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DYNAMIC PRODUCTION FUNCTIONS

by
RONALD W. SHEPHARD

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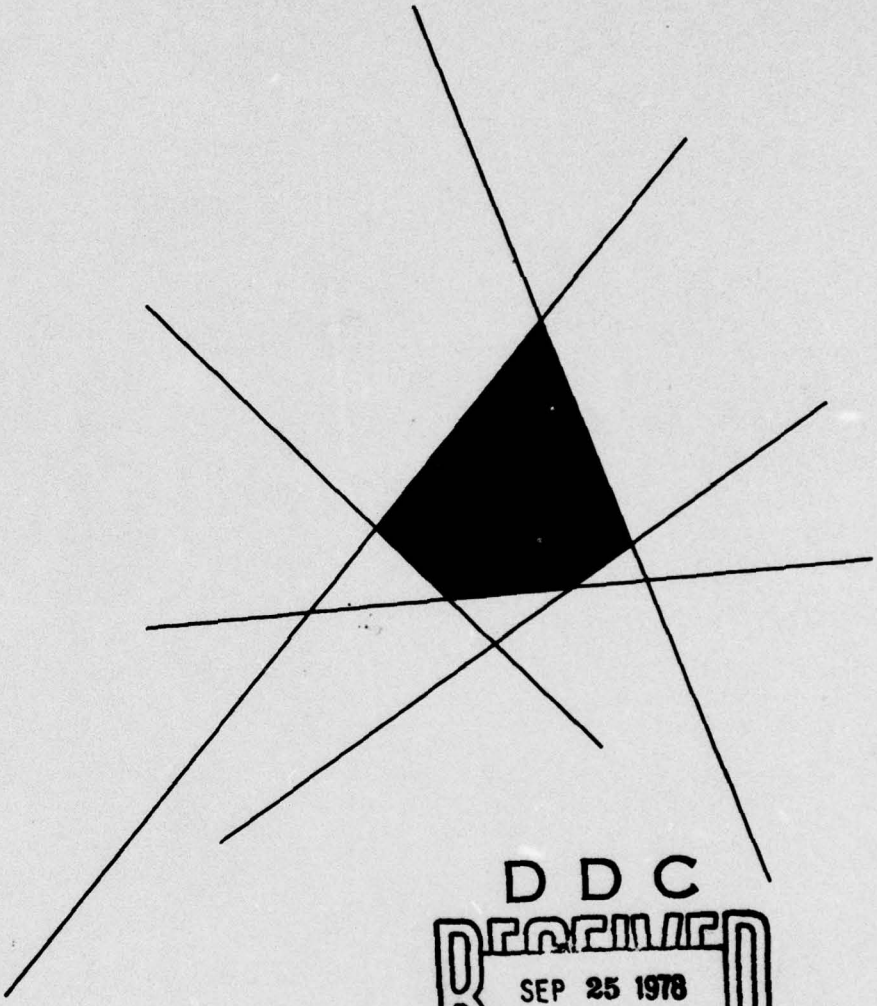

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DYNAMIC PRODUCTION FUNCTIONS

by

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ABSTRACT

An overview is presented of a dynamic theory of production correspondences with an example of the construction of a dynamic activity analysis production correspondence.

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DYNAMIC PRODUCTION FUNCTIONS

by

Ronald W. Shephard

1. INTRODUCTION

The typical production function relates constant input rates of certain factors of production to a *maximal* constant output rate of a single good or service thereby obtainable, serving primarily to define factor substitution for steady state operation of a technology. As a model of production such structures often fail to be useful because in many circumstances the elements of the phenomena modeled are inherently dynamic. Take the case of shipbuilding, for example, which is carried out by the interaction in time of a number of activities, the outputs of which are phased inputs (intermediate products) for other activities. There the process of production is decidedly evolutionary, and a steady state relationship between constant input and output rates fails to display the choices which can be made by shifting in time the allocation of variable input rates, i.e., by Time Substitution as opposed to factor substitution.

