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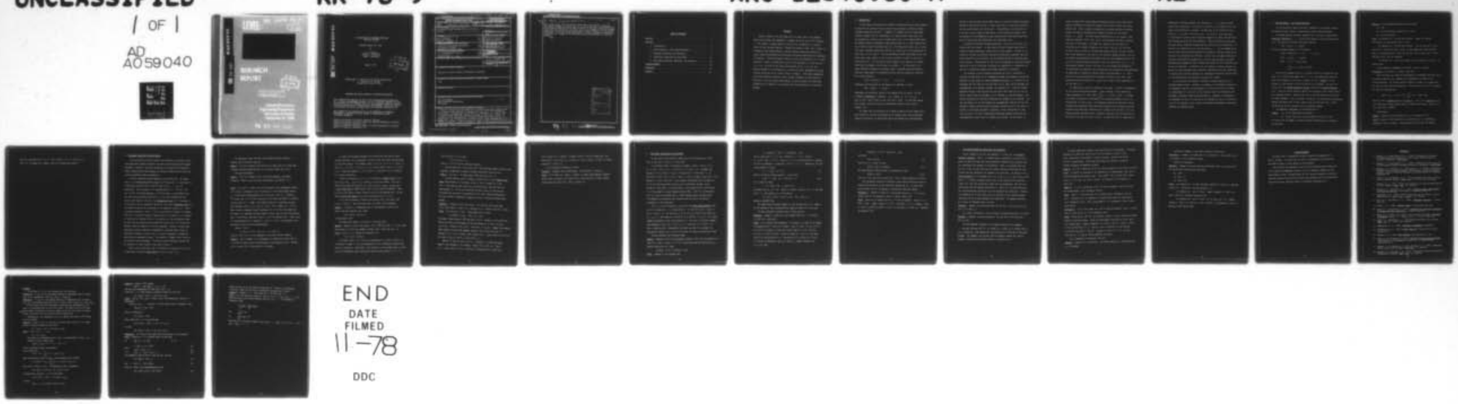

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Industrial & Systems
Engineering Department
University of Florida
Gainesville, FL 32611

ON CHARACTERIZING SUPREMUM-EFFICIENT FACILITY DESIGNS

Research Report No. 78-9

by

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August, 1978

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Define a design to be any planar set S of known area A, but unknown shape and location; more generally, a design can be any set in R^n of measure A. For example, a design might be one floor of a warehouse, or a sports arena of known seating capacity. Suppose the design to have, say, m users, or evaluators, with user/evaluator i having a design disutility function f_i, $1 \leq i \leq m$, which can be defined for all points in the plane independently of the designs of interest. Given any design S, denote by $G_i(S)$ the disutility of S to user/evaluator i where, by definition, $G_i(S)$ is the supremum of f_i over the

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set S, $1 \leq i \leq m$. Let G(S) be the vector with entries $G_i(S)$, $1 \leq i \leq m$, and define a design to be efficient if it solves the vector minimization problem obtained using the set of vectors $\langle G(S):S$ a design$\rangle$. Given mild assumptions about the disutility functions, and a slight refinement of the design definition to rule out certain pathologies, we give necessary and sufficient conditions for a design to be efficient, and study properties of efficient designs.

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ABSTRACT

Define a design to be any planar set S of known area A, but unknown shape and location; more generally, a design can be any set in R^n of measure A. For example, a design might be one floor of a warehouse, or a sports arena of known seating capacity. Suppose the design to have, say, m users, or evaluators, with user/evaluator i having a design disutility function f_i, $1 \leq i \leq m$, which can be defined for all points in the plane independently of the designs of interest. Given any design S, denote by $G_i(S)$ the disutility of S to user/evaluator i where, by definition, $G_i(S)$ is the supremum of f_i over the set S, $1 \leq i \leq m$. Let G(S) be the vector with entries $G_i(S)$, $1 \leq i \leq m$, and define a design to be efficient if it solves the vector minimization problem obtained using the set of vectors {G(S):S a design}. Given mild assumptions about the disutility functions, and a slight refinement of the design definition to rule out certain pathologies, we give necessary and sufficient conditions for a design to be efficient, and study properties of efficient designs.

1. INTRODUCTION

In this paper we consider the problem of characterizing efficient designs. To motivate the problem we first consider some examples; a precise problem statement appears in Section 2. Examples of designs we have in mind might include a single floor of a warehouse, or a sports arena, when either can be idealized as a planar set S of known total area, A, but unspecified shape. Any design S must be contained in some given planar set L, which might be the lot of land in which the design will lie. The design will have, say, m users, with user i having a disutility function f_i, where $f_i(x)$ is the disutility of the point x in S to user i. For the warehouse problem $f_i(x)$ might be the distance user i must travel, or pay to have traveled, in order to pick up an item stored at x. For the sports arena example, $f_i(x)$ might be the dissatisfaction of customer i with a seat at location x. (If the arena is to serve different purposes at different times, a customer using the arena at different times might be represented by more than one disutility function.) For a given design S,

$$G_i(S) = \sup\{f_i(x) : x \in S\} , \qquad 1 \leq i \leq m ,$$

represents the disutility of the design S to customer i, while

$$G(S) = (G_1(S), \ldots, G_m(S))$$

represents the disutility vector of the design S for all users. We call a design S* <u>efficient</u> if whenever any design S satisfies $G(S) \leq G(S^*)$, then it must be true that $G(S) = G(S^*)$. An efficient design thus solves a multiple objective optimization problem, and is Pareto optimal [10].

