_k(x)$ remains positive as $x$ decreases and becomes infinite at some finite value of $x$ is an open set, denoted by $S_1$.*

The proof of this is deferred to §3.

**Lemma 3.** *The set of $k(>0)$ for which $y_k(x)$ (which is certainly positive for sufficiently large $x$) takes negative values before (if ever) it ceases to exist is an open set, denoted by $S_2$.*

This lemma is an immediate consequence of the continuity of $y_k$ in $k$. If $y_{k_0}(x_0) < 0$ for some $k_0$ and $x_0$, then $y_k(x_0) < 0$ for all $k$ sufficiently close to $k_0$.

Lemma 4. The set $S_1$ is non-empty.

The proof is given in §4.

Lemma 5. The set $S_2$ is non-empty.

The proof is given in §5.

The proof of existence can now be completed. Since the positive semiaxis is connected, it cannot be divided into two non-empty disjoint open sets. But $S_1$ and $S_2$ are non-empty and open, and also clearly disjoint by definition, and so there exists at least one positive value of $k$ which lies in neither $S_1$ nor $S_2$. For such a value of $k$, $k^*$ say, $y_{k^*}(x)$ has the properties that it exists for all $x$ and is always positive. (It cannot take the value zero because this would have to be a minimum, and $y = y' = 0$ at any point implies from (1.1) that $y \equiv 0$.)

To obtain further properties of $y_{k^*}$, we note first that $y'_{k^*} < 0$. For suppose for contradiction that $x_0$ is the first value of $x$ (for decreasing $x$) for which $y'_{k^*}(x) = 0$. From (1.1) we certainly have

$$y^{\alpha}_{k^*}(x_0) + x_0 \leq 0 . \tag{2.2}$$

Also,

$$\frac{d}{dx}\left(y^{\alpha}_{k^*} + x\right) = \alpha y^{\alpha-1}_{k^*} y'_{k^*} + 1,$$

which is positive if $y'_{k^*} \geq 0$. Hence from (2.2) we see that, to the left of $x_0$, $y''_{k^*} < 0$ and $y'_{k^*} > 0$, contradicting $y_{k^*} > 0$ .

Now set, for $x < 0$,

$$y_{k^*}(x) = \left(-\frac{1}{2}x\right)^{1/\alpha} z(x), \tag{2.3}$$

and it is routine that $z$ satisfies

$$z'' + \frac{2}{\alpha x} z' = \left\{-x(z^{\alpha} - 1) - \frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)/x^2\right\} z . \tag{2.4}$$

This implies that, for $x < 0$, $z$ cannot have a relative maximum where

$$z^{\alpha} - 1 > \frac{1}{\alpha}\left|\frac{1}{\alpha} - 1\right|/|x|^3,$$

or a relative minimum where

$$z^{\alpha} - 1 < -\frac{1}{\alpha}\left|\frac{1}{\alpha} - 1\right|/|x|^3,$$

and so either $z \to 1$ as $x \to -\infty$, which proves that $y_{k*}$ satisfies (1.1-3), or $z$ is monotonic for large negative $x$ and so tends to infinity or to a finite limit other than 1.

If $z(-\infty) > 1$, including $z(-\infty) = \infty$, then it follows from (1.1) that ultimately (for large negative $x$)

$$y''_{k*} \geq K y_{k*}^{\alpha+1},$$

for some constant $K(>0)$, and integration of this shows that $y_{k*}(x)$ blows up at finite $x$, contradicting the fact that $k^*$ does not belong to $S_1$.

If $z(-\infty) < 1$, then it follows from (1.1) that ultimately

$$y''_{k*} \leq Kxy_{k*},$$

for a possibly different positive constant $K$, and since the equation obtained by replacing $\leq$ by $=$ is certainly oscillatory, we contradict $y_{k*} > 0$.

The proof of existence is thus complete, granted the proof of the lemmas to follow, but we can conveniently prove here the remaining properties of $y_{k*}$ stated in Theorem 1.

Clearly, $z(x) > 1$ for $x(<0)$ sufficiently small. (Indeed, $z(x) \to \infty$ as $x \to 0$.) If $\alpha \leq 1$, (2.4) shows that $z$ can have no minimum below 1, and since $z(-\infty) = 1$, it follows that always $z > 1$ and so $y''_{k*} > 0$.

If, alternatively, $\alpha > 1$, then $z(x)$ can cross the value 1 as $x$ decreases, but it cannot cross back since it can have no maximum exceeding 1. Indeed, $z$ must cross the value 1 once, since otherwise we would have $y_{k*} \geq 0$, $y'_{k*} < 0$, and so $y_{k*}(x) > K|x|$ as $x \to -\infty$, for some positive $K$. But then, with $\alpha > 1$, this implies that

$$y''_{k*} \geq y_{k*}^{\alpha+1}$$

for sufficiently large negative $x$, which causes blow-up at finite $x$. Hence $z$ crosses 1 precisely once for $x < 0$, from which the required properties for $y''_{k*}$ follow.

## 3. Proof of Lemma 2

If, for a specific value of $k$, say $k_0$, $y_{k_0}$ blows up at $x = x_0$, then clearly we can find a value $x_1$ near $x_0$ with

$$y_{k_0}^{\alpha}(x_1) > |x_1| + 1, \quad y_{k_0}'(x_1) < 0 . \tag{3.1}$$

If $k$ is sufficiently close to $k_0$, the inequalities (3.1) continue to hold with $k_0$ replaced by $k$, and it is then clear from (1.1) that, for $-1 \leq x - x_1 \leq 0$, at least as long as $y_k(x)$ is defined,

$$y_k^{\alpha}(x) > |x|, \quad y_k'(x) < 0, \quad y_k'' > y_k^{\alpha+1} .$$

But integration of the last (autonomous) inequality forces blow-up at a finite point, indeed a finite point in $x - x_1 \geq -1$ if

$$y_k(x_1) > K, \quad y_k'(x_1) < -K, \tag{3.2}$$

where $K$ is some positive constant which is independent of $x_1$. Clearly we can choose $x_1$ so that (3.2) is satisfied in addition to (3.1), and the lemma is proved.