Not in shipbuilding alone, are dynamic elements significant for economic analysis of production. All construction activity has the same character. In automated manufacturing of a variety of parts, similar dynamic interactions arise, and to some degree all production exhibits dynamic aspects. Once Time Substitution is admitted as part of production phenomena, econometric studies of productivity and technical progress are thereby complicated. Thus, one is prompted to make an effort to develop a dynamic theory of production functions (correspondences).

2. BRIEF SKETCH OF STEADY STATE PRODUCTION FUNCTIONS

Let u be a nonnegative constant output rate for a single (net) product, and $x = (x_1, x_2, \ldots, x_n)$ denote a vector of nonnegative constant input rates for n factors of production (inputs).

The traditional production function is a mapping of x into $\phi(x)$ where $\phi(x)$ denotes the maximal output rate for a single product obtainable from x. By considering the level set

$$L(u) := \left\{ x \in R_+^n : \phi(x) \geq u \right\}$$

one may exhibit factor substitution by the boundary of $L(u)$, i.e., by the

$$\text{ISOQUANT } L(u) := \{ x \in L(u) : (\lambda x) \notin L(u) \text{ for } 0 \leq \lambda < 1 \},$$

as illustrated in Figure 1.

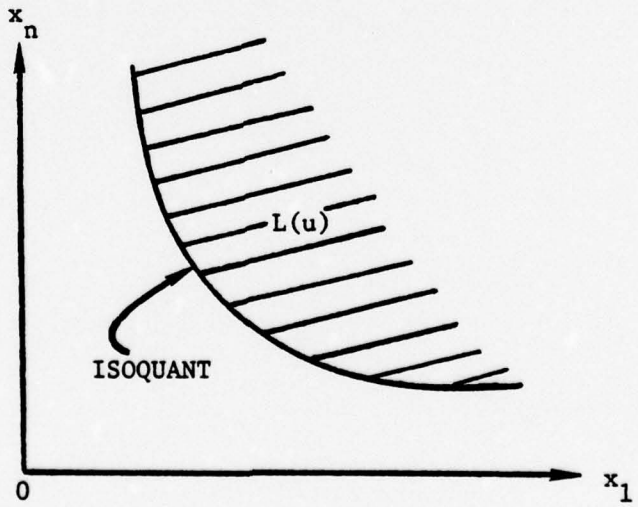

FIGURE 1

This production function may be regarded as a smooth idealization of k independent, parallel alternative processes, say the correspondence $u \to L(u)$ expressed by

$$L(u) = \left\{ x \in R_+^n : \sum_{i=1}^{n} \xi_i \geq u \ , \ x \geq \xi A \right\},$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_k)$ is a vector of k nonnegative output rates for the processes and $A = ||a_{ij}||$, $(i = 1,2, \ldots, k)$, $(j = 1,2, \ldots, n)$, is a matrix of nonnegative input coefficients in which a_{ij} is the input of the j^{th} factor per unit output of the i^{th} (activity) process. Typically, one or other, or variants, of the following two functions are used:

$$\phi(x) = \phi_o \prod_{1}^{n} x_i^{\alpha_i} \ , \ \alpha_i \geq 0 \ , \ \sum_{1}^{n} \alpha_i = 1 \ , \ \phi_o > 0 \quad \text{Cobb-Douglas}$$

$$\phi(x) = \phi_o \left[\sum_{1}^{n} a_i x_i^{-\rho} \right]^{-\frac{1}{\rho}} \ , \ \rho \geq -1 \ , \ \rho \neq 0 \ , \ a_i \geq 0 \quad \text{CES}$$

to represent the production function. In the case of the activity analysis expression,

$$\phi(x) = \text{Max} \left\{ \sum_{1}^{k} \xi_i : \xi A \leq x \ , \ \xi \in R_+^k \right\},$$

and in the case of a single process $(k = 1)$,

$$\phi(x) = \min_j \left\{ \left(\frac{x_j}{a_{1j}} \right) ; \ j \in \{1,2, \ldots, n\} \right\} \quad \text{LEONTIEF.}$$

All of these models emphasize factor substitution arising from independent technical alternatives, and they suffer from the restriction to a single net output.

