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SPECIAL REPORT

STATISTICAL ANALYSIS OF RADIAL ERROR IN TWO DIMENSIONS

BY

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ARMY MATERIEL TEST AND EVALUATION DIRECTORATE
US ARMY WHITE SANDS MISSILE RANGE
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is an expository report on the statistics used in an analysis of two-dimensional radial error (or radial miss-distance). The Rayleigh distribution is the basis of many of the estimators or formulas. The Circular Probable Error (CPE), because of its frequent use in esimating impact accuracy of weapon systems, is given prominence in the following areas:

(1) Estimators for the CPE under various conditions
(2) Confidence intervals and hypothesis testing on the CPE
(3) The OC curve for the CPE — next page

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(4.) Sample size problems associated with the CPE
(5.) Proportion of hits within a specified CPE circle when a bias exists.

An estimator for the CPE using the Type I Asymptotic Distribution of Minimum Values (Gumbel distribution) is derived. Several numerical examples are given for various types of problems.

1.0 DEFINITION OF THE RAYLEIGH DISTRIBUTION AND THE CIRCULAR PROBABLE ERROR (CPE)

1.0.1 If X and Y are independent and are normally distributed random variables with zero means and equal variances, that is,

$$\mu_X = \mu_Y = 0$$

and

$$\sigma_X^2 = \sigma_Y^2 = \sigma^2$$

then the random variable $R = \sqrt{X^2 + Y^2}$ is Rayleigh distributed. The density function of the Rayleigh distribution is

$$p(R) = \left(\frac{R}{\sigma^2}\right) \exp\left[-\tfrac{1}{2}\left(\frac{R}{\sigma}\right)^2\right], \quad R \geq 0 \qquad (1)$$

and the cumulative distribution function is

$$F(r) = \text{Prob}\ (R \leq r) = 1 - \exp\left[-\tfrac{1}{2}\left(\frac{r}{\sigma}\right)^2\right] \qquad (2)$$

It is seen that the variable R is a radial distance from the origin of a rectangular coordinate system, X and Y. In an analysis of radial error (or radial miss-distance), the X component is the azimuth or deflection error, and the Y component is the range or pitch error. Azimuth and range errors are associated with ground or horizontal targets, and deflection and pitch errors are associated with vertical targets.

1.0.2 If $F(r) = 0.50$, then $r = 1.1774\sigma$ is the CPE, a measure of impact accuracy often used in a missile performance evaluation. The CPE is then the radius of the circle, centered at the target, which is expected to include 50% of the impact points. Figure 1 is a graph of equation 1, the density function of R. By definition of the CPE, the latter is the median of the Rayleigh distribution. Estimators for the CPE under conditions of existing biases in the means μ_X and μ_Y and/or large difference in the component variances, σ_X^2 and σ_Y^2, are given in paragraph 2.1. These estimators have been derived in various books and journals, some of which are referenced in Appendix A.

1.1 Tests on the Validity of the Rayleigh Distribution Assumption

a. Graphical Tests

(1) Since the Raleigh distribution is the Weibull with shape parameter $\beta = 2$, the ranked radial errors $r_{(1)} \leq r_{(2)} \leq \ldots \leq r_n$ can be plotted versus the ordinate

$$100 \left(\frac{i - 0.5}{n} \right) \%$$

$i = 1, 2 \ldots, n$ on Weibull graph paper. If the plotted points cluster about a straight line and β is estimated to be close to 2.0, then the data can be considered to be sufficiently well Rayleigh distributed.

(2) Another test, using ordinary graph paper, is as follows: Compute the estimate of the scale parameter of the Weibull distribution by the estimator

$$\hat{\eta} = \left[\frac{1}{N} \sum_{1}^{N} \ell n(r_i)^{\beta} \right]^{\frac{1}{\beta}} \qquad \beta = 2.0$$

This is a maximum likelihood estimator. Then plot the ranked radial errors versus the expected values of the Weibull order statistics multiplied

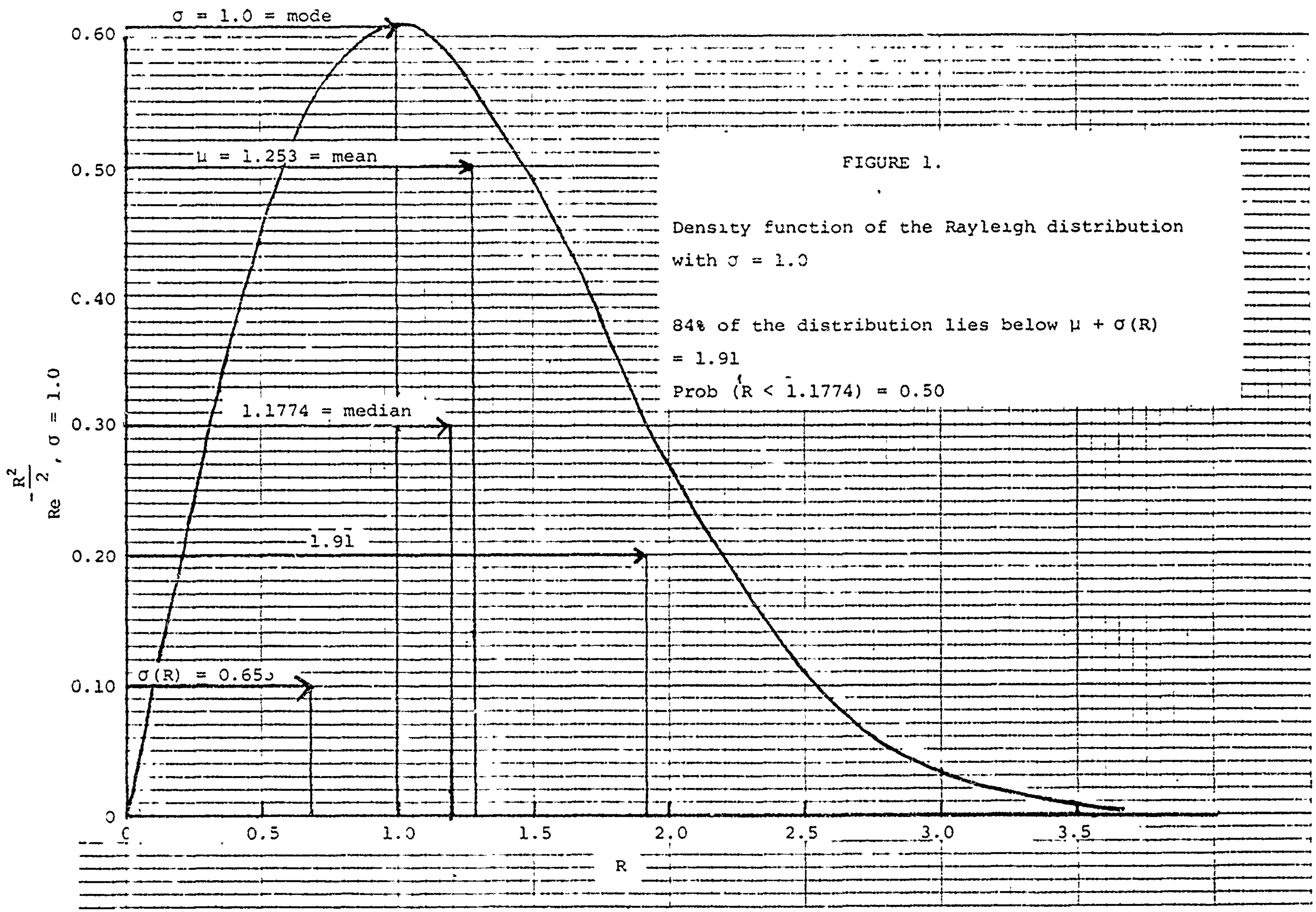

FIGURE 1.

Density function of the Rayleigh distribution with $\sigma = 1.0$

84% of the distribution lies below $\mu + \sigma(R) = 1.91$

Prob $(R < 1.1774) = 0.50$

by $\hat{r}_i$. The expected values of the order statistics are given in Table 1.1 for N = 1(1)32. If these plotted points cluster are about a straight line, then the radial errors are Rayleigh distributed. Graphical tests on distributional assumptions are always subjective; and if the plotted points do not appear to cluster too well or a skewness is disturbing, analytic tests yield more decisive results. Two such tests are given below.

b. Analytical Tests

(1) If the radial errors r_i are Rayleigh distributed, then

$$Y_i = r_i^2$$

$i = 1, 2, \ldots, N$ are exponentially distributed. The Shapiro-Wilk test (Ref 1) can be used to test for exponentiality if $7 \leq N \leq 35$.

(2) $N > 30$ (Ref 2). This test is based on the von Mises circular distribution. Only the angles α_i are given as the data $(i = 1, 2, \ldots, N)$ (Fig. 2).

The hypothesis that the α_i are uniformly distributed within the interval (0°, 360°), that is if $f(\alpha) = 1/360°$, is equivalent to the concentration parameter, K, being zero in the von Mises distribution. If $s^2 (\sin \alpha)$ and $s^2 (\cos \alpha)$ be the sample variances of $\sin \alpha_i$ and $\cos \alpha_i$, respectively, then the test statistic is $2N\bar{V}^2$, where

$$\bar{V}^2 = 1 - \left(\frac{N-1}{N}\right) \left[s^2 (\sin \alpha) + s^2 (\cos \alpha)\right]$$

If

$$2N\bar{V}^2 \geq \chi^2_\alpha (2)$$

TABLE 1.1

EXPECTED VALUES OF WEIBULL ORDER STATISTICS

SHAPE PARAMETER K = 2.0

M/N	1	2	3	4	5	6	7	8
1	0.88623	0.62666	0.51166	0.44311	0.39633	0.36180	0.33496	0.31333
2		1.14580	0.85664	0.71731	0.63024	0.56899	0.52283	0.48640
3			1.29037	0.99598	0.84793	0.57272	0.68440	0.63213
4				1.38851	0.09467	0.94313	0.84382	0.77152
5					1.46196	1.17045	1.01762	0.91612
6						1.52027	1.23158	1.07852
7							1.56838	1.28260
8								1.60921

