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Upper Bounds for Ratios of Lp Norms On Finite Dimensional Spaces Via Spectral Estimates

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PREFACE

This study was accepted in June 1978 as a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics at the University of Rhode Island.

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The L_{2p} norm, for positive integers p, of a real (complex) polynomial π_n of degree at most n is shown to be the positive 2p-th root of a constrained quadratic (hermitian) form of certain linear operator. The 2p-th root of the spectral radius of this linear operator is shown to give an upper bound for the supremum of the ratio of the L_{2p} norm to the L_2 norm of π_n, where the supremum is taken over arbitrary real (complex) polynomials π_n. For the case of real polynomials, estimates of the spectral radius of the

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linear operator are obtained for all n and p for a general class of weighted L_{2p} norms of π_n. For the case of complex polynomials defined on the unit circle, the spectral radius of the linear operator is determined explicitly. All the foregoing is extended to study the supremum of the ratio of the L_{2p} norm of the k-th derivative of π_n to the L_2 norm of π_n. The underlying technique is not restricted to polynomials, and a generalization of these results to arbitrary finite dimensional function spaces which satisfy a certain Non-negativity Condition is presented. pi sub n

In an entirely different direction, the algebraic properties of the linear operator mentioned above are studied and a Representation Theorem for the L_{2p} norm of polynomials π_n can be expressed as the 2p-th root of a finite linear combination of p-th powers of quadratic (hermitian) forms in the coefficients of π_n. Because of the strictly algebraic proof, the Representation Theorem is generalized to arbitrary finite dimensional function spaces on which an L_{2p} norm is defined.

Finally, an algorithm called the Quadratic Relaxation Algorithm is presented for the numerical computation of polynomials for which the ratio of L_{2p} norm to L_2 norm of π_n is a maximum. Although convergence is not proved, numerical evidence indicates that the Quadratic Relaxation Algorithm is rapidly convergent.

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Chapter I

INTRODUCTION

For extended real numbers $1 \le p \le \infty$, let $\| \cdot \|_p$ denote the *norm* of some classical Banach space L_p of real (complex) functions. Let $\mathcal{P}_n$ be any subspace of L_p of dimension $n + 1$. The starting point of this thesis is the apparently new observation that $(\|\pi_n\|_{2p})^{2p}$, $\pi_n \in \mathcal{P}_n$, $p = 1, 2, 3, \ldots$, is algebraically identical to a constrained quadratic (hermitian) form in the coefficients of π_n. Specifically, if $x \in \mathbb{R}^{n+1}$ $(\mathbb{C}^{n+1})$ is the vector of coefficients of π_n, then there exists a symmetric (hermitian) matrix M of dimension $N \times N$, $N = (n+1)^p$, such that we have the identity

$$(\|\pi_n\|_{2p})^{2p} \equiv (\underbrace{x \otimes \cdots \otimes x}_{p \text{ factors}},\ M\,\underbrace{x \otimes \cdots \otimes x}_{p \text{ factors}}),\quad \pi_n \in \mathcal{P}_n,\quad p = 1, 2, 3, \ldots \tag{1.1}$$

where $(\cdot,\cdot)$ on the right hand side of (1.1) denotes the usual Euclidean inner product on $\mathbb{R}^N$ $(\mathbb{C}^N)$, and $x \otimes \cdots \otimes x \in \mathbb{R}^N$ $(\mathbb{C}^N)$ denotes the Kronecker product (see Chapter II, or Marcus and Minc [24, Section 1.9]) of the vector x with itself p times. The identity (1.1) represents a constrained quadratic (hermitian) form because, in general, not every vector in $\mathbb{R}^N$ $(\mathbb{C}^N)$ can be expressed in the form $x \otimes \cdots \otimes x$ (p factors of x). Thus, (1.1) is the quadratic (hermitian) form of M evaluated only for vectors in $\mathbb{R}^N$ ($\mathbb{C}^N$) of the special form $x \otimes \cdots \otimes x$. Furthermore, the matrix M can always be exhibited explicitly. The explicit form for M and the identity (1.1), together, are developed in several different directions in this thesis.

Chapter II is devoted exclusively to the study of the space P_n of complex polynomials of degree at most n, defined on the unit circle, and equipped with the norms

$$\|\pi_n\|_p = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |\pi_n(e^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}}, & 1 \le p < \infty \\ \max\limits_{0\le\theta\le 2\pi} |\pi_n(e^{i\theta})|, & p = \infty \end{cases} \tag{1.2}$$

The integral in (1.2), like every integral in this thesis, is the Lebesgue integral. (Hence, the norms (1.2) are the norms of the Hardy H_p spaces.) In this space, the matrix M of the identity (1.1) happens to have an especially nice structure which allows us to determine explicitly all its eigenvalues and eigenvectors (see Lemma 2.3). We exploit this fact to show that the ratio

$$R_{n,2p} \equiv \max_{0\neq\pi_n\in P_n} \left\{\frac{\|\pi_n\|_{2p}}{\|\pi_n\|_2}\right\}, \quad p = 1, 2, 3, \ldots \tag{1.3}$$

is, in effect, a constrained Rayleigh quotient which is therefore bounded above by the spectral radius of M. Thus, from Theorem 2.1, we have

$$R_{n,2p} \le \{\lambda_{n,p}\}^{\frac{1}{2p}} \le (n+1)^{\frac{1}{2}-\frac{1}{2p}} \tag{1.4}$$

where the integer $\lambda_{n,p}$ is the spectral radius of M and is also precisely the largest coefficient in the power series expansion of

$$(1 + z + z^2 + \cdots + z^n)^p \tag{1.5}$$

into ascending powers of z. More generally, this technique is applied to the problem

$$R_{n,2p}^{(k)} = \max_{0 \neq \pi_n \in P_n} \left\{ \frac{\|\pi_n^{(k)}\|_{2p}}{\|\pi_n\|_2} \right\}, \quad p = 1, 2, 3, \ldots \tag{1.6}$$

where $\pi_n^{(k)}$, k = 0, 1, 2, ..., denotes the k-th derivative of π_n. An extension to ratios of the form (1.6) is possible because $(\|\pi_n^{(k)}\|_{2p})^{2p}$ is a constrained hermitian form of the type (1.1). As before, the matrix, denoted now by $M^{(k)}$, of this hermitian form is such that its eigenstructure can be written down explicitly. Thus, we get from Theorem 2.7

$$R_{n,2p}^{(k)} \leq k!\{\lambda_{n,p}^{(k)}\}^{\frac{1}{2p}}, \quad k = 0, 1, \ldots, n \tag{1.7}$$

where the integer $(k!)^{2p}\lambda_{n,p}^{(k)}$ is the spectral radius of $M^{(k)}$, and $\lambda_{n,p}^{(k)}$ is precisely the largest coefficient in the power series expansion of

$$\left\{ \sum_{\ell=k}^{n} \binom{\ell}{k}^2 z^{\ell-k} \right\}^p, \quad k = 0, 1, \ldots, n \tag{1.8}$$

into ascending powers of z.

In Chapter III, we develop some general consequences of identities of the kind (1.1). In this chapter, P_n is an n+1 dimensional subspace of $L_{2p}^{\omega}[a,b]$, the space of measurable real (complex) functions defined on the interval (a,b), $-\infty \leq a < b \leq +\infty$, and equipped with the norms

$$\|\pi_n\|_p^{\omega} = \begin{cases} \left(\int_a^b |\pi_n(x)|^p \omega(x)dx\right)^{\frac{1}{p}}, & 1 \le p < \infty \\ \operatorname*{ess\,sup}_{a<x<b} |\pi_n(x)|, & p = \infty \end{cases} \tag{1.9}$$

where the real measurable function $\omega(x) > 0$ almost everywhere on (a,b), and satisfies

$$0 < \int_a^b \omega(x)dx < +\infty \tag{1.10}$$

Let $\{h_0, h_1, \ldots\}$ be an orthonormal basis for $L_2^{\omega}[a,b]$ with $\{h_0, h_1, \ldots, h_n\}$ an orthonormal basis of P_n. Then it is shown that if

$$\int_a^b h_j(x)h_k(x)\overline{h_\ell(x)}\omega(x)dx \ge 0, \quad j,k,\ell = 0, 1, 2, \ldots \tag{1.11}$$

then

$$\max_{0\neq\pi_n\in P_n} \left\{\frac{\|\pi_n\|_{2p}^{\omega}}{\|\pi_n\|_2^{\omega}}\right\} \le \max_{0\le k\le n} \sqrt{\|h_k S_n\|_p^{\omega}}, \quad p = 1, 2, 3, \ldots \tag{1.12}$$

where

$$S_n(x) = h_0(x) + h_1(x) + \cdots + h_n(x) \tag{1.13}$$

It might seem that (1.11) is a very restrictive condition on the basis; however, roughly half the Jacobi polynomials, all the generalized Laguerre polynomials (properly normalized), and the Hermite polynomials satisfy the condition (1.11). Furthermore, if $\phi(x)$ is defined on (c,d) in a manner analogous to the definition of $\omega(x)$ on (a,b), and if

$$\int_c^d h_j(x)h_k(x)\overline{h_\ell(x)}\phi(x)dx \geq 0, \quad j,k,\ell = 0, 1, 2, \ldots \tag{1.14}$$

then

$$\max_{0\neq\pi_n\in P_n}\left\{\frac{\|\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}}\right\} \leq \max_{0\leq k\leq n}\sqrt{\|h_k S_n\|_p^{\phi}}\ , \ p = 1, 2, 3, \ldots \tag{1.15}$$

where $S_n(x)$ is given by (1.13). Still further, if D is any linear operator on P_n (e.g., a derivative of some order) such that

$$\int_c^d Dh_j(x)Dh_k(x)\overline{Dh_\ell(x)}\phi(x)dx \geq 0, \quad j,k,\ell = 0, 1, 2, \ldots \tag{1.16}$$

then

$$\max_{0\neq\pi_n\in P_n}\left\{\frac{\|D\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}}\right\} \leq \max_{0\leq k\leq n}\sqrt{\|Dh_k \cdot DS_n\|_p^{\phi}}$$

$$p = 1, 2, 3, \ldots \tag{1.17}$$

where $S_n(x)$ is given by (1.13). Theorem 3.5 establishes the bound (1.17) under a weaker hypothesis than (1.16) which we have called the Nonnegativity Condition. See (3.20). It is not hard to see that (1.16) implies that the Nonnegativity Condition holds for the functions $\{Dh_0, Dh_1, \ldots, Dh_n\}$. The effect of the Nonnegativity Condition on the appropriate quadratic (hermitian) form of the kind (1.1) is that it forces the matrix of this form to have only nonnegative entries.

Chapter IV develops some of the consequences of the general results of Chapter III for the space of real (or complex) polynomials P_n of degree at most n defined on various real intervals. We are most successful, however, on the interval (-1,+1). For example, adopting the notation

$$\|\pi_n\|_p^{(\alpha,\beta)} = \left(\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}|\pi_n(x)|^p dx\right)^{\frac{1}{p}}, \quad p \geq 1 \tag{1.18}$$

and defining

$$T_{n,p}^{(\alpha,\beta)} = \max_{0\neq\pi_n\in P_n}\left\{\frac{\|\pi_n\|_p^{(\alpha,\beta)}}{\|\pi_n\|_2^{(\alpha,\beta)}}\right\}, \quad p \geq 1 \tag{1.19}$$

we show that, for $\alpha \geq \beta > -1$ and $\alpha \geq 0$,

$$T_{n,2p}^{(\alpha,\beta)} < A_0\left(n + \frac{\alpha+\beta+3}{2}\right)^{(1+\alpha)(1-\frac{1}{p})}, \quad p = 1,\ 2,\ 3,\ \ldots \tag{1.20}$$

and

$$T_{n,\infty}^{(\alpha,\beta)} < A_0\left(n + \frac{\alpha+\beta+3}{2}\right)^{1+\alpha} \tag{1.21}$$

where the constant A_0 is independent of both n and p. It is clear that the choice of weight function in (1.19) affects the exponent of n in (1.20) and (1.21). Therefore, it is reasonable to expect the use of different weight functions in numerator and denominator to have an effect on the exponent of n. For example, for $\alpha \geq 0$ define

$$U_{n,p}^{(\alpha)} = \max_{0\neq\pi_n\in P_n}\left\{\frac{\|\pi_n'\|_p^{(\alpha+1,\alpha+1)}}{\|\pi_n\|_2^{(\alpha,\alpha)}}\right\}, \quad p \geq 1 \tag{1.22}$$

and

$$V_{n,p}^{(\alpha)} = \max_{0\neq\pi_n\in P_n}\left\{\frac{\|\pi_n'\|_p^{(\alpha,\alpha)}}{\|\pi_n\|_2^{(\alpha,\alpha)}}\right\}, \quad p \geq 1 \tag{1.23}$$

where the prime denotes differentiation. The weight function in the numerator of (1.22) differs from the weight function in the numerator of (1.23) by a factor of $(1-x^2)$. Since $0 < 1-x^2 \le 1$ on $(-1,+1)$, we must have

$$U_{n,p}^{(\alpha)} \le V_{n,p}^{(\alpha)} \quad , \qquad p \ge 1$$

More to the point, however, we show that

$$U_{n,2p}^{(\alpha)} < A_1 (n+2\alpha+2)^{2+(1+\alpha)(1-\frac{1}{p})-\frac{1}{p}} \;, \; p = 2,3,4,\ldots \qquad (1.24)$$

$$V_{n,2p}^{(\alpha)} < A_2 (n+2\alpha+2)^{2+(1+\alpha)(1-\frac{1}{p})} \qquad , \; p = 2,3,4,\ldots \qquad (1.25)$$

where the constants A_1 and A_2 are independent of both n and p. Note that as $p \to \infty$, both (1.24) and (1.25) give the same exponent for n, which is to be expected considering the definitions (1.22) and (1.23). We emphasize that all these results are, in essence, corollaries of (1.17), that is, Theorem 3.5.

Chapter V studies the operator M defined via the identity (1.1) and leads to a new representation theorem for a special class of homogeneous polynomials. Following Hardy, Littlewood, and Polya [18, Appendix I], any homogeneous polynomial with real coefficients is called a "form." A form $F(a_0, a_1, \ldots, a_n)$ is said to be strictly positive if and only if $F(a_0, a_1, \ldots, a_n) > 0$ unless $a_0 = a_1 = \ldots = a_n = 0$. It can be shown [18] that every strictly positive form F can be written

$$F = \frac{\sum_i M_i^2}{\sum_j N_j^2} \qquad (1.26)$$

where M_i and N_j are suitably chosen forms, and each sum in (1.26) has a finite number of terms. Now, for integer $p \geq 1$, define $G_p : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$G_p(a_0, a_1, \ldots, a_n) = (\|a_0 + a_1 x + \cdots + a_n x^n\|_{2p}^{\omega})^{2p} \qquad (1.27)$$

where $\|\cdot\|_{2p}^{\omega}$ is given by (1.9) for some fixed $\omega(x)$. Clearly, G_p is a strictly positive homogeneous polynomial of degree 2p in the variables $a_0, a_1, \ldots, a_n$ and so has the representation (1.26). We prove that G_p also has the representation

$$G = \begin{cases} \sum_t (Q_t)^p & \text{if } p \text{ odd} \\ \sum_r (Q_r)^p - \sum_s (Q_s)^p & \text{if } p \text{ even} \end{cases} \qquad (1.28)$$

where Q_r, Q_s, and Q_t are suitably chosen quadratic forms, and each summation in (1.28) being finite. We also show that every quadratic form in (1.28) can be taken to have full rank n+1. Theorem 5.8 proves these assertions. (Its proof is easily modified to give a natural extension to complex variables $a_0, a_1, \ldots, a_n$.) We remark that Hilbert's 17-th problem (see [28]) concerns arbitrary, i.e., not necessarily homogeneous, polynomials F satisfying only $F(a_0, a_1, \ldots, a_n) \geq 0$ for all real $a_0, a_1, \ldots, a_n$.

Finally, in Chapter VI, we present an algorithm for the computation of $\pi_n^* \in P_n$ such that

$$\frac{\|\pi_n^*\|_{2p}}{\|\pi_n^*\|_2} = \max_{0 \neq \pi_n \in P_n} \left\{ \frac{\|\pi_n\|_{2p}}{\|\pi_n\|_2} \right\}, \quad p = 1, 2, 3, \ldots \qquad (1.29)$$

(Note that π_n^* depends on p.) The space P_n in (1.29) can be any finite dimensional space of functions defined on a measure space, provided every element in P_n has a finite L_{2p} norm. The algorithm, called the Quadratic Relaxation Algorithm (QRA), is attractive because it is as easy to apply to the general problem (1.29) as it is to apply to ratios like (1.3), or (1.12), or (1.17). Unfortunately, convergence of the QRA is at present unproved. Although the QRA is shown to have a convergent subsequence, the limit of this convergent subsequence is not proved to be a solution of (1.29). The QRA is easy to program on a computer, appears to be numerically stable, and has (so far) always converged rapidly to what appears to be the solution of (1.29).

Most of the topics discussed in this thesis appear to be little studied in the literature. The pivotal identity (1.1), after a literature search and personal correspondence with both algebraists and analysts, does seem to be original to this thesis. Its subsequent application to bounding ratios of norms and to representations of norms is, therefore, original as well. Although the particular applications can be studied by other methods, this does not appear to have been done in the literature, except possibly in special cases.

All of Chapters II, III, V, and VI seem to be original, except of course where otherwise noted. Certain special cases of topics studied in Chapter IV have been studied, however, and we now proceed to summarize them. Since all these papers restrict themselves to algebraic polynomials, we let P_n denote the space of real polynomials of degree at most n throughout the following discussion of the literature.

Amir and Ziegler [2] study the problem

$$\max_{0 \neq \pi_n \in P_n} \frac{\|\pi_n\|_\infty}{\|\pi_n\|_q}, \qquad q \geq 1 \tag{1.30}$$

where the norms are defined as in (1.9) with $\omega(x) = 1$ on $(a,b) = (0,1)$. They give a characterization theorem for extremal polynomials, π_n^*, of (1.30). (They also give a characterization theorem when the maximum in (1.30) is taken over various subclasses of P_n.) This characterization result is used to show that for $q = 1$ or $q = 2$, the zeros of the extremal π_n^* and the zeros of the extremal π_{n+1}^* interlace.

Gilbert and Slepian [15] study the problem

$$\lambda_0^{(n)} \equiv \max_{0 \neq \pi_n \in P_n} \frac{\int_\gamma^\delta |\pi_n(x)|^2 dx}{\int_\alpha^\beta |\pi_n(x)|^2 dx} \tag{1.31}$$

with emphasis on asymptotic results for n large. They employ asymptotic methods for an equivalent differential equation eigenvalue problem to obtain results for two cases of (1.31). Specifically, they obtain for the case $(\alpha,\beta) = (-1,+1)$ and $(\gamma,\delta) = (\tilde{a},a)$ with $1 \leq \tilde{a} < a$, the asymptotic expansion

$$\lambda_0^{(n)} = \frac{(a + \sqrt{a^2-1})^{2n+2}}{8\pi n\sqrt{a^2-1}}\left[1 + O\left(\frac{1}{n}\right)\right], \quad n \to \infty \tag{1.32}$$

In the other case, $(\alpha,\beta) = (-1,+1)$ and $(\gamma,\delta) = (-a,a)$ with $0 < a < 1$, they obtain

$$1 - \lambda_0^{(n)} = \frac{4\sqrt{\pi a n}}{1 + a}\left(\frac{1-a}{1+a}\right)^n\left[1 + O\left(\frac{1}{n}\right)\right], \quad n \to \infty \tag{1.33}$$

In both (1.32) and (1.33), the notation $O(\frac{1}{n})$, $n \to \infty$, is used in place of some function, say $f(n)$, which has the property that there exists a constant B, independent of n, such that for all n we have $f(n) \leq B/n$. See [7, Section 1.2].

Turan [32] and Schmidt [35] study problems similar to (1.31), but for infinite intervals with weight functions e^{-x} and e^{-x^2}, respectively. See Chapter IV, equations (4.82) and (4.91), for details.

Handelsman and Lew [17], as well as Bleistein and Handelsman [7], study the rate at which $\|g\|_p$ converges to $\|g\|_\infty$ as $p \to \infty$, where the norms are defined by (1.9) with $\omega(x) = 1$ and (a,b) arbitrary. In [7, Problem 5.9], it is shown that if g has 2k+1 continuous derivatives on [a,b], and if g attains a unique maximum at the point $\gamma \in (a,b)$, and if $g^{(j)}(\gamma) = 0$, $j = 1, 2, \ldots, 2k-1$, and if $g^{(2k)}(\gamma) < 0$, then

$$\|g\|_p = \|g\|_\infty\left[1 - \frac{\log p}{kp} + o\left(\frac{\log p}{p}\right)\right], \quad p \to \infty \tag{1.34}$$

where the notation $o(p^{-1} \log p)$, $p \to \infty$, is used in place of some function, say $f(p)$, which has the property that for any $\varepsilon > 0$, there exists N such that for all $p \geq N$ we have $f(p) \leq \varepsilon p^{-1} \log p$. Dividing (1.34) by $\|g\|_2$ and replacing g by $\pi_n \neq 0$ gives

$$\frac{\|\pi_n\|_p}{\|\pi_n\|_2} = \frac{\|\pi_n\|_\infty}{\|\pi_n\|_2}\left[1 - \frac{\log p}{kp} + o\left(\frac{\log p}{p}\right)\right], p \to \infty \tag{1.35}$$

provided only that π_n attains a unique maximum interior to $[a,b]$. Now take $[a,b] = [-1,+1]$, and let π_n^* be an extremal polynomial for $T_{n,\infty}^{(0,0)}$ defined in (1.19). If π_n^* attains a unique maximum at $\gamma \in (-1,+1)$ with the property that $(\pi_n^*)^{(j)}(\gamma) = 0$, $j = 1, 2, \ldots, 2k-1$, then

$$T_{n,p}^{(0,0)} > T_{n,\infty}^{(0,0)}\left[1 - \frac{\log p}{kp} + o\left(\frac{\log p}{p}\right)\right], \ p \to \infty \tag{1.36}$$

Jackson [36] contains many interesting inequalities, only two of which seem related to this thesis. Jackson proves that if

$$\left\{\int_a^b (b-x)^{-\frac{1}{2}} |\pi_n(x)|^p \, dx\right\}^{\frac{1}{p}} = 1, \qquad p > 0 \tag{1.37}$$

then

$$|\pi_n(x)| \leq \frac{Cn^{\frac{1}{p}}}{(x-a)^{N/2}}, \qquad a < x \leq b \tag{1.38}$$

where C is a constant independent of n and x, and N is the smallest integer greater than or equal to $1/p$. (A similar result is derived by replacing x, a, b by -x, -b, -a.) Another inequality gives an upper bound in n for the ratio of the sup norm to the weighted L_2 norm of π_n. Let $\omega(x)$ be a nonnegative integrable function on (a,b) such that

$$\int_a^b \frac{(b-x)^{-(r+1)/2}}{[\omega(x)]^r}\,dx < \infty \tag{1.39}$$

for some $r > 0$. Then Jackson [36, Theorem 12] proves that

$$\frac{|\pi_n(x)|}{\|\pi_n\|_2^{\omega}} \le O\left(n^{\frac{1}{2}+\frac{1}{2r}}\right), \qquad n \to \infty \tag{1.40}$$

throughout any interval $a+\delta \le x \le b$, $\delta > 0$.