## 4. Proof of Lemma 4

Since the integral term in (2.1) is positive for sufficiently large $x$ (from the asymptotic expressions for Ai,Bi), it follows that $y_k(x) > k\mathrm{Ai}(x)$ *for sufficiently* large $x$. Differentiating (2.1), we can similarly show that $y_k'(x) < k\mathrm{Ai}'(x) < 0$, and so we can choose $x_1$ and $k$ so that the inequalities (3.1) (with $k_0 = k$) and (3.2) are satisfied. But then we know from the analysis of §3 that $y_k$ blows up at a finite point, proving the lemma.

## 5. Proof of Lemma 5

Since $k = 0$ makes $y_k$ identically zero, we can use the continuity of $y_k$ in $k$ (proved in Lemma 1) to choose $k$ sufficiently small that $y_k(-1)$, $y_k'(-1)$ are in turn so small that, for $-1 \geq x \geq -(1 + \pi\sqrt{2})$,

$$x - 2|y_k(x)|^\alpha \leq -\frac{1}{2}.$$

But then comparison of (1.1) with

$$y'' = -\frac{1}{2}y$$

shows that $y_k$ must vanish somewhere in $[-1 - \pi\sqrt{2}, -1]$, completing the proof of the lemma.

## 6. Uniqueness of the solution

The proof of uniqueness proceeds in the following stages. We first prove that any solution of (1.1) which satisfies (1.3) is asymptotic to $kAi(x)$ for some $k$. We then prove a limited uniqueness theorem, that there is only one solution of (1.1-3) for which $y' < 0$ (and so $y > 0$).

We then use this limited result to prove that $(0<)k < k^*$ implies that $y_k(x) \to 0$ as $x \to -\infty$, while $k > k^*$ implies that $y_k$ blows up at a finite point. This, together with the already established fact that any solution satisfying (1.3) is of the form $y_k$, completes the proof of uniqueness.

To establish that any solution satisfying (1.3) is of the form $y_k$, we note that $y(x) \to 0$ as $x \to \infty$ implies that the coefficient of $y$ in (1.1) is necessarily positive for sufficiently large positive $x$, and so the equation is non-oscillatory, and to obtain $y \to 0$, we must have ultimately $y > 0$, $y' < 0$, $y'' > 0$ (or $y < 0$, $y' > 0$, $y'' < 0$, although this is not significant since the negative of a solution is also a solution). Indeed, since, for any fixed $r$ with $0 < r < 1$,

$$y'' \geq rxy$$

for $x$ sufficiently large, we can conclude, as in §8.2 of [10], that, for some constant $C$,

$$y(x) \leq C \exp\left(-\frac{2}{3}r^{1/2}x^{3/2}\right),$$

and then the formal process that renders (1.1) and (1.3) equivalent to the integral equation (2.1) is certainly justified, and the required asymptotic form follows.

We now establish the limited uniqueness theorem. Suppose that $y_1$ and $y_2$ satisfy (1.1-3) with $y_i > 0$, $y_i' < 0$ for $i = 1,2$, and suppose that $y_1, y_2$ correspond to $k_1, k_2$, with $k_1 > k_2$, so that ultimately (for $x$ sufficiently large and positive) $y_1 > y_2$ and $y_1' < y_2'$. In fact, $y_1(x) > y_2(x)$ for all $x$, as we can prove by considering the expression

$$V_i(x) = \frac{1}{2}y_i'^2(x) - \frac{1}{2}xy_i^2(x) - \frac{2}{2+\alpha}|y_i(x)|^{2+\alpha}, \tag{3.1}$$

for which it is easy to verify that

$$V_i' = -\frac{1}{2}y_i^2 . \tag{3.2}$$

Now let us suppose for contradiction that $y_1(x) = y_2(x)$ first (for decreasing $x$) at $x = x_0$. Then we have

$$y_1(x_0) = y_2(x_0), \quad 0 > y_1'(x_0) \geq y_2'(x_0), \quad y_1'^2(x_0) \leq y_2'^2(x_0) ,$$

so that, from (3.1),

$$V_1(x_0) \leq V_2(x_0) .$$

But also $V_1(\infty) = V_2(\infty)$, and (3.2) then implies that $V_1(x_0) > V_2(x_0)$, giving the required contradiction.

Hence $y_1 > y_2$. Using the mean value theorem to give

$$y_1^{\alpha+1} - y_2^{\alpha+1} > (\alpha + 1)y_2^{\alpha}(y_1 - y_2),$$

we see from (1.1) that

$$(y_1 - y_2)'' > \{2(\alpha + 1)y_2^{\alpha} + x\}(y_1 - y_2) .$$

Since

$$2y_2^{\alpha}(x) \sim -x \quad \text{as} \quad x \to -\infty,$$

we can conclude that, for large negative $x$,

$$(y_1 - y_2)'' \geq -\frac{\alpha x}{2}(y_1 - y_2) ,$$

so that, as $x \to -\infty$, either $y_1 - y_2$ is exponentially large, which contradicts $(y_1 - y_2)/(-x)^{1/\alpha} \to 0$, or both $y_1 - y_2$ and $(y_1 - y_2)'$ are exponentially small, which makes $(V_1 - V_2)(-\infty) = 0$ and is again a contradiction. So limited uniqueness is established.