M/N	9	10	11	12	13	14	15	16
1	0.29541	0.28025	0.26721	0.25583	0.24580	0.23685	0.22882	0.22156
2	0.45669	0.43184	0.41067	0.39234	0.37627	0.36203	0.34929	0.33781
3	0.59040	0.55605	0.52712	0.50232	0.48073	0.46172	0.44481	0.42964
4	0.71559	0.67054	0.63319	0.60154	0.57427	0.55043	0.52936	0.51056
5	0.84144	0.78316	0.73590	0.69648	0.66291	0.63386	0.60837	0.58577
6	0.97586	0.89971	0.83989	0.79108	0.75018	0.71521	0.68483	0.65809
7	1.12985	1.02663	0.94957	0.88869	0.83880	0.79681	0.76078	0.72939
8	1.32624	1.17408	1.07066	0.99305	0.93146	0.88078	0.83799	0.80115

M/N	9	10	11	12	13	14	15	16
9	1.64458	1.36428	1.21286	1.10947	1.03154	0.96947	0.91822	0.87482
10		1.67572	1.39792	1.24733	1.14410	1.06603	1.00364	0.95198
11			1.70350	1.42804	1.27830	1.17533	1.09722	1.03464
12				1.72855	1.45527	1.30638	1.20373	1.12567
13					1.75132	1.48009	1.33204	1.22975
14						1.77218	1.50286	1.35564
15							1.79142	1.52389
16								1.80926

M/N	17	18	19	20	21	22	23	24
1	0.21494	0.20889	0.20331	0.19817	0.19339	0.18894	0.18479	0.18090
2	0.32740	0.31789	0.30917	0.30113	0.29368	0.28676	0.28031	0.27428
3	0.41593	0.40345	0.39204	0.38155	0.37185	0.36286	0.35450	0.34669
4	0.49364	0.47830	0.46432	0.45151	0.43970	0.42878	0.41863	0.40918
5	0.56555	0.54730	0.53073	0.51559	0.50168	0.48885	0.47696	0.46591
6	0.63432	0.61299	0.59370	0.57615	0.56009	0.54531	0.53165	0.51898
7	0.70168	0.67698	0.65477	0.63466	0.61631	0.59950	0.58400	0.56967
8	0.76896	0.74049	0.71506	0.69214	0.67134	0.65235	0.63491	0.61882

M/N	17	18	19	20	21	22	23	24
9	0.83736	0.80455	0.77547	0.74943	0.72594	0.76458	0.68504	0.66709
10	0.90812	0.87017	0.83687	0.80729	0.78077	0.75679	0.73496	0.71497
11	0.98268	0.93847	0.90015	0.86645	0.83646	0.80954	0.78516	0.76294
12	1.06297	1.01082	0.96634	0.92772	0.89370	0.86809	0.83613	0.81142
13	1.15180	1.08905	1.03676	0.99209	0.95324	0.91896	0.88838	0.86084
14	1.25374	1.17503	1.11319	1.06081	1.01600	0.97697	0.94249	0.91168
15	1.37748	1.27597	1.19834	1.13564	1.08322	1.08881	0.99913	0.96449
16	1.54341	1.39778	1.29668	1.21924	1.15661	1.10417	1.05920	1.01992

Table 1.1 (cont)

M/N	17	18	19	20	21	22	23	24
17	1.82587	1.56162	1.41674	1.31604	1.23881	1.17627	1.12385	1.07884
18		1.84141	1.57866	1.43451	1.33421	1.25720	1.19477	1.14239
19			1.85601	1.59468	1.45123	1.35132	1.27454	1.21224
20				1.86977	1.60978	1.46700	1.36749	1.29094
21					1.88277	1.62406	1.48193	1.38280
22						1.89508	1.63759	1.49609
23							1.90679	1.65046
24								1.91793

M/N	25	26	27	28	29	30	31	32
1	0.17725	0.17380	0.17055	0.16748	0.16457	0.16180	0.15917	0.15666
2	0.26862	0.26330	0.25828	0.25353	0.24904	0.24478	0.24074	0.23688
3	0.33937	0.33250	0.32603	0.31993	0.31415	0.30868	0.30348	0.29854
4	0.40034	0.39205	0.38426	0.37691	0.36997	0.36340	0.35717	0.35125
5	0.45559	0.44592	0.43686	0.42833	0.42828	0.41267	0.40546	0.39861
6	0.50718	0.49615	0.48582	0.47612	0.46697	0.45834	0.45016	0.44242
7	0.55635	0.54393	0.53231	0.52142	0.51117	0.50151	0.49238	0.48374
8	0.60392	0.59005	0.57711	0.56500	0.55362	0.54292	0.53281	0.52326

M/N	25	26	27	28	29	30	31	32
9	0.65050	0.63511	0.62078	0.60740	0.59486	0.58307	0.57196	0.56148
10	0.69658	0.67956	0.66376	0.64904	0.63527	0.62236	0.61021	0.59876
11	0.74257	0.72380	0.70642	0.69027	0.67520	0.66110	0.64786	0.63540
12	0.78887	0.76817	0.74908	0.73138	0.71492	0.69956	0.68516	0.67164
13	0.83585	0.81302	0.79204	0.77267	0.75470	0.73798	0.72235	0.70770
14	0.88391	0.85868	0.83561	0.81439	0.79478	0.77658	0.75962	0.74376
15	0.93350	0.90554	0.88011	0.85683	0.83540	0.81558	0.79717	0.78001
16	0.98514	0.95401	0.92588	0.90028	0.97683	0.85522	0.83522	0.81663

M/N	25	26	27	28	29	30	31	32
17	1.03948	1.00460	0.97335	0.94508	0.91934	0.89573	0.87397	0.85381
18	1.09735	1.05795	1.02299	0.99163	0.96326	0.93739	0.91365	0.89176
19	1.15990	1.11487	1.07543	1.04041	1.00897	0.98051	0.95453	0.93068
20	1.22876	1.17649	1.13148	1.09202	1.05695	1.02546	0.99691	0.97085
21	1.30648	1.24444	1.19225	1.14726	1.10779	1.07270	1.04116	1.01255
22	1.39733	1.32125	1.25936	1.20734	1.16229	1.12283	1.08772	1.05614
23	1.50956	1.41117	1.33531	1.27357	1.22154	1.17664	1.13720	1.10208
24	1.66271	1.52239	1.42436	1.34874	1.28714	1.23521	1.19036	1.15094

M/N	25	26	27	28	29	30	31	32
25	1.92857	1.67440	1.53465	1.43696	1.36157	1.30013	1.24829	1.20350
26		1.93873	1.68558	1.54637	1.44902	1.37386	1.31257	1.26083
27			1.94847	1.69629	1.55760	1.46059	1.38564	1.32451
28				1.95781	1.70657	1.56888	1.47169	1.39697
29					1.96678	1.71644	1.57874	1.48236
30						1.97542	1.72593	1.58871
31							1.98373	1.73508
32								1.99175

the hypothesis of the uniformity of the angles is rejected, and, hence, the hypothesis of a Rayleigh being the underlying distribution of the radial errors is also rejected. The angles α_i are originally angles of inclination of N unit vectors in the unit circle. To use the test statistics $2N\overline{R}^2$, in the notation of [2], $2N\overline{R}^2$ is derived from

$$\overline{R} = \left[\left(\frac{\sum_i^N \cos \alpha_i}{N}\right)^2 + \left(\frac{\sum_i^N \cos \alpha_i}{N}\right)^2\right]^{\frac{1}{2}}$$

The N vectors in Figure 2 are normalized to unity using the relations

$$\sum_i^N \cos^2 \alpha_i + \sum_i^N \sin^2 \alpha_i = N = \sum_i^N \left(\frac{X_i}{r_i}\right)^2 + \sum_i^N \left(\frac{Y_i}{r_i}\right)^2$$

$$r_i^2 = X_i^2 + Y_i^2$$

1.2 Expected Value and Variance of a Rayleigh Distributed Variable; Other Relationships; Table of Radial Probabilities

$$R = \sqrt{X^2 + Y^2}$$

R is Rayleigh distributed with cumulative distribution function

$$\text{Prob } (R \leq r) = 1 - \exp\left[-\tfrac{1}{2}\left(\frac{r}{\sigma}\right)^2\right]$$

where

$$\sigma_X^2 = \sigma_Y^2 = \sigma^2$$

$$\mu_X = \mu_Y = 0$$

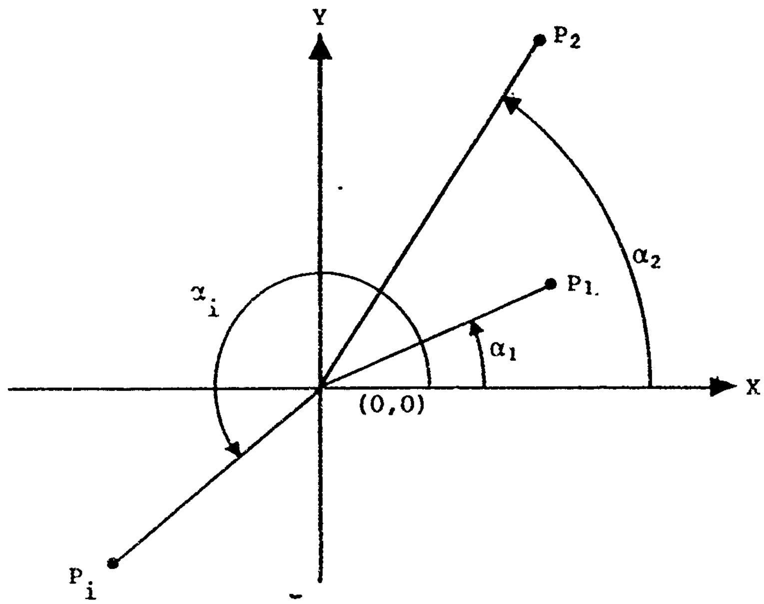

FIGURE 2. Polar Coordinates of Impact Points

$\rho_{XY} = 0$

X and Y are bivariate normal

Other relationships are as follows:

$E(R) = \text{expected value} = 1.2533\sigma = 1.913\sigma(R)$

$\sigma(R) = \text{standard deviation} = 0.655\sigma = 0.5227E(R)$

$\sigma^2(R) = 0.429\sigma^2$

$CPE = \text{median} = 1.1774\sigma = 1.798\sigma(R) = 0.9398E(R)$

$\sigma = 1.527\sigma(R)$

$\frac{\sigma(R)}{E(R)} = 0.523 = \text{coefficient of variation}$

If E(R) is a function of the ratio $\frac{\sigma\ (min)}{\sigma\ (max)}$, then

$$E(R) = \sqrt{\frac{2}{\pi}}\,[\sigma\ (max) \cdot E(k)]$$

where

$E(k)$ = Legendre's complete elliptic integral of the second kind

$\frac{\sigma\ (min)}{\sigma\ (max)} = d \leq 1.0$

$k = \sqrt{1 - d^2} = \sin \alpha$

TABLE 1.2. RADIAL PROBABILITIES

$\text{Prob}\ (R < 1\sigma) = 0.394$	$\text{Prob}\ [R < E(R)] = 0.542$
$\text{Prob}\ (R < 2\sigma) = 0.865$	$\text{Prob}\ [R < E(R) + 1\sigma(R)] = 0.838$
$\text{Prob}\ (R < 3\sigma) = 0.989$	$\text{Prob}\ [R < E(R) + 2\sigma(R)] = 0.962$
	$\text{Prob}\ [R < E(R) + 3\sigma(R)] = 0.994$
$\text{Prob}\ [R < 1\sigma(R)] = 0.193$	$\text{Prob}\ [R < E(R) + 1\sigma] = 0.921$
$\text{Prob}\ [R < 2\sigma(R)] = 0.576$	$\text{Prob}\ [R < E(R) + 2\sigma] = 0.995$
$\text{Prob}\ [R < 3\sigma(R)] = 0.855$	$\text{Prob}\ [R < E(R) + 3\sigma] = 0.999...$

2.0 CPE

2.1 Estimators for the CPE and its Variance. There are six estimators for the CPE given in this section. The first four estimators are used under the following conditions, respectively.

a. There is no significant difference in the (population) component variances, and the component means are both zero (no significant bias).

b. There is a significant difference in the variances but no bias in the means.

c. There is a significant bias in one or both means but no significant difference in the variances.

d. A significant difference exists between the variances and a significant bias exists in one or both means.

The other two estimators of the CPE depend upon the sample size explicitly. In five cases, the target is located at the center of the CPE circle and the correlation between X and Y is zero; in Case d, the CPE circle is centered at $(\overline{X}, \overline{Y})$ and $\rho_{XY} = 0$.

<u>Case a</u>. $\sigma_X^2 = \sigma_Y^2 = \sigma^2$, $\mu_X = \mu_Y = 0$

$$\widehat{CPE} = 1.1774\hat{\sigma}$$

where $\hat{\sigma}^2 = \frac{1}{2}(s_X^2 + s_Y^2)$. s_X^2 and s_Y^2 are the unbiased sample variances in the X and Y directions, respectively. If $\overline{r}$ equals the sample mean of the radial errors, another CPE estimator is $\widehat{CPE} = 0.9394\overline{r}$.

Case b. $\sigma_X^2 \neq \sigma_Y^2$, $\mu_X = \mu_Y = 0$

$\widehat{CPE} \approx 0.614s\ (min) + 0.563s\ (max)$ if $c = \dfrac{s\ (min)}{s\ (max)} > 0.33$. Here, s (min) is the smaller of the two component sample standard deviations and s (max) is the larger (Ref 7, page 214).

For a more accurate estimate of the CPE when $0.05 \leq c \leq 0.95$, use Table 1, page 170, Reference 17. In this table,

$$\widehat{CPE} = Ks\ (max),\ P = 0.50$$

See also References 12 and 13. Figure 3 is a graph of K versus c.

Case c. Either $\mu_X \neq 0$ or $\mu_Y \neq 0$, or both are not equal to zero, $\sigma_X^2 = \sigma_Y^2 = \sigma^2$.

$$\widehat{CPE} \approx R\hat{\sigma}$$

where $D = \sqrt{\dfrac{\bar{X}^2 + \bar{Y}^2}{\hat{\sigma}}} \leq 1.0$

Values of R for $0.2 \leq D \leq 1.0$ are given below.

D	R	D	R
0.2	1.198	0.6	1.296
0.3	1.216	0.7	1.341
0.4	1.234	0.8	1.386
0.5	1.252	0.9	1.431
		1.0	1.476

See Table 1-B, page 353, Reference 16, for exact table values. In that table, P = 0.50 for the CPE and $0.10 \leq D \leq 120.0$.

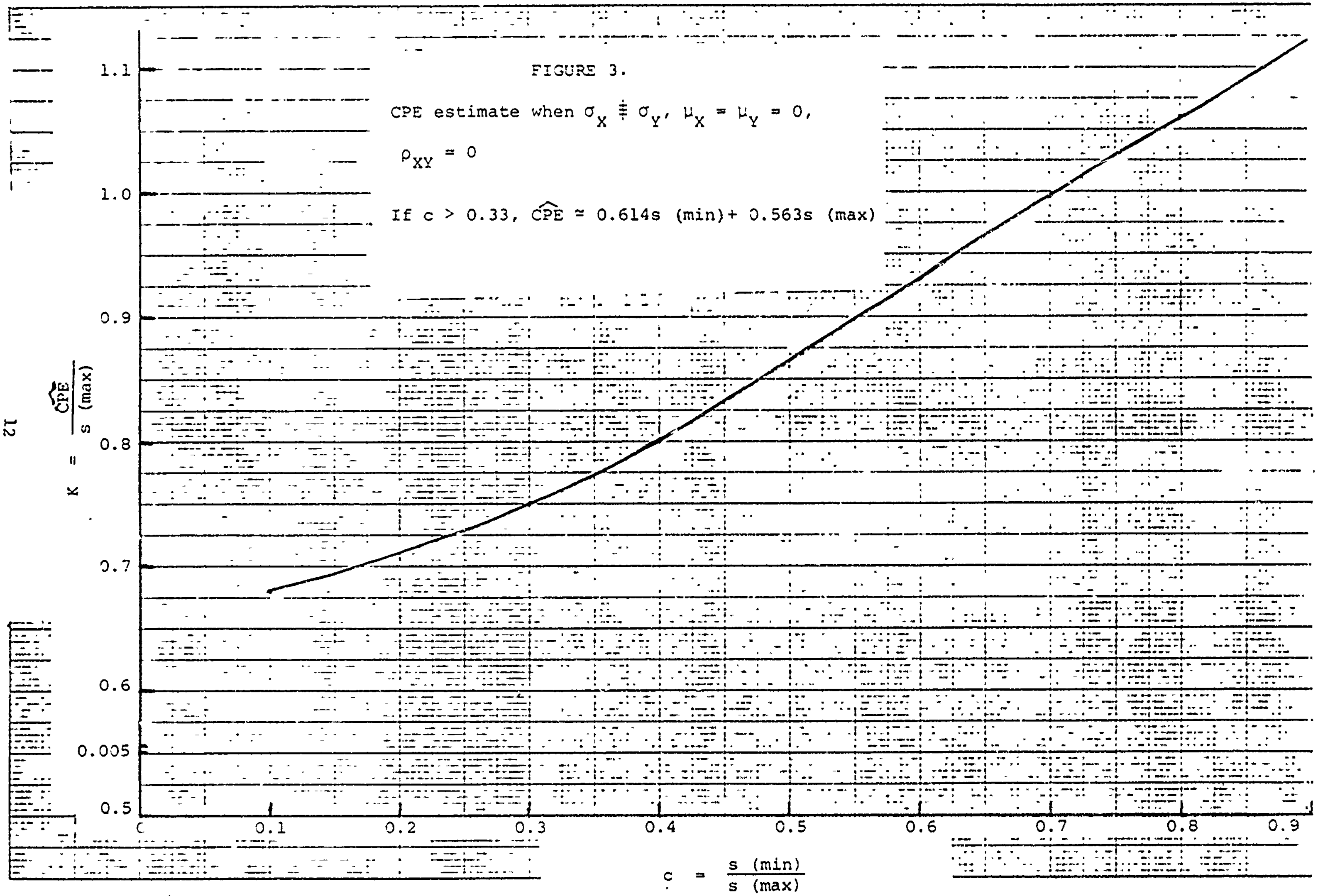
FIGURE 3.
CPE estimate when $\sigma_X \neq \sigma_Y$, $\mu_X = \mu_Y = 0$,
$\rho_{XY} = 0$
If $c > 0.33$, $\widehat{CPE} \approx 0.614s\ (min) + 0.563s\ (max)$
$K = \frac{\widehat{CPE}}{s\ (max)}$
1.1
1.0
0.9
0.8
0.7
0.6
0.005
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$c = \frac{s\ (min)}{s\ (max)}$

Case d. $\sigma_X^2 \neq \sigma_Y^2$, $\mu_X \neq 0$ or $\mu_Y \neq 0$, or both means are different from zero (Ref 3 and 4).

$$\widehat{CPE} \simeq s\sqrt{m}\left(1 - \frac{v}{9m^2}\right)^{\frac{3}{2}}$$

where

$$s^2 = s_X^2 + s_Y^2$$

$$m = 1 + \frac{1}{s^2}\left(\overline{X}^2 + \overline{Y}^2\right)$$

$$v = \left(\frac{2}{s^4}\right)\left[\left(s_X^4 + s_Y^4\right) + 2\left(\overline{X}^2 s_X^2 + \overline{Y}^2 s_Y^2\right)\right]$$

The center of the CPE circle is at $(\overline{X}, \overline{Y})$.

Case e. The CPE as a function of the sample size n, where $5 \leq n \leq 20$

a. Let

$$\sqrt{s_X^2 + s_Y^2}$$

equal the radial standard deviation (RSD) (Ref 4). The variances s^2 are biased estimates, i.e., calculated with n in the denominator instead of (n - 1). Then,

$$\widehat{CPE} \simeq k(n)\sqrt{s_X^2 + s_Y^2}$$

See table below.