Černyh [9] studies the problem

$$\max_{0\neq\pi_n\in P_n}\left\{\frac{\|\pi_n^{(k)}\|_p^{\chi_1}}{\|\pi_n\|_2^{\chi_2}}\right\}, \qquad 1 \le p \le \infty \tag{1.41}$$

where $\pi_n^{(k)}$ denotes the k-th derivative of π_n, $k = 0, 1, \ldots$, and χ_1 and χ_2 are characteristic functions of the intervals (α,β) and $(-1,+1)$, respectively, and (α,β) is not a subset of $(-1,+1)$. Černyh proves [9, Corollary 3 of Theorem 1] that if $\alpha \neq -\beta$, then, for $1 \le p \le \infty$,

$$\max_{0\neq\pi_n\in P_n}\left\{\frac{\|\pi_n^{(k)}\|_p^{\chi_1}}{\|\pi_n\|_2^{\chi_2}}\right\} = n^{k-\frac{1}{p}}\, g^n(\eta)\left[H_{p,2}^{(k)} + o(1)\right], \qquad n \to \infty \tag{1.42}$$

where $\eta = \max\{|\alpha|,\beta\}$, $g(\eta) = \eta + \sqrt{\eta^2-1}$, and

$$H_{p,2}^{(k)} = \frac{g^{\frac{3}{2}}(\eta)}{2\sqrt{\pi}\, p^{1/p}\sqrt{g^2(\eta) - 1}} (\eta^2 - 1)^{\frac{1}{2p} - \frac{2k+1}{4}} \tag{1.43}$$

Černyh's results are not directly comparable to any derived in this thesis.

Certain analogous problems for the trigonometric polynomials have also been studied. See, for example, Jackson [36], Bari [37], and Videnskii [38].

For future reference, we record the following bound which, although apparently new, is easily derived. For even integer p, the bound (1.45) will be shown to be identically the 2p-th root of the trace of a matrix E of an identity analogous to (1.1). (See Lemma 3.9, Theorem 3.4, and Corollary 3.11.)

<u>Theorem 1.1</u> Let $p \geq 1$ be a real number. Let $-\infty \leq a < b \leq +\infty$ and $-\infty \leq c < d \leq +\infty$. Let P_n be a subspace of $L_2^{\omega}[a,b] \cap L_{2p}^{\phi}[c,d]$ with a basis $\{h_0, h_1, \ldots, h_n\}$ which is orthonormal with respect to the inner product

$$(f,g)_{\omega} = \int_a^b f(x)\, \overline{g(x)}\, \omega(x)dx \tag{1.44}$$

where the real measurable functions $\omega(x) > 0$ and $\phi(x) \geq 0$ almost everywhere on (a,b) and (c,d), respectively, and

$$0 < \int_a^b \omega(x)dx < +\infty$$

$$0 < \int_c^d \phi(x)dx < +\infty$$

Let $D:\mathcal{P}_n \to L_{2p}^{\phi}[c,d]$ be a linear transformation.
Then,

$$\max_{0\neq\pi_n\in\mathcal{P}_n}\left\{\frac{\|D\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}}\right\}\leq \left\{\int_c^d\left[K_n^{(D)}(x)\right]^p\phi(x)\,dx\right\}^{\frac{1}{2p}} \tag{1.45}$$

where

$$K_n^{(D)}(x) = \sum_{k=0}^{n}|Dh_k(x)|^2 \tag{1.46}$$

Proof Without loss of generality, we restrict our attention to those π_n such that $\|\pi_n\|_2^{\omega} = 1$. Let

$$\pi_n(x) = \sum_{k=0}^{n} a_k h_k(x)$$

Then, by the orthonormality of $\{h_k\}$,

$$\sum_{k=0}^{n}|a_k|^2 = 1$$

Thus, by the Cauchy-Schwarz inequality,

$$|D\pi_n(x)|^2 \leq \sum_{k=0}^{n}|a_k|^2 \sum_{k=0}^{n}|Dh_k(x)|^2$$

$$= K_n^{(D)}(x) \tag{1.47}$$

Raising both sides to the p-th power, multiplying by ϕ, and integrating completes the proof.

Finally, we remark that every maximum taken of ratios similar to (1.6), or (1.19), or (1.45), is attained. Since $S = \{\pi_n \in \mathcal{P}_n : \|\pi_n\|_2^{\omega} = 1\}$ is a closed and bounded subset of the finite dimensional normed linear space $\mathcal{P}_n$, S is compact.

Since f : $P_n \to R$ defined by $f(\pi_n) = \|D\pi_n\|_{2p}^{\phi}$ is continuous on S, f attains its maximum on S.

Chapter II

COMPLEX POLYNOMIALS DEFINED ON THE UNIT CIRCLE

A. Preliminaries

In this chapter attention is confined to P_n, the collection of all polynomials with complex coefficients of degree at most n, equipped with the norms

$$\|\pi_n\|_q = \left\{\frac{1}{2\pi}\int_0^{2\pi} |\pi_n(e^{i\theta})|^q \, d\theta\right\}^{\frac{1}{q}}, \qquad q \geq 1 \tag{2.1}$$

$$\|\pi_n\|_\infty = \max_{0\leq\theta\leq 2\pi} |\pi_n(e^{i\theta})| \tag{2.2}$$

where $\pi_n \in P_n$. The norms (2.1) and (2.2) are the norms of Hardy H_p spaces. If $q = 2p \geq 2$, then (2.1) can be written, by setting $z = e^{i\theta}$ and letting C be the unit circle, as

$$\|\pi_n\|_{2p} = \left\{\frac{1}{2\pi i}\int_C \left[\pi_n(z)\overline{\pi_n(z)}\right]^p \frac{dz}{z}\right\}^{\frac{1}{2p}}, \quad p \geq 1 \tag{2.3}$$

An inner product is defined for all f and g in P_n by

$$(f,g) = \frac{1}{2\pi i}\int_C f(z)\overline{g(z)}\,\frac{dz}{z} \tag{2.4}$$

Theorem 1.1 is easily extended to complex polynomials defined on the unit circle. Such an extension gives

$$\frac{\|\pi_n\|_{2p}}{\|\pi_n\|_2} \leq \sqrt{\|K_n\|_p}\ , \qquad p \geq 1 \tag{2.5}$$

where

$$K_n(z) = \sum_{k=0}^{n} |h_k(z)|^2$$

and $\{h_0(z), h_1(z), \ldots, h_n(z)\}$ form an orthonormal basis for P_n with respect to the inner product (2.4). Since $\{1, z, z^2, \ldots, z^n\}$ form an orthonormal basis here, $K_n(z) \equiv n + 1$ and (2.5) gives

$$\frac{\|\pi_n\|_{2p}}{\|\pi_n\|_2} \leq \sqrt{n+1}, \quad p \geq 1 \tag{2.6}$$

Note that (2.6) is valid for all real $p \geq 1$.

The bound (2.6) can be improved upon considerably in the case of even integer norms. Throughout the rest of this chapter we restrict attention to the norms (2.2) and (2.3) with $p = 1, 2, 3, \ldots$. Before getting to the general result (Theorem 2.1), we first examine the special case $n = 2$ and $p = 2$. Let $\pi_2(z) = a + bz + cz^2$. From (2.3) we have

$$\begin{aligned}
\left(\|\pi_2\|_4\right)^4 &= \frac{1}{2\pi i}\int_C \left[\pi_n(z)\overline{\pi_n(z)}\right]^2 \frac{dz}{z} \\
&= \frac{1}{2\pi i}\int_C \left[\left(a + bz + cz^2\right)\left(\bar{a} + \frac{\bar{b}}{z} + \frac{\bar{c}}{z^2}\right)\right]^2 \frac{dz}{z} \\
&= \frac{1}{2\pi i}\int_C \frac{1}{z^5}\Big[a\bar{c} + (a\bar{b} + b\bar{c})z \\
&\qquad + (|a|^2 + |b|^2 + |c|^2)z^2 \\
&\qquad + (\bar{a}b + \bar{b}c)z^3 + \bar{a}cz^4\Big]^2 dz
\end{aligned}$$

since $\bar{z} = 1/z$ for $z \in C$. By the residue theorem of elementary complex analysis, the value of the last given integral is clearly the coefficient of z^4 after the square in the integrand has been taken, so that

$$\|\pi_2\|_4^4 = (|a|^2 + |b|^2 + |c|^2)^2$$

$$+ 2(\bar{a}b + \bar{b}c)(a\bar{b} + b\bar{c}) + 2|ac|^2$$

$$= |a|^4 + |b|^4 + |c|^4 + 4|ab|^2 + 4|bc|^2 + 4|ac|^2$$

$$+ 2(\bar{a}\bar{c}b^2 + ac\bar{b}^2) \tag{2.7}$$

The last expression can be written as the hermitian form of a certain matrix evaluated at a certain vector. Explicitly,

$$\|\pi_2\|_4^4 = \begin{bmatrix} \overline{aa} \\ \overline{ab} \\ \overline{ac} \\ \overline{ba} \\ \overline{bb} \\ \overline{bc} \\ \overline{ca} \\ \overline{cb} \\ \overline{cc} \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} aa \\ ab \\ ac \\ ba \\ bb \\ bc \\ ca \\ cb \\ cc \end{bmatrix}$$

$$\equiv \bar{u}^T M u \tag{2.8}$$

with the vector u and matrix M having the obvious definitions.

What is the eigenstructure of M? A tedious computation would show that

$$\det(M - \lambda I) = -\lambda^4(\lambda - 1)^2(\lambda - 2)^2(\lambda - 3) \tag{2.9}$$

However, this computation can be avoided because all the eigenvalues and a complete set of eigenvectors can be found for M by means of a very simple observation: the rows of M are orthogonal eigenvectors of M and have corresponding eigenvalues equal to the row sums. We now verify that the eigenvalues must be 0, 1, 2, and 3 with multiplicities given by (2.9), where 0 is included because M clearly does not have full row rank. The eigenvalue $\lambda = 3$ has at least one eigenvector, namely,

$$v_0 = \langle 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \rangle^T \tag{2.10.1}$$

The eigenvalue $\lambda = 2$ has at least two eigenvectors,

$$v_1 = \langle 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \rangle^T \tag{2.10.2}$$

$$v_2 = \langle 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \rangle^T \tag{2.10.3}$$

and the eigenvalue $\lambda = 1$ has at least two also

$$v_3 = \langle 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \rangle^T \tag{2.10.4}$$

$$v_4 = \langle 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \rangle^T \tag{2.10.5}$$

Finally, the eigenvalue $\lambda = 0$ has four easily found linearly independent eigenvectors,

$$v_5 = \langle 0 \quad 0 \quad 1 \quad 0 \quad -2 \quad 0 \quad 1 \quad 0 \quad 0 \rangle^T \tag{2.10.6}$$

$$v_6 = \langle 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad -2 \quad 0 \quad 0 \rangle^T \tag{2.10.7}$$

and

$$v_7 = \langle 0 \quad 1 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \rangle^T \tag{2.10.8}$$

$$v_8 = \langle 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0 \rangle^T \tag{2.10.9}$$

Since nine linearly independent eigenvectors have been found, we have them all.

How does knowledge of the eigenstructure of M help to improve the bound (2.6)? Since

$$\|\pi_2\|_2^4 = (|a|^2 + |b|^2 + |c|^2)^2$$

$$= |a|^4 + |b|^4 + |c|^4 + 2|ab|^2 + 2|ac|^2 + 2|bc|^2$$

$$= \begin{bmatrix} \overline{aa} \\ \overline{ab} \\ \overline{ac} \\ \overline{ba} \\ \overline{bb} \\ \overline{bc} \\ \overline{ca} \\ \overline{cb} \\ \overline{cc} \end{bmatrix}^T \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} aa \\ ab \\ ac \\ ba \\ bb \\ bc \\ ca \\ cb \\ cc \end{bmatrix}$$

$$= \bar{u}^T u, \tag{2.11}$$

we can write from (2.8) and (2.11),

$$\frac{\|\pi_2\|_4}{\|\pi_2\|_2} = \left\{\frac{\|\pi_n\|_4^4}{\|\pi_n\|_2^4}\right\}^{\frac{1}{4}}$$

$$= \left\{\frac{\bar{u}^T M u}{\bar{u}^T u}\right\}^{\frac{1}{4}}$$

$$\le \max_{v \neq 0} \left\{\frac{\bar{v}^T M v}{\bar{v}^T v}\right\}^{\frac{1}{4}} \tag{2.12}$$

$$= \left\{\lambda_{max}\right\}^{\frac{1}{4}} = 3^{\frac{1}{4}} \tag{2.13}$$

where v is an arbitrary nonzero vector in $\mathbf{C}^9$ and λ_{max} is the largest eigenvalue of M. The key equality (2.13) follows from the well-known fact (see, e.g., Gantmacher [13]) that the ratio of hermitian forms in (2.12) is bounded above by the largest eigenvalue of M and that this bound is attained if and only if v is an eigenvector of M corresponding to the largest eigenvalue. Hence

$(\bar{v}^T M v/\bar{v}^T v) = \lambda_{max}$ if and only if $v = cv_0$, where v_0 is given in (2.10.1) and $c \neq 0$ is an arbitrary constant. However,

$$v_0 \neq \langle aa\ ab\ ac\ ba\ bb\ bc\ ca\ cb\ cc\rangle^T$$

for any choice of a, b, and c, so that inequality (2.12) is a strict inequality. Therefore,

$$\frac{\|\pi_2\|_4}{\|\pi_2\|_2} < 3^{\frac{1}{4}} \tag{2.14}$$

This estimate is considerably better than (2.6), which gives $3^{1/2}$.

It will be shown that the hermitian forms (2.8) and (2.11) can be generalized, that the eigenstructure of these hermitian forms can be written down explicitly, and that the inequality (2.14) can be extended to even integer norms of polynomials in π_n. Before doing this, however, some definitions are in order. These definitions are simply notational devices. At this point, nothing deeper than notational formalism is intended because only the notation is required for the proofs in this chapter.

Let $p \geq 1$ and $n \geq 0$ be integers. Let the p indices $\alpha_1, \ldots, \alpha_p$ each run independently over the common index set $\{0, 1, \ldots, n\}$. Define

$$\Gamma = \Gamma_{n,p} = \{\alpha | \alpha = (\alpha_1, \ldots, \alpha_p)\} \tag{2.15}$$

so that Γ has $(n+1)^p$ elements and each element is a p-tuple of nonnegative integers. We will always assume that Γ is lexicographically ordered; that is, if $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$ and $\beta = (\beta_1, \ldots, \beta_p) \in \Gamma$, then $\alpha < \beta$ if and only if there exists an integer t, $1 \le t \le p$, such that

$$\alpha_1 = \beta_1, \ldots, \alpha_{t-1} = \beta_{t-1}, \quad \alpha_t < \beta_t \tag{2.16}$$

It is easy to see that (2.16) defines a linear ordering on Γ with $(0, 0, \ldots, 0)$ and $(n, n, \ldots, n)$ as the first and last elements, respectively. (Lexicographic ordering is not novel. See, e.g., Marcus and Minc [24, p 10].)

Let $x = \langle x_0\ x_1\ \cdots\ x_n \rangle^T \in \mathbb{C}^{n+1}$. Now, for each $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$, we can compute the product

$$x_{\alpha_1} \cdots x_{\alpha_p} \in \mathbb{C} \tag{2.17}$$

and this number is uniquely defined for each $\alpha \in \Gamma$. The collection of all $(n+1)^p$ products of the form (2.17), linearly ordered by the linear ordering in Γ, defines the Kronecker product (see, e.g., Marcus and Minc [24, Section 1.9]) $x \otimes \cdots \otimes x$ of the vector x with itself p times. Explicitly, the Kronecker product $x \otimes \cdots \otimes x$ (p factors of x) is defined by

$$\underbrace{x \otimes \cdots \otimes x}_{p\ \text{factors}} = \langle x_{\alpha_1} \cdots x_{\alpha_p} \rangle_{\alpha \in \Gamma} \in \mathbb{C}^{(n+1)^p} \tag{2.18}$$

where $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$. For example, if $x = \langle x_0\ x_1\ x_2 \rangle^T \in \mathbb{C}^3$, then

$$x \otimes x = \begin{bmatrix} x_0x_0 \\ x_0x_1 \\ x_0x_2 \\ x_1x_0 \\ x_1x_1 \\ x_1x_2 \\ x_2x_0 \\ x_2x_1 \\ x_2x_2 \end{bmatrix} \in \mathbb{C}^9 \tag{2.19}$$

For $\alpha \in \Gamma$, we may speak of the α-th component of the Kronecker product $x \otimes \cdots \otimes x$ (p factors of x). For example, the (1,2) component of (2.19) is x_1x_2 while the (0,1) component is x_0x_1. Also, if

$$u = \langle u_1\ u_2\ \cdots\ u_9\rangle^T \in \mathbf{C}^9,$$

we can regard, for example, u_6 as the (1,2) component of u and u_2 as the (0,1) component of u using the lexicographical ordering on Γ. Thus, any vector u of dimension $(n + 1)^p$ can be written in the general form

$$u = \langle u_\alpha\rangle, \quad \alpha \in \Gamma \tag{2.20}$$

Similarly, any complex matrix M of dimension $(n + 1)^p \times (n + 1)^p$ can be expressed in the general form

$$M = [m_{\alpha,\beta}], \qquad \alpha,\beta \in \Gamma \tag{2.21}$$

where $m_{\alpha,\beta}$ is the entry in the row corresponding to $\alpha \in \Gamma$ and the column corresponding to $\beta \in \Gamma$, and where the rows and columns of M are ordered lexicographically. For n = 1 and p = 2, the general form is

$$M = \begin{bmatrix} m_{(00),(00)} & m_{(00),(01)} & m_{(00),(10)} & m_{(00),(11)} \\ m_{(01),(00)} & m_{(01),(01)} & m_{(01),(10)} & m_{(01),(11)} \\ m_{(10),(00)} & m_{(10),(01)} & m_{(10),(10)} & m_{(10),(11)} \\ m_{(11),(00)} & m_{(11),(01)} & m_{(11),(10)} & m_{(11),(11)} \end{bmatrix} \tag{2.22}$$

Finally, the Kronecker product of an $r \times s$ matrix A with the $p \times q$ matrix B is denoted by $A \otimes B$ and is defined (in partitioned form) to be the $sp \times rq$ matrix

$$A \otimes B \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1r}B \\ a_{21}B & a_{22}B & \cdots & a_{2r}B \\ \vdots & \vdots & & \vdots \\ a_{s1}B & a_{s2}B & \cdots & a_{sr}B \end{bmatrix}$$

(See, e.g., [24, Section 1.9].) Proceeding inductively by defining $A \otimes B \otimes C = A \otimes (B \otimes C)$, one can define the Kronecker product of any number of matrices each of arbitrary dimension. Note that the earlier definition (2.18) of the Kronecker product $x \otimes \cdots \otimes x$ (p factors of x) is merely a special case and, in fact, is now extended in a meaningful manner to Kronecker products of the form $x \otimes y \otimes \cdots \otimes z$, where $x, y, \cdots, z$ all lie in $\mathbf{C}^{n+1}$. Many elementary algebraic properties of Kronecker products are known, but will not be given here. See [24], for example.

B. The Underlying Matrix and Its Eigenstructure

With this notation, we now generalize the hermitian form (2.8).