We remark that an evaluation of a design as given by $G_i(S)$ might occur when disutility can only be measured on an ordinal scale (thus precluding adding disutilities), or when users employ the design only intermittently (as with a sports arena) and so might adopt the relatively simple evaluation implied by the entries in G(S), or when users adopt a conservative approach and are worried about the "worst case". Alternatively, the disutility of S to user i, $G_i(S)$, might be appropriate, in the warehouse context, when the operator of the warehouse is more concerned with providing "quick service" to users than with minimizing total operating costs.

Allowing for different design shapes may or may not be realistic, depending upon the context. In some cases efficient designs should best be viewed as design guidelines, rather than as final answers. Further, as is typically the case with multiple objective optimization problems, many different designs can be efficient, and the problem remains of choosing among such solutions. Nevertheless we feel that the knowledge of efficient designs should be of value in helping to delimit the comparison of alternatives as well as to define more sharply the design problem(s) of interest.

For purposes of the literature discussion we find it convenient to consider a single point to be a degenerate design; the designs we study subsequently in this paper are then nondegenerate designs. To the best of our knowledge the only work for the case when m is at least two and designs are nondegenerate is by Chalmet, Francis, and Lawrence [3]. Given any (nondegenerate) design S they consider the case where each term $G_i(S)$ is the integral, rather than the supremum, of the function f_i over the design S, and obtain necessary and sufficient conditions for designs to be efficient. For the case where m is one and designs are nondegenerate, Francis [5] has considered a stadium design problem, discussed in more detail in [6], which is related to the design problems we consider. Corley and Roberts [4] and Lowe and Hurter [12] have investigated minimizing integral functions over (nondegenerate) planar sets in a market area context. In the latter of these two papers the authors make an assumption which we also make subsequently (although for different reasons) that certain contour lines have measure zero. When a design is degenerate the design problem we consider becomes the same as finding Pareto optimal solutions for the vector $(f_1(x), \ldots, f_m(x))$. For this latter problem there is, of course, a substantial literature ([10], [16], [18]), the discussion of which is beyond the scope of this paper. We single out, however, the case where the functions f_i are "planar" distances, say $f_i(x) = d(x, p_i)$, as this case is quite close in spirit to the warehouse design problem, and has provided much of the impetus for the study of the efficient design problem. For the case where the distances are Euclidean, Kuhn [11] has demonstrated that the set of all Pareto optimal solutions is just the convex hull of $p_1, \ldots, p_m$. Subsequently Wendell, Hurter, and Lowe [17] have studied the problem for the case where the distances are ℓ_p distances, and concentrate upon developing an algorithm for finding all Pareto optimal solutions when $p = 1$. Their work has in turn motivated work by Chalmet and Francis, who give a geometrical solution procedure [1], as well as an order m log m algorithm [2] for the case $p = 1$.

At this point we give an overview of the paper. Section 2 concentrates on the contour-envelopes of designs. Given a design S, E(S) denotes the contour-envelope of S, where $E(S) = \bigcap_{i=1}^{m} \{x \in L : f_i(x) \leq G_i(S)\}$. A design S is always contained (or enveloped) in its contour-envelope, which is an intersection of contour sets. The remaining sections are devoted to establishing that conditions called the contour-envelope (C-E) conditions provide necessary and sufficient conditions for a design to be efficient. For the design problems discussed above, a design S satisfies the C-E conditions if and only if the area of E(S)-S is zero. In order for the C-E conditions to characterize efficient designs, the functions $f_1, \ldots, f_m$ must be sufficiently well structured. In Section 3 we take L to be a subset of R^n, and refine the definition of a design. We show that if the disutility functions $f_1, \ldots, f_m$ are measurable, bounded above, and have no "flat spots" (contour lines of positive measure, or area) then any efficient design satisfies the C-E conditions. When the disutility functions have flat spots, efficient designs need not satisfy the C-E conditions. In Section 4 we establish that if the functions $f_1, \ldots, f_m$ are bounded above, and are lower semicontinuous (implied by continuity) then any design satisfying the C-E conditions is efficient. Also we give an example showing if the lower semicontinuity assumption is omitted (and replaced by an upper semicontinuity assumption) that a design can satisfy the C-E conditions and yet not be efficient. In Section 5 we tie results of earlier sections together to obtain necessary and sufficient conditions for the C-E conditions to characterize efficient designs, and investigate additional results which can be obtained when the disutility functions are convex.

In the analysis of Sections 3 through 5 we make extensive use of Lebesgue measure theory. However, by thinking of a measurable function as an integrable function, and the measure of a set as the area (or volume, or hypervolume) of a set, most of the analysis should be accessible with knowledge of measure theory being unnecessary. Likewise, by thinking of the supremum (or sup) of a function as the maximum (or max) of a function, little or no insight is lost, since the maximum and supremum are identical whenever the maximum exists.