Let us consider next, then, an activity analysis model of production with m (net) outputs at nonnegative constant rates $u = (u_1, u_2, \ldots, u_m) \in R_+^m$. Further, the activities need not be independent alternatives for final output. Indeed, if we are to display the potential for dynamic phenomena some of the activities must serve to produce intermediate products for others, as in shipbuilding. As nonnegative technical coefficients, retain A and add $B = ||b_{ij}||$, $(i = 1, \ldots, k)$, $(j = 1, \ldots, m)$, $\bar{A} = ||\bar{a}_{ij}||$, $(i = 1,2, \ldots, k)$, $(j = 1,2, \ldots, m)$, where b_{ij} is the output of the j^{th} product per unit of ξ_i and $\bar{a}_{ij}$ is the input of the j^{th} product required per unit of ξ_i, and $\xi = (\xi_1, \ldots, \xi_k)$ is interpreted as a vector of intensities of operation. Then the production function is expressed by two inversely related correspondences

$$u \to L(u) := \left\{x \in R_+^n : \xi(B - \bar{A}) \geq u , x \geq \xi A\right\}$$

$$x \to P(x) := \left\{u \in R_+^m : \xi A \leq x , \xi(B - \bar{A}) \geq 0 , 0 \leq u \leq \xi(B - \bar{A})\right\} .$$

Here the production functions express steady state transfer of intermediate products. The coefficients A and B are assumed to satisfy

$$\sum_{i=1}^{k} a_{ij} > 0 , (j = 1,2, \ldots, n) , \quad \sum_{j=1}^{n} a_{ij} > 0 , (i = 1,2, \ldots, k)$$

$$\sum_{j=1}^{m} b_{ij} > 0 , (i = 1,2, \ldots, k) , \quad \sum_{i=1}^{k} b_{ij} > 0 , (j = 1,2, \ldots, m) .$$

In the case of shipbuilding, we may regard each activity as yielding a single product, with one of them, say the k^{th}, outputing a ship while all the others provide intermediate products. Then $B = ||b_{ii}||$ (diagonal matrix) and $\bar{a}_{ik} = 0$ for $(i = 1,2, \ldots, k)$. The Leontief open model is obtained by stipulating further that $k = m$ and allowing each product to be both intermediate and final.

In general (abstract) terms one may express all steady state (models) production functions by two inversely related correspondences:

$$x \in R_+^n \to P(x) \in 2^{R_+^m} \; ; \; u \in R_+^m \to L(u) \in 2^{R_+^n}$$

$$u \in P(x) \iff x \in L(u)$$

having the following properties (see Reference [3]):

P.1 $P(0) = \{0\}$, NOTHING FROM NOTHING.

P.2 $P(x)$ bounded, OUTPUT IS FINITE.

P.3 $P(\lambda x) \supset P(x)$, $\lambda \geq 1$, WEAK NONDECREASING OUTPUT.

P.3SS $P(x') \supset P(x)$, $x' \geq x$, STRONG NONDECREASING OUTPUT.

P.4.1 For each $i \in \{1, \ldots, m\}$ there is some $x^{(i)}$, such that $u \in P(x^{(i)})$ with $u_i > 0$, POSSIBILITY OF PRODUCING ALL OUTPUTS.

P.4.2 If $u \in P(x)$, $u \neq 0$, $(\theta u) \in P(\lambda_\theta x)$ for some $\lambda_\theta > 0$, $\theta \in (0,+\infty)$, SCALED OUTPUTS OBTAINABLE BY SCALED INPUTS.

P.5 $x \to P(x)$ is closed, THE SET OF ALL FEASIBLE VECTOR PAIRS (x,u) IS CLOSED.