Lemma 2.1 Let $\pi_n(z) = a_0 + a_1 z + \cdots + a_n z^n \in P_n$, and let $x = \langle a_0\ a_1\ \cdots\ a_n \rangle^T \in \mathbf{C}^{n+1}$. Then for positive integer p,

$$\|\pi_n\|_{2p} \equiv \{\bar{u}^T M_{n,p} u\}^{\frac{1}{2p}} \tag{2.23}$$

where $u = x \otimes \cdots \otimes x \in \mathbf{C}^{(n+1)^p}$, the matrix $M_{n,p} = [m_{\alpha,\beta}]$ of dimension $(n+1)^p \times (n+1)^p$ is given by

$$m_{\alpha,\beta} = \delta_{\alpha_1+\cdots+\alpha_p,\beta_1+\cdots+\beta_p} \tag{2.24}$$

where $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$, $\beta = (\beta_1, \ldots, \beta_p) \in \Gamma$, and

$$\delta_{\alpha_1+\cdots+\alpha_p, \beta_1+\cdots+\beta_p} = \begin{cases} 1, & \text{if } \alpha_1+\cdots+\alpha_p = \beta_1+\cdots+\beta_p \\ 0, & \text{if otherwise} \end{cases} \tag{2.25}$$

<u>Proof</u> Since

$$[\pi_n(z)]^p = \sum_{\alpha\in\Gamma} a_{\alpha_1} \cdots a_{\alpha_p} z^{\alpha_1+\cdots+\alpha_p} \tag{2.26}$$

we have from (2.3) and the fact that $\bar{z} = \frac{1}{z}$ on C,

$$\|\pi_n\|_{2p}^{2p} = \frac{1}{2\pi i}\int_C [\pi_n(z)\overline{\pi_n(z)}]^p \frac{dz}{z} = \frac{1}{2\pi i}\int_C \pi_n^p(z)\overline{\pi_n^p(z)} \frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_C \left[\sum_{\beta\in\Gamma} a_{\beta_1}\cdots a_{\beta_p} z^{\beta_1+\cdots+\beta_p}\right]$$

$$\cdot \overline{\left[\sum_{\alpha\in\Gamma} a_{\alpha_1}\cdots a_{\alpha_p} z^{\alpha_1+\cdots+\alpha_p}\right]} \frac{dz}{z}$$

$$= \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma} \overline{a_{\alpha_1}\cdots a_{\alpha_p}}\, a_{\beta_1}\cdots a_{\beta_p}$$

$$\cdot \left\{\frac{1}{2\pi i}\int_C z^{(\beta_1+\cdots+\beta_p)-(\alpha_1+\cdots+\alpha_p)} \frac{dz}{z}\right\}$$

$$= \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma} \overline{a_{\alpha_1}\cdots a_{\alpha_p}}\, a_{\beta_1}\cdots a_{\beta_p}$$

$$\cdot\, \delta_{\alpha_1+\cdots+\alpha_p, \beta_1+\cdots+\beta_p}$$

$$= \bar{u}^T M_{n,p} u$$

This concludes the proof.

The matrix $M_{n,p}$ reduces to the identity matrix for $p = 1$. Note, too, that the matrix in (2.8) is just $M_{2,2}$.

Since $M_{n,p}$ is a real symmetric matrix, its eigenvalues must be real. Also, $M_{n,p}$ is both nonnegative and positive semidefinite. The positive semidefiniteness of $M_{n,p}$ will follow from Lemma 2.3 in which it is established that all the eigenvalues are nonnegative. (An easier and much more general proof of positive semidefiniteness is given in Chapter III and applies to $M_{n,p}$ also.)

Lemma 2.2 Let $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$. Then the integer

$$N(\alpha) = \sum_{j_1,\ldots,j_p=0}^{n} 1 \tag{2.27}$$

where the sum is taken subject to the constraint

$$j_1 + \cdots + j_p = \alpha_1 + \cdots + \alpha_p$$

is the coefficient of $z^{\alpha_1+\cdots+\alpha_p}$ in the expansion of

$$(1 + z + z^2 + \cdots + z^n)^p \tag{2.28}$$

into ascending powers of z. Conversely, every coefficient in the expansion of (2.28) has the form (2.27) for some $\alpha \in \Gamma$.

Proof Let $x = \langle 1\ z\ z^2\ \cdots\ z^n\rangle^T \in \mathbb{C}^{n+1}$. Then from (2.18), we have for $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$,

$$\underbrace{x \otimes \ldots \otimes x}_{p\ \text{factors}} = \langle z^{\alpha_1} \cdots z^{\alpha_p}\rangle_{\alpha\in\Gamma}$$

$$= \langle z^{\alpha_1+\cdots+\alpha_p}\rangle_{\alpha\in\Gamma} \in \mathbb{C}^{(n+1)^p}$$

For each $\alpha \in \Gamma$, how many different components of $x \otimes \cdots \otimes x$

are identically equal to the α-th component? Clearly the answer is $N(\alpha)$. Therefore, summing the components of $x \otimes \cdots \otimes x$ and collecting terms gives a power series in z with coefficients of the form $N(\alpha)$ for some $\alpha \in \Gamma$. Since every integer $N(\alpha)$ must occur as a coefficient, and since

$$\sum_{\alpha_1,\ldots,\alpha_p=0}^{n} z^{\alpha_1+\cdots+\alpha_p} \equiv (1 + z + \cdots + z^n)^p \qquad (2.29)$$

the proof is complete.

<u>Lemma 2.3</u> Let $M_{n,p}$ be the matrix (2.24). Then

i. The null space of $M_{n,p}$ has dimension $(n + 1)^p - (np + 1)$.

ii. The nonzero eigenvalues of $M_{n,p}$ appear with the correct multiplicity as the coefficients in the expansion of $(1 + z + z^2 + \cdots + z^n)^p$ into ascending powers of z, and every such coefficient is an eigenvalue of $M_{n,p}$.

iii. The largest eigenvalue of $M_{n,p}$ has multiplicity 1 if np is even and multiplicity 2 if np is odd. All other nonzero eigenvalues have multiplicity 2.

iv. Every column of $M_{n,p}$ is an eigenvector corresponding to a nonzero eigenvalue of $M_{n,p}$.

v. Any two columns of $M_{n,p}$ are either orthogonal or identical.

vi. The orthogonal columns of $M_{n,p}$ form a basis for the range of $M_{n,p}$.

vii. The eigenspace of the largest eigenvalue of $M_{n,p}$ does not contain a vector of the form $x \otimes \cdots \otimes x$, $x \neq 0$, $x \in \mathbb{C}^{n+1}$, unless $n = 0$ or $p = 1$.

<u>Proof</u> Throughout this proof, let $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$, $\beta = (\beta_1, \ldots, \beta_p) \in \Gamma$, and $\gamma = (\gamma_1, \ldots, \gamma_p) \in \Gamma$. Also, denote the α-th column of $M_{n,p}$ by

$$R_\alpha = \langle m_{\gamma,\alpha} \rangle_{\gamma\in\Gamma}$$

Now, to prove (v), note that the inner product between the α-th and β-th column is

$$\bar{R}_\beta{}^T R_\alpha = \sum_{\gamma\in\Gamma} m_{\gamma,\beta} m_{\gamma,\alpha}$$

$$= \sum_{\gamma\in\Gamma} \delta_{\gamma_1+\cdots+\gamma_p,\beta_1+\cdots+\beta_p} \delta_{\gamma_1+\cdots+\gamma_p,\alpha_1+\cdots+\alpha_p}$$

$$= \begin{cases} N(\alpha), & \text{if } \alpha_1+\cdots+\alpha_p = \beta_1+\cdots+\beta_p \\ 0, & \text{if otherwise} \end{cases} \tag{2.30}$$

so that $R_\alpha \perp R_\beta$ or $\alpha_1 + \cdots + \alpha_p = \beta_1 + \cdots + \beta_p$. In the latter case, the definition of $M_{n,p}$ implies that $R_\alpha = R_\beta$. This proves (v).

To prove (iv), we show that

$$M_{n,p} R_\alpha = N(\alpha) R_\alpha \tag{2.31}$$

for each $\alpha \in \Gamma$. Fix α. Then the matrix equation (2.31) is equivalent to the system of linear equations

$$R_\beta{}^T R_\alpha = N(\alpha) m_{\beta,\alpha}, \quad \text{for all } \beta \in \Gamma \tag{2.32}$$

since $M_{n,p}$ is real symmetric and the β-th row is the transpose of the column R_β. If $m_{\beta,\alpha} = 1$, then $R_\alpha = R_\beta$ and (2.32) follows from (2.30). On the other hand, if $m_{\beta,\alpha} = 0$, then $R_\alpha \neq R_\beta$ and (2.32) follows again from (2.30). In either case, the proof of (iv) is complete. Actually, (2.31) together with Lemma 2.2 also proves that every coefficient in the expansion (2.28) is a nonzero eigenvalue of $M_{n,p}$. This statement is half of (ii). To prove the other half of (ii), recall that $M_{n,p}$ is symmetric and the column rank of $M_{n,p}$ equals the number of nonzero eigenvalues, counted with correct multiplicity (see [24]). From (v) and $R_\alpha = R_\beta$ iff $\alpha_1 + \cdots + \alpha_p = \beta_1 + \cdots + \beta_p$, the column rank is $np + 1$ (the number of distinct sums of the form $\alpha_1 + \cdots + \alpha_p$) and (2.31) has $np + 1$ solutions. Therefore, every nonzero eigenvalue of $M_{n,p}$ is of the form $N(\alpha)$. From Lemma 2.2, $N(\alpha)$ is the coefficient of $z^{\alpha_1 + \cdots + \alpha_p}$ in the expansion of (2.28). Since the mapping $R_\alpha \to z^{\alpha_1 + \cdots + \alpha_p}$ takes distinct columns of $M_{n,p}$ into distinct powers of z, $N(\alpha)$ occurs with correct multiplicity in the expansion of (2.28). This proves (ii). From (ii) follow (i) and (vi), since $M_{n,p}$ is symmetric and the range is orthogonal to the null space (see [24]). Also, (iii) follows directly from (ii) by examination of the coefficients in the expansion (2.28). For $n = 0$ or $p = 1$, the assertion of (vii) is clear. For $n > 0$ and $p > 1$, the proof proceeds by showing that if $x \otimes \cdots \otimes x$ (p factors) is in the eigenspace of the largest eigenvalue, then $x = 0 \in \mathbb{C}^{n+1}$. This will establish (vii). By (ii), the largest eigenvalue is the largest coefficient in the expansion of (2.28). First, let np be even. Then the largest coefficient occurs only once and is the coefficient of z^k, where $k = np/2$. Thus, the eigenspace consists of constant multiples of the column

$$R_\alpha = \langle \delta_{\gamma_1 + \cdots + \gamma_p,\, k} \rangle_{\gamma \in \Gamma}$$

where α is fixed and $\alpha_1 + \cdots + \alpha_p = k$. Suppose that for some $x = \langle x_0\ x_1\ \cdots\ x_n \rangle^T \in \mathbb{C}^{n+1}$ and constant $c \neq 0$,

$$cR_\alpha = x \otimes \cdots \otimes x$$

Now the γ-th components must be equal, so

$$c\delta_{\gamma_1 + \cdots + \gamma_p,\, k} = x_{\gamma_1} \cdots x_{\gamma_p}$$

For $\gamma_1 = \cdots = \gamma_p = t$, $t = 0, 1, \ldots, n$, we have

$$c\delta_{pt,k} = (x_t)^p$$

or

$$x_t = \begin{cases} (c)^{\frac{1}{p}}, & \text{if } t = n/2 \\ 0, & \text{if } t \neq n/2 \end{cases}$$

since $k/p = n/2$. If n is odd, then $x_t = 0$ for all t, and we are done. If n is even, then x has precisely one nonzero component, so that $x \otimes \cdots \otimes x$ has only one nonzero component and cR_α must have only one nonzero component. But the last statement is false for $p > 1$ and $n > 0$. Therefore, $c = 0$ and $x = 0 \in \mathbb{C}^{n+1}$, and we are done if np is even. On the other hand, if np is odd the largest eigenvalue of $M_{n,p}$ occurs as the coefficient of both z^k and z^{k+1}, where $k = (np - 1)/2$. Let α and β be fixed so that $\alpha_1 + \cdots + \alpha_p = k$ and $\beta_1 + \cdots + \beta_p = k + 1$. Then the general element of the eigenspace of the largest eigenvalue can be written, for arbitrary constants c_1 and c_2,

$$c_1R_\alpha + c_2R_\beta = \langle c_1\delta_{\gamma_1+\cdots+\gamma_p,k} + c_2\delta_{\gamma_1+\cdots+\gamma_p,k+1}\rangle_{\gamma\in\Gamma}$$

Suppose $x = \langle x_0\ x_1\ \cdots\ x_n\rangle^T \in \mathbb{C}^{n+1}$ is such that

$$c_1R_\alpha + c_2R_\beta = x \otimes \cdots \otimes x$$

for some choice of constants $c_1 \neq 0$ and $c_2 \neq 0$. Then the γ-th coordinates are equal, so

$$c_1\delta_{\gamma_1+\cdots+\gamma_p,k} + c_2\delta_{\gamma_1+\cdots+\gamma_p,k+1} = x_{\gamma_1}\cdots x_{\gamma_p}$$

For $\gamma_1 = \cdots = \gamma_p = t$, $t = 0, 1, \ldots, n$, we have

$$c_1\delta_{pt,k} + c_2\delta_{pt,k+1} = (x_t)^p$$

and so

$$x_t = \begin{cases} (c_1)^{\frac{1}{p}} & \text{if } pt = k \\ (c_2)^{\frac{1}{p}} & \text{if } pt = k+1 \\ 0 & \text{if otherwise} \end{cases}$$

Now x can have only one nonzero component since p cannot divide both k and k + 1. Hence, $x \otimes \cdots \otimes x$ has only one nonzero component, and so $c_1R_\alpha + c_2R_\beta$ has only one nonzero component. This can't happen for $n > 0$ and $p > 1$, so it must be that $c_1 = c_2 = 0$. Then $x = 0 \in \mathbb{C}^{n+1}$. This proves (vii) and concludes the proof of Lemma 2.3.

<u>Corollary 2.4</u> $M_{n,p}$ is positive semidefinite.

<u>Proof</u> By a standard result (see [24, Section 4.12]), a symmetric matrix is positive semidefinite iff all its

eigenvalues are nonnegative. Lemma 2.3 shows that all the eigenvalues of $M_{n,p}$ are nonnegative.

<u>Lemma 2.5</u> Let $\pi_n(z) = a_0 + a_1 z + \cdots + a_n z^n \in P_n$, and let $x = \langle a_0\ a_1\ \cdots\ a_n\rangle^T \in \mathbf{C}^{n+1}$. Then

$$\|\pi_n\|_2 = \left\{\bar{u}^T I u\right\}^{\frac{1}{2p}} = \left\{\bar{u}^T u\right\}^{\frac{1}{2p}}$$

where $u = x \otimes \cdots \otimes x \in \mathbf{C}^{(n+1)^p}$ and I is the identity matrix of dimension $(n+1)^p \times (n+1)^p$.

<u>Proof</u> Certainly

$$\|\pi_n\|_2^2 = |a_0|^2 + |a_1|^2 + \cdots + |a_n|^2$$

$$= \sum_{k=0}^{n} a_k\,\bar{a}_k$$

so that

$$\|\pi_n\|_2^{2p} = \left(\sum_{k=0}^{n} a_k\,\bar{a}_k\right)^p$$

$$= \sum_{\alpha_1,\ldots,\alpha_p=0}^{n} (a_{\alpha_1}\,\bar{a}_{\alpha_1}) \cdots (a_{\alpha_p}\,\bar{a}_{\alpha_p})$$

$$= \sum_{\alpha_1,\ldots,\alpha_p=0}^{n} (a_{\alpha_1} \cdots a_{\alpha_p})(\overline{a_{\alpha_1} \cdots a_{\alpha_p}})$$

$$= \bar{u}^T I u$$

since

$$u = x \otimes \cdots \otimes x = \langle a_{\alpha_1} \cdots a_{\alpha_p}\rangle_{\alpha\in\Gamma}$$

where $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$. This concludes the proof.

C. Bound for Ratio of L_{2p} Norm to L_2 Norm

For extended real numbers $p \geq 1$, define

$$R_{n,p} = \max_{0 \neq \pi_n \in P_n} \left\{ \frac{\|\pi_n\|_p}{\|\pi_n\|_2} \right\} \tag{2.33}$$

It is easy to see that the maximum in (2.33) can be taken over the closed and bounded set $S = \{\pi_n \in P_n : \|\pi_n\|_2 = 1\}$ without changing $R_{n,p}$. Since $\|\cdot\|_p$ is a continuous function on the compact set S, $\|\cdot\|_p$ attains its maximum. Thus the maximum in (2.33) is attained. Any polynomial for which $R_{n,p}$ is attained is called an extremal polynomial of $R_{n,p}$. Any nonzero constant multiple of an extremal polynomial of $R_{n,p}$ gives another extremal polynomial of $R_{n,p}$. Thus, extremal polynomials are not unique. Normalizing extremal polynomials by the requirement that they have unit L_2 norm does not necessarily give uniqueness. For example, every polynomial is an extremal polynomial of $R_{n,2}$.

Note that $R_{n,p} \leq R_{n,q}$ whenever $1 \leq p \leq q$. By Hölder's inequality, for $r \geq 1$, $s \geq 1$, and $\frac{1}{r} + \frac{1}{s} = 1$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |\pi_n(e^{i\theta})|^p \, 1 d\theta \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\pi_n(e^{i\theta})|^{pr} \, d\theta \right\}^{\frac{1}{r}} \cdot \left\{ \frac{1}{2\pi} \int_0^{2\pi} 1^s \, d\theta \right\}^{\frac{1}{s}}$$

so that

$$\|\pi_n\|_p \leq \|\pi_n\|_{pr}, \qquad r \geq 1$$

and this establishes our claim. The last inequality also shows that $\|\pi_n\|_2 \leq \|\pi_n\|_p$ whenever $p \geq 2$, so that

$$r_{n,p} \equiv \min_{\pi_n \neq 0} \left\{\frac{\|\pi_n\|_p}{\|\pi_n\|_2}\right\} = 1, \qquad p \geq 2$$

since $\pi_n(z) \equiv 1$ gives $\|\pi_n\|_p / \|\pi_n\|_2 = 1$.

In this section, we obtain estimates for $R_{n,2p}$ in terms of the spectral radius of $M_{n,p}$ defined in Lemma 2.1.

Notation: For $n \geq 0$ and $p \geq 1$, define the integer $\lambda_{n,p}$ to be the largest coefficient in the power series expansion of $(1 + z + z^2 + \cdots + z^n)^p$ into ascending powers of z. Thus, $\lambda_{n,p}$ is the coefficient z^N, $N = [\frac{np}{2}]$, in this expansion.

The multinomial coefficients $\lambda_{n,p}$ will be seen to play an important role in this chapter. For example, Lemma 2.3 shows that $\lambda_{n,p}$ is identically the spectral radius of $M_{n,p}$. Another example is provided by the next lemma.

Lemma 2.6 $\|1 + z + z^2 + \cdots + z^n\|_{2p} = \{\lambda_{n,2p}\}^{\frac{1}{2p}}$, $p = 1, 2, 3, \ldots$.

Proof From (2.3) we have

$$(\|1 + z + \cdots + z^n\|_{2p})^{2p} = \left\{\frac{1}{2\pi i}\int_C (1 + z + \cdots + z^n)^p \cdot (1 + \bar{z} + \cdots + \bar{z}^n)^p \frac{dz}{z}\right\}^{\frac{1}{2p}}$$

$$= \left\{\frac{1}{2\pi i}\int_C (1 + z + \cdots + z^n)^{2p} \frac{dz}{z^{np+1}}\right\}^{\frac{1}{2p}}$$

$$= \{\lambda_{n,2p}\}^{\frac{1}{2p}}$$

We note that, for fixed integer $n \geq 0$, the sequence

$$\{\lambda_{n,2p}\}^{\frac{1}{2p}}, \qquad p = 1, 2, 3, \ldots$$

is monotone increasing. This follows from Lemma 2.6, and the fact that

$$\|1 + z + z^2 + \cdots + z^n\|_{2p} \leq \|1 + z + z^2 + \cdots + z^n\|_{2p+2}$$

The next theorem generalizes the bound (2.14) of the earlier example. It is the main theorem of this chapter.

Theorem 2.1 For $n = 0, 1, 2, 3, \ldots$ and for $p = 1, 2, \ldots$,

$$\frac{\{\lambda_{n,2p}\}^{\frac{1}{2p}}}{\sqrt{n+1}} \leq R_{n,2p} \leq \{\lambda_{n,p}\}^{\frac{1}{2p}} \leq (n+1)^{\frac{1}{2}-\frac{1}{2p}} \tag{2.34}$$

The second inequality in (2.34) is strict if and only if $n \geq 1$ and $p \geq 2$, while the third inequality in (2.34) is strict if and only if $n \geq 1$ and $p \geq 3$. Furthermore,

$$R_{n,\infty} = \sqrt{n+1} \tag{2.35}$$

and an extremal polynomial of $R_{n,\infty}$ is just

$$1 + z + z^2 + \cdots + z^n$$

Proof From Lemma 2.6, we have

$$R_{n,2p} \geq \frac{\|1 + z + z^2 + \cdots + z^n\|_{2p}}{\|1 + z + z^2 + \cdots + z^n\|_2} = \frac{\{\lambda_{n,2p}\}^{\frac{1}{2p}}}{\sqrt{n+1}}$$

which proves the first inequality in (2.34). Let $\pi_n(z) = a_0 + a_1 z + \cdots + a_n z^n$, and let $x = \langle a_0\ a_1\ \cdots\ a_n\rangle^T \in \mathbf{C}^{n+1}$. From Lemma 2.1 and Lemma 2.5,

$$\frac{\|\pi_n\|_{2p}}{\|\pi_n\|_2} = \left\{\frac{\bar{u}^T M_{n,p} u}{\bar{u}^T u}\right\}^{\frac{1}{2p}} \le \max_{v \neq 0}\left\{\frac{\bar{v}^T M_{n,p} v}{\bar{v}^T v}\right\}^{\frac{1}{2p}} \tag{2.36}$$

$$= \{\lambda_{n,p}\}^{\frac{1}{2p}}$$

where v is an arbitrary nonzero vector in $\mathbf{C}^{(n+1)^p}$ and $u = x \otimes \cdots \otimes x$ (p factors). Therefore,

$$R_{n,2p} = \max_{\pi_n \neq 0} \frac{\|\pi_n\|_{2p}}{\|\pi_n\|_2} \le \{\lambda_{n,p}\}^{\frac{1}{2p}} \tag{2.37}$$

which is the second inequality of (2.34). Finally, from the identity

$$(1 + z + \cdots + z^n)^p = (1 + z + \cdots + z^n) \cdot (1 + z + \cdots + z^n)^{p-1}$$

follows immediately the inequality

$$\lambda_{n,p} \le (n + 1)\lambda_{n,p-1}, \qquad p \ge 1 \tag{2.38}$$

Since $\lambda_{n,1} = 1$, (2.38) implies

$$\lambda_{n,p} \le (n + 1)^{p-1}, \qquad p \ge 1$$

which proves the third inequality in (2.34). Next, note that inequality (2.36) is an equality if and only if u is in the eigenspace of the largest eigenvalue, namely $\lambda_{n,p}$, of the matrix $M_{n,p}$. From Lemma 2.3 (vii), there exists an element of the form $x \otimes \cdots \otimes x$, $x \neq 0$, in the eigenspace of $\lambda_{n,p}$ if and only if $n = 0$ or $p = 1$. Thus, Lemma 2.1 implies inequality (2.36) is strict if and only if $n \ge 1$ and $p \ge 2$. Thus, the second inequality in (2.34) is strict if and only if $n \geq 1$ and $p \geq 2$. Also, the third inequality of (2.34) is an equality if and only if (2.38) is an equality, which is the case if and only if $n = 0$ or $p = 1$ or 2. Hence, we have left to prove only (2.35). From the Cauchy-Schwarz inequality,

$$\begin{aligned} \|\pi_n\|_\infty &= |\pi_n(z_0)|, \qquad \text{some } z_0 \in C, \\ &= |a_0 + a_1 z_0 + \cdots + a_n z_0^n| \\ &\leq \left\{\sum_{k=0}^{n} |a_k|^2\right\}^{\frac{1}{2}} \left\{\sum_{k=0}^{n} |z_0^k|^2\right\}^{\frac{1}{2}} \\ &= \|\pi_n\|_2 \sqrt{n+1} \end{aligned}$$

Equality is possible with, for example, $\pi_n(z) = 1 + z + \cdots + z^n$, so that (2.35) follows. This concludes the proof.