$\underline{n}$	$\underline{k(n)}$	$\underline{n}$	$\underline{k(n)}$
5	0.9608	13	0.8757
6	0.9354	14	0.8724
7	0.9184	15	0.8695
8	0.9062	16	0.8671
9	0.8971	17	0.8649
10	0.8899	18	0.8630
11	0.8842	19	0.8613
12	0.8796	20	0.8598

$$k(n) \approx 1.1774 \left[\frac{1}{\sqrt{\frac{2(n-1)}{n}} \left(1 - \frac{0.125}{n-1} \right)} \right]$$

As $n \to \infty$, $k(n) \to \frac{1.1774}{\sqrt{2}}$. Hence,

$$\widehat{CPE} \to 1.1774 \sqrt{\frac{s_X^2 + s_Y^2}{2}}$$

b. The maximum likelihood estimator for the CPE is

$$\widehat{CPE} = 1.1774 \left[\frac{\Gamma(n)}{\Gamma(n + 0.5)} \sqrt{\frac{1}{2} \sum_1^n \left(X_i^2 + Y_i^2 \right)} \right]$$

where $\sqrt{X_i^2 + Y_i^2} = r_i$ = the radial error (Ref 5). A variance estimator for the CPE is given by

$$s^2(\widehat{CPE}) = 0.5198 \frac{\widehat{CPE}^2}{n}, \text{ for large } n$$

Hence,

$$s(\widehat{CPE}) = 0.72097 \frac{\widehat{CPE}}{\sqrt{n}}$$

or

$$s(\widehat{CPE}) = \frac{0.84898\hat{\sigma}}{\sqrt{n}}$$

Here,

$$\widehat{CPE} = 1.1774\hat{\sigma}$$

2.2 Confidence Intervals and Hypothesis Testing on the CPE. The table below gives the factors C_1 and C_2 used in calculating a two-sided 90% confidence interval on the CPE for sample sizes N = 2(1)15, N being the number of rounds.

N	C_1	C_2
2	0.680	5.16
3	0.765	3.44
4	0.812	2.79
5	0.845	2.46
6	0.870	2.26
7	0.890	2.11
8	0.906	2.01
9	0.919	1.94
10	0.931	1.88
11	0.940	1.83
12	0.949	1.78
13	0.955	1.75
14	0.961	1.72
15	0.968	1.69

$$C_1 = \frac{1.1774}{\sqrt{\frac{\chi^2_{0.95}}{2(N-1)}}}, \quad C_2 = \frac{1.1774}{\sqrt{\frac{\chi^2_{0.05}}{2(N-1)}}}$$

The lower limit is $C_1\hat{\sigma}$, the upper limit $C_2\hat{\sigma}$, where $\hat{\sigma}^2 = \frac{1}{2}(s_X^2 + s_Y^2)$. An upper one-sided 100 $(1 - \alpha)$% confidence limit on the CPE is given by the formula

$$\sqrt{\frac{1.386C^2}{\chi^2_{1-\alpha}\ (2N-2)}}$$

where

$$C^2 = \sum_{1}^{N} \left[r_i^2 - N(\overline{X}^2 + \overline{Y}^2) \right] \quad \text{(Ref 6)}$$

r = the radial miss-distances

$\overline{X}$ = sample mean of the X components

$\overline{Y}$ = sample mean of the Y components

2.2.1 Hypothesis Testing on the CPE

a. The standardized Rayleigh distribution has a mean of 1.0 and a coefficient of variation $= \frac{\sigma(R)}{E(R)} = 0.523$, where $E(R) = 1.0$ (Fig. 4). Since $E(R) = 1.0 = 1.253\sigma$, $\sigma = 0.7981$. Then CPE = 1.1774 (0.7981) = 0.9397, and if

$$\text{Prob } (R > r) = \exp\left[-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right] = P = 0.05$$

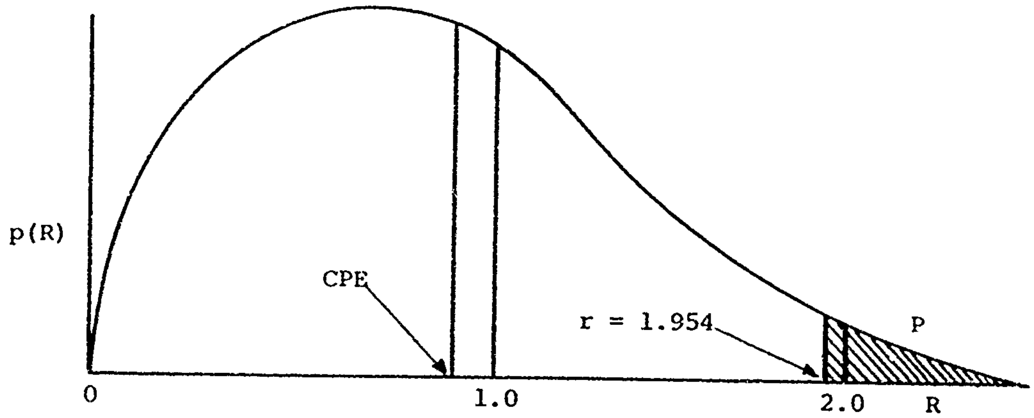

FIGURE 4. The Standardized Rayleigh Distribution With Mean 1.0 and Coefficient of Variation 0.523.

then r equals 1.954. The specified value of P and its ˈation to r and E(R) is utilized below in a hypothesis test on the Cʳ ʟet the test hypothesis be H_T : CPE = CPE_0 and the alternative H_A : CPE > CPE_0, where CPE_0 is a specified value. If

$$r = 1 + z_\alpha \left(\frac{0.523}{\sqrt{N}}\right)$$

is the test statistic, where z_α is defined by

$$\text{Prob } (Z > z_\alpha) = \frac{1}{\sqrt{2\pi}} \int_{z_\alpha}^{\infty} e^{-\frac{t^2}{2}} dt = \alpha$$

and if $\widehat{CPE} = 1.1774\hat{\sigma} < r(CPE_0)$, do not reject H_T; otherwise reject.

b. Another test for H_T : CPE = CPE_0, H_A : CPE > CPE_0. Suppose the CPE criterion is 15.0, N = 19, and the significance level is 0.05 for testing the hypothesis CPE = CPE_0 = 15.0 versus the alternative CPE > 15.0. Now,

$$\sqrt{\frac{\chi^2_{0.95}}{2N - 2}} = \sqrt{1.4166} = 1.1902$$

$\widehat{CPE}$ = 14.305 from example 10.2. Since $\frac{\widehat{CPE}}{CPE_0} = \frac{14.305}{15} = 0.954 < 1.1902$, do not reject that CPE = 15. Using the test statistic $r = 1 + 1.645 \times (\frac{0.523}{\sqrt{19}}) = 1.1974$ and since $\widehat{CPE} = 14.305 < 1.1974(15) = 17.96$, again do not reject that CPE = 15.0.

2.3 The Generalized CPE. One CPE is by definition the radius of the circle centered at the target which includes, or covers, 50% of the impact points. Hence, the CPE corresponds to the 50% coverage circle. The following table gives multiple values of the CPE that yield various coverage circles of P percent. The formula for this generalized CPE is $k(CPE) = c\sigma$.

k	c	P(%)
1.000	1.1774	50
1.150	1.3537	60
1.414	1.6651	75
1.524	1.7941	80
1.823	2.1460	90
2.015	2.3721	94
2.079	2.4477	95
2.578	3.0349	99
2.994	3.5255	99.8
3.157	3.7169	99.9

3.0 OUTLIER TEST FOR THE RADIAL ERROR

Suppose that in a set of radial errors there is an abnormally large value which is suspected to be an outlier, a value which may have come from a different distribution than the others. If an engineering analysis cannot decide whether such a maximum value is one belonging to the same distribution or an outlier, the following statistical test can be used.

Assumptions

$$\sigma_X^2 = \sigma_Y^2 = \sigma^2$$

and

$$\mu_X = \mu_Y = 0$$

If R equals the random variable of maximum radial errors and N is the total sample size, then

$$\text{Prob } (R \leq r) = \left[1 - \exp\left(-\tfrac{1}{2}\frac{r^2}{\sigma^2}\right)\right]^N$$

Hence,

$$\text{Prob } (R > r) = 1 - \left[1 - \exp\left(-\tfrac{1}{2}\frac{r^2}{\sigma^2}\right)\right]^N$$

Put $r = k\sigma$. Then,

$$\text{Prob } (R > k\sigma) = 1 - \left[1 - \exp\left(-\frac{k^2}{2}\right)\right]^N \leq \alpha$$

where α = significance level of the test

If, then in a sample of N radial errors the maximum value is greater than $k\hat{\sigma}$, reject this value as an outlier. Table 3.0 gives values of $k\hat{\sigma}$ for $\alpha = 0.10(0.05)(0.01)$; $N = 2(1)10,\ 12(2)20,\ 30(10)50$.

TABLE 3.0

Values of $k\hat{\sigma}$ such that if the maximum radial error is greater than $k\hat{\sigma}$ in a sample of N, the maximum is rejected as an outlier at α = 0.10, 0.05, 0.01

N	α = 0.10	α = 0.05	α = 0.01
2	2.4371$\hat{\sigma}$	2.7115$\hat{\sigma}$	3.2545$\hat{\sigma}$
3	2.5948	2.8556	3.3765
4	2.7018	2.9539	3.4605
5	2.7822	3.0281	3.5244
6	2.8464	3.0875	3.5757
7	2.8996	3.1368	3.6185
8	2.9450	3.1789	3.6552
9	2.9844	3.2157	3.6873
10	3.0193	3.2482	3.7157
12	3.0789	3.3037	3.7644
14	3.1283	3.3500	3.8051
16	3.1706	3.3895	3.8401
18	3.2074	3.4240	3.8706
20	3.2400	3.4546	3.8977
30	3.3625	3.5699	4.0004
40	3.4469	3.6496	4.0717
50	3.5110	3.7102	4.1261

4.0 SAMPLE SIZE PROBLEMS

4.1 Sample Size Required Such That Prob $[\mathrm{CPE}\,(1 - \varepsilon) \leq \widehat{\mathrm{CPE}} \leq \mathrm{CPE}\,(1 + \varepsilon)] = 1 - \alpha$

Determine the sample size n (of missiles, say) such that the relative error in the CPE = $\left|\frac{\widehat{\mathrm{CPE}} - \mathrm{CPE}}{\mathrm{CPE}}\right|$ does not exceed a specified value ε with probability $1 - \alpha$. That is,

$$\mathrm{Prob}\left(\left|\frac{\widehat{\mathrm{CPE}} - \mathrm{CPE}}{\mathrm{CPE}}\right| \leq \varepsilon\right) = 1 - \alpha$$

where $\varepsilon < 1.0$ (Ref 7). Since CPE equals 1.1774σ,

$$\left(\frac{\widehat{\mathrm{CPE}}}{\mathrm{CPE}}\right)^2 = \left(\frac{\hat{\sigma}}{\sigma}\right)^2$$

where $\hat{\sigma}^2 = \frac{1}{2}(s_X^2 + s_Y^2)$, the estimator for σ^2. Then,

$$\left(\frac{\widehat{\mathrm{CPE}}}{\mathrm{CPE}}\right)^2 = \frac{\frac{1}{2}\left(s_X^2 + s_Y^2\right)}{\sigma^2}$$

and

$$2n\left(\frac{\widehat{\mathrm{CPE}}}{\mathrm{CPE}}\right)^2 = \frac{ns_X^2}{\sigma^2} + \frac{ns_Y^2}{\sigma^2} = \chi^2(2n - 2)$$