P.6 If $u \in P(x)$, $(\theta u) \in P(x)$, $\theta \in [0,1]$, WEAK DISPOSABILITY OF OUTPUTS.

P.6SS If $u \in P(x)$ then $v \in P(x)$ for all $0 \leqq v \leqq u$, STRONG DISPOSABILITY OF OUTPUTS.

E. Eff $L(u)$ is bounded, i.e., BOUNDED EFFICIENT SUBSTITUTION OF FACTORS $\left(\text{Eff } L(u) := \left\{x \in R_+^n : x \in L(u),\ y \notin L(u) \text{ for } y \leqq x,\ y \neq x\right\}\right)$.

With this abstract structure one may develop a steady state theory of production. The properties for the inverse correspondence are implied by P.1, ..., P.6SS. See Reference [3] for details.

3. GENERAL ABSTRACT STRUCTURE OF DYNAMIC PRODUCTION FUNCTIONS

Here the primitive elements of production are not constant input and output rates (n,m-tuples of real numbers) but input and output rate histories for the factors and products (nonnegative functions of time defined on $[0,+\infty)$). Let x_i denote an input rate history for the i^{th} input, and u_j denote an output rate history for the j^{th} product. Two histories for the same factor or product are not distinguishably different for the purposes of this theory, if they differ only on a subset of measure zero, since when summed by integration they will yield the same value. Further we shall restrict the histories to be bounded and measurable functions of time. In mathematical terms then, each input and output rate history is an element of a function space (L_∞). The vectors x and u of such histories are elements of the product spaces $(L_\infty)_+^n$, $(L_\infty)_+^m$, respectively. Two histories can be compared, added and multiplied by a constant, pointwise in time. The norms $||x_i||$, $||u_j||$ of an input and output rate history are taken as the essential suprema of the history, i.e., the supremal value except for those on a subset of values of time $t \in [0,+\infty)$ of measure zero. The norms of vectors x and u are taken as

$$||x|| = \operatorname{Max}_i \{||x_i||\} , \quad ||u|| = \operatorname{Max}_j \{||u_j||\} ,$$

and the distance between any two vectors x and y in $(L_\infty)_+^n$, or u and v in $(L_\infty)_+^m$, is defined by the metric

$$d(x,y) = ||x - y|| , \quad d(u,v) = ||u - v|| .$$

Thus, with points being vectors of time histories we have metric spaces for expressing the theory by operations which are analogous to those used when only constant rates are considered.

With the foregoing conventions, a dynamic production function (correspondences) is defined by

$$x \in (L_\infty)_+^n \to \mathbb{P}(x) \in 2^{(L_\infty)_+^m} , \quad u \in (L_\infty)_+^m \to \mathbb{L}(u) \in 2^{(L_\infty)_+^n}$$

$$u \in \mathbb{P}(x) \iff x \in \mathbb{L}(u) ,$$

as a mapping relating points in one product function space to a subset of points in another product function space. The abstraction used has the same apparent structure in the abstract spaces as that of the steady state model in real coordinate spaces. Indeed the same axioms P.1, ..., P.6SS used for the steady state model may be carried over for the map sets $\mathbb{P}(x)$ $(\mathbb{L}(u))$, except that now one must keep in mind that a point u of the output set $\mathbb{P}(x)$ is a vector of output rate histories, and the point x is a vector of input rate histories. On this account one may add two additional possibilities for axioms $\mathbb{P}.3$ and $\mathbb{P}.6$. They are:

P.3S $\mathbb{P}(\lambda_1 x_1, \ldots, \lambda_n x_n) \supset \mathbb{P}(x)$ for $\lambda_i \geq 1$, $i \in \{1, \ldots, n\}$, NONDECREASING OUTPUT RATE HISTORIES UNDER UPWARD SCALING OF INPUT RATE HISTORIES.