Corollary 2.7 For $n = 0, 1, 2, \ldots,$

$$\lim_{p\to\infty} \{\lambda_{n,p}\}^{\frac{1}{p}} = n + 1 \tag{2.39}$$

Proof Let $\pi_n^*(z) = 1 + z + \cdots + z^n$ in Theorem 2.1. Then, using Lemma 2.6,

$$\begin{aligned} \frac{\{\lambda_{n,2p}\}^{\frac{1}{2p}}}{\sqrt{n+1}} &= \frac{\|\pi_n^*\|_{2p}}{\|\pi_n^*\|_2} \\ &\leq \{\lambda_{n,p}\}^{\frac{1}{2p}} \leq \sqrt{n+1} \end{aligned}$$

so that taking the limit as $p \to \infty$ finishes the proof since

$$\frac{\|\pi_n^*\|_\infty}{\|\pi_n^*\|_2} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$$

Note that Corollary 2.7 shows that for fixed n, both the upper and the lower bounds of $R_{n,2p}$ in (2.34) go to $\sqrt{n+1}$ as p goes to infinity.

At this point, it is appropriate to point out that the author communicated the result

$$R_{n,2p} \le \{\lambda_{n,p}\}^{\frac{1}{2p}} \tag{2.40}$$

(without proof) privately to D. J. Newman who discovered the following short proof of this result: Write

$$\pi_n(z) = \sum_{k=0}^{n} a_k z^k \not\equiv 0 \tag{2.41}$$

$$(1 + z + \cdots + z^n)^p = \sum_{j=0}^{np} \lambda_j z^j$$

so that

$$\lambda_{n,p} = \max_{0\le j\le np} \lambda_j$$

Now, since the powers of z are orthonormal on the unit circle C, with $z = e^{i\theta}$,

$$\|\pi_n\|_{2p}^{2p} = \frac{1}{2\pi}\int_C |\pi_n(z)|^{2p}\, d\theta$$

$$= \frac{1}{2\pi}\int_C |[\pi_n(z)]^p|^2\, d\theta$$

$$= \frac{1}{2\pi}\int_C \left|\sum_{j=0}^{np}\left(\sum_{\alpha_1+\cdots+\alpha_p=j} a_{\alpha_1}\cdots a_{\alpha_p}\right) z^j\right|^2 d\theta$$

$$= \sum_{j=0}^{np} \left| \sum_{\alpha_1+\cdots+\alpha_p=j} a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_p} \right|^2 \tag{2.42}$$

Schwarz's Inequality gives

$$\left| \sum_{\alpha_1+\cdots+\alpha_p=j} a_{\alpha_1} \cdots a_{\alpha_p} \right|^2 \le \left(\sum_{\alpha_1+\cdots+\alpha_p=j} 1^2 \right)$$

$$\cdot \left(\sum_{\alpha_1+\cdots+\alpha_p=j} \left| a_{\alpha_1} \cdots a_{\alpha_p} \right|^2 \right)$$

$$= \lambda_j \sum_{\alpha_1+\cdots+\alpha_p=j} \left| a_{\alpha_1} \cdots a_{\alpha_p} \right|^2 \tag{2.43}$$

so that

$$\|\pi_n\|_{2p}^{2p} \le \sum_{j=0}^{np} \left(\lambda_j \sum_{\alpha_1+\cdots+\alpha_p=j} \left| a_{\alpha_1} \cdots a_{\alpha_p} \right|^2 \right)$$

$$\le \lambda_{n,p} \left(\sum_{j=0}^{np} \sum_{\alpha_1+\cdots+\alpha_p=j} \left| a_{\alpha_1} \cdots a_{\alpha_p} \right|^2 \right)$$

$$= \lambda_{n,p} \left(\sum_{j=0}^{n} |a_j|^2 \right)^p$$

$$= \lambda_{n,p} \|\pi_n\|_2^{2p} \tag{2.44}$$

The inequality (2.44) immediately implies (2.40).

Newman's proof gains in brevity over the algebraic approach of Theorem 2.1. It depends heavily on the fact that the powers of z are orthonormal. In spaces where the powers of z are not orthonormal, it is not clear how to modify Newman's proof in order to get useful results. (The difficulty is caused by the fact that the product of two orthonormal polynomials is not, generally, orthonormal to either of the polynomials in the product.) The algebraic approach of Theorem 2.1 is, however, generalized without too much additional difficulty to situations in which the powers of z are not orthonormal. See Chapter III.

It is interesting to note that the constants λ_j of the Schwarz Inequality (2.43) each turn out to be eigenvalues of the matrix $M_{n,p}$ of Lemma 2.1.

D. The Spectral Radius $\lambda_{n,p}$

The integers $\lambda_{n,p}$ have a geometrical significance in $\mathbb{R}^p$. (This result is apparently new.) Consider first the case (p = 2) of a square lattice with n + 1 points on a side. What is the largest number of points that can lie on any line perpendicular to a major diagonal of the square? We easily see the answer is n + 1 (= $\lambda_{n,2}$). Next, consider the case (p = 3) of a cubic lattice with n + 1 points on a side. What is the largest number of points that can lie on any plane perpendicular to a major diagonal of the cube? In this case the answer is not so clear, but we will show that the answer is just $\lambda_{n,3}$. More generally, and more carefully, the set Γ can be considered as a finite "hypercube" lattice in $\mathbb{R}^p$ with n + 1 points on a side. Let T be the hyperplane in $\mathbb{R}^p$ consisting of all vectors in $\mathbb{R}^p$ orthogonal (with the usual inner product) to the vector $\alpha_0 = (1, 1, \ldots, 1) \in \mathbb{R}^p$. Thus, dim T = p - 1. Let $T_\alpha \equiv \alpha + T$, $\alpha \in \mathbb{R}^p$. Then, we show that

$$\max_{\alpha \in \mathbb{R}^p} o(T_\alpha \cap \Gamma) = \lambda_{n,p} \qquad (2.45)$$

where the small "o" notation in (2.45) denotes the number of elements in the set. Let $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p$ with $\alpha_1 + \cdots + \alpha_p = s$. With $\beta = (\beta_1, \ldots, \beta_p) \in \mathbb{R}^p$, we have $\beta \in T_\alpha$ if and only if $(\beta - \alpha) \perp \alpha_0$ if and only if $\beta_1 + \cdots + \beta_p = \alpha_1 + \cdots + \alpha_p = s$. Therefore, $T_\alpha \cap \Gamma \neq \emptyset$ if and only if $s \in \{0, 1, 2, \ldots, np\}$. Hence, by Lemma 2.2,

$$o(T_\alpha \cap \Gamma) = \begin{cases} N(\tilde{\alpha}), & \text{if there exists } \tilde{\alpha} \in T_\alpha \cap \Gamma \\ 0, & \text{if not} \end{cases}$$

An inspection of the sum (2.27) shows that $N(\tilde{\alpha}) = N(\tilde{\beta})$ if $\tilde{\alpha}$ and $\tilde{\beta}$ both lie in $T_\alpha \cap \Gamma$, so that $o(T_\alpha \cap \Gamma)$ is unambiguously defined. This proves (2.45) by definition of $\lambda_{n,p}$.

In certain cases, the integers $\lambda_{n,p}$ possess a generating function. Polya-Szegö [29, Part III, Chapter 5, Problems 217-218] derive the expansions

$$\frac{1}{\sqrt{1 - 2\omega - 3\omega^2}} = \sum_{p=0}^{\infty} \lambda_{2,p}\omega^p$$

$$= 1 + \omega + 3\omega^2 + 7\omega^3 + 19\omega^4 + \cdots$$

$$\frac{1}{\sqrt{1 - 4\omega}} = \sum_{p=0}^{\infty} \lambda_{1,2p}\omega^p$$

$$= 1 + 2\omega + 6\omega^2 + 20\omega^3 + 70\omega^4 + \cdots$$

$$\frac{1}{2\omega}\left[\frac{1}{\sqrt{1 - 4\omega}} - 1\right] = \sum_{p=0}^{\infty} \lambda_{1,2p+1}\omega^p$$

$$= 1 + 3\omega + 10\omega^2 + 35\omega^3 + 126\omega^4 + \cdots$$

Unfortunately, the methods used to derive these expansions are not easily extended to the general case.

The integers $\lambda_{n,p}$ are also related to a certain problem in probability. See, for example, Feller [11, Chapter 11, Problem 11].

Finally, the integers $\lambda_{n,p}$ have a combinatorial significance that is explored by MacMahon [22, Section IV]. (This reference was pointed out by George Andrews in a private communication.) A composition of an integer I is a partition of I in which the order of occurrence of the parts is important. For example, there are three partitions of I = 3, namely,

$$3 = 3$$
$$3 = 2 + 1$$
$$3 = 1 + 1 + 1$$

while there are four compositions, namely,

$$3 = 3$$
$$3 = 2 + 1$$
$$3 = 1 + 2$$
$$3 = 1 + 1 + 1$$

MacMahon shows that the number of compositions of I into exactly $s \geq 1$ parts, with each part restricted not to exceed $t \geq 1$ in magnitude, is precisely the coefficient of z^I in the expansion of

$$z^s\left(\frac{1 - z^t}{1 - z}\right)^s \equiv z^s(1 + z + \cdots z^{t-1})^s$$

Hence, the integer $\lambda_{n,p}$ is the number of compositions of

$$I = \left[\frac{(n + 2)p}{2}\right] = \left[\frac{np}{2}\right] + p$$

into exactly p parts, each part being restricted not to exceed $n + 1$ in magnitude. For $n = 1$ and $p = 2$, we have $I = 3$ and $\lambda_{1,2} = 2$ which is precisely the number of compositions of 3 into $n + 1 = 2$ parts with each part not exceeding 2 in magnitude.

The combinatorial interpretation gives a bound for $\lambda_{n,p}$. Define the denumerant $D(n)$ of the integer n to be the number of p-tuples $(x_1, \ldots, x_p)$ of solutions of the equation

$$x_1 + x_2 + \cdots + x_p = n \tag{2.46}$$

where $x_1, \cdots, x_p$ are required to be nonnegative integers. By a theorem of Bell [6, 30], $D(n)$ is a polynomial in n of degree $p - 1$. Specifically, Bell [6] states that

$$D(n) = \frac{1}{(p-1)!} \prod_{r=1}^{p-1} (n + r) \tag{2.47}$$

Since $\lambda_{n,p}$ is the coefficient of z^N, $N = [np/2]$, in the expansion of $(1 + z + z^2 + \cdots + z^n)^p$ into ascending powers of z, and since we require $0 \le x_k \le n$ in (2.46) to compute $\lambda_{n,p}$ [see (2.27)], we have

$$\lambda_{n,p} \le D(N) \le D\left(\frac{np}{2}\right) \tag{2.48}$$

It is not hard to show that

$$D\left(\frac{np}{2}\right) \le \frac{p^{p-1}(n + 1)^{p-1}}{2^{p-1}(p - 1)!} \tag{2.49}$$

Considering (2.49), (2.48), and (2.34), gives

$$R_{n,2p} \le \left(\frac{p^{p-1}}{2^{p-1}(p - 1)!}\right)^{\frac{1}{2p}} (n + 1)^{\frac{1}{2} - \frac{1}{2p}} \tag{2.50}$$

Unfortunately, (2.50) does not improve (2.34) since

$$\frac{p^{p-1}}{2^{p-1}(p - 1)!} \geq 1, \qquad p = 1, 2, 3, \ldots \tag{2.51}$$

as can be seen by showing that the left hand side of (2.51) is strictly increasing in p.

It is possible to give explicit expressions for $\lambda_{n,p}$. Define, for integer $p \geq 1$ and for all real x, the polynomial

$$a_p(x) = \begin{cases} \dfrac{(x + 1)(x + 2)\cdots(x + p - 1)}{(p - 1)!}, & p \geq 2 \\ 1, & p = 1 \end{cases} \tag{2.52}$$

The polynomial $a_p(x)$ has degree $p - 1$ in x which, on the nonnegative integers, is just

$$a_p(k) = \binom{p + k - 1}{k}, \quad k = 0, 1, 2, \ldots \tag{2.53}$$

(For a connection between Stirling numbers of the first kind and the polynomials $a_p(x)$, see [29, Part I, Chapter 4, Problem 199].) The next theorem shows that, for fixed p, $\lambda_{n,p}$ is "almost" a polynomial of degree $p - 1$ in n.

<u>Theorem 2.2</u> For integer $p \geq 1$,

$$\lambda_{n,2p} = \sum_{j=0}^{p-1} (-1)^j \binom{2p}{j} a_{2p}\big((p - j)n - j\big), \tag{2.54}$$

$$n = 0, 1, 2, 3, \ldots$$

$$\lambda_{n,2p-1} = \begin{cases} \displaystyle\sum_{j=0}^{p-1} (-1)^j \binom{2p-1}{j} a_{2p-1}\big((p - j - \tfrac{1}{2})n - j\big), n=0,2,4,\ldots \\ \displaystyle\sum_{j=0}^{p-1} (-1)^j \binom{2p-1}{j} a_{2p-1}\big((p - j - \tfrac{1}{2})n - j - \tfrac{1}{2}\big), n=1,3,5,\ldots \end{cases} \tag{2.55}$$

Proof For p = 1, the theorem is easily verified. Let $p \geq 2$. From the binomial theorem and the binomial series (see, e.g., [1, Equation (3.6.8)])

$$(1 + z + z^2 + \cdots + z^n)^p$$

$$= \left\{\frac{1 - z^{n+1}}{1 - z}\right\}^p$$

$$= (1 - z^{n+1})^p (1 - z)^{-p}$$

$$= \left(\sum_{j=0}^{p} (-1)^j \binom{p}{j} z^{j(n+1)}\right)\left(\sum_{k=0}^{\infty} a_p(k) z^k\right) \tag{2.56}$$

where $a_p(k)$ is defined by (2.52). Let $N = [np/2]$. Since $\lambda_{n,p}$ is the coefficient of z^N, (2.56) implies

$$\lambda_{n,p} = \sum_{\substack{j(n+1)+k=N \\ j\geq 0,\ k\geq 0}} (-1)^j \binom{p}{j} a_p(k)$$

$$= \sum_{j=0}^{\left[\frac{N}{n+1}\right]} (-1)^j \binom{p}{j} a_p\big(N - j(n+1)\big)$$

It is easy to see that

$$\left[\frac{N}{n + 1}\right] \leq \left[\frac{p - 1}{2}\right], \qquad n \geq 0 \tag{2.57}$$

If (2.57) is a strict inequality for some $n = n'$, then for each integer j such that

$$\left[\frac{N}{n' + 1}\right] < j \leq \left[\frac{p - 1}{2}\right]$$

we have

$$-(p - 1) \leq -\left[\frac{p - 1}{2}\right] \leq N - j(n' + 1) < 0$$

so that by (2.52)

$$a_p(N - j(n' + 1)) = 0$$

Therefore, whether or not (2.57) is a strict inequality,

$$\lambda_{n,p} = \sum_{j=0}^{\left[\frac{p-1}{2}\right]} (-1)^j \binom{p}{j} a_p(N - j(n + 1)), \quad n = 0, 1, 2, \ldots \tag{2.58}$$

Specializing (2.58) proves the various cases of the Theorem.

Table II.1 lists a few of the multinomial coefficients $\lambda_{n,p}$. Table II.2 gives values for the bound (2.34). The entries in Table II.2 have been rounded, so to construct a bound in (2.34) from this table the last digit might need to be increased by 1. Both tables were constructed by means of a simple synthetic multiplication scheme. A scaling trick and double precision arithmetic were both required to compute

$$\lambda_{100,256} \approx 1.09169 \times 10^{510}$$

on the UNIVAC 1108 in just under 39 minutes. Utilization of the symmetry of the expansion coefficients of (2.28) would have reduced the computation time by nearly half. Alternatively, Theorem 2.2 could have been used directly. This approach requires more care because of the cancellation inherent in (2.54) and (2.55).

<u>Corollary 2.8</u> The following equations hold:

i. For all $n \geq 0$, $\lambda_{n,1} = 1$.

ii. For all $n \geq 0$, $\lambda_{n,2} = n + 1$.

iii. For $n = 1, 3, 5, \ldots$, $\lambda_{n,3} = \frac{3}{4}(n^2 + 2n + 1)$, and for $n = 0, 2, 4, \ldots$, $\lambda_{n,3} = \frac{3}{4}(n^2 + 2n + \frac{4}{3})$.

iv. For all $n \geq 0$,

$$\lambda_{n,4} = \frac{2}{3}\left(n^3 + 3n^2 + \frac{7}{2} n + \frac{3}{2}\right)$$

v. For $n = 1, 3, 5, \ldots,$

$$\lambda_{n,5} = \frac{115}{192}\left(n^4 + 4n^3 + \frac{142}{23} n^2 + \frac{100}{23} n + \frac{27}{23}\right)$$

and for $n = 0, 2, 4, \ldots,$

$$\lambda_{n,5} = \frac{115}{192}\left(n^4 + 4n^3 + \frac{148}{23} n^2 + \frac{112}{23} n + \frac{192}{115}\right)$$

vi. For all $n \geq 0$,

$$\lambda_{n,6} = \frac{11}{20}\left(n^5 + 5n^4 + \frac{115}{11} n^3 + \frac{125}{11} n^2 + \frac{74}{11} n + \frac{20}{11}\right)$$

vii. For all $n \geq 0$,

$$\lambda_{n,8} = \frac{151}{315}\left(n^7 + 7n^6 + \frac{3241}{151} n^5 + \frac{5635}{151} n^4 + \frac{6034}{151} n^3 + \frac{4018}{151} n^2 + \frac{1599}{151} n + \frac{315}{151}\right)$$

viii. For all $n \geq 0$,

$$\lambda_{n,10} = \frac{15619}{36288}\left(n^9 + 9n^8 + \frac{569634}{15619} n^7 + \frac{1363446}{15619} n^6 + \frac{2127531}{15619} n^5 + \frac{2251179}{15619} n^4 + \frac{1625216}{15619} n^3 + \frac{780804}{15619} n^2 + \frac{234288}{15619} n + \frac{36288}{15619}\right)$$

ix. The coefficient of n^6 in $\lambda_{n,7}$ is $\frac{5887}{11520}$, for all $n \geq 0$.

x. The coefficient of n^8 in $\lambda_{n,9}$ is $\frac{259723}{573440}$, for all $n \geq 0$.

