LEVEL II

ORC 78-5
APRIL 1978

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This research has been partially supported by the Air Force Office of Scientific Research (AFSC), USAF, under Grant AFOSR-77-3213 and the Office of Naval Research under Contract N00014-77-C-0299 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

78 08 10

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: ORC-78-5	2. GOVT ACCESSION NO.
	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): ON THE FIRST TIME A SEPARATELY MAINTAINED PARALLEL SYSTEM HAS BEEN DOWN FOR A FIXED TIME	5. TYPE OF REPORT & PERIOD COVERED: Research Report
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Sheldon M. Ross and Jack Schechtman	8. CONTRACT OR GRANT NUMBER(s): AFOSR-77-3213
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Operations Research Center, University of California, Berkeley, California 94720	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS: United States Air Force, Air Force Office of Scientific Research, Bolling AFB, D.C. 20332	12. REPORT DATE: April 1978
	13. NUMBER OF PAGES: 15
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Also supported by the Office of Naval Research under Contract N00014-77-C-0299. AFOSR-77-3213

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Parallel System
New Better Than Used
Down for Fixed Time
First Passage

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(SEE ABSTRACT)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

270 750

ABSTRACT

Consider a system consisting of n separately maintained independent components where the components alternate between intervals in which they are "up" and in which they are "down." When the i^{th} component goes up [down] then, independent of the past, it remains up [down] for a random length of time having distribution $F_i[G_i]$ and then goes down [up]. We say that component i is failed at time t if it has been "down" at all time points $s \in [t - A,t]$; otherwise it is said to be working. Thus a component is failed if it is down and has been down for the previous A time units. Assuming that all components initially start "up" let T denote the first time they are all failed, at which point we say the system is failed. We obtain the moment generating function of T when $n = 1$, for general F and G , thus generalizing previous results which assumed that at least one of these distributions be exponential. In addition we present a condition under which T is an NBU (new better than used) random variable. Finally we assume that all the up and down distributions F_i , G_i , $i = 1, \ldots, n$, are exponential and we obtain an exact expression for $E(T)$ for general n ; in addition we obtain bounds for all higher moments of T by showing that T is NBU.

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0. INTRODUCTION AND SUMMARY

In considering a system that works for a random time and when failed is fixed in a length of time that is also random an important question is the study of the first time the system is not working for an interval of time longer than some prespecified value. For instance in a nuclear reactor, when the safety system is out for some critical time it is necessary to shut down the complete system with all the problems this entails. In the food industry where food must in general be kept at a certain temperature, an important question when the refrigeration system goes down is how long this situation can be maintained before the food becomes spoiled.

In this paper we consider a system consisting of n separately maintained independent components where the components alternate between intervals in which they are "up" and in which they are "down." When the i^{th} component goes up [down] then, independent of the past, it remains up [down] for a random length of time having distribution $F_i[G_i]$ and then goes down [up]. We say that component i is failed at time t if it has been "down" at all time points $s \in [t - A,t]$; otherwise it is said to be working. Thus a component is failed if it is down and has been down for the previous A time units. Assuming that all components initially start "up" let T denote the first time they are all failed, at which point we say the system is failed.

In Section 1 we obtain the moment generating function of T when $n = 1$, for general F and G, thus generalizing results in [2] and [3] which assumed that at least one of these distributions be exponential. In Section 2 we present a condition under which T is an NBU (new better than used) random variable. In Section 3 we assume that all the up and down distributions F_i, G_i, $i = 1, \ldots, n$, are exponential and we obtain an exact expression for $E(T)$ for general n; in addition we obtain bounds for all higher moments of T by showing that T is NBU.