P.6S If $u \in \mathbb{P}(x)$, $(\theta_1 u_1, \theta_2 u_2, \ldots, \theta_m u_m) \in \mathbb{P}(x)$ for $\theta_i \in [0,1]$, $i \in \{1,2, \ldots, m\}$, SCALED DISPOSABILITY OF OUTPUT RATE HISTORIES.

Also, certain additional axioms (properties) need be applied.

Let $\bar{t}_u$ ($\underline{t}_u$) denote the latest (earliest) time at which a history u_i is positive, except for time points on $[0,+\infty)$ of measure zero. Let $\bar{t}_x$ ($\underline{t}_x$) have similar meaning. Then

$\mathbb{P}.T.1$ $\underline{t}_u > \underline{t}_x$ NONINSTANTANEOUS OUTPUT.

$\mathbb{P}.T.2$ $\bar{t}_u \leq \bar{t}_x$ NO OUTPUT WHEN INPUTS CEASE.

$\mathbb{L}.T.1$ If $\int_0^\infty u_j d\mu_j(t) < +\infty$, $j = 1,2, \ldots, m$, the vector u of such output rate histories can be obtained by a vector x of input rate histories such that $\int_0^\infty x_i d\nu_i(t) < +\infty$, $i = 1,2, \ldots, n$. BOUNDED TOTAL OUTPUTS CAN BE OBTAINED BY BOUNDED TOTAL INPUTS.

$\mathbb{L}.T.2$ If $\bar{t}_u < +\infty$ and x yields u , then y yields u where $y_i(t) = x_i(t)$, $t \in [0,\bar{t}_u]$ $y_i(t) = 0$, $t > \bar{t}_u$. INPUTS ARE NOT REQUIRED WHEN ALL OUTPUTS CEASE.

$\mathbb{E}$. Eff $\mathbb{L}(u) := \left\{x \in (L_\infty)_+^n : x \in \mathbb{L}(u) , y \notin \mathbb{L}(u) \text{ if } y \leq x , y \neq x\right\}$ is bounded - UNBOUNDED INPUT RATE HISTORIES ARE NOT EFFICIENT.

Either of two topologies may be used: the norm topology, or a weak* topology generated by price vectors from products of spaces for bounded, measurable and absolute summable functions. Boundedness of $\mathbb{P}(x)$ and Eff $\mathbb{L}(u)$ does not imply these sets to be compact when closed, unless the weak* topology is used. For the norm topology properties $\mathbb{P}.2$ and $\mathbb{E}$ can be strengthened to "totally bounded." Details on these matters need not concern us here. See Reference [1] for details on the dynamic structure of production.

4. POSSIBILITY OF A DYNAMIC NEOCLASSICAL PRODUCTION FUNCTION

In the dynamic framework it is clear that an output rate history can vary in both time pattern and magnitude. For example, consider the case of $(k - 1)$ distinct single intermediate product producing activities, with the k^{th} activity yielding a single final output, as described at the end of Section 2, above, for shipbuilding. The assignment of an available vector x of input rate histories to the m activities, say $x_{o\alpha}$, $(\alpha = 1,2, \ldots, k)$, $\sum_{\alpha=1}^{k} x_{o\alpha} \leq x$, can be made in a very large number of ways indeed, and for each such assignment there will be an output pattern of final product. Off hand, one cannot guarantee that among all such possible patterns there is one output pattern which dominates all, i.e., it is a maximal output rate history. Thus, in general a neoclassical dynamic production function

$$x \in (L_\infty)_+^n \rightarrow \mathbb{F}(x) \in (L_\infty)_+ ,$$

relating maximal output rate history to a vector of input rate histories, *does not exist*. TIME SUBSTITUTION for output rate histories exists even when there is no product substitution. This phenomena of time substitution is inherent in dynamic phenomena of production. Immediately, one is confronted with the fact that pseudo dynamic production functions like $F(x(t),t)$ entail serious problems of interpretation and may lead to spurious correlations with time series data. First, this function implies instantaneous effect of inputs. Second, an apparent variation of relationship between output at time t and $x(t)$ may be merely a result of time substitution and not imply anything at all about productivity and technical

progress. The use of time lags to relieve the situation is of no avail, since actual lags in production are endogenous and merely part of the changes involved in time substitution.