TABLE II.1. The Spectral Radius $\lambda_{n,p}$

n \ p	2	3	4	5	6	7	8	9	10
1	2	3	6	10	20	35	70	126	252
2	3	7	19	51	141	393	1107	3139	8953
3	4	12	44	155	580	2128	8092	30276	1 16304
4	5	19	85	381	1751	8135	38165	1 80325	8 56945
5	6	27	146	780	4332	24017	1 35954	7 67394	43 95456
6	7	37	231	1451	9331	60691	3 98567	26 36263	175 38157
7	8	48	344	2460	18152	1 34512	10 12664	76 35987	581 99208
8	9	61	489	3951	32661	2 73127	23 06025	196 10233	1677 29959
9	10	75	670	6000	55252	5 12365	48 16030	454 33800	4324 57640
10	11	91	891	8801	88913	9 08755	93 77467	974 64799	10188 72811
11	12	108	1156	12435	1 37292	15 28688	172 32084	1951 70310	22281 54512
12	13	127	1469	17151	2 04763	24 73325	301 62301	3704 87485	45771 27763
13	14	147	1834	23030	2 96492	38 52919	506 51498	6694 80588	89143 09964
14	15	169	2255	30381	4 18503	58 32765	820 73295	11632 05475	1 65814 20835
15	16	192	2736	39280	5 77744	85 82336	1289 12240	19476 52092	2 96326 04816
16	17	217	3281	50101	7 82153	123 54469	1970 18321	31645 88407	5 11256 45317
17	18	243	3894	62910	10 40724	173 95119	2938 97718	49959 76968	8 55016 36868
18	19	271	4579	78151	13 63573	240 72133	4290 42211	77021 89345	13 90719 24069
19	20	300	5340	95875	17 62004	327 26960	6142 99660	1 16038 17450	22 06336 15280
20	21	331	6181	1 16601	22 48575	438 74139	8642 87973	1 71489 49027	34 22376 34221
21	22	363	7106	1 40360	28 37164	579 71221	11968 54978	2 48704 46964	52 01360 84072

TABLE II.1. The Spectral Radius $\lambda_{n,p}$ (continued)

n \ p	2	3	4	5	6	7	8	9	10
22	23	397	8119	1 67751	35 43035	757 15487	16335 86615	3 55000 63501	77 59386 66273
23	24	432	9224	1 98780	43 82904	977 02640	22003 65864	4 98824 04555	113 80110 20024
24	25	469	10425	2 34131	53 75005	1248 53275	29279 84825	6 91619 90275	164 31511 28475
25	26	507	11726	2 73780	65 39156	1579 24585	38528 12366	9 46257 75270	233 85833 73776
50	51	1951	88451	40 52751	1897 97061	89937 13821	$\approx 4.3026 \times 10^{11}$	$\approx 2.0733 \times 10^{13}$	$\approx 1.0048 \times 10^{15}$
100	101	7651	686901	623 30501	57808 12871	$\approx 5.4249 \times 10^{11}$	$\approx 5.1397 \times 10^{13}$	$\approx 4.9047 \times 10^{15}$	$\approx 4.7076 \times 10^{17}$

TABLE II.2. The Upper Bound $\{\lambda_{n,p}\}^{\frac{1}{2p}}$

n \ p	2	3	4	5	6	7	8	9
1	1.18921	1.20094	1.25103	1.25893	1.28357	1.28911	1.30412	1.30824
2	1.31607	1.38309	1.44492	1.48169	1.51043	1.53219	1.94974	1.56411
3	1.41421	1.51309	1.60484	1.65590	1.69936	1.72867	1.75491	1.77399
4	1.49535	1.63352	1.74252	1.81173	1.86326	1.90244	1.93354	1.95886
5	1.56509	1.73205	1.86442	1.94630	2.00936	2.05539	2.09333	2.12298
6	1.62658	1.82544	1.97447	2.07094	2.14204	2.19610	2.23888	2.27364
7	1.68179	1.90637	2.07525	2.18320	2.26418	2.32456	2.37324	2.41202
8	1.73205	1.98406	2.16852	2.28913	2.37777	2.44519	2.49850	2.54178
9	1.77828	2.05357	2.25559	2.38680	2.48425	2.55757	2.61618	2.66324
10	1.82116	2.12084	2.33741	2.48001	2.58472	2.66442	2.72744	2.77859
11	1.86121	2.18225	2.41474	2.56723	2.68001	2.76527	2.83316	2.88788
12	1.89883	2.24199	2.48816	2.65112	2.77079	2.86196	2.93404	2.99256
13	1.93434	2.29731	2.55814	2.73042	2.85760	2.95402	3.03066	3.09257
14	1.96799	2.35134	2.62509	2.80712	2.94086	3.04282	3.12347	3.18895
15	2.00000	2.40187	2.68930	2.88016	3.02096	3.12793	3.21287	3.28159
25	2.25810	2.82381	3.22585	3.49736	3.69794	3.85127	3.97293	4.07172
50	2.67235	3.53489	4.15277	4.57905	4.89618	5.14039	5.33468	5.49306
100	3.17015	4.43901	5.36552	6.01825	6.50877	6.88923	7.19339	7.44219
	$(=\sqrt[4]{n+1})$							

TABLE II.2 (continued)

n \ p	10	11	16	32	64	128	256	∞
1	1.31847	1.32166	1.34408	1.37142	1.38893	1.39963	1.40595	1.41421
2	1.57615	1.58641	1.62131	1.66687	1.69459	1.71091	1.72029	1.73205
3	1.79176	1.80527	1.85389	1.91532	1.95195	1.97317	1.98520	2.00000
4	1.97992	1.99775	2.05758	2.13356	2.17835	2.20404	2.21850	2.23607
5	2.14857	2.16941	2.24073	2.33032	2.38275	2.41263	2.42935	2.44949
6	2.30250	2.32688	2.40837	2.51083	2.57049	2.60433	2.62319	2.64575
7	2.44481	2.47200	2.56374	2.67850	2.74505	2.78267	2.80356	2.82843
8	2.57769	2.60800	2.70913	2.83568	2.90885	2.95009	2.97293	3.00000
9	2.70270	2.73564	2.84619	2.98410	3.06365	3.10838	3.13309	3.16228
10	2.82102	2.85682	2.97615	3.12505	3.21077	3.25887	3.28540	3.31663
11	2.93358	2.97187	3.09999	3.25954	3.35124	3.40261	3.43090	3.46410
12	3.04109	3.08204	3.21846	3.38837	3.48589	3.54043	3.57043	3.60555
13	3.14416	3.18748	3.33218	3.51218	3.61537	3.67301	3.70467	3.74166
14	3.24326	3.28908	3.44167	3.63152	3.74024	3.80090	3.83418	3.87298
15	3.33879	3.38688	3.54736	3.74682	3.86095	3.92456	3.95942	4.00000
25	4.15382	4.22304	4.45376	4.74010	4.90308	4.99335	5.04250	5.09902
50	5.62477	5.73611	6.10765	6.56916	6.83093	6.97504	7.05297	7.14143
100	7.64963	7.82531	8.41345	9.14635	9.56173	9.78955	9.91216	10.04988
								$(=\sqrt{n+1})$

Proof Use finite differences in Table II.1, in light of Theorem 2.2.

For integer $p \geq 1$, let c_p be the coefficient of n^{p-1} in the polynomial expression for $\lambda_{n,p}$. (The next theorem will show that c_p is well defined.) The preceding corollary gives the following table.

TABLE II.3. The Coefficient c_p

p	c_p	c_p rounded	$(c_p)^{\frac{1}{2p}}$ rounded	$\left(\frac{1}{c_p}\right)^{\frac{1}{p-1}}$ rounded
1	1	1.00000	1.00000	1.00000
2	1	1.00000	1.00000	1.00000
3	3/4	.75000	.95318	1.15470
4	2/3	.66667	.95058	1.14471
5	115/192	.59896	.95004	1.13671
6	11/20	.55000	.95140	1.12701
7	5887/11520	.51102	.95318	1.11839
8	151/315	.47937	.95508	1.11076
9	259723/573440	.45292	.95695	1.10407
10	15619/36288	.43042	.95873	1.09819

The extra columns are included for later reference. Also, since (2.49) holds for all n no matter how large, we see that

$$c_p \leq \frac{p^{p-1}}{2^{p-1}(p-1)!} < \sqrt{\frac{2}{\pi p}}\left(\frac{e}{2}\right)^p \qquad (2.59)$$

where the second inequality follows from Stirling's inequality [see equation (4.21)]. An explicit form for c_p is given in the next theorem.

<u>Theorem 2.3</u> The numbers c_p, $p \geq 1$, are well defined and are given explicitly by

$$c_p = \frac{1}{2^{p-1}(p-1)!} \sum_{k=0}^{\left[\frac{p-1}{2}\right]} (-1)^k \binom{p}{k} (p-2k)^{p-1}, \quad p=1,2,3,\ldots \tag{2.60}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^p dx > 0, \quad p = 1, 2, 3, \ldots \tag{2.61}$$

Therefore, the polynomials (2.54) and (2.55) for $\lambda_{n,p}$ are of degree precisely $p - 1$ in n.

<u>Proof</u> By Theorem 2.2 and (2.52), it is clear that c_p is well defined if p is even. If p is odd, then the two expansions of (2.55) corresponding to n even and n odd, respectively, show that c_p is the same in both. Therefore, c_p is well defined. Then (2.60) follows by using (2.52) in Theorem 2.2 and examining the leading coefficient. Also, the integral (2.61) is given explicitly by Jolley [19] (see also Bromwich [8, page 518] where it is attributed without reference to Wolstenholme) and is seen to be identical to (2.60). Finally, from the integral expression (2.61), it is easy to see that c_p must be positive. This completes the proof.

We remark that the sum (2.60) seems to be related to Stirling numbers of the second kind. See [29, Part I, Chapter 4, Problem 189].

We remark also that Nuttall [27] states that the integral (2.61) is important in an electrical engineering application (where it appears in certain "nonlinear systems subject to input processes with rectangular spectra"). From the integral (2.61), Nuttall derives the asymptotic expansion

$$c_p = \sqrt{\frac{6}{\pi p}}\left\{1 - \frac{3}{20} \cdot \frac{1}{p} - \frac{13}{1120} \cdot \frac{1}{p^2} + o\left(\frac{1}{p^3}\right)\right\}, \; p \to \infty \tag{2.62}$$

and shows how to compute c_p rapidly to high accuracy (i.e., 18 significant decimal digits) for any positive integer p.

Finally, based on the preceding theorems and Corollary 2.8, we make the following Conjectures:

A. The polynomials (2.54) and (2.55) for $\lambda_{n,p}$ have positive coefficients.

B. For each fixed integer $p \geq 1$, the polynomial expressions (2.54) and (2.55) for $\lambda_{n,p}$ each have an asymptotic expansion of the form

$$\lambda_{n,p} = c_p(n+1)^{p-1}\left\{1 + o\left(\frac{1}{n^2}\right)\right\}, \; n \to \infty$$

Conjecture B is really a conjecture about the coefficient of n^{p-2} since from Theorem 2.3 we clearly have the asymptotic expansion

$$\lambda_{n,p} = c_p(n+1)^{p-1}\left\{1 + o\left(\frac{1}{n}\right)\right\}, \; n \to \infty \tag{2.63}$$

We now discuss $R_{n,2p}$ for p fixed as n goes to infinity. From Theorem 2.1 and (2.63), for each fixed $p \geq 1$, we have

$$R_{n,2p} \leq (c_p)^{\frac{1}{2p}}(n+1)^{\frac{1}{2}-\frac{1}{2p}}\left\{1 + o\left(\frac{1}{n}\right)\right\}, \; n \to \infty$$

and also

$$R_{n,2p} \geq (c_{2p})^{\frac{1}{2p}}(n+1)^{\frac{1}{2}-\frac{1}{2p}}\left\{1 + o\left(\frac{1}{n}\right)\right\}, \quad n \to \infty$$

Therefore, it is very tempting to make Conjecture C:

$$R_{n,2p} = A_p(n+1)^{\frac{1}{2}-\frac{1}{2p}}\left\{1 + o\left(\frac{1}{n}\right)\right\}, \quad n \to \infty$$

for each $p = 1, 2, \ldots$, where A_p is a constant satisfying

$$(c_{2p})^{\frac{1}{2p}} \leq A_p \leq (c_p)^{\frac{1}{2p}}$$

If Conjecture B is true, then we would replace $o(1/n)$ in Conjecture C by $o(1/n^2)$. However, at this time we prove only the following theorem.

<u>Theorem 2.4</u> For $p = 2, 3, 4$, and 5, and for all $n \geq 0$,

$$(c_{2p})^{\frac{1}{2p}}(n+1)^{\frac{1}{2}-\frac{1}{2p}} < R_{n,2p} \leq (c_p)^{\frac{1}{2p}}\left\{n + \left(\frac{1}{c_p}\right)^{\frac{1}{p-1}}\right\}^{\frac{1}{2}-\frac{1}{2p}}, \tag{2.64}$$

Furthermore, for $p = 2$ and 3,

$$\left(\frac{c_{2p}}{c_p}\right)^{\frac{1}{2p}}\{\lambda_{n,p}\}^{\frac{1}{2p}} < R_{n,2p} < \{\lambda_{n,p}\}^{\frac{1}{2p}}, \quad n \geq 0 \tag{2.65}$$

where

$$\left(\frac{c_4}{c_2}\right)^{\frac{1}{4}} = \left(\frac{2}{3}\right)^{\frac{1}{4}} \approx .90360, \quad \left(\frac{c_6}{c_3}\right)^{\frac{1}{6}} = \left(\frac{11}{15}\right)^{\frac{1}{6}} \approx .94962$$

<u>Proof</u> Direct computations in Corollary 2.8 show that for $p = 2, 3, 4$, and 5, we have

$$\lambda_{n,2p} > c_{2p}(n+1)^{2p-1} \tag{2.66}$$

$$\lambda_{n,p} \le c_p\left[n + \left(\frac{1}{c_p}\right)^{\frac{1}{p-1}}\right]^{p-1} \tag{2.67}$$

Applying (2.66) and (2.67) in (2.34) proves (2.64). The lower bound in (2.65) is proved similarly, since

$$\frac{\{\lambda_{n,4}\}^{\frac{1}{4}}}{\sqrt{n+1}} = \left\{\frac{\frac{2}{3}(n^3 + 3n^2 + \frac{7}{2}\, n + \frac{3}{2})}{(n+1)^2}\right\}^{\frac{1}{4}}$$

$$> \left\{\frac{2}{3}(n+1)\right\}^{\frac{1}{4}} = \left(\frac{c_4}{c_2}\right)^{\frac{1}{4}} \{\lambda_{n,2}\}^{\frac{1}{4}}$$

and

$$\frac{\{\lambda_{n,6}\}^{\frac{1}{6}}}{\sqrt{n+1}} = \left\{\frac{\lambda_{n,6}}{(n+1)^3}\right\}^{\frac{1}{6}}$$

$$> \left\{\frac{11}{20} \cdot \frac{4}{3} \cdot (n^2 + 2n + \frac{4}{3})\right\}^{\frac{1}{6}}$$

$$\ge \left(\frac{c_6}{c_3}\right)^{\frac{1}{6}} \{\lambda_{n,3}\}^{\frac{1}{6}}$$

This completes the proof.

Computation has shown that the first inequality in (2.65) is not valid for $p = 4$, so it cannot be generalized for larger p in a direct manner. The inequality (2.64) follows directly from (2.34). Therefore, the difficulty in generalizing (2.64) lies in extending (2.66) and (2.67) for all $p \ge 1$.

The next result is essentially a corollary of Corollary 2.8. We state it as a theorem because of its interesting form, as well as the fact that we conjecture it to hold for all $p \geq 2$ and not merely the cases cited.

Theorem 2.5 For all integer $n \geq 0$, and for $p = 2, 3, 4, 5, 6, 8$, and 10,

$$\|\pi_n\|_{2p} \leq (c_p)^{\frac{1}{2p}} \left\{ n + \left(\frac{1}{c_p}\right)^{\frac{1}{p-1}} \right\}^{\frac{1}{2} - \frac{1}{2p}} \|\pi_n\|_2 \qquad (2.68)$$

for all $\pi_n \in P_n$ with equality if and only if $\pi_n = 0$.

Proof For $p = 2, 3, 4$, and 5, this is merely a restatement of Theorem 2.4. For $p = 6, 8$, and 10, (2.68) follows as in the proof of Theorem 2.4 by noting that (2.67) is valid for $p = 6, 8$, and 10 as well. This completes the proof.

A particularly nice result is (2.68) with $p = 2$; that is,

$$\|\pi_n\|_2 \leq \|\pi_n\|_4 < (n + 1)^{\frac{1}{4}} \|\pi_n\|_2 \qquad (2.69)$$

The left hand inequality follows, as mentioned earlier, from Hölder's inequality. In particular, if π_n is restricted to have unit modulus coefficients, then $\|\pi_n\|_2 = \sqrt{n + 1}$, and

$$(n + 1)^{\frac{1}{2}} \leq \|\pi_n\|_4 < (n + 1)^{\frac{3}{4}} \qquad (2.70)$$

The lower bound in Theorem 2.4 can be used to estimate how well the extremal polynomial of $R_{n,\infty}$, namely, $\pi_n^*(z) = 1 + z + \cdots + z^n$, approximates an extremal polynomial of $R_{n,2p}$. For example, since $c_{10} > (.91915)^{10}$ and

$c_5 < (.95004)^{10}$, we have for $p = 5$

$$.91915(n+1)^{2/5} < \frac{\|\pi_n^*\|_{10}}{\|\pi_n^*\|_2}$$

$$\leq R_{n,10}$$

$$< .95004(n + 1.13672)^{2/5}$$

In particular, for $n = 100$, we have

$$5.82257 < \frac{\|\pi_{100}^*\|_{10}}{\|\pi_{100}^*\|_2} \leq R_{100,10} < 6.02152$$

This result is, of course, not as sharp as could be had from (2.34) using Corollary 2.8 (or Table II.2); that is,

$$5.82264 < \frac{(\lambda_{100,10})^{\frac{1}{10}}}{\sqrt{101}} = \frac{\|\pi_{100}^*\|_{10}}{\|\pi_{100}^*\|_2}$$

$$\leq R_{100,10} < (\lambda_{100,5})^{\frac{1}{10}} < 6.01825$$

with the first and last inequalities due merely to rounding the lower bound down and the upper bound up.

E. Extension to Derivatives

So far, bounds of the ratios of norms of π_n have been investigated. We now show that bounds similar in spirit to that of Theorem 2.1 can be given for ratios

$$\frac{\|\pi_n'\|_{2p}}{\|\pi_n\|_2} \tag{2.71}$$

where the prime denotes differentiation. An algebraic proof of such a bound requires new theorems very similar in content and proof to Lemmas 2.1-2.3. The matrix identity (2.23) of Lemma 2.1 becomes

$$\|\pi_n'\|_{2p} \equiv \left\{\bar{u}^T M_{n,p}^{(1)} u\right\}^{\frac{1}{2p}} \tag{2.72}$$

where $M_{n,p}^{(1)} = [m_{\alpha,\beta}^{(1)}]$ with

$$m_{\alpha,\beta}^{(1)} = \left(\prod_{k=1}^{p} \alpha_k \beta_k\right) \delta_{\alpha_1+\cdots+\alpha_p,\beta_1+\cdots+\beta_p} \tag{2.73}$$

The integers (2.27) in Lemma 2.2 become

$$N^{(1)}(\alpha) = \sum_{j_1,\cdots,j_p=0}^{n} \left(\prod_{k=1}^{p} j_k\right)^2 \tag{2.74}$$

with the sum taken subject to the constraint $j_1 + \cdots + j_p = \alpha_1 + \cdots + \alpha_p$, and the polynomial (2.28) becomes

$$(1 + 2^2 z + 3^2 z^2 + \cdots + n^2 z^{n-1})^p \tag{2.75}$$

As in Lemma 2.3, the coefficients of the expansion of (2.75) can be shown to constitute all the nonzero eigenvalues of $M_{n,p}^{(1)}$. Defining $\lambda_{n,p}^{(1)}$ to be the largest coefficient in the expansion of (2.75), it is then easy to show that (2.71) is bounded above by $\lambda_{n,p}^{(1)}$.

This procedure gave the original proof of the next theorem. Fortunately, however, Donald J. Newman's short proof of part of Theorem 2.1 can be adapted to prove the same theorem with less work.

<u>Notation</u>: As mentioned above, let $\lambda_{n,p}^{(1)}$ be the largest coefficient in the expansion of (2.75) into ascending powers of z.

<u>Theorem 2.6</u> For all $0 \neq \pi_n \in P_n$, and for $p = 1, 2, 3, \ldots$,

$$\frac{\|\pi_n'\|_{2p}}{\|\pi_n\|_2} \leq \{\lambda_{n,p}^{(1)}\}^{\frac{1}{2p}} \leq n^{\frac{1}{p}}(1^2 + 2^2 + 3^2 + \cdots + n^2)^{\frac{1}{2} - \frac{1}{2p}} \tag{2.76}$$

Furthermore,

$$\frac{\|\pi_n'\|_\infty}{\|\pi_n\|_2} \leq \sqrt{\frac{1}{6}\, n(n+1)(2n+1)} \tag{2.77}$$

and equality is attained in (2.77) for

$$\pi_n(z) = z(1 + z + z^2 + \cdots + z^n)'$$

<u>Proof</u> With $\pi_n(z) = a_0 + a_1 z + \cdots + a_n z^n \neq 0$,

$$z\pi_n'(z) = \sum_{k=0}^{n} k a_k z^k \tag{2.78}$$

Now, let

$$z^p(1^2 + 2^2 z + 3^2 z^2 + \cdots + n^2 z^{n-1})^p = \sum_{j=0}^{np} \lambda_j^{(1)} z^j \tag{2.79}$$

so that

$$\lambda_{n,p}^{(1)} = \max_{0 \leq j \leq np} \lambda_j^{(1)}$$

With $z = e^{i\theta}$,

$$\begin{aligned}\|\pi_n'\|_{2p}^{2p} &= \frac{1}{2\pi}\int_C |\pi_n'(z)|^{2p}|z|^{2p}\, d\theta \\ &= \frac{1}{2\pi}\int_C |[z\pi_n'(z)]^p|^2\, d\theta \\ &= \frac{1}{2\pi}\int_C \left|\left(\sum_{j=0}^{n} j a_j z^j\right)^p\right|^2 d\theta\end{aligned}$$

$$= \sum_{j=0}^{np} \left| \sum_{\alpha_1+\cdots+\alpha_p=j} (\alpha_1 a_{\alpha_1}) \cdots (\alpha_p a_{\alpha_p}) \right|^2 \tag{2.80}$$

with the sum over $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$. By Schwarz's Inequality,

$$\left| \sum_{\alpha_1+\cdots+\alpha_p=j} (\alpha_1 \cdots \alpha_p)(a_{\alpha_1} \cdots a_{\alpha_p}) \right|^2$$

$$\leq \left(\sum_{\alpha_1+\cdots+\alpha_p=j} |\alpha_1 \cdots \alpha_p|^2 \right)$$

$$\cdot \left(\sum_{\alpha_1+\cdots+\ p=j} |a_{\alpha_1} \cdots a_{\alpha_p}|^2 \right)$$

$$= \lambda_j^{(1)} \sum_{\alpha_1+\cdots+\alpha_p=j} |a_{\alpha_1} \cdots a_{\alpha_p}|^2$$

Hence, we have

$$\|\pi_n'\|_{2p}^{2p} \leq \sum_{j=0}^{np} \left(\lambda_j^{(1)} \sum_{\alpha_1+\cdots+\alpha_p=j} |a_{\alpha_1} \cdots a_{\alpha_p}|^2 \right)$$

$$\leq \lambda_{n,p}^{(1)} \left(\sum_{j=0}^{n} |a_j|^2 \right)^p$$

$$= \lambda_{n,p}^{(1)} \|\pi_n\|_2^{2p} \tag{2.81}$$

This proves the first inequality in (2.76). From the identity

$$(1^2 + 2^2 z + \cdots + n^2 z^{n-1})^p$$

$$= (1^2 + 2^2 z + \cdots + n^2 z^{n-1})$$

$$\cdot (1^2 + 2^2 z + \cdots + n^2 z^{n-1})^{p-1}, \ p \geq 1$$

we have the inequality

$$\lambda_{n,p}^{(1)} \le (1^2 + 2^2 + \cdots + n^2)\lambda_{n,p-1}^{(1)}$$

Since $\lambda_{n,1}^{(1)} = n^2$, this implies

$$\lambda_{n,p}^{(1)} \le n^2(1^2 + 2^2 + \cdots + n^2)^{p-1} \tag{2.82}$$

Extracting the 2p-th roots yields the second inequality in (2.76).