Now suppose that $0 < k < k^*$, and consider the solution $y_k$ of (1.1). This must have the property that $y_k'$ vanishes at some finite point, since otherwise we could prove as above that $y_{k^*} > y_k > 0$ everywhere, so that $y_k$ exists and is positive everywhere, from which it follows by the analysis in the existence proof that $y_k$ satisfies (1.2), contradicting limited uniqueness.

We can also show that always $|y_k| < y_{k*}$. (It is of course possible that $y_k$ oscillates; indeed, the asymptotic behaviour proved in §7 shows that it does oscillate.) Certainly, $|y_k(x)| < y_{k*}(x)$ for $x$ sufficiently large and positive. Consider the function $V_k$ defined in (3.1), and suppose for contradiction that, as $x$ decreases, $|y_k(x)| = y_{k*}(x)$ first at $x = x_0$. Then we have

$$|y_k(x_0)| = y_{k*}(x_0), \quad |y_k'(x_0)| \geq y_{k*}(x_0), \quad V_k(x_0) \geq V_{k*}(x_0),$$

and we can obtain a contradiction to this by integrating (3.2), as in the proof of limited uniqueness.

Further, $y_k$ is bounded. This follows from the fact that if $x = x_1$ is the first maximum of $y_k(x)$ as $x$ decreases, then $|y_k(x)| < y_k(x_1)$ for $x < x_1$. To prove this, suppose for contradiction that $x_2$ is the first value of $x$ as $x$ decreases from $x_1$ for which $|y_k(x)| = y_k(x_1)$. Then

$$\begin{aligned} V_k(x_2) &= \tfrac{1}{2} y_k'^2(x_2) - \tfrac{1}{2} x_2 y_k^2(x_2) - \tfrac{2}{2+\alpha} |y_k(x_2)|^{2+\alpha} \\ &\geq V_k(x_1) - \tfrac{1}{2}(x_2 - x_1) y_k^2(x_1), \end{aligned}$$

since $y_k'(x_1) = 0$. But also

$$|y_k(x)| < y_k(x_1) \quad \text{for} \quad x_2 < x < x_1,$$

and so

$$\begin{aligned} V_k(x_2) &= V_k(x_1) - \tfrac{1}{2} \int_{x_1}^{x_2} y_k^2(x)\,dx \\ &< V_k(x_1) - \tfrac{1}{2}(x_2 - x_1) y_k^2(x_1), \end{aligned}$$

giving the required contradiction.

It is now clear that $y_k$ cannot satisfy (1.2), which is all that is necessary for the present uniqueness proof. The detailed asymptotic behaviour of $y_k$ is given in §7.

Now suppose that $k > k^*$. We wish to prove that $y_k$ blows up at a finite point.

If $y'_k < 0$, we are done, for then we can prove, as in the proof of limited uniqueness, that $y_k > y_{k*}$ so long as both continue to exist, and so, if $y_k$ does not blow up at a finite point, it is a solution of (1.1) which exists and is positive for all $x$, and then the analysis in the existence proof shows that $y_k$ satisfies (1.2), *contradicting limited uniqueness.*

In the case $\alpha \leq 1$, we saw in §2 that $y''_{k*} > 0$, i.e. $2y^{\alpha}_{k*} + x > 0$; and so $y''_k > 0$ as long as $y_k > y_{k*}$, so that, if $y_k$ meets $y_{k*}$, we must have $y'_k < 0$ at the first point of meet (for decreasing $x$). But then the use of the energy function $V_k$ shows as before that there is no point of meet, so that $y'_k < 0$ and $y_k$ must blow up at a finite point.

The only difficulty occurs if $\alpha > 1$, for then it is not clear that $y'_k$ may not vanish. Of course, $y'_k$ can only vanish for the first time (as $x$ decreases) where $y''_k \leq 0$, i.e. where $2y^{\alpha}_k + x \leq 0$. But if $\alpha > 1$, it is true that $2y^{\alpha}_{k*} + x < 0$ for all $x$ sufficiently large and negative, and so for values of $k$ sufficiently close to $k*$, it must be true that $y_k$ crosses the curve $2y^{\alpha} + x = 0$.

Now consider two subsets of the semiaxis $k > k*$. The set $T_1$ is the set of $k$ such that $y'_k$ becomes positive at some finite point. The set $T_2$ is the set of $k$ such that $y'_k < 0$ for so long as the solution exists.

The set $T_1$ is clearly open. Let us suppose for contradiction that it is non-empty. (If it is empty, $y'_k < 0$ for $k > k*$, and we are done.) It is certainly disjoint from $T_2$.

The set $T_2$ is also open. For, if $k_0$ lies in $T_2$, we have already seen that the solution blows up at a finite point or we would contradict limited uniqueness, and it blows up with $y'_{k_0} \to -\infty$. Lemma 2 and its proof then assure us that for any $k$ sufficiently close to $k_0$, $y_k$ also blows up at a finite point with $y'_k \to -\infty$, and that we can suppose (for $k$ sufficiently close to $k_0$) that $y'_k < 0$ everywhere that $y_k$ exists, i.e. $k$ is in $T_2$. Lemma 4 and its proof show that $T_2$ is non-empty.

As usual, there must be some $k > k^*$ which is in neither $T_1$ nor $T_2$. The corresponding solution $y_k$ must have the property that, at some finite point, $y_k' = y_k'' = 0$. But then it is easy to check from (1.1) that $y_k''' > 0$, which forces $y_k'$ to take positive values and places $k$ in $T_1$. This contradiction leads to the conclusion that $T_1$ is empty and completes the proof of uniqueness.