Every scheduler of production is acutely aware of this dynamic phenomena of Time Substitution, and his production plan must allow for such variations otherwise the work flow would be too rigid and would entail irregularities in loading.

Only when there is a maximal output rate history $\mathbb{F}(x)$ associated with a vector x of input rate histories can one justifiably associate a single output rate history to x. Even then a function $F(x(t),t)$, $t \in [0,+\infty)$, cannot evidently be used to represent $\mathbb{F}(x)$, since the value of $\mathbb{F}(x)$ at time t is not likely to be related to the input rate history x merely by the value $x(t)$ at the time t.

5. POSSIBILITY OF STEADY STATE PRODUCTION FUNCTIONS

Among all the output rate functions $u \in (L_\infty)_+$ consider the subset of these functions which have constant value consistent with axiom $\mathbb{P}$.T.1. Let C denote this subset. For $v \in C$, consider any vector y of input rate functions belonging to $\mathbb{L}(v)$. Let C' denote the subset of $(L_\infty)_+^n$ such that each component of a vector of $(L_\infty)_+^n$ has constant value. Now for $v \in C$, it is not true necessarily that $\mathbb{L}(v) \cap C'$ is not empty. If $\mathbb{L}(v) \cap C'$ is not empty, then a steady state correspondence $u \to L(u)$ is definable in the following way

$$u := ||v|| ,$$

$$L(u) := \left\{x \in R_+^n : x = (||y_1||, \ldots, ||y_n||) , y \in \mathbb{L}(v) \cap C'\right\} ,$$

and inversely for $y \in C'$

$$x := (||y_1||, ||y_2||, \ldots, ||y_n||)$$

$$P(x) := \{u \in R_+ : u = ||v|| , v \in \mathbb{P}(y) \cap C\} = [0, \phi(x)]$$

where

$$\phi(x) = \text{Max}\ \{u \in R_+ : u = ||v|| , v \in \mathbb{P}(y) \cap C\}$$

is the familiar neoclassical production function. This construction will hold, only if $\mathbb{L}(v) \cap C'$ and $\mathbb{P}(y) \cap C$ are not empty. In the case of the input correspondence, this is assured if input histories are strongly disposable, i.e., property $\mathbb{L}$.3SS holds, and under such circumstances the steady state model in effect replaces each input rate

history by the largest value in time, which may be what is being done in many cases to model a production system as a steady state. One so to speak takes care of the largest requirement in time. The same applies to production correspondences with more than one output. Interestingly, the axioms for the dynamic model imply that those for steady state model hold for the above construction.

There is still another way in which a "steady state" production function may be defined, namely as a *long run average* of the output rate history $\mathbb{F}(x)$. It is shown that the function constructed in this way satisfies the properties implied by the axioms for steady state models.

See Reference [1] for details on the existence of steady state production functions.

6. AN EXAMPLE OF A DYNAMIC ACTIVITY ANALYSIS PRODUCTION FUNCTION

Consider again the steady state activity analysis production function described at the end of Section 2 above, for a single final product with all activities except the k^{th} yielding a single distinct intermediate product. Instead of defining the operation of the activities by a vector ξ of constant intensities in time, let $z_\alpha : z_\alpha(t)$, $t \in [0,+\infty)$, $(\alpha = 1,2, \ldots, k)$ denote nonnegative time variable intensity functions. Replace A by $\mathbb{A} = ||a_{ij}(t)||$, B by $\mathbb{B} = ||b_{ii}(t)||$, a matrix of only diagonal coefficients, and $\bar{A}$ by $\bar{\mathbb{A}} = ||\bar{a}_{ij}(t)||$. The dependence of these coefficients upon time is not essential for time substitution phenomena. Learning effects are in this way incorporated.