Finally, the Schwarz Inequality implies

$$\begin{aligned} \|\pi_n'\|_\infty &= |\pi_n'(z_0)|, \qquad \text{some } z_0 \in C \\ &= \left| \sum_{k=0}^{n} k a_k z_0^{\,k} \right| \\ &\le \left\{ \sum_{k=0}^{n} |k z_0^{\,k}|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{k=0}^{n} |a_k|^2 \right\}^{\frac{1}{2}} \\ &= \left(\sum_{k=0}^{n} k^2 \right)^{\frac{1}{2}} \|\pi_n\|_2 \end{aligned} \tag{2.83}$$

with equality possible in (2.83) for $n \ge 0$, e.g., with $\bar{\pi}_n(z) = z(1 + z + \cdots + z^n)'$. Using the identity

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{2.84}$$

in (2.83) completes the proof.

Corollary 2.9 For all $0 \ne \pi_n \in P_n$, and for $p = 1, 2, 3, \ldots$,

$$\frac{\|\pi_n'\|_{2p}}{\|\pi_n\|_2} < \left(\frac{1}{3}\right)^{\frac{1}{2} - \frac{1}{2p}} (n+1)^{\frac{3}{2} - \frac{1}{2p}}, \quad n = 0, 1, 2, \ldots \tag{2.85}$$

Proof Follows from (2.76) and (2.84).

Corollary 2.10 For all $0 \neq \pi_n \in P_n$,

$$\frac{\|\pi_n'\|_2}{\|\pi_n\|_2} \leq n \tag{2.86}$$

with equality possible in (2.86) for all $n \geq 0$.

Proof Use (2.76) since $\lambda_{n,1}^{(1)} = n^2$. Let $\pi_n(z) = z^n$ to prove equality in (2.86).

The inequality (2.86) can be obtained directly. For $\pi_n(z) = a_0 + a_1 z + \cdots + a_n z^n$,

$$\begin{aligned} \|\pi_n'\|_2 &= \left\| \sum_{k=0}^{n} k a_k z^{k-1} \right\|_2 \\ &= \left[\sum_{k=0}^{n} k^2 |a_k|^2 \right]^{\frac{1}{2}} \\ &\leq n \|\pi_n\|_2 \end{aligned}$$

This method shows easily that the maximum in (2.86) is attained only for nonzero constant multiples of $\pi_n(z) = z^n$.

These results are easily generalized to higher order derivatives. Define $\lambda_{n,p}^{(k)}$ for $k = 0, 1, \ldots, n$, to be the largest coefficient in the expansion of

$$\left\{ \sum_{\ell=k}^{n} \binom{\ell}{k}^2 z^{\ell-k} \right\}^p \tag{2.87}$$

into ascending powers of z. Denote the k-th derivative of π_n by $\pi_n^{(k)}$.

Theorem 2.7 Let $0 \leq k \leq n$. For all $0 \neq \pi_n \in P_n$, and for $p = 1, 2, 3, \ldots$,

$$\frac{\|\pi_n^{(k)}\|_{2p}}{\|\pi_n\|_2} \le k!\left\{\lambda_{n,p}^{(k)}\right\}^{\frac{1}{2p}} \tag{2.88}$$

$$\le k!\binom{n}{k}^{\frac{1}{p}} Q^{\frac{1}{2}-\frac{1}{2p}} \tag{2.89}$$

$$< k!\ \sqrt{Q} \tag{2.90}$$

where

$$Q = \sum_{\ell=k}^{n}\binom{\ell}{k}^2 \tag{2.91}$$

Furthermore,

$$\frac{\|\pi_n^{(k)}\|_{\infty}}{\|\pi_n\|_2} \le k!\ \sqrt{Q} \tag{2.92}$$

$$< k!\ \sqrt{n-k+1}\ \binom{n}{k}$$

with equality attained in (2.92) by

$$\pi_n(z) = z^k(1 + z + z^2 + \cdots + z^n)^{(k)}$$

The proof of this theorem so closely follows the proof of Theorem 2.6 that it is not given here. Alternatively, the proof could proceed algebraically by proving results analogous to Lemmas 2.1-2.3. We emphasize that (2.88) is the spectral radius of an operator $M_{n,p}^{(k)}$ which can be defined in a manner analogous to $M_{n,p}$ in (2.23) and $M_{n,p}^{(1)}$ in (2.72).

A more natural bound than (2.88) is given later in Theorem 3.7. The result given there is not, however, as good as (2.88).

Corollary 2.11 Under the conditions of Theorem 2.7,

$$\frac{\|\pi_n^{(k)}\|_{2p}}{\|\pi_n\|_2} < \left(\frac{1}{2k+1}\right)^{\frac{1}{2}-\frac{1}{2p}} (n+1)^{k+\frac{1}{2}-\frac{1}{2p}} \tag{2.93}$$

Proof From (2.91),

$$Q < \left(\frac{1}{k!}\right)^2 \sum_{j=k}^{n} j^{2k} < \left(\frac{1}{k!}\right)^2 \int_k^{n+1} t^{2k}\, dt = \frac{(n+1)^{2k+1}}{(2k+1)(k!)^2}$$

so that in (2.89),

$$k!\binom{n}{k}^{\frac{1}{p}} Q^{\frac{1}{2}-\frac{1}{2p}} < (k!)^{1-\frac{1}{p}} n^{\frac{k}{p}} \left\{\frac{(n+1)^{2k+1}}{(2k+1)(k!)^2}\right\}^{\frac{1}{2}-\frac{1}{2p}}$$

which proves (2.93).

Better estimates than (2.93) can, of course, be found by using better estimates of Q in the proof of Corollary 2.11.

Corollary 2.12 Under the conditions of Theorem 2.7,

$$\lim_{p\to\infty} \{\lambda_{n,p}^{(k)}\}^{\frac{1}{p}} = Q \tag{2.94}$$

Proof Define

$$\pi_n(z) = \sum_{j=0}^{n} j(j-1)\cdots(j-k+1)z^j$$

Then

$$\frac{\|\pi_n^{(k)}\|_{2p}}{\|\pi_n\|_2} \le k!\{\lambda_{n,p}^{(k)}\}^{\frac{1}{2p}} \le k!\,\sqrt{Q}$$

Since

$$\frac{\|\pi_n^{(k)}\|_\infty}{\|\pi_n\|_2} = k!\ \sqrt{Q}$$

(2.94) follows immediately.

Chapter III

COMPLEX FINITE DIMENSIONAL FUNCTION SPACES

A. General Spaces

The method of the preceding chapter can be generalized considerably. One direction is to replace the unit circle by a rectifiable Jordan curve as in Szegö [31, Chap. XVI]. Another direction is to replace the unit circle by a real interval and change the integral to some Lebesgue-Stieltjes integral as in Szegö [31, Chap. I]. A modified version of this latter direction is taken here because of the nature of the examples in Chapter IV; however, the last part of this chapter deals again with complex polynomials on the unit circle.

Let $\omega(t)$ and $\phi(t)$ be nonnegative Lebesgue measurable functions on the real intervals (a,b) and (c,d), respectively, such that $-\infty \le a < b \le +\infty$, $-\infty \le c < d \le +\infty$, and

$$0 < \int_a^b \omega(t)dt < +\infty, \qquad 0 < \int_c^d \phi(t)dt < +\infty \tag{3.1}$$

We further assume that $\omega(t) > 0$ almost everywhere on (a,b). For extended real numbers $p \ge 1$, let $L_p^\omega[a,b]$ be the class of measurable complex valued functions f on (a,b) such that

$$\|f\|_p^\omega = \begin{cases} \left\{\int_a^b |f(t)|^p \,\omega(t)dt\right\}^{\frac{1}{p}} < \infty, & 1 \le p < +\infty \quad (3.2) \\ \inf_g \sup_{a<t<b} |g(t)|, & p = \infty \quad (3.3) \end{cases}$$

where the inf in (3.3) is taken over all bounded measurable functions g on (a,b) which equal f almost everywhere. Note that $\|f\|_\infty^\omega$ is not necessarily equal to $\|f\|_\infty^\phi$ because these norms depend on the interval of definition of ω and ϕ, respectively. Note also, that $\omega(t) > 0$ almost everywhere on (a,b) implies that $\|f\|_p^\omega = 0$ if and only if $f(t) = 0$ almost everywhere on (a,b). From this point on, we will consider two measurable functions equivalent on (a,b) if they are equal almost everywhere on (a,b). As is customary, we regard $L_p^\omega[a,b]$ and $L_p^\phi[c,d]$ as equivalence classes of functions. Therefore,

$$(f,g)_\omega = \int_a^b f(t)\overline{g(t)}\ (t)dt \tag{3.4}$$

defines an inner product on $L_2^\omega[a,b]$.

<u>Lemma 3.1</u> Let $p \geq 2$ be an integer. Let $g_k \in L_p^\phi[c,d]$, $k = 1, \ldots, p$. Then

$$\int_c^d \left| \prod_{k=1}^p g_k(t) \right| \phi(t)dt \leq \prod_{k=1}^p \|g_k\|_p^\phi \tag{3.5}$$

If $g_k \neq 0$, $k = 1, \ldots, p$, almost everywhere on (c,d), and if $\phi(t) > 0$ almost everywhere on (c,d), then (3.5) is an equality if and only if there exist nonzero constants $c_1, \ldots, c_p$ such that $c_1|g_1| = c_2|g_2| = \cdots = c_p|g_p|$ almost everywhere on (c,d).

<u>Proof</u> For p = 2, the Cauchy-Schwarz Inequality gives

$$\int_c^d |g_1(t)g_2(t)|\phi(t)dt \leq \|g_1\|_2^\phi \|g_2\|_2^\phi$$

If $g_1 \neq 0$, $g_2 \neq 0$, and $\phi(t) > 0$ almost everywhere on (c,d), then we have equality iff there exist nonzero constants c_1 and c_2 such that $c_1|g_1| = c_2|g_2|$ almost everywhere on (c,d). Now, suppose the result holds for some $p \geq 2$. With $\frac{1}{p+1} + \frac{1}{q} = 1$,

$$\int_c^d \left| \prod_{k=1}^{p+1} g_k \right| \phi = \int_c^d |g_{p+1}| \left| \prod_{k=1}^{p} g_k \right| \phi$$

$$\leq \left(\int_c^d |g_{p+1}|^{p+1} \phi \right)^{\frac{1}{p+1}} \left(\int_c^d |g_1 \cdots g_p|^q \phi \right)^{\frac{1}{q}} \tag{3.6}$$

$$\leq \|g_{p+1}\|^{\phi}_{p+1} \left(\prod_{k=1}^{p} \|(|g_k|)^q\|^{\phi}_{p} \right)^{\frac{1}{q}} \tag{3.7}$$

$$= \prod_{k=1}^{p+1} \|g_k\|^{\phi}_{p+1} < +\infty \tag{3.8}$$

where (3.6) is Hölder's inequality, (3.7) is the induction hypothesis, and (3.8) follows from $pq = p + 1$. If $g_k \neq 0$, $k = 1, \ldots, p$, and $\phi(t) > 0$ almost everywhere on (c,d), then (3.6) is an equality iff there exist nonzero constants α and β such that

$$\alpha|g_{p+1}|^{p+1} = \beta|g_1 \cdots g_p|^q \quad \text{a.e.} \tag{3.9}$$

By the induction hypothesis, equality holds in (3.7) iff there exist nonzero constants $c_1, \ldots, c_p$ such that

$$c_1|g_1| = \cdots = c_p|g_p|, \qquad \text{a.e.} \tag{3.10}$$

Therefore,

$$\int_c^d \left|\prod_{k=1}^{p+1} g_k\right| \phi = \prod_{k=1}^{p+1} \|g_k\|_{p+1}^{\phi}$$

if and only if (3.9) and (3.10) both hold. Now (3.9) and (3.10) imply

$$\begin{aligned} \alpha|g_{p+1}|^{p+1} &= \beta\left[\left(\frac{c_p}{c_1}|g_p|\right)\left(\frac{c_p}{c_2}|g_p|\right)\cdots\left(|g_p|\right)\right]^q, && \text{a.e.} \\ &= \delta|g_p|^{pq}, && \text{a.e.} \\ &= \delta|g_p|^{p+1}, && \text{a.e.} \end{aligned}$$

where δ is the obvious nonzero constant. With

$$c_{p+1} = \left(\frac{\alpha c_p}{\delta}\right)^{\frac{1}{p+1}}$$

we see that

$$c_1|g_1| = \cdots = c_p|g_p| = c_{p+1}|g_{p+1}|, \qquad \text{a.e.} \tag{3.11}$$

Since (3.11) also implies (3.9) and (3.10), this completes the proof.

Lemma 3.2 Let $p \geq 1$ be an integer. Let P_n be a subspace of $L_2^{\omega}[a,b] \cap L_{2p}^{\phi}[c,d]$ with a basis $\{h_0, h_1, \ldots, h_n\}$ which is orthonormal with respect to the inner product $(f,g)_{\omega}$. Let

$$\pi_n(t) = a_0h_0(t) + a_1h_1(t) + \cdots + a_nh_n(t) \in P_n$$

and let $x = \langle a_0\ a_1\ \cdots\ a_n \rangle^T \in \mathbb{C}^{n+1}$. Then

$$\|\pi_n\|_{2p}^{\phi} \equiv \{\bar{u}^T L_{n,p}^{\phi}\ u\}^{\frac{1}{2p}} \tag{3.12}$$

and

$$\|\pi_n\|_2^{\omega} \equiv \{\bar{u}^T\ I\ u\}^{\frac{1}{2p}} = \{\bar{u}^T\ u\}^{\frac{1}{2p}} \tag{3.13}$$

where $u = x \otimes \cdots \otimes x$ (p factors of x) $\in \mathbb{C}^{(n+1)^p}$, I is the identity matrix of order $(n+1)^p$, and $L_{n,p}^{\phi} = [\mu_{\alpha,\beta}]$ is the hermitian matrix of dimension $(n+1)^p \times (n+1)^p$ given by

$$\mu_{\alpha,\beta} = (h_{\beta_1} \cdots h_{\beta_p}, h_{\alpha_1} \cdots h_{\alpha_p})_{\phi} < +\infty \tag{3.14}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$, $\beta = (\beta_1, \cdots, \beta_p) \in \Gamma$.

<u>Proof</u> Since

$$[\pi_n(t)]^p = \sum_{\alpha_1,\ldots,\alpha_p=0}^{n} a_{\alpha_1} \cdots a_{\alpha_p} h_{\alpha_1}(t) \cdots h_{\alpha_p}(t) \tag{3.15}$$

we have

$$\begin{aligned}
(\|\pi_n\|_{2p}^{\phi})^{2p} &= \int_c^d [\pi_n(t)]^p [\overline{\pi_n(t)}]^p \phi(t)dt \\
&= \int_c^d \left[\sum_{\beta_1,\ldots,\beta_p=0}^{n} a_{\beta_1} \cdots a_{\beta_p} h_{\beta_1} \cdots h_{\beta_p}\right] \\
&\quad \cdot \left[\sum_{\alpha_1,\ldots,\alpha_p=0}^{n} \overline{a_{\alpha_1} \cdots a_{\alpha_p} h_{\alpha_1} \cdots h_{\alpha_p}}\right] \phi \\
&= \sum_{\alpha_1,\ldots,\alpha_p} \sum_{\beta_1,\ldots,\beta_p} \overline{a_{\alpha_1} \cdots a_{\alpha_p}} a_{\beta_1} \cdots a_{\beta_p} \mu_{\alpha,\beta} \\
&= \bar{u}^T L_{n,p}^{\phi}\ u
\end{aligned}$$

This proves (3.12). That $L_{n,p}^{\phi}$ is hermitian follows from $\mu_{\alpha,\beta} = \overline{\mu_{\beta,\alpha}}$. The finiteness of (3.14) is an immediate consequence of Lemma 3.1. Finally, since the basis is orthonormal,

$$(\|\pi_n\|_2^{\omega})^2 = |a_0|^2 + |a_1|^2 + \cdots + |a_n|^2$$

$$= \sum_{k=0}^{n} a_k \overline{a_k}$$

so that

$$(\|\pi_n\|_2^{\omega})^{2p} = \left(\sum_{k=0}^{n} a_k \overline{a_k}\right)^p$$

$$= \sum_{\alpha_1,\ldots,\alpha_p=0} (a_{\alpha_1}\cdots a_{\alpha_p})(\overline{a_{\alpha_1}\cdots a_{\alpha_p}})$$

$$= \bar{u}^T u$$

This concludes the proof.

<u>Lemma 3.3</u> Under the hypotheses of Lemma 3.2, the matrix $L_{n,p}^{\phi}$ is positive semidefinite.

<u>Proof</u> Let $v = \langle v_\alpha \rangle$, $\alpha = (\alpha_1,\ldots,\alpha_p) \in \Gamma$, be an arbitrary vector in $C^{(n+1)^p}$. Then, with $\beta = (\beta_1,\ldots,\beta_p) \in \Gamma$,

$$L_{n,p}^{\phi} v = \left\langle \sum_{\beta\in\Gamma} \mu_{\alpha,\beta}\, v_\beta \right\rangle_{\alpha\in\Gamma}$$

$$= \left\langle \sum_{\beta_1,\ldots,\beta_p} (h_{\beta_1}\cdots h_{\beta_p}, h_{\alpha_1}\cdots h_{\alpha_p})_\phi v_{\beta_1,\ldots,\beta_p} \right\rangle_{\alpha\in\Gamma}$$

$$= \left\langle (v_0, h_{\alpha_1}\cdots h_{\alpha_p})_\phi \right\rangle_{\alpha\in\Gamma}$$

where

$$v_0 = \sum_{\beta_1,\ldots,\beta_p=0}^{n} v_{\beta_1,\ldots,\beta_p} h_{\beta_1} \cdots h_{\beta_p}$$

Therefore,

$$\bar{v}^T L_{n,p}^{\phi} v = \sum_{\alpha_1,\ldots,\alpha_p=0}^{n} \bar{v}_{\alpha_1,\ldots,\alpha_p} (v_0, h_{\alpha_1} \cdots h_{\alpha_p})_{\phi}$$

$$= (v_0, v_0)_{\phi} \geq 0$$

Since a hermitian matrix is positive semidefinite if and only if its hermitian form is nonnegative, the proof is complete.

The next result uses the term "reproducing kernel." This terminology is not universally accepted. For example, Szegö [31, equation (3.19)] uses simply "kernel polynomial" when discussing algebraic polynomials. In any event, all that is needed here is the definition embodied in (3.17).

Theorem 3.1 Under the hypotheses of Lemma 3.2, the trace of $L_{n,p}^{\phi}$ is given by

$$\text{Trace } (L_{n,p}^{\phi}) = \left\{ \| K_n(t,t) \|_p^{\phi} \right\}^p \tag{3.16}$$

where

$$K_n(t,s) \equiv \sum_{k=0}^{n} h_k(t) \overline{h_k(s)} \tag{3.17}$$

is the "reproducing kernel" of P_n in $L_2^{\omega}[a,b]$.

Proof Trace $(L_{n,p}^{\phi}) = \sum_{\alpha \in \Gamma} \mu_{\alpha,\alpha}$

$$= \sum_{\alpha_1,\ldots,\alpha_p=0} \int_c^d h_{\alpha_1}\cdots h_{\alpha_p}\overline{h_{\alpha_1}\cdots h_{\alpha_p}}\,\phi$$

$$= \int_c^d \left\{ \sum_{\alpha_1,\ldots,\alpha_p=0}^{n} |h_{\alpha_1}\cdots h_{\alpha_p}|^2 \right\} \phi$$

$$= \int_c^d \left\{ \sum_{k=0}^{n} |h_k|^2 \right\}^p \phi$$

$$= \left\{ \|K_n(t,t)\|_p^{\phi} \right\}^p$$

This completes the proof.

Corollary 3.4 Under the conditions of Theorem 3.1, for all $0 \neq \pi_n \in P_n$,

$$\frac{\|\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}} \leq \sqrt{\|K_n(t,t)\|_p^{\phi}} \tag{3.18}$$

Proof Put $\pi_n(t) = a_0h_0(t) + a_1h_1(t) + \cdots + a_nh_n(t)$. By Lemma 3.2,

$$\frac{\|\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}} = \left\{ \frac{\bar{u}^T L_{n,p}^{\phi} u}{\bar{u}^T u} \right\}^{\frac{1}{2p}}$$

$$\leq \left\{ \max_v \frac{\bar{v}^T L_{n,p}^{\phi} v}{\bar{v}^T v} \right\}^{\frac{1}{2p}}$$

$$= \{\lambda\}^{\frac{1}{2p}} \tag{3.19}$$

where v is an arbitrary vector in $\mathbb{C}^{(n+1)^p}$ and λ is the largest eigenvalue of $L_{n,p}^{\phi}$. Since the trace is the sum of the eigenvalues and $L_{n,p}^{\phi}$ is positive semidefinite and so has only nonnegative eigenvalues (see, e.g., [24, Section 4.12]),

$$\lambda \leq \text{Trace } (L^{\phi}_{n,p})$$

and this proves the corollary.