2. INITIAL RESULTS: THE CONTOUR-ENVELOPE

In this section we give an abstract statement of the design problem, and present initial results, concentrating on the contour-envelope.

We assume given an arbitrary nonempty set L, as well as real-valued disutility functions $f_1, \ldots, f_m$ which have domain L and are bounded above on L. For each $x \in L$ we define the vector function

$$f(x) = (f_1(x), \ldots, f_m(x)) .$$

For every nonempty subset S of L we define

$$G_i(S) = \sup\{f_i(x) : x \in S\} , \qquad 1 \leq i \leq m;$$

$$G(S) = (G_1(S), \ldots, G_m(S));$$

$$E(S) = \{x \in L : f(x) \leq G(S)\}$$

$$= \bigcap_{i=1}^{m} \{x \in L : f_i(x) \leq G_i(S)\} .$$

We note the assumption that f_i is bounded above on L guarantees that $G_i(S)$ is a (finite) real number for $1 \leq i \leq m$. We assume given some property, say Property P, such that a subset S of L is a design if S has Property P, and denote by D the collection of all designs. For each $S \in D$ we call G(S) the design disutility vector, and E(S) the contour-envelope of S. Given any $S \in D$ we denote by DE(S) the collection of all designs which are contained in E(S). We shall see that if $S \in D$ then $S \in DE(S)$ and so DE(S) is always nonempty. Also, we define a design S* to be efficient if whenever any design S satisfies $G(S) \leq G(S^*)$, then it must be true that $G(S) = G(S^*)$, and denote by D* the collection of all efficient designs.

An immediate consequence of the definitions is

Lemma 1. (a) If $S \in D$ then $S \subset E(S)$ and $S \in DE(S)$.

(b) Given S and $S' \in D$, we have $S' \in DE(S)$ iff $G(S') \leq G(S)$.

We can now use the lemma to obtain equivalent conditions for a design to be efficient.

Theorem 1. The following conditions are equivalent:

(a) $S \in D^*$

(b) $S \in D$ and $S' \in DE(S)$ implies $G(S') = G(S)$,

(c) $S \in D$ and $DE(S) \subset D^*$.

Proof. (a) implies (b): Let $S \in D$ and $S' \in DE(S)$. Lemma 1(b) implies $G(S') \leq G(S)$, so $S \in D^*$ implies $G(S') = G(S)$.

(b) implies (c): Let $S \in D$ and $S' \in DE(S)$. If $S'' \in D$ and $G(S'') \leq G(S')$, then $G(S') = G(S)$ gives $G(S'') \leq G(S)$, so that Lemma 1(b) gives $S'' \in DE(S)$. Hence, by hypothesis, $G(S'') = G(S)$ and so $G(S) = G(S')$ gives $G(S'') = G(S')$, implying $S' \in D^*$, that is, $DE(S) \subset D^*$.

(c) implies (a): Given $S \in D$, Lemma 1(a) gives $S \in DE(S)$, so $DE(S) \subset D^*$ implies $S \in D^*$.

As an immediate consequence of Theorem 1, part b, we have

Corollary 1. If $S \in D$ and $\{S\} = DE(S)$, then $S \in D^*$.

This corollary will guide the development in subsequent sections, where we obtain conditions for a design to be essentially the same as its contour-envelope. Of closely related interest is Lemma 2, in preparation for which we need several definitions. Given any vector k in R^m, we denote by C(k) the following set:

$$C(k) = \{x \in L : f(x) \leq k\} = \bigcap_{i=1}^{m} \{x \in L : f_i(x) \leq k_i\}$$

and call C(k) a contour set of f of value k. A set C is a contour set (of f) if and only if there exists some k in R^m such that $C=C(k)$. We note, for $S \in D$, that $E(S) = C(G(S))$.

We now have

Lemma 2. Given $S \in D$, we have $S=E(S)$ iff S is a contour set of f.

Proof. Clearly $S = E(S)$ implies S is a contour set of f. Conversely, suppose $S = C(k)$ for some k. The definition of C(k) and G(S) implies

$G(S) \leq k$, and thus $E(S) = \{x \in L : f(x) \leq G(S)\} \subset \{x \in L : f(x) \leq k\} =$ $C(k) = S$, By Lemma 1(a), $S \subseteq E(S)$, and so it follows that $S=E(S)$.

3. NECESSARY CONDITIONS FOR EFFICIENCY

In this section we further refine our definition of a design, introduce additional problem structure, and then, after developing the needed machinery, establish necessary conditions for designs to be efficient. These conditions show that designs are efficient when they are their own contour-envelopes "almost everywhere."