Given a function u of output history, the correspondence (production function) $u \to \mathbb{L}(u)$ may be constructed in the following way (see Reference [2]).

(1) Order the activities so that all intermediate product transfers required by the i^{th} activity are obtainable from those indexed $1,2, \ldots, (i-1)$.

(2) By some convenient unit of time, consider the time grid $\sigma = (T - t)$ for $t = T, T-1, T-2$, etc., counting time backward from a time T at which the total output is to be available.

(3) Let $\hat{\hat{V}}_\alpha(t)$ denote the cumulative output required from the α^{th} activity. $\hat{\hat{V}}_k(t)$ is the cumulative end product required by t, determined from the given output history u.

(4) The constraints of the system are:

(a) $0 \leq z_\alpha(\sigma) \leq \bar{z}_\alpha(\sigma)$, $(\alpha = 1,2, \ldots, k)$, $(\sigma = 1,2, \ldots)$.

(b)
$$\begin{cases} b_{kk}(1)z_k(1) \leq \hat{\hat{V}}_k(0) - \hat{\hat{V}}_k(1) \\ b_{kk}(\sigma)z_k(\sigma) + \sum_{\tau=1}^{(\sigma-1)} b_{kk}(\tau)z_k(\tau) \leq \hat{\hat{V}}_k(0) - \hat{\hat{V}}_k(\sigma) \quad (\sigma = 2,3, \ldots) \\ b_{\alpha\alpha}(1)z_\alpha(1) \leq \sum_{i=(\alpha+1)}^{k} \bar{a}_{i\alpha}(1)z_i(1) \quad (\alpha = 1,2, \ldots, (k-1)) \\ b_{\alpha\alpha}(\sigma)z_\alpha(\sigma) + \sum_{\tau=1}^{(\sigma-1)} b_{\alpha\alpha}(\tau)z_\alpha(\tau) \leq \sum_{\tau=1}^{\sigma} \sum_{i=(\alpha+1)}^{k} \bar{a}_{i\alpha}(\tau)z_i(\tau) \end{cases}$$

$(\alpha = 1,2, \ldots, (k-1))$, $(\sigma = 2,3, \ldots)$.

(c) $\hat{v}_i^\alpha(\sigma) \geq \hat{v}_i^\alpha(1) - \sum_{\tau=1}^{(\sigma-1)} z_\alpha(\tau)\bar{a}_{\alpha i}(\tau)$ $(i = 1,2, \ldots, (\alpha-1))$.

(d) $(x_{o\alpha}(\sigma))_j \geq a_{\alpha j}(\sigma)z_\alpha(\sigma)$ $(j = 1,2, \ldots, n)$.

Here $\bar{z}_\alpha(\sigma)$ denotes an upper bound to the intensity of the α^{th} node at time σ . The quantity $\hat{v}_i^\alpha(\sigma)$ is the cumulative transfer required at time σ of the output of the i^{th} activity to the α^{th} activity.

(5) The intensity functions $z_\alpha(\sigma)$ are driven by (b) while the constraints (c) and (d) drive intermediate product transfers and exogenous inputs respectively. A GREEDY solution to these contraints is obtained by choosing for each time σ the maximal value of $z_\alpha(\sigma)$ for $\alpha = 1,2, \ldots, k$.