Note that Corollary 3.4 is merely a special case of Theorem 1.1. See also Theorem 3.4 and Corollary 3.11.

B. Spaces Satisfying a Nonnegativity Condition

We now restrict our attention to those spaces for which $L^{\phi}_{n,p}$ is a nonnegative matrix, i.e., has only nonnegative entries.

Definition The functions $\{f_0, f_1, \ldots, f_n\} \in L^{\omega}_2[a,b] \cap L^{\phi}_{2p}[c,d]$ satisfy the Nonnegativity Condition in $L^{\phi}_{2p}[c,d]$ if and only if

$$0 \leq (f_{\beta_1} \cdots f_{\beta_p}, f_{\alpha_1} \cdots f_{\alpha_p})_{\phi} < +\infty \tag{3.20}$$

for every choice of $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$ and $\beta = (\beta_1, \ldots, \beta_p) \in \Gamma$.

Note that the finiteness of the inner products in (3.20) is implied by the requirement that each f_k be in $L^{\phi}_{2p}[c,d]$ and Lemma 3.1.

It is clear that the matrix $L^{\phi}_{n,p}$ defined in (3.12) is nonnegative if and only if the orthonormal basis $\{h_0, h_1, \ldots, h_n\}$ of P_n in $L^{\omega}_2[a,b]$ satisfies the Nonnegativity Condition in the space $L^{\phi}_{2p}[c,d]$. This condition may seem to be very restrictive, but an inspection of the examples in Chapter IV shows that a great many of the classical orthonormal polynomials in $L_2^{\omega}[-1,+1]$ satisfy the Nonnegativity Condition in $L_{2p}^{\phi}[-1,+1]$ for many different weight functions ϕ.

<u>Theorem 3.2</u> Let $p \geq 1$ be an integer. Let P_n be a subspace of $L_2^{\omega}[a,b] \cap L_{2p}^{\phi}[c,d]$ with a basis $\{h_0, h_1, \ldots, h_n\}$ which is orthonormal with respect to the inner product $(f,g)_{\omega}$ and satisfies the Nonnegativity Condition in $L_{2p}^{\phi}[c,d]$. Then, for all $0 \neq \pi_n \in P_n$,

$$\frac{\|\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}} \leq \max_{0 \leq k \leq n} \sqrt{\|h_k S_n\|_p^{\phi}} \tag{3.21}$$

where

$$S_n(t) = h_0(t) + h_1(t) + \cdots + h_n(t) \tag{3.22}$$

<u>Proof</u> Let $\pi_n(t) = a_0 h_0(t) + \cdots + a_n h_n(t) \neq 0$. First, suppose that there does not exist h_k with $\|h_k\|_{2p}^{\phi} > 0$. Then $\|h_k\|_{2p}^{\phi} = 0$ for all k, and Minkowski's inequality gives

$$\|\pi_n\|_{2p}^{\phi} \leq \sum_{k=0}^{n} |a_k| \, \|h_k\|_{2p}^{\phi} = 0$$

Hence the left hand side of (3.21) is identically zero and (3.21) necessarily true. On the other hand, suppose there does exist k such that $\|h_k\|_{2p}^{\phi} > 0$. Then, via (3.20),

$$\begin{aligned} 0 < \|h_k\|_{2p}^{\phi} &= \left\{\int_c^d (h_k)^p (\bar{h}_k)^p \phi\right\}^{\frac{1}{2p}} \\ &\leq \left\{\int_c^d (S_n)^p (\bar{S}_n)^p \phi\right\}^{\frac{1}{2p}} \\ &= \|S_n\|_{2p}^{\phi} \end{aligned} \tag{3.23}$$

Therefore,

$$0 < \int_c^d |S_n(t)|^p \phi(t)\,dt < +\infty \qquad (3.24)$$

Now, from (3.12) and (3.13),

$$\frac{\|\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}} = \left\{\frac{\bar{u}^T L_{n,p}^{\phi} u}{\bar{u}^T u}\right\}^{\frac{1}{2p}}$$

$$\leq \max_v \left\{\frac{\bar{v}^T L_{n,p}^{\phi} v}{\bar{v}^T v}\right\}^{\frac{1}{2p}} \leq \{\lambda\}^{\frac{1}{2p}}$$

where v is an arbitrary nonzero vector in $\mathbf{C}^{(n+1)^p}$, and λ is the largest eigenvalue of the hermitian positive semidefinite matrix $L_{n,p}^{\phi}$. Furthermore, $L_{n,p}^{\phi}$ is nonnegative because of the Nonnegativity Condition in $L_{2p}^{\phi}[c,d]$. Now Gershgorin's theorem (see, e.g., Marcus and Minc [24, Section 2.2]) applied to any nonnegative matrix implies that the largest row sum is an upper bound for all the eigenvalues. Thus, from (3.14),

$$\lambda \leq \max_{\alpha\in\Gamma}\left\{\sum_{\beta\in\Gamma} \mu_{\alpha,\beta}\right\}$$

$$= \max_{\alpha_1,\ldots,\alpha_p}\left\{\sum_{\beta_1,\ldots,\beta_p=0}^{n} (h_{\beta_1}\cdots h_{\beta_p}, h_{\alpha_1}\cdots h_{\alpha_p})_{\phi}\right\}$$

$$= \max_{\alpha_1,\ldots,\alpha_p} \left(\sum_{\beta_1,\ldots,\beta_p} h_{\beta_1}\cdots h_{\beta_p}, h_{\alpha_1}\cdots h_{\alpha_p}\right)_{\phi}$$

$$= \max_{\alpha_1,\ldots,\alpha_p} ((S_n)^p, h_{\alpha_1}\cdots h_{\alpha_p})_{\phi} \qquad (3.25)$$

$$\leq \max_{\alpha_1,\ldots,\alpha_p} \int_c^d |h_{\alpha_1}(t)\cdots h_{\alpha_p}(t)|\,|S_n(t)|^p \phi(t)\,dt \qquad (3.26)$$

Now let $W(t) = \phi(t)|S_n(t)|^p$. Then (3.24) implies that Lemma 3.1 can be applied in (3.26) to get

$$\lambda \le \max_{\alpha_1,\dots,\alpha_p} \int_c^d |h_{\alpha_1}(t)\cdots h_{\alpha_p}(t)|W(t)dt$$

$$= \max_{0\le k\le n} \int_c^d |h_k(t)|^p W(t)dt$$

$$= \max_{0\le k\le n} \int_c^d |h_k(t)S_n(t)|^p \phi(t)dt$$

Since (3.21) follows immediately, this concludes the proof.

Corollary 3.5 Under the conditions of Theorem 3.2,

$$\frac{\|\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}} \le \inf \sqrt{B_{rp}^{\phi}\|S_n\|_{sp}^{\phi}} < +\infty \qquad (3.27)$$

where

$$B_{rp}^{\phi} = \max_{0\le k\le n} \|h_k\|_{rp}^{\phi} \qquad (3.28)$$

and the infimum in (3.27) is taken over all extended real numbers $r \ge 1$ and $s \ge 1$ satisfying $\frac{1}{r} + \frac{1}{s} = 1$.

Proof From Hölder's inequality,

$$\int_c^d |h_k S_n|^p \phi = \int_c^d |h_k^p||S_n^p|\phi$$

$$\le \left\{\int_c^d |h_k^p|^r \phi\right\}^{\frac{1}{r}} \left\{\int_c^d |S_n^p|^s \phi\right\}^{\frac{1}{s}}$$

$$= \left(\|h_k\|_{rp}^{\phi}\|S_n\|_{sp}^{\phi}\right)^p, \quad k = 0, 1, \dots, n$$

which proves the first inequality (3.27). The finiteness of the bound follows from the case $r = s = 2$ and the fact that $P_n \subset L_{2p}^{\phi}[c,d]$. This completes the proof.

Corollary 3.6 Under the conditions of Theorem 3.2, for all $\pi_n \in P_n$,

$$\|\pi_n\|_{2p}^{\phi} \le \|S_n\|_{2p}^{\phi} \|\pi_n\|_2^{\omega} \tag{3.29}$$

and

$$\|\pi_n\|_{2p}^{\phi} \le \sqrt{n+1}\, B_{2p}^{\phi} \|\pi_n\|_2^{\omega} \tag{3.30}$$

Proof From Minkowski's Inequality,

$$\|S_n\|_{2p}^{\phi} \le \sum_{k=0}^{n} \|h_k\|_{2p}^{\phi} \le (n+1) B_{2p}^{\phi}$$

On the other hand, (3.23) proves that

$$B_{2p}^{\phi} = \max_{0 \le k \le n} \|h_k\|_{2p}^{\phi} \le \|S_n\|_{2p}^{\phi}$$

Also, for $r = s = 2$ in Corollary 3.5, we have

$$\|\pi_n\|_{2p}^{\phi} \le \sqrt{B_{2p}^{\phi} \|S_n\|_{2p}^{\phi}}\; \|\pi_n\|_2^{\omega} \tag{3.31}$$

The last three inequalities prove (3.29) and (3.30) immediately.

Corollary 3.7 Under the conditions of Theorem 3.2,

$$\frac{\|\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}} \le \left\{B_{\infty}^{\phi} \|S_n\|_{\infty}^{\phi}\right\}^{\frac{1}{2}} \left\{\frac{\|S_n\|_2^{\phi}}{\|S_n\|_{\infty}^{\phi}}\right\}^{\frac{1}{p}}, \quad p \ge 2 \tag{3.32}$$

provided the norms $\|h_k\|_{\infty}^{\phi}$, $k = 0, 1, \ldots, n$, are finite.

<u>Proof</u> For each $k = 0, 1, \ldots, n$,

$$\left(\|h_k s_n\|_p^{\phi}\right)^p = \int_c^d |h_k s_n|^p \, \phi$$

$$\leq \left(\|h_k s_n\|_{\infty}^{\phi}\right)^{p-2} \int_c^d |h_k s_n|^2 \, \phi$$

$$\leq \left(\|h_k\|_{\infty}^{\phi}\|s_n\|_{\infty}^{\phi}\right)^{p-2}\left(\|h_k\|_{\infty}^{\phi}\right)^2 \int_c^d |s_n|^2 \, \phi$$

$$= \left(\|h_k\|_{\infty}^{\phi}\right)^p\left(\|s_n\|_{\infty}^{\phi}\right)^{p-2}\left(\|s_n\|_2^{\phi}\right)^2$$

Extracting 2p-th roots proves (3.32).

The examples studied in Chapter IV will show that (3.32) gives (in some spaces) the same order of magnitude bounds for all 2p norms that the next theorem gives for all 2^p norms. For example, when $(a,b) = (c,d) = (-1,+1)$ and $\phi(t) = \omega(t) = 1$, Lemma 4.5 will show that the Non-negativity Condition required in Theorem 3.3 is satisfied. Therefore, from (3.36), for all polynomials π_n of degree at most n with real or complex coefficients, we have

$$\|\pi_n\|_{2^p}^{\omega} \leq \left[(n+1)(n+\tfrac{1}{2})\right]^{\frac{1}{2}-\frac{1}{2^p}} \|\pi_n\|_2^{\omega}$$

$$\leq (n+1)^{1-\frac{1}{2^{p-1}}} \|\pi_n\|_2^{\omega}, \quad p = 1, 2, 3, \ldots \qquad (3.33)$$

since

$$\left\{\sqrt{k+\tfrac{1}{2}}\, P_k(t)\right\}$$

form the orthonormal basis $\{h_k\}$, where $P_k(t)$ is the k-th degree Legendre polynomial, and $B_\infty^\omega = \sqrt{n + \frac{1}{2}}$. On the other hand, Corollary 3.7 will lead to Theorem 4.2, which in this space implies that

$$\|\pi_n\|_{2p}^\omega \le A\left(n + \frac{3}{2}\right)^{1 - \frac{1}{p}} \|\pi_n\|_2^\omega, \quad p = 1, 2, 3, \ldots$$

where the constant A can be taken equal to $\sqrt{3/2}\,\exp(1/12)$.

<u>Theorem 3.3</u> Let $p \ge 2$ be an integer. Let P_n be a subspace of $L_2^\omega[a,b] \cap L_{2^p}^\phi[c,d]$ with a basis $\{h_0, h_1, \ldots, h_n\}$ which is orthonormal with respect to the inner product $(f,g)_\omega$ and satisfies the Nonnegativity Condition in the spaces $L_{2^k}^\phi[c,d]$, $k = 2, \ldots, p$. If $B_\infty^\phi < \infty$, then for all $\pi_n \in P_n$,

$$\|\pi_n\|_{2^p}^\phi \le \left\{\sqrt{n+1}\; B_\infty^\phi\right\}^{1 - \frac{1}{2^{p-1}}} \left\{\frac{\|S_n\|_2^\phi}{\sqrt{n+1}}\right\}^{\frac{1}{2^{p-1}}} \|\pi_n\|_2^\omega \tag{3.34}$$

and

$$\|\pi_n\|_{2^p}^\phi \le \left\{\|S_n\|_\infty^\phi\right\}^{1 - \frac{1}{2^{p-1}}} \left\{B_2^\phi\right\}^{\frac{1}{2^{p-1}}} \|\pi_n\|_2^\omega \tag{3.35}$$

<u>Proof</u> (By induction on p) Let $p = 2$. For $r = \infty$ and $s = 1$ in (3.27) gives, for all $\pi_n \in P_n$,

$$\|\pi_n\|_4^\phi \le \sqrt{B_\infty^\phi \|S_n\|_2^\phi}\; \|\pi_n\|_2^\omega$$

which proves (3.34) for $p = 2$. Now, suppose (3.34) holds for some $p \ge 2$. Then, put $\pi_n = S_n$ in (3.34) to get

$$\|s_n\|^{\phi}_{2^p} \le \left\{\sqrt{n+1}\; B^{\phi}_{\infty}\right\}^{1-\frac{1}{2^{p-1}}} \left\{\frac{\|s_n\|^{\phi}_2}{\sqrt{n+1}}\right\}^{\frac{1}{2^{p-1}}} \sqrt{n+1}$$

From (3.27), with $r = \infty$ and $s = 1$,

$$\|\pi_n\|^{\phi}_{2^{p+1}} \le \sqrt{B^{\phi}_{\infty}\|s_n\|^{\phi}_{2^p}}\; \|\pi_n\|^{\omega}_2$$

$$\le \left[B^{\phi}_{\infty}\left\{\sqrt{n+1}\; B^{\phi}_{\infty}\right\}^{1-\frac{1}{2^{p-1}}} \left\{\frac{\|s_n\|^{\phi}_2}{\sqrt{n+1}}\right\}^{\frac{1}{2^{p-1}}} \sqrt{n+1}\right]^{\frac{1}{2}} \|\pi_n\|^{\omega}_2$$

$$= \left\{\sqrt{n+1}\; B^{\phi}_{\infty}\right\}^{1-\frac{1}{2^p}} \left\{\frac{\|s_n\|^{\phi}_2}{\sqrt{n+1}}\right\}^{\frac{1}{2^p}} \|\pi_n\|^{\omega}_2$$

and this completes the proof of (3.34). The proof of (3.35) is similar. Let $r = 1$ and $s = \infty$ in (3.27), so that for all $\pi_n \in P_n$,

$$\|\pi_n\|^{\phi}_4 \le \sqrt{B^{\phi}_2\|s_n\|^{\phi}_{\infty}}\; \|\pi_n\|^{\omega}_2$$

which proves (3.35) for $p = 2$. Now, suppose that (3.35) holds for some $p \ge 2$. Then put $\pi_n = h_k$ in (3.35) to get

$$\|h_k\|^{\phi}_{2^p} \le \left\{\|s_n\|^{\phi}_{\infty}\right\}^{1-\frac{1}{2^{p-1}}} \left\{B^{\phi}_2\right\}^{\frac{1}{2^{p-1}}} \cdot 1$$

so that

$$B^{\phi}_{2^p} \le \left\{\|s_n\|^{\phi}_{\infty}\right\}^{1-\frac{1}{2^{p-1}}} \left\{B^{\phi}_2\right\}^{\frac{1}{2^{p-1}}}$$

Therefore, by (3.27) with $r = 1$ and $s = \infty$,

$$\|\pi_n\|^{\phi}_{2^{p+1}} \le \sqrt{B^{\phi}_{2^p}\|s_n\|^{\phi}_{\infty}}\; \|\pi_n\|^{\omega}_2$$

$$\le \left\{ \left(\|S_n\|_\infty^\phi \right)^{1-\frac{1}{2^{p-1}}} \left(B_2^\phi \right)^{\frac{1}{2^{p-1}}} \|S_n\|_\infty^\phi \right\}^{\frac{1}{2}} \|\pi_n\|_2^\omega$$

$$= \left(\|S_n\|_\infty^\phi \right)^{1-\frac{1}{2^p}} \left(B_2^\phi \right)^{\frac{1}{2^p}} \|\pi_n\|_2^\omega$$

which completes the proof of (3.35).

<u>Corollary 3.8</u> Under the conditions of Theorem 3.3, if $\phi(t) = \omega(t)$ almost everywhere on (a,b), then

$$\|\pi_n\|_{2^p}^\omega \le \left(\sqrt{n+1}\ B_\infty^\omega \right)^{1-\frac{1}{2^{p-1}}} \|\pi_n\|_2^\omega \tag{3.36}$$

and

$$\|\pi_n\|_{2^p}^\omega \le \left(\|S_n\|_\infty^\omega \right)^{1-\frac{1}{2^{p-1}}} \|\pi_n\|_2^\omega \tag{3.37}$$

<u>Proof</u> Follows from Theorem 3.3, since $B_2^\omega = 1$ and $\|S_n\|_2^\omega = \sqrt{n+1}$.

We remark that (3.36) and (3.37) give the same bound as (3.30) and (3.29), respectively, as $p \to \infty$ for the case $\phi = \omega$.

C. <u>Extension to Linear Transformations on the Space</u>

The preceding development can be extended easily to finding upper bounds for ratios of the form

$$\frac{\|D\pi_n\|_{2p}^\phi}{\|\pi_n\|_2^\omega} \tag{3.38}$$

where D is some suitable linear transformation on P_n. Typically, D will be a derivative of some order. The next lemma generalizes Lemma 3.2 and does not require the

Nonnegativity Condition.

Lemma 3.9 Let $p \geq 1$ be an integer. Let P_n be a subspace of $L_2^{\omega}[a,b] \cap L_{2p}^{\phi}[c,d]$ with a basis $\{h_0, h_1, \ldots, h_n\}$ which is orthonormal with respect to the inner product $(f,g)_{\omega}$. Let $D: P_n \to L_{2p}^{\phi}[c,d]$ be any linear transformation on P_n. Let

$$\pi_n(t) = a_0 h_0(t) + a_1 h_1(t) + \cdots + a_n h_n(t) \in P_n$$

and let $x = \langle a_0\ a_1\ \cdots\ a_n\rangle^T \in \mathbf{C}^{n+1}$. Then

$$\|D\pi_n\|_{2p}^{\phi} \equiv \{\bar{u}^T\ E_{n,p}^{\phi}\ u\}^{\frac{1}{2p}} \tag{3.39}$$

where $u = x \otimes \cdots \otimes x$ (p factors of x) $\in \mathbf{C}^{(n+1)^p}$, and $E_{n,p}^{\phi} = [\nu_{\alpha,\beta}]$ is the hermitian matrix of dimension $(n+1)^p \times (n+1)^p$ given by

$$\nu_{\alpha,\beta} = (Dh_{\beta_1} \cdots Dh_{\beta_p}, Dh_{\alpha_1} \cdots Dh_{\alpha_p})_{\phi} < +\infty \tag{3.40}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$, $\beta = (\beta_1, \ldots, \beta_p) \in \Gamma$.

Proof Since D is linear,

$$[D\pi_n(t)]^p = \sum_{\alpha_1,\ldots,\alpha_p=0}^{n} a_{\alpha_1}\cdots a_{\alpha_p} Dh_{\alpha_1}(t)\cdots Dh_{\alpha_p}(t) \tag{3.41}$$

The rest of the proof is too similar to the proof of (3.12) to repeat here.

Lemma 3.10 Under the hypotheses of Lemma 3.9, the matrix $E_{n,p}^{\phi}$ is positive semidefinite.

Proof Follow the proof of Lemma 3.3 using (3.40) instead of (3.14).

Theorem 3.4 Under the hypotheses of Lemma 3.9, the trace of $E^{\phi}_{n,p}$ is given by

$$\mathrm{Trace}(E^{\phi}_{n,p}) = \left\{\|K_n^{(D)}(t,t)\|^{\phi}_{p}\right\}^{p} \tag{3.42}$$

where

$$K_n^{(D)}(t,s) = \sum_{k=0}^{n} Dh_k(t)\,\overline{Dh_k(s)} \tag{3.43}$$

Proof Follow the proof of Theorem 3.1.

Corollary 3.11 Under the conditions of Theorem 3.4, for all $\pi_n \in P_n$, $\pi_n \neq 0$,

$$\frac{\|D\pi_n\|^{\phi}_{2p}}{\|\pi_n\|^{\omega}_{2}} \le \sqrt{\|K_n^{(D)}(t,t)\|^{\phi}_{p}} \tag{3.44}$$

Proof Follow the proof of Corollary 3.4.