We first establish needed assumptions and definitions. We assume that L is a Lebesgue-measurable subset of R^n, with $\mu(L) < \infty$, and that f is a measurable function from L into R^m, that is, $f_1, \ldots, f_m$ are measurable. Also we assume f is bounded above on L. Given any set T in R^n, of positive measure, if $z \in T$ and $T \cap N(z, \varepsilon)$ has measure zero for some positive epsilon (here $N(z, \varepsilon)$ is an open neighborhood with center z and radius ε) we call z an <u>inessential point</u> of T. Any point y in T which is not an inessential point of T we call an <u>essential point</u> of T, and denote by ess(T) the collection of all essential points of T. Given a positive constant A, with $A < \mu(L)$, we now define a design S by stating that for S to satisfy property P means that S is a subset of L, of measure A, such that $S = \text{ess}(S)$, that is, every point in S is an essential point. We shall see that deleting the inessential points from a set of positive measure leaves the measure of the set unchanged. Further, in many cases inessential points constitute a mathematical pathology which cannot be physically realized, in which case it is quite reasonable to assume designs have no inessential points. The purpose of Lemmas 3 and 4 below is to eliminate such pathologies. For most purposes "essential points" may be thought of as just "points", and ess(T) as just T.

Given a set Q in R^n and a set I of indices, a collection $F = \{T_i : i \in I\}$ of open sets is called an <u>open cover</u> of Q if $Q \subset \cup\{T_i : i \in I\}$.

In reference 8 (pps. 106-108), the Lindelöf theorem is given, a special case of which we state as

Lemma 3. If Q is a subset of R^n, and F is an open cover of Q, then some countably infinite subcollection of F is also an open cover of Q.

We can now establish

Lemma 4. Let T be a subset of R^n, of positive measure, and define $T'=ess(T)$. Then T' and T have the same measure, and $ess(T') = T'$.

Proof. Let $I=T-T'$, so that I is the collection of all inessential points of T, and so I consists of all the points x in T for which there exists $\varepsilon_x > 0$ such that $T \cap N_x$ has measure zero, where $N_x = N(x, \varepsilon_x)$. Since N_x is an open set, if F is the collection of all such sets then F is clearly an open cover of I. Lemma 3 then implies there is a countably infinite open subcover, say F′, of I, with $F' = \{N_x : x \in I'\}$, where $I' \subset I$. It is a basic result in measure theory (reference 15, p. 46) that the measure of the union of a countably infinite number of sets, each of which has measure zero, is also zero, and hence $\cup\{N_x \cap T : x \in I'\}$ has measure zero. Thus, since F' an open subcover of I implies $I \subset \cup\{N_x \cap T : x \in I'\}$, I has measure zero, and hence T' and T have the same measure.

Since $T = T' \cup I$,

$$T \cap N(x, \varepsilon) = \{T' \cap N(x, \varepsilon)\} \cup \{I \cap N(x, \varepsilon)\}$$

and $\mu[I \cap N(x, \varepsilon)] = 0$, the remainder of the proof follows readily.

Remark 1. Due to Lemma 4 it is reasonable to assume that every point in L, as well as every point in each design, is an essential point. Further, if $S \subset L$ and $\mu(S) = A$, then ess(S) is a design.

In order for efficient designs to be effectively the same as their contour-envelopes, it is necessary to rule out the case where the functions f_i have "flat spots." For example, take $L = [0, \infty)$, $A = 1$, $m = 1$, $f(x) = x$ for $0 \leq x \leq .5$, $f(x) = .5$ for $.5 \leq x \leq 2$ (a flat spot), $f(x) = x - 1.5$ for $2 \leq x$. Given the design $S = [0, 1]$, $G(S) = .5$ and $E(S) = [0, 2]$, and yet S is clearly efficient.

Given the functions f_i, $1 \leq i \leq m$, we define a <u>contour line</u> of f_i to be the set $\{x \epsilon L_i : f_i(x) = k_i\}$, where k_i is any real number. (We note a contour line need not actually be a line or even a curve.) We say that f_i <u>has a flat spot</u> if some contour line of f_i has positive measure. Thus f_i has no flat spots if every contour line of f_i has measure zero. We say that f <u>has no flat spots</u> if f_i has no flat spots for $1 \leq i \leq m$.

It is also convenient to define the function $\omega(y)$, for every $y \epsilon R^m$, by $\omega(y) = \mu[C(y)]$. The following result is proven in the appendix.

<u>Lemma 5</u>. If every f_i is measurable and if the function f has no flat spots, then the function $\omega(y)$, where

$$\omega(y) = \mu\{x \epsilon L : f(x) \leq y\},$$

is continuous on R^m.

We remark that Lemma 5 is false if f has flat spots.

<u>Lemma 6</u>. Suppose f has no flat spots. If $y, z \epsilon R^m$ with $\omega(y) < A < \omega(z)$, then there exists x on the line segment joining y and z such that $\omega(x) = A$.

<u>Proof</u>. For $0 \leq \lambda \leq 1$, define the function h by

$$h(\lambda) = \omega(\lambda z + (1-\lambda)y).$$

For fixed z and y, $\lambda z + (1-\lambda)y$ is continuous in λ, and so, since ω is continuous by Lemma 5, h is the composition of two continuous functions and hence is continuous. Since $h(0) = \omega(y)$ and $h(1) = \omega(z)$ we have $h(0) < A < h(1)$, and the intermediate value theorem now implies there exists λ, $0 < \lambda < 1$, such that $h(\lambda) = A$, so that

$$\omega(\lambda z + (1-\lambda)y) = A.$$

With $x = \lambda z + (1-\lambda)y$ the conclusion follows.