(6) Formulas for the Greedy Solution are:

$$z_k(1) = \text{Min}\left[\frac{\hat{\bar{V}}_k(0) - \hat{\bar{V}}_k(1)}{b_{kk}(1)}\ ,\ \bar{z}_k(1)\right]$$

$$z_k(\sigma) = \text{Min}\left[\frac{\hat{\bar{V}}_k(0) - \hat{\bar{V}}_k(\sigma) - \sum_{\tau=1}^{(\sigma-1)} b_{kk}(\tau)z_k(\tau)}{b_{kk}(\sigma)}\ ,\ \bar{z}_k(\sigma)\right] \quad (\sigma = 2,3,\ \ldots)$$

$$z_\alpha(1) = \text{Min}\left[\frac{\sum_{i=(\alpha+1)}^{k} \bar{a}_{i\alpha}(1)z_i(1)}{b_{\alpha\alpha}(1)}\ ,\ \bar{z}_\alpha(1)\right] \quad (\alpha = 1,2,\ \ldots,\ (k-1))$$

$$z_\alpha(\sigma) = \text{Min}\left[\frac{\sum_{\tau=1}^{\sigma}\ \sum_{i=(\alpha+1)}^{k} \bar{a}_{i\alpha}(\tau)z_i(\tau) - \sum_{\tau=1}^{(\sigma-1)} b_{\alpha\alpha}(\tau)z_\alpha(\tau)}{b_{\alpha\alpha}(\sigma)}\ ,\ \bar{z}_\alpha(\sigma)\right]$$

$$(\alpha = 1,2,\ \ldots,\ (k-1))\ ,\ (\sigma = 2,3,\ \ldots)\ .$$

(7) Steps for constructing the solution are:

(i) From the given cumulative final output schedule $\hat{\bar{V}}_k(\sigma)$, calculate $z_k(\sigma)$ for $\sigma = 1,2,3,\ \ldots$ until no further positive intensities are needed.

(ii) With the results of (i) calculate successively $z_{k-1}(1), z_{k-1}(2),\ \ldots$.

(iii) With the results for $z_k(\sigma)$, $z_{k-1}(\sigma)$, $\sigma = 1,2,\ \ldots$, calculate successively $z_{k-2}(1), z_{k-2}(2),\ \ldots$.

(iv) Continue in the same fashion until $z_1(1), z_1(2),\ \ldots$ is determined.

(v) Use (c) and (d) to find intermediate product transfers and exogenous inputs required.

(vi) THE EVOLUTIONARY CHARACTER OF THIS DYNAMIC SOLUTION IS EVIDENT.

The upper bounds $\bar{z}_\alpha(\sigma)$, $\sigma = 1,2, \ldots$ reflect physical limitations in production. For the production model discussed there are not any alternative processes and all services of equipment and facilities have been preallocated, in time, to the various activities, and reflected by the upper bounds $\bar{z}_\alpha(\sigma)$, $\sigma = 1,2,3, \ldots$. Even so, to obtain the given output schedule $\hat{\hat{V}}_k(\sigma)$, $\sigma = 1,2, \ldots$ there is substitution permitted.

Let $z^*_\alpha(\sigma)$, $\sigma = 1,2, \ldots$, $\alpha = 1,2, \ldots, k$ denote the intensity functions obtained for the greedy solution. Not all of the activities at each point of time need be critical. An activity is critical if the total production time for $\hat{\hat{V}}_k(0)$ is increased if its intensity upper bound is decreased. In fact many values of $\bar{z}_\alpha(\sigma)$, $\alpha = 1,2, \ldots, k$, $\sigma = 1,2, \ldots$ can be reduced without increasing the total production time for $\hat{\hat{V}}_k(0)$. Any combination of such reductions leads to different exogenous input histories, i.e., TIME SUBSTITUTIONS.

It is this property of time substitution which is the overriding aspect of the dynamics of production. Operationally one would like to use exogenous input histories $(x_{o\alpha})_j$ which are as smooth (constant) as possible, or at least have $\sum_{\alpha=1}^{k} (x_{o\alpha})_j$ constant in time. This problem of smooth loading is the production scheduler's burden, and will not be discussed in detail here. The foregoing phenomena of time substitution arises even though the given output function is constant in time.

In so far as the upper bounds $\bar{z}_\alpha(\sigma)$, $\sigma = 1,2, \ldots$ result from shared allocation of fixed resources of real capital available to the production system, there will arise a factor-time substitution possible by altering the shared allocations to the activities.

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