1. The Case $n = 1$

Let us denote by N the number of "up" intervals that occur before the component fails. Then given $N = k$, we can represent T by

$$T = X_1 + \cdots + X_k + Y_1^A + \cdots + Y_{k-1}^A + A \tag{1}$$

where X_i denotes the length of the i^{th} up cycle and Y_i^A the length of the i^{th} down cycle. All the random variables in the representation (1) are independent with the X_i having distribution F and the Y_i^A having distribution

$$P\left\{Y_i^A \le x\right\} = P\{Y \le x \mid Y < A\} = \begin{cases} \dfrac{G(x)}{G(A)} & x < A \\ 1 & x \ge A \end{cases}$$

where F is the distribution of an up cycle and G that of a down cycle. As

$$P\{N = k\} = \bar{G}(A)(G(A))^{k-1}, \quad k = 1, \ldots$$

where $\bar{G} = 1 - G$, we obtain the moment generating function of T by conditioning on N as follows.

$$\begin{aligned} E[e^{sT}] &= E[E[e^{sT} \mid N]] \\ &= E\left[e^{sA}(\phi_X(s))^N\left(\phi_{Y^A}(s)\right)^{N-1}\right] \\ &= e^{sA}\phi_X(s)\bar{G}(A) \sum_{k=1}^{\infty} \left(\phi_X(s)\phi_{Y^A}(s)G(A)\right)^{k-1} \qquad (2) \\ &= \frac{e^{sA}\phi_X(s)\bar{G}(A)}{1 - G(A)\phi_X(s)\phi_{Y^A}(s)} \end{aligned}$$

where

$$\phi_X(s) = E[e^{sX}] = \int_0^\infty e^{sx} dF(x)$$

$$\phi_{Y^A}(s) = E\left[e^{sY^A}\right] = E\left[e^{sY} \mid Y < A\right]$$

$$= \int_0^A \frac{e^{sx} dG(x)}{G(A)} .$$

For the special case in which X is exponential with mean $1/\lambda$ and Y is exponential with mean $1/\mu$ we have

$$E\left[e^{sT}\right] = \frac{\lambda(\mu - s)e^{-(\mu-s)A}}{s^2 - (\lambda + \mu)s + \lambda\mu e^{-(\mu-s)A}}$$

a result previously obtained in [2] and [3].

All of the moments can now be obtained by successive differentiation of (2), or by a direct conditioning argument. For instance we obtain

$$E[T] = E[E[T \mid N]]$$

$$= E[NE[X] + (N - 1)E[Y \mid Y < A] + A]$$

$$(3) \qquad = \frac{E[X]}{G(A)} + \frac{\int_0^A x dG(x)}{\bar{G}(A)} + A .$$

By viewing the working-failed system as an alternating renewal process it follows that the long run proportion of time the component is failed is

$$\frac{E[Y - A \mid Y > A]}{E[T] + E[Y - A \mid Y > A]} = \text{proportion of time failed}$$

which can be shown to equal

$$\frac{\int_A^\infty \bar{G}(y)dy}{E[Y] + E[X]} = \text{proportion of time failed.}$$

2. WHEN IS T NBU, $n = 1$

The nonnegative random variable W is said to be new better than used (written NBU) if

$$P\{W > s + t \mid W > s\} \leq P\{W > t\} \quad \forall s, t \geq 0 .$$

If we think of W as representing the life of some object then W NBU means that the additional remaining life of any s year old (i.e., used) item is stochastically smaller than that of a new item, for all s.

If W is NBU and has distribution function H then we also say that H is NBU.

Proposition 1:

If X, the length of an up time, is NBU then so is T.

Proof:

Suppose failure has not yet occurred by time s. Now there are 2 possibilities:

Case 1:

At time s the component is up and has been up for a time t. In this case the remaining time to failure has the distribution of the convolution of F_t and H, where F_t is the distribution of remaining up time for a component that has been up for a time t and H is the distribution of time to failure starting with the component initially down. But since F_t is stochastically smaller than F (definition of X being NBU) this distribution is stochastically smaller than the convolution of F and H, which is the distribution of T.

Case 2:

At time s the component is down and has been down for a time t (necessarily, $t < A$). In this case the remaining time to failure has some distribution call it D. However the distribution of T can be written as the convolution of D and the distribution of the first time that the component has been down for t consecutive time units. This latter convolution distribution is clearly stochastically larger than D.