Let

$$1 - \varepsilon \leq \frac{\widehat{\mathrm{CPE}}}{\mathrm{CPE}} \leq 1 + \varepsilon$$

and

$$\mathrm{Prob}\left[(1 - \varepsilon)^2 \leq \left(\frac{\widehat{\mathrm{CPE}}}{\mathrm{CPE}}\right)^2 \leq (1 + \varepsilon)^2\right] = 1 - \alpha$$

Then,

$$\text{Prob}\ [2n(1-\varepsilon)^2 \le \chi^2(2n-2) \le 2n(1+\varepsilon)^2] = 1-\alpha$$

For selected values of n (the sample size) and $1-\alpha$, the value of ε can be determined. If $n = 10$, $1-\alpha = 0.95$, $20(1-\varepsilon)^2 = 8.231$, $20(1+\varepsilon)^2 = 31.526$ since Prob $[8.231 \le \chi^2(18) \le 31.526] = 0.95$. Hence,

$$\begin{aligned} (1-\varepsilon)^2 &= 0.41155 & \qquad (1+\varepsilon)^2 &= 1.57630 \\ 1-\varepsilon &= 0.64152 & \qquad 1+\varepsilon &= 1.25550 \end{aligned}$$

Consequently, $2\varepsilon = 0.61398$, $\varepsilon = 0.307 \approx 31\%$. Therefore, 10 missiles are required to estimate the CPE within ±31% of its true value. If the true value of the CPE is two units, then the estimated CPE is within the interval (1.38, 2.62) with 95% probability. A table can then be constructed giving values of ε for various values of n and $1-\alpha$. Table 4.1 gives, for values of n = 3(1)15, 20, 25, 30, 35, 40, the values of ε for $1-\alpha = 0.90$ and 0.75. If the true CPE is two units, then $\widehat{CPE} = 2(1 \pm \varepsilon)$ with $(1-\alpha)$ probability. Figure 5 is the graph of $\varepsilon = \Gamma$ versus N (sample size) for $1-\alpha = 0.90$. $(1-\alpha)$ is often interpreted as a confidence.

4.2 Sample size such that the confidence interval on the mean radial error shall not exceed a specified value (Ref 8). Assume the radial errors r_i are known, but not the components of the radial errors. Now,

$$\frac{\sum_1^N r_i^2}{\sigma^2} = \chi^2(2N)$$

and

$$[E(R)]^2 = \mu^2 = (1.253\sigma)^2 = 1.5700\sigma^2$$

TABLE 4.1

N	$\varepsilon(1 - \alpha = 0.90)$	$\varepsilon(1 - \alpha = 0.75)$
3	58	35
4	48	38
5	42	24
6	38	21
7	34	19
8	32	18
9	30	16
10	28	15.5
11	26	15
12	25	14
13	24	13.5
14	23	13
15	22	12.5
20	19	10.5
25	17	10
30	15	9
35	14	8
40	13	7^{+}

The percent ε within which the true CPE is estimated with 90(75) percent probability. N is the sample size required.

$$\text{Prob}\left(\left|\frac{\widehat{CPE} - CPE}{CPE}\right| \leq \varepsilon\right) = \begin{cases} 0.95 \\ 0.75 \end{cases}$$

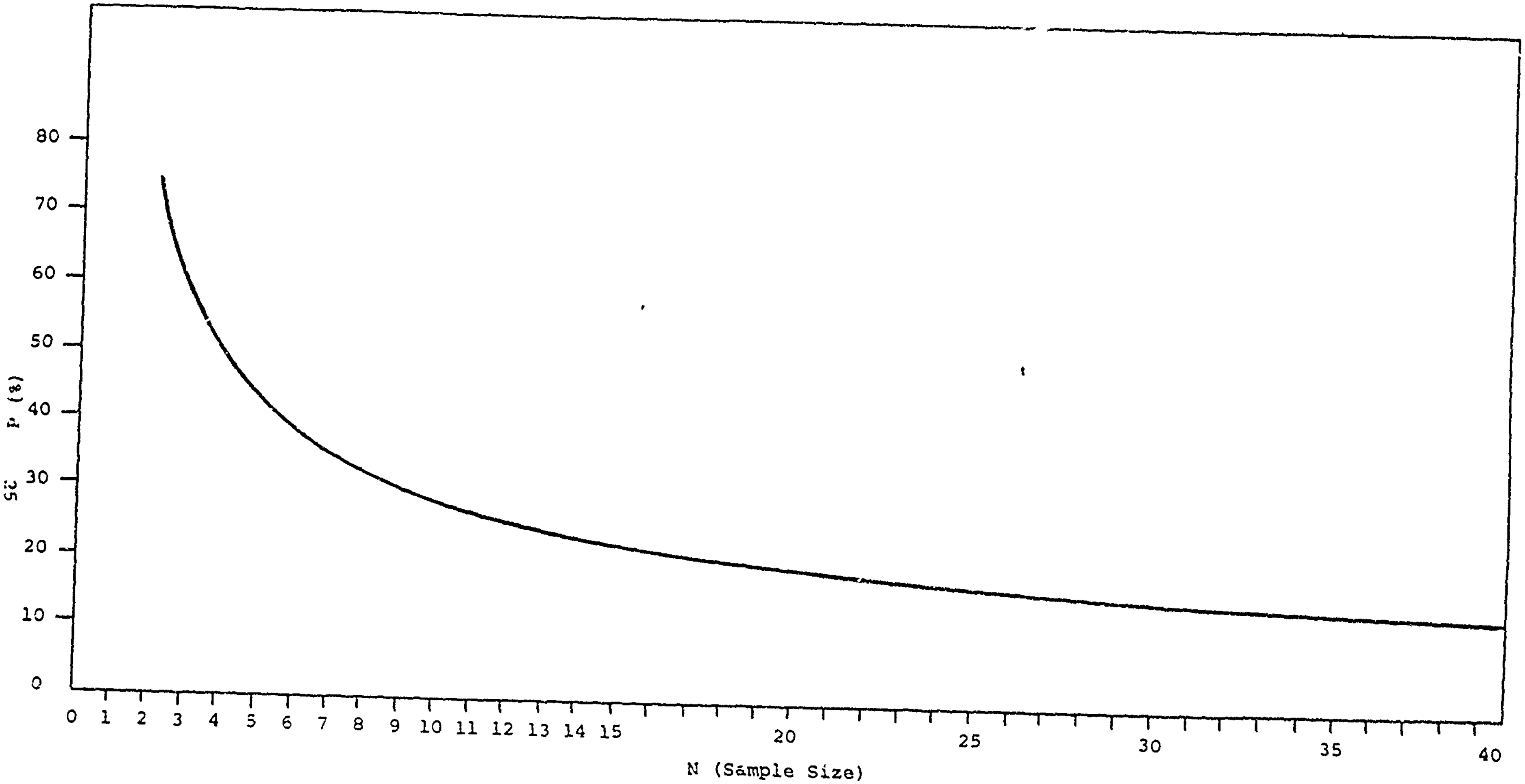

FIGURE 5. Number of Rounds Required to Estimate the CPE to Within P-Percent of Its True Value With 90% Confidence

N such that Prob $[CPE(1 - P) < \widehat{CPE} < CPE(1 + P)] = 0.90$

Hence,

$$1.5700 \sum_{1}^{N} r_i^2/\mu^2 = \chi^2(2N)$$

$$1.5700 \sum_{1}^{N} r_i^2/\chi^2(2N) = \mu^2$$

The 100(1 - α)% confidence interval on the mean radial error is then

$$\left[\sqrt{1.5700 \sum_{1}^{N} r_i^2/\chi^2_{1-\frac{\alpha}{2}}(2N)} \qquad \sqrt{1.5700 \sum_{1}^{N} r_i^2/\chi^2_{\frac{\alpha}{2}}(2N)} \right]$$

If $N \geq 30$, then the mean $\bar{r}$ should be nearly normally distributed and the confidence interval could be computed from

$$\left[\bar{r} - \frac{ts(r)}{\sqrt{N}} \qquad \bar{r} + \frac{ts(r)}{\sqrt{N}} \right]$$

where

t = the student t distribution with N - 1 degrees of freedom at the $\frac{\alpha}{2}$ percentage point

$s(r)$ = the sample standard deviation of the radial errors:

$$s^2(r) = \frac{\sum_{1}^{N} r_i^2 - N\bar{r}^2}{N - 1}$$

Or, more simply, since $\sigma(R) = 0.5227E(R)$, assume $s(r) = 0.5227\bar{r}$. To determine the sample size N, assuming $N < 30$,

a. Estimate some value for

$$\sum_{1}^{N} r_i^2 \Big| N$$

Call this estimate m, i.e.,

$$\sum r_i^2 \Big| N \hat{=} m, \ m > N$$

b. Select a value α and a width w such that the length of the confidence interval does not exceed w.

c. Set up the following relationship and proceed by trial and error until the desired interval has not been exceeded.

$$\sqrt{\frac{1.5700 \times N \times m}{\chi^2_{1-\frac{\alpha}{2}}(2N)}} \leq \mu \leq \sqrt{\frac{1.5700 \times N \times m}{\chi^2_{\frac{\alpha}{2}}(2N)}}$$

Examples: Set $m = 10$, $w = 2$, $1 - \alpha = 0.95$. Let $N = 5$.