7. Asymptotic behaviour

We want to investigate the behaviour as $x \to -\infty$ of $y_k(x)$ for $0 < k < k^*$. We have already seen that $y_k$ is bounded. To obtain further information, we make the substitutions

$$t = \frac{2}{3}(-x)^{3/2} + f(-x), \quad y_k = (-x)^{-1/4}w,$$

where the function $f$ will be chosen suitably later. Then (1.1), after some routine calculations, becomes (with $u = -x$)

$$\frac{d^2w}{dt^2} + \left\{\frac{f''}{\left(u^{1/2} + f'\right)^2} - \frac{1}{2}\frac{f'}{u\left(u^{1/2} + f'\right)^2}\right\}\frac{dw}{dt} + \frac{5}{16}\frac{w}{u^2\left(u^{1/2} + f'\right)^2}$$

(7.1)

$$= -\left\{\frac{u}{\left(u^{1/2} + f'\right)^2} - \frac{2|w|^\alpha}{|u|^{\alpha/4}\left(u^{1/2} + f'\right)^2}\right\} w\ .$$

If, in the first place, we take $f \equiv 0$, we have

(7.2)
$$\frac{d^2w}{dt^2} + \frac{5}{16}\frac{w}{u^3} = -\left(1 - \frac{2|w|^\alpha}{u|u|^{\alpha/4}}\right) w\ .$$

Now multiply by $w' \equiv dw/dt$ and integrate. Recalling that $y_k(x)$ is bounded as $x \to -\infty$, so that $w(t) = O(t^{1/6})$ as $t \to \infty$, we see that we can, by integration by parts, estimate

$$\int_T^\infty \frac{ww'}{t^2}\, dt = O(T^{-5/3}),$$

and this (and other similar estimates) leads to

$$w'^2 + w^2 = \text{constant} + O(t^{-1/3})\ .$$

It follows from this that $w'$ and $w$ are both bounded, so that $y_k = O(|x|^{-1/4})$, $y_k' = O(|x|^{1/4})$, and (7.2) is then oscillatory and asymptotically the distance between successive zeros of $w(t)$ is $\pi$.

In the case $\alpha > 2$ we can now quickly complete the argument. For (7.2) can be written as

$$w'' + w = O(t^{-1-\delta}),$$

for some $\delta > 0$, and since $t^{-1-\delta} \in L^1(1,\infty)$, a routine application of the variation of constants formula shows that

$$w = d \sin(t - c_2) + O(t^{-\delta}),$$

proving (1.11).

If $\alpha \leq 2$, the leading terms on the right-hand side of (7.1) (assuming $f'$ to be small) are

$$- w + \frac{2f'}{u^{1/2}} w + \frac{2|w|^\alpha}{u|u|^{\alpha/4}} w,$$

and it turns out that we have to choose $f$ so that the second and third of these are of the same order. So for $\alpha = 2$ we take $f(u) = c \log u$, and for $\alpha < 2$ we take $f(u) = cu^{\frac{1}{2}-\frac{1}{4}\alpha} \left(\frac{1}{2} - \frac{1}{4} \alpha\right)$, where the constant $c$ (which may depend on $\alpha$) has still to be determined.

Let us first take the case $\alpha = 2$. Then with the given choice of $f$ the equation (7.1) reduces to

$$\frac{d^2w}{dt^2} + w = \frac{4}{3t} (c + w^2)w + O(t^{-2}\log t)$$

Set $w = \rho \cos\theta$, $w' = \rho \sin\theta$, so that

$$\rho^2 = w^2 + w'^2, \quad \theta = \tan^{-1}(w'/w) .$$

Then

$$\rho\rho' = ww' + w'w'' = \frac{4}{3t} (c + w^2)ww' + O(t^{-2}\log t),$$

and so, for some constant $d$, by an integration by parts,

$$\rho^2 = d^2 + \frac{2}{3t} (2cw^2 + w^4) + O(t^{-1}\log t) .$$

Also,

$$\frac{d\theta}{dt} = \frac{ww'' - w'^2}{w^2 + w'^2} = -1 + \frac{4}{3t\rho^2} (cw^2 + w^4) + O(t^{-2}\log t)$$

$$= -1 + \frac{4}{3t} (c \cos^2\theta + d^2\cos^4\theta) + O(t^{-2}\log t)$$

$$= -1 + \frac{4}{3t} (c \cos^2\theta + d^2\cos^4\theta) \frac{d\theta}{dt} + O(t^{-2}\log t) .$$

Hence, integrating by parts and choosing $c$ so that

$$\int_0^{\frac{1}{2}\pi} (c\cos^2\theta + d^2\cos^4\theta)d\theta = 0,$$

i.e. choosing $c = -\frac{3}{4}d^2$, we have

$$\theta = -t + \text{constant} + O(t^{-1}\log t),$$

from which (1.10) follows.

The argument for the case $\alpha < 2$ is similar. The equation (7.1) reduces to

$$\frac{d^2w}{dt^2} + w = 2\left(\frac{2}{3t}\right)^{\frac{2}{3}+\frac{1}{6}\alpha}(c + |w|^\alpha)w + O(t^{-\frac{4}{3}-\frac{1}{3}\alpha}) .$$

Introducing $\rho$ and $\theta$ as before, we see that

$$\rho^2 = d^2 + O(t^{-\frac{1}{3}-\frac{1}{3}\alpha}) ,$$

and so

$$\begin{aligned}\frac{d\theta}{dt} &= -1 + \frac{2}{\rho^2}\left(\frac{2}{3t}\right)^{\frac{2}{3}+\frac{1}{6}\alpha}(cw^2 + |w|^{2+\alpha}) + O(t^{-\frac{4}{3}-\frac{1}{3}\alpha})\\ &= -1 + 2\left(\frac{2}{3t}\right)^{\frac{2}{3}+\frac{1}{6}\alpha}(c\cos^2\theta + |d|^\alpha|\cos\theta|^{2+\alpha})\frac{d\theta}{dt} + O(t^{-1-\frac{1}{2}\alpha}) .\end{aligned} \tag{7.3}$$