Thus in all cases the distribution of T is stochastically larger than the distribution of remaining time until failure. Hence T is NBU.||

3. EXPONENTIAL LIFETIMES, GENERAL n

In this section we suppose there are n components and the distribution of up [down] time for the i^{th} component is exponential with rate $\lambda_i[\mu_i]$, $i = 1, \ldots, n$. We start by deriving $E[T]$, the expected time until the system fails, that is until all components are failed, starting with all components up.

We can write T as the sum of independent random variables as follows

$$T = T_{A=0} + Z \tag{4}$$

where $T_{A=0}$ denotes the first time that all components are down (it is thus equal to T in the special case $A = 0$) and Z the extra (or additional) time from $T_{A=0}$ until all components are failed. Now Brown in [1] has computed $E[T_{A=0}]$ and showed it to equal

$$E[T_{A=0}] = \sum_{k=1}^{n} \sum_{i_1<i_2<\cdots<i_k} \frac{\left[\prod_{j=1}^{k} \frac{\mu_{i_j}}{\lambda_{i_j}} - (-1)^k\right]}{\sum_{j=1}^{k} \left(\lambda_{i_j} + \mu_{i_j}\right)} .$$

Thus it remains to compute $E[Z]$. Let M denote an exponential random variable with rate $\mu \equiv \sum_{1}^{n} \mu_i$. Then by conditioning on whether or not all components remain down in the A time units following time $T_{A=0}$ we obtain

$$E[Z] = Ae^{-\mu A} + (1 - e^{-\mu A})[E[M \mid M < A] + E[D] + E[Z]]$$

where D is the time until all components are down given that they were all down and one has just gone up. Thus, from the above, we obtain

$$\text{(5)} \qquad E[Z] = A + (e^{\mu A} - 1) \frac{\int_0^A \mu x e^{-\mu x} dx}{1 - e^{-\mu A}} + E[D] \quad .$$

However, Ross in [5] has shown that

$$\text{(6)} \qquad E[D] = \frac{1 - \prod_{j=1}^{n} \frac{\lambda_j}{\mu_j + \lambda_j}}{\sum_{i=1}^{n} \mu_i \prod_{j=1}^{n} \frac{\lambda_j}{\mu_j + \lambda_j}}$$

and thus the expression for $E[T]$ follows from (4), (5) and (6).

The next proposition partly characterizes the distribution of T and will enable us to obtain bounds on all higher moments of T .

Proposition 2:

T is NBU.

Proof:

Suppose that all components have never been simultaneously failed by time s . There are 2 cases.

Case 1:

At time s all components are down, the one that has been down for the shortest time having been down for a time t (where necessarily $t < A$). Since T can be expressed as T_{A-t} (the first time all components have been down for the past t time units) plus a random variable having the same distribution as the remaining time to failure of the system, it follows that T is stochastically larger than the remaining time to system failure in this case.

Case 2:

Not all components are down at time s. In this case the remaining time to system failure can be written as the time until all components are down plus an independent random variable having the same distribution as Z in the representation (4). Now Ross in [6] has shown that the time until all components are down is stochastically larger starting with all being initially up than starting in any other position. Hence, from the representation (4), it follows that the remaining time to system failure at time s is stochastically smaller than T.

Hence, in all cases T is stochastically larger than the remaining time to system failure; thus proving the result.||

The above result is particularly useful as it enables us to obtain bounds on $E[f(T)]$ whenever f is an increasing convex function, by use of the following special case of Theorem 4.6 of Marshall and Proschan [4].

Proposition 3:

If X is NBU with mean $1/\lambda$, then

$$E[f(X)] \leq \int_0^\infty f(x)\lambda e^{-\lambda x} d\lambda$$

for all increasing convex functions f .

In words Proposition 3 says that if X is NBU then $E[f(X)] \leq F[f(M)]$ for all increasing convex f , where M is an exponential random variable having the same mean as X .

<u>Corollary 1</u>:

$$\mathrm{Var}(T) \leq (E[T])^2 \ .$$

<u>Proof</u>:

Follows immediately from Propositions 2 and 3 by use of the function $f(x) = x^2$.

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