$$\sqrt{\frac{1.5700(50)}{20.5}} \quad , \quad \sqrt{\frac{1.5700(50)}{3.25}}$$

or 1.957 4.915 $w = 2.96$

Let N = 7

$$\sqrt{\frac{1.5700(70)}{26.1}} \quad , \quad \sqrt{\frac{1.5700(70)}{5.63}}$$

or 2.052 4.418 w = 2.37

Let N = 8

$$\sqrt{\frac{1.5700(80)}{28.8}} \quad , \quad \sqrt{\frac{1.5700(80)}{6.91}}$$

or 2.088 4.263 w = 2.18

Let N = 9

$$\sqrt{\frac{1.5700(90)}{31.5}} \quad , \quad \sqrt{\frac{1.5700(90)}{8.23}}$$

or 2.118 4.144 w = 2.03

If N = 10, the squa.. roots are 2.142, 4.046, and w = 1.904. Hence, N = 10. If N is expected to be large, more than 30,

a. Estimate μ by ρ and assume $\sigma(\rho) = 0.5227\rho$

b. Set a width w. Then

$$N = \left(t_{\frac{\alpha}{2}} \frac{2\sigma(\rho)}{w} \right)^2$$

or

$$N = \left(t_{\frac{\alpha}{2}} \cdot \frac{1.0454\rho}{w} \right)^2$$

Examples: Let $\alpha = 0.05$, $\rho = 5$, $w = 0.60$. Then $N = (1.96 \times 8.7116)^2 = 292$. This sample size could be verified as follows: Suppose $N = 292$, $\alpha = 0.05$, $\bar{r} = 5$, $s(r) = 0.5227\bar{r} = 2.614$. Then the 95% confidence interval on $\mu(\rho)$ is

$$\bar{r} - \frac{1.96 \times 2.614}{\sqrt{292}} \qquad \bar{r} + \frac{1.96 \times 2.614}{\sqrt{292}}$$

or $4.70 \leq \mu(\rho) \leq 5.30$ with $w = 0.60$.

4.3 Sample Size for the CPE Utilizing the α, β Risks and CPE Criteria (Definitions and Equations). Let

CPE_E = the maximum acceptable or essential CPE

CPE_D = the design CPE

$CPE_E > CPE_D$

β = the user's probability of the missile passing the accuracy test if the true CPE = CPE_E

α = the producer's probability of the missile failing this test if the true CPE = CPE_D

$100(1 - \beta)$% is interpreted as the apriori confidence that the true CPE $\geq$ CPE_E if the $\widehat{CPE}$ satisfies

$$\widehat{CPE} \leq \sqrt{\frac{\chi^2_\beta(2N - 2)}{2N - 2}}\,(CPE_E)$$

If the true CPE = CPE_D, there is a probability $1 - \alpha$ that the missile will be accepted. Table 4.3 lists the (α, β) risks, the ratio $\frac{CPE_E}{CPE_D}$, and

TABLE 4.3

The sample size required such that the CPE criteria are met with assumed α, β risks. CPE_D = design CPE, CPE_E = maximum acceptable CPE.

α, β	CPE_E/CPE_D	N	$\sqrt{\chi^2_\beta(2N-2)/2N-2}$
α = β = 0.25	1.365	6	0.8208
	1.326	7	0.8386
	1.298	8	0.8521
	1.275	9	0.8628
	1.257	10	0.8716
	1.242	11	0.8790
	1.229	12	0.8853
	1.218	13	0.8906
	1.208	14	0.8953
	1.200	15	0.8995
	1.192	16	0.9033
α = 0.50, β = 0.20	1.205	7	0.8067
	1.187	8	0.8224
	1.173	9	0.8367
	1.162	10	0.8466
	0.8961	15	0.8783
	0.9094	20	0.8959

N. In forming this table, N, α, and β were selected first and the ratio $\frac{CPE_E}{CPE_D}$ was computed last. The equation

$$\frac{\chi^2_{\alpha}(2N - 2)}{\chi_{1-\beta}(2N - 2)} = \left(\frac{CPE_E}{CPE_D}\right)^2$$

connects these quantities. The upper percentage points of chi-square are used with 2N - 2 degrees of freedom.

4.3.1 OC Curve for the CPE. The probability that a missile will pass an accuracy test (the probability of acceptance) is

$$P_a = \text{Prob}\left[\chi^2(2N = 2) \leq \chi^2_{\beta}(2N - 2)\left(\frac{CPE_O}{CPE_T}\right)^2\right]$$

where

CPE_T = the true CPE

CPE_O = the specified CPE

If a sample size of seven missiles is allocated and $\beta = 0.20$, $\chi^2_{0.20}(12) = 7.81$. Suppose $CPE_O = 300$ meters. Then the coordinates of the OC curve are below.

CPE_T	$P_a \approx$
180	0.96
200	0.87
220	0.74
240	0.57
260	0.42
280	0.29
300	0.20
320	0.14
340	0.09
350	0.07

5.0 ESTIMATOR FOR THE CPE USING THE GUMBEL DISTRIBUTION OF SMALLEST VALUES (Ref 9)

If the shape parameter, β, of the Weibull distribution

$$F(w) = 1 - \exp\left[-\left(\frac{w}{\eta}\right)^{\beta}\right]$$

is equal to 2.0, then this distribution reduces to that of the Rayleigh with

$$\eta = \sigma\sqrt{2}$$

Furthermore, if the random variable W is Weibull distributed, $U = \ln W$ is distributed as the Gumbel distribution of smallest values given by

$$F_U(u) = 1 - \exp\,[-\exp(u)]$$

where

$$u = \frac{\ln W - m}{b}$$

is the standardized variable; m = the mode = $\ln\eta$; and b = the scale parameter = β^{-1} = 0.50. Hence, if R = the random variable of the radial error,

$$\text{Prob}\ (R \leq r) = 1 - \exp\left\{-\exp\left[2\ln\left(\frac{r}{\eta}\right)\right]\right\}$$

If Prob $(R \leq r) = 0.50$, r is the population CPE and is equal to anti$|$n $(-0.18326 + \ln\eta)$, where $-0.18326 = \frac{1}{2}\ln[-\ln(\frac{1}{2})]$. Hence, $\widehat{CPE}$ = anti$|$n $(-0.18326 + \ln\hat{\eta})$, where

$$\hat{\eta} = \left(\frac{1}{N}\sum_{1}^{N} r_i^2\right)^{\frac{1}{2}}$$

equals the maximum likelihood estimator of η*, r_i equals the radial errors.

5.1 Example. The following 40 radial errors are from a Weibull distribution with $\beta = 2.0$, $\eta = 100$. These were generated as pseudo-random variables.

5, 10, 17, 32, 32, 33, 34, 36, 54, 55, 55, 58, 58, 61, 64, 65, 65, 66, 67, 68, 82, 85, 90, 92, 92, 102, 103, 106, 107, 114, 114, 116, 117, 124, 139, 142, 143, 151, 158, 195

$$\hat{\eta} = \left(\frac{1}{40}\sum_{1}^{40} r_i^2\right)^{\frac{1}{2}} = 93.32$$

$\tilde{\eta}$ = unbiased MLE = $1.003130\hat{\eta} = 93.61$. Then, $\widehat{CPE}$ = anti$\ln$ $(-0.18326 + \ln\tilde{\eta}) = 77.935$, and CPE (population) $= \frac{68 + 82}{2} = 75.00$ = the median. If the estimator $\widehat{CPE} = 1.1774\hat{\theta}$ is used with $\hat{\theta} = \frac{\hat{\eta}}{\sqrt{2}} = \frac{\tilde{\eta}}{\sqrt{2}}$,

$$\widehat{CPE} = (1.1774)\left(\frac{93.61}{\sqrt{2}}\right) = 77.935$$

If the sample mean = $\bar{r}$ = 82.675 is substituted for E(R) in the equation CPE = 0.9396E(R) (Table 3.0),

$$\widehat{CPE} = 77.698$$

*Unbiasing factors for $\hat{\eta}$ can be found in Reference 10. η is the scale parameter of the Weibull distribution. In Reference 10, $\eta = \theta$.

6.0 PROPORTION OF HITS WITHIN A SPECIFIED CPE CIRCLE WHEN A BIAS EXISTS

The following problem is sometimes of interest in accuracy studies. Suppose the first CPE circle $(CPE)_1$ is the one which is designated as an accuracy requirement. The impact points of a series of missile firings are distributed such that 50% of these impacts are located within the second CPE circle $(CPE)_2$. The variance in range is assumed equal to the variance in azimuth, but a bias exists in the center of impact which is equal to λr (Fig. 6). What proportion of the impact points are within $(CPE)_1$? This proportion is related to the common area, A, of the two CPE circles. From a formula in plane geometry,

$$A = r^2 \left[2 \cos^{-1} \left(\frac{\lambda}{2} \right) - \frac{\lambda}{2} \sqrt{4 - \lambda^2} \right]$$

where the bias $= \lambda r$, $0 \leq \lambda \leq 1$. Then the proportion in question is given by:

$$p = 0.50 \times \frac{2 \cos^{-1} \left(\frac{\lambda}{2} \right) - \frac{\lambda}{2} \sqrt{4 - \lambda^2}}{\pi = 3.1416} \qquad (1)$$

Using the method of least squares, the following estimate of p is given as a linear equation in $\lambda (0 < \lambda < 1)$.

$$p = 0.497 - 0.305\lambda, \text{ bias } = \lambda r$$

Ninety-five percent confidence limits on the slope of this line are (-0.3097, -0.3005). Ninety-five percent confidence limits on the intercept are (0.4946, 0.5000). More generally*, if the bias is in two dimensions instead of just one and the circles have different radii corresponding to different component variances, then

*Mr. Karl L. Koski derived the formula and established the conditions for the general case.