We now choose $c$ so that

$$\int_0^{\frac{1}{2}\pi} (c\cos^2\theta + |d|^\alpha\cos^{2+\alpha}\theta)d\theta = 0, \tag{7.4}$$

and note that, by the transformation $\cos^2\theta = z$, and with the usual notation for the beta function,

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} \cos^{2+\alpha}\theta d\theta &= \frac{1}{2}\int_0^1 z^{\frac{1}{2}+\frac{1}{2}\alpha}(1-z)^{-1/2}dz\\ &= \frac{1}{2}B\left(\frac{3}{2}+\frac{1}{2}\alpha, \frac{1}{2}\right)\\ &= \frac{1}{2}\Gamma\left(\frac{3}{2}+\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}\right)\Big/\Gamma\left(2+\frac{1}{2}\alpha\right) .\end{aligned}$$

The choice of $c$ from (7.4) is thus $c = -c_1$, where $c_1$ is given by (1.9), and with this choice of $c$ we integrate (7.3) by parts to obtain

$$\theta = -t + \text{constant} + O(t^{-\alpha/2}),$$

from which (1.8) follows.

## 8. The value of $k^*(2)$

From now on we are concerned with (1.1) only in the case $\alpha = 2$. Using the relationship between (1.1) and the Korteweg-de Vries equation that was mentioned in the introduction, Ablowitz and Segur [4] have shown a connection between the solution of (1.1) which is asymptotic to $k\mathrm{Ai}(x)$ as $x \to \infty$, i.e. $y_k(x)$ in our notation, and the solution of the pair of integral equations (in which $x$ is effectively a parameter)

$$K_1(x,y) = k\mathrm{Ai}\left(\frac{x+y}{2}\right) - \frac{1}{2}k\int_x^\infty K_2(x,s)\mathrm{Ai}\left(\frac{s+y}{2}\right)ds, \tag{8.1}$$

$$K_2(x,y) = -\frac{1}{2}k\int_x^\infty K_1(x,s)\mathrm{Ai}\left(\frac{s+y}{2}\right)ds . \tag{8.2}$$

Ablowitz and Segur prove the following results about the solution of these equations.

*For $x$ sufficiently large (depending on the choice of $k$), say $x \geq x_0$, there exists a unique solution of (8.1-2) that is square-integrable on $[x_0,\infty)$. Further, for $x \geq x_0$, we have*

$$K_1(x,x) = y_k(x) .$$

This result clearly suggests the importance of studying the operator $L_x$, where, for any $f \in L^2(x,\infty)$, $L_x$ is defined by

$$(L_x f)(y) = \frac{1}{2}\int_x^\infty \mathrm{Ai}\left(\frac{y+s}{2}\right)f(s)ds . \tag{8.3}$$

Theorem 2 in §1, that $k^*(2) = 1$, follows from the following series of lemmas, the proofs of most of which are fairly immediate. One part of the proof is deferred to §9.

**Lemma 8.1.** *$L_x$ is a compact, indeed Hilbert-Schmidt, self-adjoint operator in $L^2(x,\infty)$.*

**Proof.** This comes immediately from the observation that the kernel $\frac{1}{2}\mathrm{Ai}\left(\frac{y+s}{2}\right)$ of $L_x$ is symmetric with

$$\int_x^\infty\int_x^\infty \left\{\mathrm{Ai}\left(\frac{y+s}{2}\right)\right\}^2 dy\, ds < \infty,$$

the convergence of the integral being a consequence of the exponential decay of $\mathrm{Ai}(t)$ as $t \to \infty$.

Before stating the next lemma, we note that, at least in a formal sense,

$$L_{-\infty}^2 = I, \tag{8.4}$$

where $I$ is the identity operator in $L^2(-\infty,\infty)$. This is nothing more than the Titchmarsh-Kodaira form of the *eigenfunction expansion* (*or resolution of the identity*) associated with the operator

$$-\frac{d^2}{dt^2} + \frac{1}{8}t \tag{8.5}$$

in $L^2(-\infty,\infty)$, for, as we shall see in §9, that expansion can be written formally as

$$f(y) = \frac{1}{4}\int_{-\infty}^{\infty} Ai\left(\frac{y+s}{2}\right)\left\{\int_{-\infty}^{\infty} Ai\left(\frac{s+z}{2}\right)f(z)dz\right\}dy, \tag{8.6}$$

which is just (8.4). Since the eigenfunctions associated with the operator (8.5) are obtained by solving the equation

$$-\frac{d^2w}{dt^2} + \frac{1}{8}tw = \lambda w,$$

or, with $s = -8\lambda$,

$$\frac{d^2w}{dt^2} = \frac{1}{8}(t+s)w,$$

we see that these eigenfunctions must be multiples of $Ai\left(\frac{t+s}{2}\right)$, which is in accordance with (8.6). (They are only "generalized eigenfunctions" since the behaviour of $Ai(t)$ as $t \to -\infty$ prevents the eigenfunctions lying in $L^2(-\infty,\infty)$.)

The next lemma gives an analytical statement of at least part of these formal ideas.

<u>Lemma 8.2.</u> <u>For any</u> $f \in L^2(-\infty,\infty)$, <u>the function</u> $L_x f$, <u>which can be regarded as a function defined on</u> $(-\infty,\infty)$, <u>converges in mean as</u> $x \to -\infty$, <u>and</u>

$$\lim_{x\to-\infty} \| L_x f \| = \| f \|,$$

<u>the norms being the norms in</u> $L^2(-\infty,\infty)$.

<u>Proof.</u> This is just the Parseval theorem corresponding to the expansion (8.6), and it can be proved formally by multiplying both sides of (8.6) by $f(y)$ and integrating.