We note that $\omega(y) \geq 0$ for all $y \in R^m$. The following lemma, proven in the appendix, establishes ω takes on values arbitrarily close to zero.

Lemma 7. For any $\varepsilon > 0$ there exists $z \in R^m$ such that $\omega(z) < \varepsilon$.

In what follows, given a design S, when we say $S = E(S)$ almost everywhere (abbreviated a.e.) we mean, since $S \subset E(S)$ by Lemma 1(a), that $E(S) - S$ has measure zero, which in turn means that S and E(S) have the same measure, A. We can now establish the main result of this section. We remark that the proof is constructive and, when combined subsequently with the results of Theorem 6, suggests one way of constructing efficient designs.

Theorem 2. Assume f has no flat spots. For any $S \in D$, $\exists S' \in D$ such that $G(S') \leq G(S)$ and $S' = E(S')$ a.e.; further, if $A < \mu[E(S)]$ then $G(S') \neq G(S)$.

Proof. Let $S \in D$ and $v = G(S)$. Using Lemma 1(a), we have

$$A \leq \mu[E(S)] = \mu\{x \in L : f(x) \leq v\} = \omega(v).$$

By Lemma 7 there exists $u' \in R^m$ such that $\omega(u') < A$. Define u to be the vector whose ith entry is the minimum of the ith entries in u' and v, so that $u \leq u'$, $u \leq v$. Clearly $\omega(u) \leq \omega(u')$. Thus $\omega(u) < A \leq \omega(v)$. Lemma 6 now implies there exists w on the line segment joining u and v such that $\omega(w) = A$. Further, if $A < \omega(v) = \mu[E(S)]$, then clearly $w \neq v$. Also, since $w = \lambda u + (1-\lambda)v$ for some λ, $0 \leq \lambda \leq 1$, and $u \leq v$, we have $w \leq v$.

Define $\hat{S} = C(w)$, so that $\mu(\hat{S}) = A$. Letting $S' = \mathrm{ess}(\hat{S})$ and using Remark 1 we conclude S' is a design. Since $S' \subset \hat{S}$, $G(S') \leq w$. Since $w \leq v = G(S)$, and $w \neq v$ if $A < \mu[E(S)]$, we conclude $G(S') \leq G(S)$, and

$G(S') \neq G(S)$ if $A < \mu[E(S)]$. Further, since $S' \subset E(S')$ by Lemma 1(a), and since $E(S') \subset \hat{S}$ as $G(S') \leq w$, we have $A = \mu(S') \leq \mu[E(S')] \leq \mu(\hat{S}) = A$, which implies $S' = E(S')$ a.e.

As a corollary of Theorem 2 we have

<u>Theorem 3</u>. Assume f has no flat spots. If $S \epsilon D^*$ then $S = E(S)$ a.e.

<u>Proof</u>. If $S \neq E(S)$ a.e. then $A < \mu[E(S)]$, in which case Theorem 2 implies there exists a design S' for which $G(S') \leq G(S)$ and $G(S') \neq G(S)$, contradicting the fact that $S \epsilon D^*$. Thus $S = E(\bar{S})$ a.e.

4. SUFFICIENT CONDITIONS FOR EFFICIENCY

In this section we establish conditions on the functions in f such that if $S \in D$, and $S = E(S)$ a.e., then $S \in D^*$.

It is helpful first to consider an example, Take $L = [0,\infty)$, $A = 1$, $m = 1$, $f(x) = x$ for $0 \leq x < 1$ and $f(x) = x + 1$ for $1 \leq x$. Consider the designs $S = [0,1]$ and $S' = [0,1)$, for which $G(S) = 2$ and $G(S') = 1$, so that $E(S) = [0,1]$, $E(S') = [0,1)$. Here $S = E(S)$, but $S \notin D^*$, since $G(S') < G(S)$. However $S' = E(S')$ and $S' \in D^*$. This example illustrates that a design can be its own contour-envelope and yet not be efficient; further, such a situation occurs when f is discontinuous. More specifically, the function f for this example is upper semicontinuous (USC). (For f to be USC means that -f is LSC, as defined below.) Hence imposing upper semicontinuity is inadequate to guarantee that designs which are their own contour-envelopes a.e. are efficient.

For $1 \leq i \leq m$ the function f_i is said to be <u>lower semicontinuous</u> (LSC) at $y \in L$ if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $f_i(y) < f_i(x) + \varepsilon$ for $x \in N(y,\delta) \cap L$. The function f_i is LSC if f_i is LSC at every point in L, $1 \leq i \leq m$. The function f is LSC if f_i is LSC for $1 \leq i \leq m$. For a good summary discussion of LSC, see reference 13. (We note if the above example were changed so that $f(x) = x$ for $0 \leq x \leq 1$, and $f(x) = x + 1$ for $1 < x$, that f would be LSC.) Fortunately, we shall see that it is enough, for the general problem, for f to be LSC in order to obtain the results we seek.