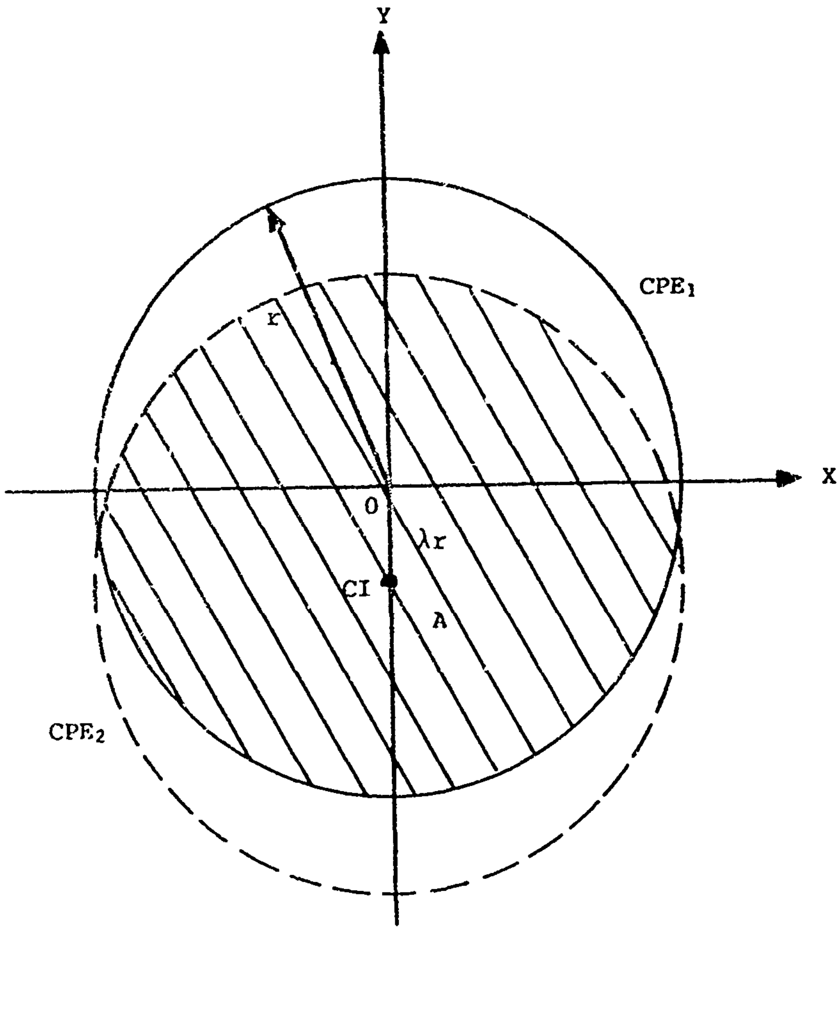

FIGURE 6. CPE$_1$ = accuracy requirement
CPE$_2$ = biased CPE circle containing 50% of impact points

$$p = 0.50\left[\frac{a_1^2\left(\frac{\alpha_1}{2}\right) + a_2^2\left(\frac{\alpha_2}{2}\right) - \sqrt{(h^2 + k^2)(a_1^2 - a^2)}}{\pi a_1^2}\right] \tag{2}$$

where

a_1 = radius of CPE circle centered at 0

a_2 = radius of CPE circle centered at (h, k)

$\alpha_1 = 2\cos^{-1}\left(\frac{a}{a_1}\right)$, in radians

$\alpha_2 = 2\cos^{-1}\left(\frac{b}{a_2}\right)$, in radians

and

$$a = \frac{a_1^2 - a_2^2 + h^2 + k^2}{2\sqrt{h^2 + k^2}}$$

$$b = \sqrt{h^2 + k^2} - a$$

$(a + b$ = radial bias$)$

If $a < 0$, use $\frac{\alpha_1}{2} = \pi - \cos^{-1}\left(\frac{|a|}{a_1}\right)$, $a = 0$, $\frac{\alpha_1}{2} = \frac{\pi}{2}$, $b \leq 0$, reverse the circles, then a_1 becomes a_2 and a_2 is a_1. Example: Suppose the scaled $(\times 10^{-2})$ radii and center of two such CPE circles are:

$a_1 = 1.50$ $\quad a_2 = 1.75$

$h = 0.25$ $\quad k = 0.50$

Then we have the following values:

$$a = \frac{-\sqrt{5}}{5} < 0$$

$$b = \frac{9\sqrt{5}}{20}$$

$$\alpha_1 = 2\left[\pi - \cos^{-1}\left(\frac{\sqrt{5}}{7.5}\right)\right]$$

$$\alpha_2 = 2\cos^{-1}\left(\frac{9\sqrt{5}}{35}\right)$$

and

$$p = \frac{0.50}{\pi(2.25)}\ [2.25\ (\pi - 1.26805) + 3.0625\ (0.95821) - 0.80039]$$

$$p = \frac{(0.50)(6.3496)}{7.0686} = 0.45$$

Hence, the original or design CPE circle of (scaled) radius 1.50 will include approximately 45% of the impact points if the second biased CPE circle has (scaled) center (0.25, 0.50) and radius 1.75. Formula 2 reduces to 1 when $a_1 = a_2 = r$, $h = 0$, and $k = \lambda r$.

7.0 LOWER CONFIDENCE LIMITS FOR IMPACT PROBABILITY WITHIN A CIRCLE WHEN $\sigma^2(X) \neq \sigma^2(Y)$ (Ref 11)

The problem is to determine P, if Prob $(R \leq r) \geq P$, with γ^2 confidence. Here, R is the random variable of the radial error in two dimensions and r is a specified radial distance from the target, the latter being centered at (0, 0). The procedure is as follows:

a. Find the upper one-sided confidence limit for σ^2 (min), where s^2 (min) is the smaller of the two component sample variances. This upper limit is

$$\bar{\sigma}^2 = \frac{s^2(\min)}{\chi^2_{1-\gamma}(N)}$$

where

N	=	sample size of rounds
$\chi^2_{1-\gamma}(N)$	=	chi-square fractile value with N degrees of freedom
γ	=	the chosen confidence level

b. Find the upper one-sided confidence limit for a quantity called τ^2. This upper limit is, with confidence level γ,

$$\bar{\tau}^2 = \frac{s^2\ (\max)}{s^2\ (\min)}\left[F_\gamma\ (N,\ N)\right]$$

$$\frac{s^2\ (\max)}{s^2\ (\min)} > 1.0$$

c. Compute $\frac{\bar{\sigma}^2}{r^2}$ and find P by linear interpolation in the tables of Reference 11, where $\left(\frac{\bar{\sigma}}{r}\right)^2$ and $\bar{\tau}^2$ are given. There is then γ^2 confidence that at least P percent of the impact points will lie within a circle of radius r centered at (0, 0). Example: $N = 20$, $r = 2$, $\gamma = 0.95$, $\frac{s^2\ (\max)}{s^2\ (\min)} = \frac{54.4}{30.2} = 1.80$, which is close to significance at $\alpha = 0.10$.

(1) $\bar{\sigma}^2 = \frac{30.2}{10.9} = 2.771$

(2) $\bar{\tau}^2 = (1.80)(2.12) = 3.816 \approx 3.8$

(3) $\left(\frac{\bar{\sigma}}{r}\right)^2 = 0.6928$

From the table,

$\bar{\sigma}^2/r^2$	$\bar{\tau}^2$	P
0.60	3.80	0.33518
0.80	3.80	0.26648

Hence, $P \approx 0.303$. Then with $(0.95)^2 = 0.9025$ confidence Prob $(R \leq 2) \geq 0.303$.

7.1 The OC Curve for the CPE (Ref 8)

Suppose 30 rounds have been allocated for an accuracy test. The missile will be accepted if it can be verified at the 90% confidence level that the CPE ≤ 2.0. Then $\beta = 0.10$, the user's risk of accepting a missile with a CPE > 2.0. Assume $\tau^2 = \frac{\sigma^2\ (\max)}{\sigma^2\ (\min)} = 1.8$. Since $\chi^2_\beta\ (2N) = \chi^2_{0.10}\ (60) = 46.5$, the upper confidence limit on σ^2 (min) $= \frac{T}{46.5}$ where T is determined below. Let $r = \text{CPE} = 2.0$. Now $\frac{\bar{\sigma}^2}{r^2} = 0.514$ from the Zacks-Solomon tables for $P = 0.50$, $\tau^2 = 1.8$. Hence,

$$\frac{\bar{\sigma}^2\ (\min)}{(\text{CPE})^2} \approx \frac{T/46.5}{4} = 0.514$$

and $T = 95.6 = \sigma^2\ (\min)\ \chi^2_P\ (60)$

Therefore, the probability of acceptance, P, versus σ^2 (min) can be plotted for various values of P. For example,

$$P = 0.80, \sigma^2 \text{ (min)} = 1.386$$

$$P = 0.70, \sigma^2 \text{ (min)} = 1.466$$

Furthermore, since $\frac{\overline{\sigma}^2 \text{ (min)}}{(\text{CPE})^2} = 0.514$, CPE $= \left[\frac{\sigma^2 \text{ (min)}}{0.514}\right]^{1/2}$ and at P = 0.60, σ^2 (min) $= \frac{95.6}{62.1} = 1.54$. Hence, CPE = 1.73 when σ^2 (min) = 1.54. The table below gives some other values of P versus σ^2 (min) and the corresponding values of the CPE.

P	σ^2 (min)	CPE
0.99	1.10	1.46
0.89	1.30	1.59
0.66	1.50	1.71
0.50	1.61	1.77
0.30	1.78	1.86
0.10	2.055	2.00

8.0 TRUNCATED AND CENSORED SAMPLES

Suppose all the radial errors greater than a specified value r_0 are excluded from an analysis and the number of such omissions is unknown. We wish to obtain an estimate of σ from this truncated sample (Ref 14) if n equals the sample size of radial errors considered. This estimate is given by

$$\hat{\sigma} = \frac{r_0}{\xi_0}$$

where ξ_0 is obtained from the table in Reference 14 after $\sum_{1}^{N} \frac{r_i^2}{n r_0^2}$ has been computed and put equal to $G_2(\xi_0)$.

If the number of r's > r_0 is known, then we have a censored sample. In this case,

$$\hat{\sigma}^2 = \left(\frac{1}{2N}\right)\left(\sum_{1}^{N} r_i^2 + n_0 r_0^2\right) \qquad (1)$$

where n_0 = the number of radial errors for which $r > r_0$.

9.0 ESTIMATE OF σ USING QUANTILES OF THE RADIAL ERRORS (Ref 15)

Let $\bar{r}_p$ = the mean of the r's greater than the p^{th} quantile, where R is the random variable of the radial error and

$$p = \text{Prob}\ (R \leq r_p), \quad r_p \geq 0$$

$$P = \text{Prob}\ (R > r_p)$$

Then

$$\hat{\sigma} = \frac{\bar{r}_p}{\sqrt{-2\ell nP} + \frac{\sqrt{2\pi}}{P}\left[1 - \Phi(\sqrt{-2\ell nP})\right]}$$

$$0 < P \leq 1$$

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{t^2}{2}}\, dt$$

Figure 7 is a graph of $\frac{\bar{r}_p}{\sigma}$ versus P.