A proof has to follow the lines of Theorem 3.7 of [10], which gives the analysis for an operator in $L^2(0,\infty)$ with a boundary condition at $0$, the modifications for $L^2(-\infty,\infty)$ being indicated in §3.8 of [10]. In our notation, Titchmarsh shows that $L_x f$ converges in mean in $L^2(-\infty,\infty;\ dk)$, where the Lebesgue-Stieltjes measure $dk$ measures the spectral density, and (8.6) shows that in this particular case this measure is just Lebesgue measure. Furthermore, the Parseval theorem (formula (3.7.1) in [10]) states just (8.7).

Lemma 8.3. For any finite $x$, $|||\, L_x \,||| \le 1$, where $|||\, \ldots \,|||$ denotes the operator norm of $L_x$ in $L^2(x,\infty)$.

Proof. We note first that $|||\, L_x \,|||$ is a nondecreasing function of $x$. For if we estimate the eigenvalue of largest modulus of $L_x$ by the usual variational procedure, the set of trial functions increases as $x$ decreases, and so the modulus of the eigenvalue does not decrease.

Now assume for contradiction that, for some $X$, the modulus of this largest eigenvalue of $L_X$ exceeds 1. Then, if $\phi_X$ is the corresponding eigenfunction, we have

$$L_X \phi_X = \mu \phi_X,$$

with $|\mu| > 1$. The eigenfunction $\phi_X$ is defined only on $(X,\infty)$, but if we set

$$\phi(y) = \begin{cases} \phi_X(y), & y \ge X, \\ 0, & y < X, \end{cases}$$

then, for any $x \le X$,

$$(L_x \phi)(y) = \mu\phi(y) \quad \text{for} \quad y \ge X,$$

and so

$$\| L_x \phi \|_{L^2(-\infty,\infty)} \ge \| L_x \phi \|_{L^2(X,\infty)} = |\mu| \ \| \phi \|_{L^2(X,\infty)} = |\mu| \ \| \phi \|_{L^2(-\infty,\infty)} .$$

But this contradicts Lemma 8.2 and completes the proof.

Lemma 8.4. $k^*(2) \ge 1$ .

Proof. The equations (8.1-2) can be combined to give a single integral equation for $K_1$. In fact, (8.8) $$K_1(x,y) = kAi\left(\frac{x+y}{2}\right) + k^2\left(L_x^2 K_1\right)(x,y),$$

and Lemma 8.3 implies $|||\, L_x^2 \,||| \le 1$.

Now suppose for contradiction that $k^*(2) < 1$. Then (8.8) can be solved for any finite $x$ with $k = \frac{1}{2}(1 + k^*)$, since $k < 1$, and this leads to a solution $K_1(x,x) = y_k(x)$ of (1.1) which exists for all $x$. But at the same time $k > k^*$, and so we have a contradiction to Theorem 1.

Proof of Theorem 2. In view of Lemma 8.4, it only remains to prove that $k^*(2) > 1$ is impossible. So let us suppose for contradiction that $k^*(2) > 1$. Then, by Theorem 1, $y_k(x)$ exists and is bounded for all $x$ and for all $k$ with $0 < k < k^*$.

Now consider the equation

(8.9) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 K_1(x,y) = \left(\frac{x+y}{2}\right) K_1(x,y) + 2\{K_1(x,x)\}^2 K_1(x,y)$$

which Ablowitz and Segur show is satisfied by the solution $K_1(x,y)$ of (8.8) for $y \ge x$ and $x$ sufficiently large (depending on the choice of $k$). Indeed, if $k$ is sufficiently small, (8.9) holds for all $x$ and $y$. Writing

$$u = \frac{1}{2}(x + y), \quad v = \frac{1}{2}(x - y), \quad F(u,v) = K_1(x,y),$$

we see that $F$ satisfies

(8.10) $$\frac{\partial^2 F}{\partial u^2}(u,v) = uF(u,v) + 2F^2(u,0)F(u,v) .$$

For any fixed $v \le 0$, we can solve this equation for $F(u,v)$ uniquely provided that we are given $F(u,0)$, which for consistency must satisfy (1.1), and also, for some $u_0$, the values of $F(u_0,v)$ and $\frac{\partial F}{\partial u}(u_0,v)$. We can take for $F(u,0)$ just $y_k(u)$, and if $u_0$ is sufficiently large (depending on $k$), we can take $F(u_0,v) = K_1(u_0 + v, u_0 - v)$, with the corresponding value for $\frac{\partial F}{\partial u}(u_0,v)$. If $0 < k < k^*$, the solution of (8.10) then exists for the particular value of $v$ and all $u$, since it is a linear equation with continuous coefficients, and the solution is analytic in $k$. Since this is true for any one particular value of $v$, it is true for all $v \le 0$. Also, $F\left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ coincides with $K_1(x,y)$ at least for $k$ sufficiently small (when we can solve (8.8) to give $K_1$ and show that it satisfies (8.9), and (from (8.8)) $K_1(x,y)$ is analytic in $k$ except for possible singularities at those values of $k$ for which $k^{-2}$ is an eigenvalue of $L_x^2$. Hence $F\left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ and $K_1(x,y)$ coincide (and $K_1$ is analytic in $k$) for all $k$ with $0 < k < k^*$, and all $x$ and $y$.