We now establish the following lemma.

<u>Lemma 8</u>. Suppose S' and S are measurable subsets of R^n, both of measure $A > 0$, that $S' \subset S$, and $S = \text{ess}(S)$. If h is a real-valued and LSC function on S, bounded from above on S, then

$$\sup\{h(x) : x \in S'\} = \sup\{h(x) : x \in S\} .$$

<u>Proof</u>. Suppose to the contrary that

$$\sigma' \equiv \sup\{h(x) : x \in S'\} < \sigma \equiv \sup\{h(x) : x \in S\} .$$

(It is clear that $\sigma' \leq \sigma$.) Let η satisfy $0 < \eta < \sigma - \sigma'$, and let $S'' = \{x \in S : h(x) > \sigma' + \eta\}$. Since $\sigma' + \eta < \sigma$, the definition of σ implies $S'' \neq \emptyset$, as otherwise $\sigma = \sup\{h(x) : x \in S\} \leq \sigma' + \eta < \sigma$. Therefore, let $y \in S'$: we may choose ε so that

$$0 < \varepsilon < h(y) - (\sigma' + \eta). \qquad \text{(i)}$$

Since h is LSC on S there exists $\delta > 0$ such that

$$h(y) - \varepsilon < h(x) \quad \text{for} \quad x \in N(y,\delta) \cap S, \qquad \text{(ii)}$$

so (i) and (ii) imply

$$\sigma' + \eta < h(x) \quad \text{for} \quad x \in N(y,\delta) \cap S$$

and hence $\{N(y,\delta) \cap S\} \subset S''$. Since $S = \text{ess}(S)$, $\mu[N(y,\delta) \cap S] > 0$, and thus $\mu(S'') > 0$. Now clearly $S' \cap S'' = \emptyset$ and so

$$A = \mu(S') < \mu(S') + \mu(S'') = \mu(S' \cup S'') \leq \mu(S) = A,$$

giving a contradiction.

We remark that the above example of this section with h = f, where f is USC instead of LSC, illustrates the lemma is false if h is not LSC.

We next establish the main result of this section.

<u>Theorem 4</u>. Assume f is LSC on L, and bounded above on L. If $S \in D$ and $S = E(S)$ a.e., then $S \in D^*$.

<u>Proof</u>. Let S satisfy the hypotheses. By Theorem 1, we know it is enough to establish $G(S') = G(S)$ if $S' \in DE(S)$. With $T \equiv E(S) - S$, $S'S \equiv S' \cap S$, $S'T \equiv S' \cap T$, we know $S' \subset E(S)$ and $S \subset E(S)$ imply $S' = S'S \cup S'T$ and $A = \mu(S') = \mu(S'S) + \mu(S'T)$. Since $S = E(S)$ a.e., $\mu(T) = 0$, and thus $\mu(S'T) = 0$, giving $\mu(S'S) = A$. Now since $S'S \subset S$, $\mu(S'S) = A = \mu(S)$, $S = \text{ess}(S)$ by hypothesis, and f_i is LSC on L, Lemma 8 implies, for $1 \leq i \leq m$, that

$$\sup\{f_i(x) : x\epsilon S'S\} = \sup\{f_i(x) : x\epsilon S\} ,$$

and hence

$$G(S) = G(S'S). \qquad (i)$$

Since $S'S \subset S' \subset E(S)$, we have

$$G(S'S) \leq G(S') \leq G[E(S)]. \qquad (ii)$$

The definitions of $E(S)$ and $G(S)$, and Lemma 1(a), give

$$G[E(S)] = G(S). \qquad (iii)$$

From (i), (ii), and (iii) we have $G(S) = G(S')$, which completes the proof.

The following corollary of Theorem 4 gives some indication of how extensive the collection of all efficient designs can be. In particular, every contour set of f of measure A contains an efficient design.

Corollary 2. Assume f is LSC on L, and bounded above on L. If S is any contour set of f of measure A, and $S' = ess(S)$, then $S'\epsilon D^*$.

Proof. Since S is a contour set of f, $S = E(S)$ by Lemma 2. Since $S' \subset S$, $E(S') \subset E(S)$. Lemma 1 implies $S \subset E(S)$. By Remark 1, S' is a design. Thus $A = \mu(S') \leq \mu[E(S')] \leq \mu[E(S)] = \mu(S) = A$, and hence $S' = E(S)$ a.e. Theorem 4 now implies $S'\epsilon D^*$.