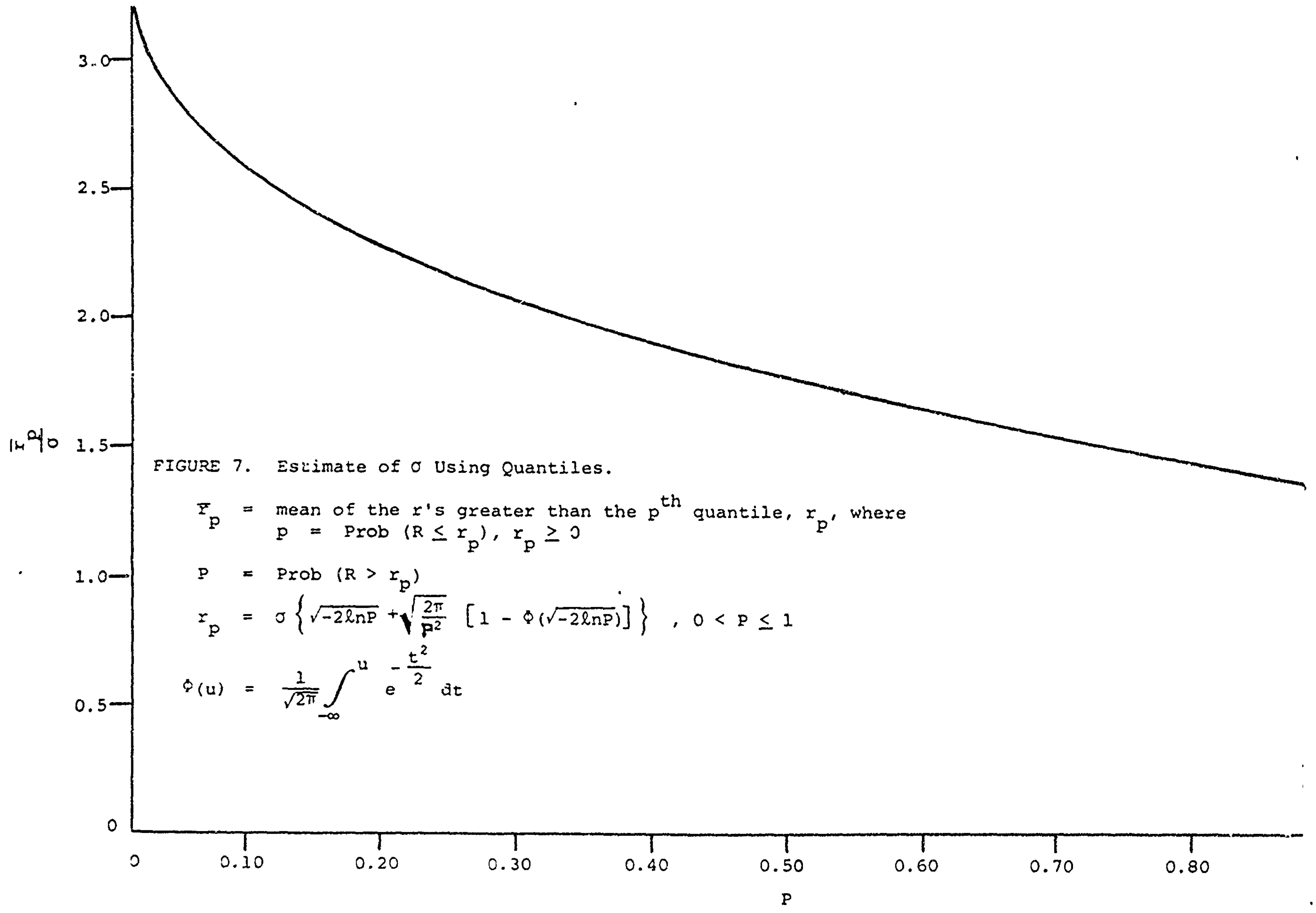

FIGURE 7. Estimate of σ Using Quantiles.

$\bar{r}_p$ = mean of the r's greater than the p^{th} quantile, r_p, where $p = \text{Prob}(R \leq r_p)$, $r_p \geq 0$

$P = \text{Prob}(R > r_p)$

$$r_p = \sigma \left\{ \sqrt{-2\ln P} + \sqrt{\frac{2\pi}{P^2}} \left[1 - \Phi(\sqrt{-2\ln P}) \right] \right\}, \quad 0 < P \leq 1$$

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{t^2}{2}} dt$$

10.0 MORE EXAMPLES OF COMPUTATIONS FOR ESTIMATES OF β, CONFIDENCE INTERVALS, CPE, σ, AND THE W TEST

10.1 Forty ranked radial errors from a Weibull distribution with $\beta = 2.0$, $\eta = 100$, γ = location parameter = 10 are as follows:

15, 20, 27, 42, 42, 43, 44, 46, 64, 65, 65, 68, 68, 71, 74, 75, 75, 76, 77, 78, 92, 95, 100, 102, 102, 112, 113, 116, 117, 124, 124, 126, 127, 134, 149, 152, 153, 161, 168, 205

A modified method of moments estimator for β is

$$\hat{\beta} = \frac{1.2826}{s(\ell n r)} \doteq \frac{1.2826}{0.57617} = 2.226$$

and an MLE is (assuming β, η, and γ unknown) = 2.199. Given $\beta = 2.0$, the MLE of η is 102.28 and the unbiased estimate is 102.60. Two-sided 95% confidence limits on β are

$$\left[\frac{2.199\ \chi^2_{0.025}(80)}{80}, \frac{2.199\ \chi^2_{0.975}(80)}{80}\right] = (1.571,\quad 2.931)$$

$\widehat{CPE}$ equals anti$|$n $(-0.18326 + \ell n\ 102.60) = 85.42$. The median of the sample is calculated to be $\frac{78 + 92}{2} = 85.0$. Estimate σ using the 75% quantile: $p = 0.75$, $\hat{P} =$ Prob $(R > r_{0.75} = 124)$, or $\hat{P} = \frac{9}{40} = 0.225$. Then we have

$$\bar{r}_p = 152.78$$

From Figure 6, we have for $\hat{P} = 0.225$,

$$\frac{\bar{r}_p}{\hat{\sigma}} = 2.2$$

from which $\hat{\sigma} = 69.4$. From the estimator $\hat{\sigma} = 1.527\hat{\sigma}(R)$, $N = 40$, $\hat{\sigma} = 1.527 \times (43.835) = 66.9$.

10.2 The X and Y component data and the calculated radial miss-distances from 19 rounds are in Table 10.2. Although the 95% confidence interval on μ_Y does not quite span zero, this will be declared practically not significant. Assume that the radial misses are Rayleigh distributed. Then the estimated CPE, using the estimator $\widehat{CPE} = 1.1774\hat{\sigma}$ with $\hat{\sigma} = 12.15$, is 14.31. Using the estimator for the CPE in paragraph 2.1, Case a,

$$\widehat{CPE} = k(n)\sqrt{s_X^2 + s_Y^2}\ , \quad N = 19$$

$$\widehat{CPE} = (0.8613)(16.730)$$

or

$$\widehat{CPE} = 14.410$$

If the maximum likelihood estimator (para 2.1, Case b) is used,

$$\widehat{CPE} = 1.1774\left[\frac{\Gamma(19)}{\Gamma(19.5)}\sqrt{\tfrac{1}{2}\sum_{1}^{19} r_i^2}\right]$$

$$\widehat{CPE} = 1.1774\left[\frac{6.402373705 \times 10^{15}}{2.7724323 \times 10^{16}} \times 56.49987\right]$$

or

$$\widehat{CPE} = 15.362$$

If $E(R) = 1.2533\sigma$ is assumed to be $\bar{R} = 1.2533\hat{\sigma}$, then $\bar{R} = 15.23$ compared to the sample mean of 15.94.

TABLE 10.2

X and Y Components of the Radial Miss Distance for 19 Rounds

i	X_i	Y_i	$r_i = \sqrt{X_i^2 + Y_i^2}$
1	0.90	22.50	22.52
2	9.10	16.90	19.19
3	-1.80	14.40	14.51
4	-16.90	21.80	27.58
5	-10.60	13.70	17.32
6	-19.0	23.00	29.83
7	0.40	7.40	7.41
8	5.20	21.40	22.02
9	7.80	-5.50	9.54
10	-9.10	-0.70	9.13
11	22.70	3.50	22.97
12	7.40	0.30	7.41
13	12.80	17.10	21.36
14	-1.80	3.30	3.76
15	5.30	6.50	8.39
16	2.70	2.00	3.36
17	5.30	0.00	5.30
18	21.40	-27.10	34.53
19	-16.70	-0.40	16.70
Mean	1.32	7.37	15.94
Std Dev	11.83	12.47	

NOTE: The CPE estimate of 14.410, obtained by using the radial standard deviation RSD $=\sqrt{s_X^2 + s_Y^2}$ and k(n), is more accurate since the RSD is a more accurate estimator of σ (Ref 4).

Suppose that only the range of X, w(X), is known and the range of Y, w(Y). Then the CPE estimate is computed from

$$\left(\frac{1.1774}{d_n}\right)\{\tfrac{1}{2}[w(X) + w(Y)]\}$$

where d_n is found in page 9 of Reference 4.

10.2.1 Test for the Rayleigh Distribution. Letting $Y_i = r_i^2$ in the test of 1.1b(1) verify that the radial misses in paragraph 10.2 are Rayleigh distributed. Let $\alpha = 0.05$. Using the Shapiro-Wilk test, the W statistic is

$$(N - 1)\left(\frac{s(Y_i)}{n\bar{Y}}\right)^2 = (18)\left(\frac{333.3819}{19 \times 336.0247}\right)^2 = 0.0491$$

Since 0.0491 is within the 95% range (0.022, 0.096), we can assert that the radial misses are Rayleigh distributed at the 95% probability level.

10.2.2 In example 10.2, there were three radial miss-distances, greater than fifty, that were excluded from the original CPE analysis. Then from Equation 1 of paragraph 8.0,

$$\sigma^2 = \frac{1}{38}(6{,}384.47 + 7{,}500.0)$$

or

$$\sigma^2 = 365.38$$

$$\sigma' = 19.11$$

If an estimate of the CPE is made using this value of θ, $\widehat{CPE} = 1.1774 \times (19.11) = 22.51$.

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