Now let $k$ increase from $0$ to $|k_{0,x}|$, where (for any fixed $x$) $k_{0,x}^{-2}$ is the largest eigenvalue of $L_x^2$. Since we are assuming that $k^* > 1$, it certainly follows from Lemma 8.2 that $|k_{0,x}| < k^*$ if $x$ is chosen sufficiently large, and so, by what we have said above, $K_1(x,y)$ remains analytic (and so in particular bounded) as $k$ increases to $|k_{0,x}|$. But since $k_{0,x}^{-2}$ is an eigenvalue of $L_x^2$, this is possible (from (8.8)) only if

$$\int_x^\infty \mathrm{Ai}\left(\frac{x+y}{2}\right)\psi_{0,x}(y)\,dy = 0, \tag{8.11}$$

where $\psi_{0,x}$ is the eigenfunction of $L_x^2$ corresponding to the eigenvalue $k_{0,x}^{-2}$. We have merely therefore to show that (8.11) is impossible, and the theorem is proved.

To show (8.11) impossible, we note first that $\psi_{0,x}$ must be an eigenfunction of $L_x$, with eigenvalue $\pm k_{0,x}^{-1}$. Thus

$$\psi_{0,x}(y) = \pm \frac{1}{2} k_{0,x} \int_x^\infty \mathrm{Ai}\left(\frac{y+s}{2}\right)\psi_{0,x}(s)\,ds, \tag{8.12}$$

and setting $y = x$, we see from (8.11) that

$$\psi_{0,x}(x) = 0 . \tag{8.13}$$

Making the transformation $s = t - y$, we write (8.12) as

$$\psi_{0,x}(y) = \pm \frac{1}{2} k_{0,x} \int_{x+y}^\infty \mathrm{Ai}\left(\frac{1}{2}t\right)\psi_{0,x}(t-y)\,dt,$$

so that

$$\psi'_{0,x}(y) = \pm \frac{1}{2} k_{0,x}\left\{-\mathrm{Ai}\left(\frac{x+y}{2}\right)\psi_{0,x}(x) - \int_{x+y}^\infty \mathrm{Ai}\left(\frac{1}{2}t\right)\psi'_{0,x}(t-y)\,dt\right\}$$

$$= \mp \frac{1}{2} k_{0,x} \int_x^\infty \mathrm{Ai}\left(\frac{y+s}{2}\right)\psi'_{0,x}(s)\,ds,$$

using (8.13) and the backward transformation $t = s + y$. Thus $\psi'_{0,x}$ is also an eigenfunction of $L_x$ with eigenvalue $\mp k_{0,x}^{-1}$, and the same argument as was applied to $\psi_{0,x}$ shows that also

$$\psi'_{0,x}(x) = 0 .$$

The process can now be repeated to conclude that $\psi_{0,x}$ and all its derivatives vanish at $x$, *implying that* $\psi_{0,x}$ *is identically zero, which is impossible for an* eigenfunction.

9. Proof of (8.6)

We want to obtain the eigenfunction expansion associated with the operator

(9.1)
$$L = -\frac{d^2}{dx^2} + \frac{1}{8}x$$

in $L^2(-\infty,\infty)$, and this can be found by following various formulae given in [10].

As in §3.1 of [10], the procedure is as follows. First consider the problem associated with the operator (9.1) in $L^2(-\infty,0)$, and let $m_1(\mu)$ be the (uniquely determined) function such that

$$\theta(x,\mu) + m_1(\mu)\phi(x,\mu) \in L^2(-\infty,0),$$

where $\theta,\phi$ are the solutions of $Ly = \mu y$ for which

$$\theta(0,\mu) = 1, \quad \theta'(0,\mu) = 0,$$

$$\phi(0,\mu) = 0, \quad \phi'(0,\mu) = -1 .$$

In fact, as one of the examples in Chapter IV of [10], §4.13, Titchmarsh works out the function $m(\lambda)$ associated with the operator

$$M = -\frac{d^2}{dt^2} - t$$

in $L^2(0,\infty)$, and the equation $My = \lambda y$, and this, by the transformation $t = -\frac{1}{2}x$, $\lambda = 4\mu$, $m(\lambda) = -2m_1(\mu)$, gives the function $m_1(\mu)$ that we want. Thus

$$m_1(\mu) = \frac{1}{2}\lambda^{1/2}\frac{H^{(1)\prime}_{1/3}\left(\frac{2}{3}\lambda^{3/2}\right)}{H^{(1)}_{1/3}\left(\frac{2}{3}\lambda^{3/2}\right)} + \frac{1}{4\lambda}$$

$$= \frac{1}{2}\frac{d}{d\lambda}\left\{\lambda^{1/2}H^{(1)}_{1/3}\left(\frac{2}{3}\lambda^{3/2}\right)\right\}\Big/\left\{\lambda^{1/2}H^{(1)}_{1/3}\left(\frac{2}{3}\lambda^{3/2}\right)\right\} .$$

Here, in standard Bessel function notation [5], $H^{(1)}_{1/3}$ is the Hankel function of the first kind, and we have the relations

$$H^{(1)}_{1/3}(z) = \frac{2}{\pi i}e^{-\pi i/6}K_{1/3}(ze^{-\pi i/2})$$

$$= \frac{2}{\pi i}e^{-\pi i/6}\{e^{-\pi i/3}K_{1/3}(ze^{-3\pi i/2}) - \pi i I_{1/3}(ze^{-3\pi i/2})\},$$

where $K_{1/3}$ (already mentioned in §1) and $I_{1/3}$ are Bessel functions of imaginary argument. Hence

$$m_1(\mu) = \frac{1}{2}\frac{\frac{d}{d\lambda}\left\{(-\lambda)^{1/2}\left[e^{-\pi i/3}K_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right) - \pi i I_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right)\right]\right\}}{(-\lambda)^{1/2}\left[e^{-\pi i/3}K_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right) - \pi i I_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right)\right]}$$

$$= \frac{1}{2}\frac{\frac{d}{d\lambda}\left\{Ai(-\lambda) - 3^{-1/2}ie^{\pi i/3}(-\lambda)^{1/2}I_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right)\right\}}{Ai(-\lambda) - 3^{-1/2}ie^{\pi i/3}(-\lambda)^{1/2}I_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right)} .$$