5. THE CONTOUR-ENVELOPE CONDITIONS, AND CONVEXITY

Given a design S, we call the condition $S = E(S)$ a.e. the contour-envelope condition. When f is bounded above, measurable, and has no flat spots, we have seen that the contour-envelope conditions are necessary (but not sufficient) for designs to be efficient. When f is bounded above and each f_i is LSC, we have seen that the contour-envelope conditions are sufficient (but not necessary) for designs to be efficient. We define f to be well-structured if f is bounded above, f_i is LSC $1 \leq i \leq m$, and f has no flat spots. It is known, because contour sets of LSC functions are relatively closed in L, and such relatively closed sets are measurable, that LSC of a real-valued function implies measurability. Hence if f is well-structured the contour-envelope conditions are both necessary and sufficient. Further, the assumption of no flat spots cannot be omitted if the contour-envelope conditions are to be both necessary and sufficient. To summarize matters, as a result of Theorems 3 and 4 we have

Theorem 5. Assume f is well-structured, and let $S \in D$. We have $S \in D^*$ if and only if $S = E(S)$ a.e.

As a result of Theorem 5, we can readily strengthen Theorem 2, to obtain

Theorem 6. Assume f is well-structured. For any $S \in D \ \exists\ S' \in D^*$ such that $G(S') \leq G(S)$.

We note Theorem 6 implies D^* is nonempty whenever D is nonempty.

We shall now see that if f is convex (i.e., every f_i is convex) and if L is a convex set, then additional conclusions may be drawn about efficient designs. For example, for any $S \in D$, $E(S)$ is always a convex set, and so Theorem 5 implies any efficient design is a convex set a.e.

To obtain additional results, some notation will be convenient. Following reference 14, given any convex set C in R^n we denote by int(C), ri(C), cl(C), and aff(C), the interior, relative interior, closure, and affine hull of C respectively. With these definitions we proceed to establish several necessary intermediate results.

<u>Lemma 9</u>. If C is a convex set in R^n, of positive measure, *then* $int(C) = ri(C) \neq \emptyset$.

<u>Proof</u>. If aff(C) is contained in an intersection of hyperplanes, then $\mu(C) = 0$, as each hyperplane has measure zero. Thus aff(C) = R^n, in which case (reference 14, p. 44, par. 6), ri(C) = int(C). By Theorem 6.2 of reference 14, ri(C) $\neq \emptyset$.

<u>Lemma 10</u>. If C is a convex set in R^n, of positive measure, then every open set intersecting C also intersects int(C).

<u>Proof</u>. Since $C \subset c\ell(C)$ any open set T intersecting C intersects $c\ell(C)$ as well. Corollary 6.3.2 of reference 14 implies every open set intersecting $c\ell(C)$ intersects ri(C). Thus, using Lemma 9, if T intersects C then T intersects int(C).

<u>Corollary 3</u>. If C is any convex set in R^n of positive measure, then C = ess(C), that is, every point in C is an essential point.

<u>Proof</u>. Given any $y \in C$ and $\delta > 0$, it suffices to show $C \cap N(y,\delta)$ contains a hypersphere of positive radius. Given the hypotheses, since $N(y,\delta)$ is an open set intersecting C, Lemma 10 implies $\exists\, u \in int(C) \cap N(y,\delta)$. Since $u \in int(C)$ $\exists\, \tau > 0$ such that $N(u,\tau) \subset C$. Since $u \in N(y,\delta)$ and $N(y,\delta)$ is an open set $\exists\, r > 0$ such that $N(u,r) \subset N(y,\delta)$. Letting $\varepsilon = \min(\tau, r) > 0$, it is clear that $N(u,\varepsilon) \subset C \cap N(y,\delta)$, which completes the proof.

As a consequence of Corollary 3 we have

<u>Remark 2</u>. Assume L is a convex set. Any convex subset of L, having measure A, is a design.

Corollary 2, Remark 2, and basic convexity results give

Corollary 4. Assume f is convex and L is a convex set. Any contour set of f of measure A is an efficient design.

Finally, we establish

Theorem 7. Assume f is convex and well-structured, and L is a convex set. For any $S \in D$ the following are equivalent:

(a) $S \in D^*$

(b) $E(S) \in D$

(c) $E(S) \in D^*$.

Proof. (a) implies (b): If $S \in D^*$, Theorem 5 implies $S = E(S)$ a.e., implying $\mu[E(S)] = A$. Corollary 4 now implies $E(S) \in D$.

(b) implies (c): Let $S' = E(S)$. Lemma 2 implies $S' = E(S')$ a.e. Thus, since $S' \in D$, Theorem 5 implies $S' \in D^*$.

(c) implies (a): Let $S' = E(S) \in D^*$, so that $\mu(S') = A$. Lemma 1 implies $S \subset E(S) = S'$, and so $A = \mu(S) \leq \mu[E(S)] = A$, implying $S = E(S)$ a.e. Theorem 5 thus implies $S \in D^*$.

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APPENDIX

In reference 9, p. 38, two theorems say the following:

Theorem A1: If $\{E_n\}$ is an increasing sequence of measurable sets for which $\lim_n \{E_n\}$ is measurable, then $\lim_n \mu(E_n) = \mu(\lim_n E_n)$.

Theorem A2: If $\{E_n\}$ is a decreasing sequence of measurable sets for which $\lim_n \{E_n\}$ is measurable and some $\mu(E_n)$ is finite, then $\mu(\lim_n E_n) = \lim_n \mu(E_n)$.

We will apply these two theorems to establish the continuity of $\omega(z)$ when f is measurable and f has no flat spots. We remark that for the case where omega is defined on the reals Lemmas A1 and A2 are related to having omega continuous from below and from above respectively.