Similarly, we define $m_2(\mu)$ to be the uniquely determined function such that

$$\theta(x,\mu) + m_2(\mu)\phi(x,\mu) \in L^2(0,\infty),$$

where $\theta,\phi$ are as before. This function is also in effect worked out by Titchmarsh in §4.12 of [10], and

$$m_2(\mu) = -\frac{1}{2}\psi_0'(0,\lambda)/\psi_0(0,\lambda),$$

where, as before, $\lambda = 4\mu$, and

$$\psi_0(t,\lambda) = (t-\lambda)^{1/2}K_{1/3}\left\{\frac{2}{3}(t-\lambda)^{3/2}\right\},$$

so that

$$m_2(\mu) = -\frac{1}{2}\ A_i'(-\lambda)/Ai(-\lambda)\ .$$

Now we note that $m_2(\mu)$ is real for real $\mu$, and this simplifies the expansion formula, as Titchmarsh points out in formula (3.1.12) of [10]. (Titchmarsh actually supposes that $m_1(\mu)$ is real for real $\mu$, but the adjustment is easily made.) Indeed, with a suitable interpretation of the infinite integrals, the expansion formula becomes

(9.2) $$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\psi_2(x,\mu)d\xi(\mu)\int_{-\infty}^{\infty}\psi_2(y,\mu)f(y)dy,$$

where

$$\psi_2(x,\mu) = \theta(x,\mu) + m_2(\mu)\phi(x,\mu)$$

and

(9.3) $$\xi(\mu) = \lim_{\delta\to 0}\int_0^{\mu} - \operatorname{im}\frac{1}{m_1(u+i\delta) - m_2(u+i\delta)}du\ .$$

Now, by definition of $m_2$, $\psi_2(x,\mu)$ is a solution of $Ly = \mu y$ which is in $L^2(0,\infty)$, and so

$$\psi_2(x,\mu) = CAi\left(\frac{1}{2}x - \lambda\right),$$

for some constant $C$. But also $\psi_2(0,\mu) = 1$, and so in fact

$$\psi_2(x,\mu) = Ai\left(\frac{1}{2}x - \lambda\right) Ai(-\lambda) .$$

To determine $\xi$, we note first that, for the well-behaved functions with which we are concerned, we can pass to the limit under the integral sign in (9.3) and obtain

$$\frac{d\xi}{d\mu} = -\mathrm{im}\,\frac{1}{m_1(\mu) - m_2(\mu)} .$$

Hence (8.6) follows from (9.2) if we can show (for real $\mu$) that

$$-\mathrm{im}\,\frac{1}{m_1(\mu) - m_2(\mu)} = 2\pi\{Ai(-\lambda)\}^2 .$$

Now

(9.4) $$2\{m_1(\mu) - m_2(\mu)\}$$

$$= \frac{-3^{-1/2}ie^{\pi i/3}\left[Ai(-\lambda)\,\frac{d}{d\lambda}\left\{(-\lambda)^{1/2}I_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right)\right\} + A_i'(-\lambda)(-\lambda)^{1/2}I_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right)\right]}{\left\{Ai(-\lambda) - 3^{-1/2}ie^{\pi i/3}(-\lambda)^{1/2}I_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right)\right\}Ai(-\lambda)} .$$

Since $Ai(-\lambda)$ and $(-\lambda)^{1/2}I_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right)$ satisfy the same second order differential equation, we know that their Wronskian (which appears in the numerator of (9.4)) is independent of $\lambda$, and letting $\lambda \to -\infty$, so that

$$Ai(-\lambda) \sim \frac{1}{2}\pi^{-1/2}(-\lambda)^{-1/4}\exp\left(-\frac{2}{3}(-\lambda)^{3/2}\right),$$

$$(-\lambda)^{1/2}I_{1/3}\left(\frac{2}{3}(-\lambda)^{3/2}\right) \sim \frac{1}{2}\,3^{1/2}\pi^{-1/2}(-\lambda)^{-1/4}\exp\left(\frac{2}{3}(-\lambda)^{3/2}\right),$$

we see that the numerator of (9.4) becomes $\frac{1}{2}\pi^{-1}ie^{\pi i/3}$. Thus

$$-\mathrm{im}\,\frac{1}{m_1(\mu) - m_2(\mu)} = 2\pi\{Ai(-\lambda)\}^2,$$

as required.

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## REPORT DOCUMENTATION PAGE

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The differential equation considered is $y'' - xy = y|y|^{\alpha}$. For general positive $\alpha$ this equation arises in plasma physics, in work of de Boer and Ludford. For $\alpha = 2$, it yields similarity solutions to the well-known Korteweg-de Vries equation. Solutions are sought which satisfy the boundary conditions

(1) $y(\infty) = 0$

(2) $y(x) \sim \left(-\frac{1}{2}x\right)^{1/\alpha}$ as $x \to -\infty$.


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It is shown that there is a unique such solution, and that it is, in a certain sense, the boundary between solutions which exist on the whole real line and solutions which, while tending to zero at plus infinity, blow up at a finite $x$. More precisely, any solution satisfying (1) is asymptotic at plus infinity to some multiple $kAi(x)$ of Airy's function. We show that there is a unique $k^*(\alpha)$ such that when $k = k^*(\alpha)$ the condition (2) is also satisfied. If $0 < k < k^*$, the solution exists for all $x$ and tends to zero as $x \to -\infty$, while if $k > k^*$ then the solution blows up at a finite $x$. For the special case $\alpha = 2$ the differential equation is classical, having been studied by Painlevé around the turn of the century. In this case, using an integral equation derived by inverse scattering techniques by Ablowitz and Segur, we are able to show that $k^* = 1$, confirming previous numerical estimates.