Subsequently, for convenience, we let e denote the vector in R^m having all unit entries.

Lemma A1: Given $y \in R^m$, if f has no flat spots then, for any $\varepsilon > 0$, there exists a positive integer N_1 such that

$$\omega(y - \frac{1}{n} e) > \omega(y) - \varepsilon \text{ for any } n \geq N_1.$$

Proof: For n = 1, 2, ..., let

$$E_n = C(y - \frac{1}{n} e).$$

Then each E_n is measurable since L and f are measurable, and $E_1 \subset E_2 \subset \cdots$ Further it can be shown that

$$\lim_n E_n = \bigcup_n E_n = E \equiv \{x \in L : f(x) < y\}$$

L and f measurable imply E measurable.

Now we note that

$$\{C(y) - E\} \subset \bigcup_{i=1}^{m} \{x \in L : f_i(x) = y_i\}$$

and, since contour lines of each f_i have measure zero, we have

$$0 \leq \mu[C(y) - E] \leq \mu[\bigcup_{i=1}^{m} \{x \in L : f_i(x) = y_i\}] = 0.$$

Thus $\mu(E) = \mu(C(y)) = \omega(y)$. By Theorem A1, when n increases,

$$\omega(y - \frac{1}{n} e) \rightarrow \mu(E), \text{ so } \omega(y - \frac{1}{n} e) \rightarrow \omega(y).$$

In particular, for any $\varepsilon > 0$, $\exists$ N_1 such that

$$|\omega(y - \frac{1}{n} e) - \omega(y)| < \varepsilon \text{ for all } n \geq N_1,$$

so that

$$\omega(y) - \varepsilon < \omega(y - \frac{1}{n} e) \text{ for all } n \geq N_1.$$

Lemma A2: Given $y \in R^m$, assume

$$\omega(y^\circ) < \infty \text{ for some } y^\circ > y,\ y^\circ \in R^m.$$

(We note this assumption is valid since $\mu(L) < \infty$.)

Given any $\varepsilon > 0$, there exists a positive integer N_2 such that

$$\omega(y + \frac{1}{n} e) < \omega(y) + \varepsilon \text{ for all } n \geq N_2.$$

Proof: Let $E_n = C(y + \frac{1}{n} e)$. Since L and f are measurable, each E_n is measurable.

Also $E_1 \supset E_2 \supset \ldots.$ Further, it can be shown, when n increases, that

$$\lim_n E_n = \bigcap_n E_n = C(y).$$

Thus, by Theorem A2,

$$\omega(y + \frac{1}{n} e) \to \omega(y).$$

Thus, given any $\varepsilon > 0$, $\exists\ N_2$ such that

$$|\omega(y + \frac{1}{n} e) - \omega(y)| < \varepsilon \text{ for all } n \geq N_2$$

so that

$$\omega(y + \frac{1}{n} e) < \omega(y) + \varepsilon \text{ for all } n \geq N_2.$$

Theorem A3: If f has no flat spots then the function ω is continuous.

Proof: Given any $\varepsilon > 0$, we shall show $\exists$ N such that

if $\quad \max_j |z_j - y_j| \leq \frac{1}{N}$, $\quad$ or, if

$$y - \frac{1}{N} e \leq z \leq y + \frac{1}{N} e, \tag{1}$$

then

$$|\omega(z) - \omega(y)| < \varepsilon, \tag{2}$$

i.e.,

$$\omega(y) - \varepsilon < \omega(z) < \omega(y) + \varepsilon. \tag{3}$$

From Lemmas A1 and A2 with $N = \max(N_1, N_2)$ we have

$$\omega(y + \frac{1}{N} e) < \omega(y) + \varepsilon \tag{4}$$

and

$$\omega(y) - \varepsilon < \omega(y - \frac{1}{N} e). \tag{5}$$

From (1), since ω is nondecreasing we have

$$\omega(y - \frac{1}{N} e) \leq \omega(z) \leq \omega(y + \frac{1}{N} e). \tag{6}$$

Using (4) and (5) in (6) gives (3) and thus (2). Hence ω is continuous. Note that f does not have to be continuous for this proof to hold.

Lemma A3: Suppose $\varepsilon > 0$. Then there is $y \in R^m$ with $\omega(y) < \varepsilon$.

Proof: For each positive integer k, let $S_k = \{x \in L : f(x) \leq (-k, \ldots, -k)\}$. Then each set S_k has finite measure, and $S_1 \supset S_2 \supset \ldots$. It follows by Theorem A2 that

$$\mu\left(\bigcap_{k=1}^{\infty} S_k\right) = \lim_{k\to\infty} \mu(S_k);$$

but
$$\bigcap_{k=1}^{\infty} S_k = \emptyset,$$

so
$$\lim_{k\to\infty} \mu(S_k) = 0.$$

Thus there is a positive integer k with $\mu(S_k) < \varepsilon$. Then, if $y = (-k, \ldots, -k)$, $\omega(y) = \mu(S_k) < \varepsilon$. □