As an aside, we note that Erdelyi [10, Section 10.6] states that Hahn [16] and Krall [20] proved that if $G = \{g_0, g_1, \ldots, g_n\}$ is an orthogonal system of polynomials, then G is a "classical" system if and only if the derivatives $\{g_0', g_1', \ldots, g_n'\}$ form an orthogonal system. The "classical" systems are defined here to comprise only the Jacobi, generalized Laguerre, and Hermite polynomials. Thus, if D is the derivative and G is a classical system, then (3.43) is related to a reproducing kernel of DP_n.

The next theorem generalizes Theorem 3.2 and does require a Nonnegativity Condition. This result is the main theorem of this chapter.

Theorem 3.5 Let $p \geq 1$ be an integer. Let P_n be a subspace of $L_2^\omega[a,b] \cap L_{2p}^\phi[c,d]$ with a basis $\{h_0, h_1, \ldots, h_n\}$ which is orthonormal with respect to the inner product $(f,g)_\omega$. Let $D: P_n \to L_{2p}^\phi[c,d]$ be a linear transformation such that $\{Dh_0, Dh_1, \ldots, Dh_n\}$ satisfy the Nonnegativity Condition in $L_{2p}^\phi[c,d]$. Then, for all $0 \neq \pi_n \in P_n$,

$$\frac{\|D\pi_n\|_{2p}^\phi}{\|\pi_n\|_2^\omega} \leq \max_{0 \leq k \leq n} \sqrt{\|Dh_k \cdot DS_n\|_p^\phi} \tag{3.45}$$

where $S_n(t)$ is given by (3.22).

Proof The proof of (3.45) is, in its essential details, analogous to the proof of Theorem 3.2 and will not be given here. We note only that (3.24) is replaced by

$$0 < \int_c^d |DS_n(t)|^p \phi(t)dt < +\infty \tag{3.46}$$

The next three corollaries are given without proof since their proofs so closely parallel the proofs of Corollaries 3.5 through 3.7.

Corollary 3.12 Under the conditions of Theorem 3.5

$$\frac{\|D\pi_n\|_{2p}^\phi}{\|\pi_n\|_2^\omega} \leq \inf \sqrt{M_{rp}^\phi \|DS_n\|_{sp}^\phi} < +\infty \tag{3.47}$$

where

$$M_{rp}^{\phi} = \max_{0\le k\le n} \|Dh_k\|_{rp}^{\phi} \tag{3.48}$$

and the infimum is taken over all extended real numbers $r \ge 1$ and $s \ge 1$ satisfying $\frac{1}{r} + \frac{1}{s} = 1$.

<u>Corollary 3.13</u> Under the conditions of Theorem 3.5, for all $\pi_n \in P_n$,

$$\|D\pi_n\|_{2p}^{\phi} \le \|DS_n\|_{2p}^{\phi} \|\pi_n\|_2^{\omega} \tag{3.49}$$

and

$$\|D\pi_n\|_{2p}^{\phi} \le \sqrt{n+1}\, M_{2p}^{\phi} \|\pi_n\|_2^{\omega} \tag{3.50}$$

<u>Corollary 3.14</u> Under the conditions of Theorem 3.5,

$$\frac{\|D\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}} \le \left\{M_{\infty}^{\phi}\|DS_n\|_{\infty}^{\phi}\right\}^{\frac{1}{2}} \left\{\frac{\|DS_n\|_2^{\phi}}{\|DS_n\|_{\infty}^{\phi}}\right\}^{\frac{1}{p}}, \quad p \ge 2 \tag{3.51}$$

provided the norms $\|Dh_0\|_{\infty}^{\phi}$, $\|Dh_1\|_{\infty}^{\phi}$, $\cdots$, $\|Dh_n\|_{\infty}^{\phi}$ are finite.

<u>Theorem 3.6</u> Let $p \ge 2$ be an integer. Let P_n be a subspace of $L_2^{\omega}[a,b] \cap L_{2p}^{\phi}[c,d]$ with a basis $\{h_0, h_1, \ldots, h_n\}$ which is orthonormal with respect to the inner product $(f,g)_{\omega}$. Let $D: P_n \to L_{2p}^{\phi}[c,d]$ be a linear transformation such that $\{Dh_0, Dh_1, \ldots, Dh_n\}$ satisfy the Nonnegativity Condition in $L_{2^k}^{\phi}[c,d]$, $k = 2, \ldots, p$. If $M_{\infty}^{\phi} < +\infty$, then for all $\pi_n \in P_n$,

$$\|D\pi_n\|_{2^p}^{\phi} \le \left\{\sqrt{n+1}\, M_{\infty}^{\phi}\right\}^{1-\frac{1}{2^{p-1}}} \left\{\frac{\|DS_n\|_2^{\phi}}{\sqrt{n+1}}\right\}^{\frac{1}{2^{p-1}}} \|\pi_n\|_2^{\omega} \tag{3.52}$$

and

$$\|D\pi_n\|_{2^p}^{\phi} \le \left\{\|DS_n\|_{\infty}^{\phi}\right\}^{1-\frac{1}{2^{p-1}}} \left\{M_2^{\phi}\right\}^{\frac{1}{2^{p-1}}} \|\pi_n\|_2^{\omega} \tag{3.53}$$

Proof Follow the proof of Theorem 3.3 replacing all references to Theorem 3.2 by references to Theorem 3.5 .

We point out that the algebraic methods developed in this chapter can also be applied to the more general problem

$$\max_{0\in\pi_n\in P_n}\left\{\frac{\|(D_1\pi_n)(D_2\pi_n)\cdots(D_p\pi_n)\|_2^{\phi}}{(\|\pi_n\|_2^{\omega})^p}\right\}$$

where $D_1, \ldots, D_p$ are different linear operators on P_n. Thus, if P_n are the algebraic polynomials of degree at most n, and $D_k\pi_n = \pi_n^{(k)}$ is the k-th derivative of π_n, and $\omega = \phi \equiv 1$, then we could obtain a bound for

$$\max_{0\neq\pi_n\in P_n}\left\{\frac{\|\pi_n\pi_n^{(1)}\pi_n^{(2)}\cdots\pi_n^{(p)}\|_2^{\omega}}{(\|\pi_n\|_2^{\omega})^p}\right\}$$

by following the proof of Theorem 3.2. [To see that the appropriate Nonnegativity Condition is satisfied for this ratio, refer to equations (4.38)-(4.41) and (4.53).]

D. Complex Polynomials Defined on the Unit Circle, Revisited

As mentioned earlier, all these results are easily translated into results for complex polynomials defined on the unit circle. The reason for this is simply that every integral appearing in this chapter can be replaced by contour integrals on the unit circle. Thus, using the notation of (2.1) and following the proof of Theorem 3.2 gives

$$\frac{\|\pi_n\|_{2p}}{\|\pi_n\|_2} \leq \max_{0\leq k\leq n} \sqrt{\|h_k S_n\|_p} \tag{3.54}$$

But $h_k(z) = z^k$ is the appropriate orthonormal basis satisfying the Nonnegativity Condition, so (3.54) is merely

$$\frac{\|\pi_n\|_{2p}}{\|\pi_n\|_2} \leq \sqrt{\|1 + z + z^2 + \cdots z^n\|_p} \tag{3.55}$$

Since

$$\|S_n\|_p = (\lambda_{n,p})^{\frac{1}{p}}, \text{ if } p = 2, 4, 6, \ldots,$$

$$\|S_n\|_p > (\lambda_{n,p})^{\frac{1}{p}}, \text{ if } p = 1, 3, 5, \ldots,$$

we see that (3.55) is almost the central result of Theorem 2.1. The cause of this deficiency is due entirely to the necessity of taking absolute values inside the integral in (3.26) in the proof of Theorem 3.2. Thus, an examination of (3.25) yields the essential inequality of Theorem 2.1, while (3.26) does not. The same phenomenon occurs in the proof of Theorem 3.5, which is easily modified to yield

Theorem 3.7 Let P_n be the collection of all complex polynomials of degree at most n with norms given by equation (2.3). For all $\pi_n \in P_n$, let $\pi_n^{(k)}(z)$ denote the k-th deriva- of π_n, $k = 1, 2, 3, \ldots$. Then, for all $\pi_n \in P_n$, $\pi_n \neq 0$,

$$\frac{\|\pi_n^{(k)}\|_{2p}}{\|\pi_n\|_2} \leq \sqrt{n(n-1)\cdots(n-k+1)}\{\Lambda_{n,p}^{(k)}\}^{\frac{1}{2p}}, \; p = 1, 2, 3, \ldots \tag{3.56}$$

where $\Lambda_{n,p}^{(k)}$ is the largest coefficient in the power series expansion of

$$\left\{\frac{d^k}{dz^k}(1 + z + z^2 + \cdots + z^n)\right\}^p \tag{3.57}$$

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Proof Follow the proof of Theorem 3.5 and consider the remarks immediately following (3.55).

Note that Theorem 3.7 is a more natural result than Theorem 2.5. The bound is, however, not as good, as the next corollary shows.

Corollary 3.15 With $\lambda_{n,p}^{(k)}$ defined via (2.87), and under the conditions of Theorem 3.7,

$$(k!)^{2p}\lambda_{n,p}^{(k)} \leq [n(n-1)\cdots(n-k+1)]^{p}\Lambda_{n,p}^{(k)} \tag{3.58}$$

Proof We only indicate the proof. As stated following Theorem 2.7, $(k!)^{2p}\lambda_{n,p}^{(k)}$ is precisely the spectral radius of the operator $M_{n,p}^{(k)}$. Since the bound (3.56) is an estimate of the spectral radius of $M_{n,p}^{(k)}$, we must have (3.58).

By example, it is seen that (3.58) can be strict. Let $n = 4$, $p = 2$, and $k = 1$. Then

$$\lambda_{n,p}^{(1)} = 288$$

while

$$\Lambda_{n,p}^{(1)} = 25$$

so that

$$(1!)^{2}\lambda_{4,2}^{(1)} = 288 < 400 = 4^{2}\Lambda_{4,2}^{(1)}$$

Chapter IV

APPLICATIONS TO CLASSICAL ORTHOGONAL POLYNOMIALS

A. Jacobi Polynomials

The Nonnegativity Condition (3.20) is satisfied by nearly half the Jacobi polynomials, all the generalized Laguerre polynomials (properly normalized), and the Hermite polynomials. The Jacobi polynomials turn out to be significantly easier to handle by the methods of Chapter III because they are essentially bounded on $(-1,+1)$. At the end of this chapter, some general results are quoted from Askey [3,4] which give some sufficient conditions for a given set of orthogonal polynomials to satisfy a Nonnegativity Condition.

Throughout this chapter, we will denote by P_n the collection of all polynomials of degree at most n, $n \geq 0$. We stress that these polynomials are allowed to have complex coefficients. The Gamma function $\Gamma(z)$ is defined as in Abramowitz and Stegun [1, Chapter 6] for all complex $z \neq \{0, -1, -2, \ldots\}$. We have the well known identity $\Gamma(1 + z) = z\Gamma(z)$. For integers $n \geq 0$, $\Gamma(1 + n) = n!$ Finally, the Pochhammer symbol is defined by

$$(z)_n = \begin{cases} z(z+1)\cdots(z+n-1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

for all complex $z \neq 0$, and the binomial coefficient

$$\binom{z}{u} = \frac{\Gamma(z+1)}{\Gamma(u+1)\Gamma(z-u+1)}$$

for all complex z and u such that z, u, and z - u are not negative integers.

Let $P_n^{(\alpha,\beta)}(x)$ be the n-th degree Jacobi polynomial of order (α,β), $\alpha > -1$, $\beta > -1$, as defined by Szegö [31, Chapter IV]. The Jacobi polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\,dx = \frac{\delta_{n,m}}{\{h_n^{(\alpha,\beta)}\}^2} \qquad (4.1)$$

where

$$h_n^{(\alpha,\beta)} = \begin{cases} \left(\dfrac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\right)^{\frac{1}{2}}, & n = 0 \\ \left(\dfrac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right)^{\frac{1}{2}}, & n \geq 1 \end{cases} \qquad (4.2).$$

Define

$$S_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} h_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(x) \qquad (4.3)$$

Define $g(k,m,n;\alpha,\beta)$ via the expansion

$$P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) = \sum_{k=0}^{n+m} g(k,m,n;\alpha,\beta) P_k^{(\alpha,\beta)}(x) \qquad (4.4)$$

The expansion (4.4) certainly exists and uniquely defines $g(k,m,n;\alpha,\beta)$. The question is, for which (α,β) is it true that

$$g(k,m,n;\alpha,\beta) \geq 0 \quad \text{for all } k,m,n = 0, 1, 2, \ldots ? \qquad (4.5)$$

Miller [26] gives $g(k,m,n;\alpha,\beta)$ explicitly, but in a form that is not useful here. Gasper [14] found necessary and sufficient conditions for (α,β) to be such that (4.5) holds, but without exhibiting the coefficients explicitly. Part of his result proves that (4.5) holds for all $\alpha \geq \beta > -1$ satisfying $\alpha + \beta + 1 \geq 0$.

In another direction, Askey [4], following Szegö [31, Equation 9.4.1], gives

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} t(k;\alpha,\beta,\gamma) P_k^{(\gamma,\beta)}(x) \tag{4.6}$$

where $\gamma \geq 0$ and

$$t(k;\alpha,\beta,\gamma)$$
$$= (2k+\gamma+\beta+1)\frac{\Gamma(n+k+\alpha+\beta+1)\Gamma(n-k+\alpha-\gamma)\Gamma(k+\gamma+\beta+1)\Gamma(n+\beta+1)}{\Gamma(n+k+\gamma+\beta+2)\Gamma(n-k+1)\Gamma(k+\beta+1)\Gamma(\alpha-\gamma)\Gamma(n+\beta+\alpha+1)} \tag{4.7}$$

An examination shows that for $\beta > -1$ and $\alpha > \gamma \geq 0$, the coefficients in the expansion (4.6) are all positive.

<u>Lemma 4.1</u> Let $\alpha \geq \beta > -1$ and $\alpha + \beta + 1 \geq 0$. The polynomials

$$\{P_0^{(\alpha,\beta)}(x),\ P_1^{(\alpha,\beta)}(x),\ \cdots,\ P_n^{(\alpha,\beta)}(x)\} \tag{4.8}$$

satisfy the Nonnegativity Condition (3.20) in the space $L_{2p}^{\omega}[-1,+1]$, where

$$\omega(x) = (1-x)^{\alpha}(1+x)^{\beta} \tag{4.9}$$

If in addition $\alpha > 0$, then the polynomials (4.8) satisfy the Nonnegativity Condition in every space $L_{2p}^{\phi}[-1,+1]$, where

$$\phi(x) = (1-x)^{\gamma}(1+x)^{\beta}, \quad 0 \leq \gamma < \alpha \tag{4.10}$$

<u>Proof</u> Gasper [14] proves that with these conditions on α and β, (4.5) always holds. Let $(i_1,\ldots,i_p) \in \Gamma$ and $(j_1,\ldots,j_p) \in \Gamma$. Then (4.5) implies that the expansions

$$F(x) \equiv P_{i_1}^{(\alpha,\beta)}(x)\cdots P_{i_p}^{(\alpha,\beta)}(x) = \sum_{i=0}^{i_1+\cdots+i_p} a_i P_i^{(\alpha,\beta)}(x)$$

$$G(x) \equiv P_{j_1}^{(\alpha,\beta)}(x)\cdots P_{j_p}^{(\alpha,\beta)}(x) = \sum_{j=0}^{j_1+\cdots+j_p} b_j P_j^{(\alpha,\beta)}(x)$$

have $a_i \geq 0$ and $b_j \geq 0$ for all i and j. Thus, using the orthogonality conditions

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}F(x)G(x)dx$$

$$= \sum_{i,j} a_i b_j \int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_i^{(\alpha,\beta)}(x)P_j^{(\alpha,\beta)}(x)dx$$

$$= \sum_{i,j}\left(a_i b_j \delta_{i,j}/h_i^{(\alpha,\beta)}h_j^{(\alpha,\beta)}\right) \geq 0$$

which proves that the polynomials (4.8) satisfy the Non-negativity Condition in $L_{2p}^{\omega}[-1,+1]$. For $\alpha > \gamma \geq 0$, the expansion (4.6) holds, so that

$$\int_{-1}^{1}(1-x)^{\gamma}(1+x)^{\beta}F(x)G(x)dx$$

$$= \sum_{i,j} a_i b_j \int_{-1}^{1}(1-x)^{\gamma}(1+x)^{\beta}P_i^{(\alpha,\beta)}(x)P_j^{(\alpha,\beta)}(x)dx$$

$$= \sum_{i,j} a_i b_j \int_{-1}^{1}(1-x)^{\gamma}(1+x)^{\beta}\left(\sum_{k=0}^{i} t(k;\alpha,\beta,\gamma)P_k^{(\gamma,\beta)}(x)\right)$$

$$\cdot\left(\sum_{r=0}^{j} t(r;\alpha,\beta,\gamma)P_r^{(\gamma,\beta)}(x)\right)dx$$

$$= \sum_{i,j} a_i b_j \sum_{k,r} \left(t(k;\alpha,\beta,\gamma)\, t(r;\alpha,\beta,\gamma)\, \delta_{k,r} / h_k^{(\gamma,\beta)} h_r^{(\gamma,\beta)} \right)$$

$$\geq 0$$

This concludes the proof.

For all $\pi_n \in P_n$, we adopt the notation

$$\|\pi_n\|_p^{(\alpha,\beta)} = \left\{ \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} |\pi_n(x)|^p dx \right\}^{\frac{1}{p}}, \quad 1 \leq p < \infty \tag{4.11}$$

where $\alpha > -1$, $\beta > -1$.

<u>Theorem 4.1</u> Let $p \geq 1$ be an integer. Let $\alpha \geq \beta > -1$ and $\alpha + \beta + 1 \geq 0$. Then, for all $0 \neq \pi_n \in P_n$,

$$\frac{\|\pi_n\|_{2p}^{(\alpha,\beta)}}{\|\pi_n\|_2^{(\alpha,\beta)}} \leq \max_{0 \leq k \leq n} \sqrt{h_k^{(\alpha,\beta)} \|P_k^{(\alpha,\beta)} S_n^{(\alpha,\beta)}\|_p^{(\alpha,\beta)}} \tag{4.12}$$

Alternatively, if $\alpha \geq \beta > -1$ and $\alpha > 0$, then

$$\frac{\|\pi_n\|_{2p}^{(\gamma,\beta)}}{\|\pi_n\|_2^{(\alpha,\beta)}} \leq \max_{0 \leq k \leq n} \sqrt{h_k^{(\alpha,\beta)} \|P_k^{(\alpha,\beta)} S_n^{(\alpha,\beta)}\|_p^{(\gamma,\beta)}} \tag{4.13}$$

provided only $0 \leq \gamma < \alpha$.

<u>Proof</u> The polynomials $\{h_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(x)\}$ form an orthonormal basis on $(-1,+1)$ with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$. Lemma 4.1 shows that they satisfy the Nonnegativity Conditions needed in order to apply Theorem 3.2 directly. This completes the proof.

Lemma 4.2 Let $\alpha \geq \beta > -1$ and $\alpha \geq 0$. For all $\gamma > -1$ and $\delta > -1$,

$$\max_{0\leq k\leq n} \|h_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}\|_\infty^{(\gamma,\delta)} = h_n^{(\alpha,\beta)}\binom{n+\alpha}{n}, \quad n = 0, 1, 2, \ldots \tag{4.14}$$

Furthermore,

$$h_n^{(\alpha,\beta)}\binom{n+\alpha}{n} < M\left(n + \frac{\alpha+\beta+1}{2}\right)^{\alpha+\frac{1}{2}}, \quad n = 0, 1, 2, \ldots \tag{4.15}$$

where

$$M = \left\{\frac{1}{2^{\alpha+\beta+1}\Gamma(1+\alpha)} \max\left(\frac{\Gamma(\alpha+\beta+2)}{\Gamma(1+\beta)}, \frac{2e^C}{\Gamma(1+\alpha)}\left[\frac{(1+\alpha)(1+\alpha+\beta)}{1+\beta}\right]^{\alpha+\frac{1}{2}}\right)\right\}^{\frac{1}{2}} \tag{4.16}$$

$$C = \frac{1}{12}\left(\frac{1}{1+\alpha+\beta} + \frac{1}{1+\alpha}\right)$$

Proof From Szegö [31, Equation 7.32.2], since $\alpha \geq \beta$,

$$\max_{-1\leq x\leq 1}\left|P_k^{(\alpha,\beta)}(x)\right| = P_k^{(\alpha,\beta)}(1) = \binom{k+\alpha}{k}, \quad k = 0, 1, 2, \ldots \tag{4.17}$$

Now, (4.2) and the fact that the supremum norm of a polynomial is independent of (γ,δ) in this case, implies

$$\|h_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}\|_\infty^{(\gamma,\delta)} = h_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(1), \quad k \geq 0$$

$$= \frac{1}{\Gamma(1+\alpha)}\left\{\frac{2k+\alpha+\beta+1}{2^{\alpha+\beta+1}} \frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}\right\}^{\frac{1}{2}}, \quad k \geq 1 \tag{4.18}$$

Since for all $x > y > 0$

$$\frac{\Gamma(1+x)}{\Gamma(1+y)} = \frac{x}{y}\frac{\Gamma(x)}{\Gamma(y)} > \frac{\Gamma(x)}{\Gamma(y)} \tag{4.19}$$

(4.18) implies

$$h_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(1) < h_{k+1}^{(\alpha,\beta)} P_{k+1}^{(\alpha,\beta)}(1), \quad k = 1, 2, 3, \ldots \tag{4.20}$$

By inspection, (4.20) is seen to hold for $k = 0$ as well, and this proves (4.14). Now, one form of Stirling's approximation [1, equation 6.1.38] is

$$\Gamma(1+x) = \sqrt{2\pi}\, x^{x+\frac{1}{2}} \exp\left(-x + \frac{\theta(x)}{12x}\right), \quad 0 < \theta(x) < 1, \; x > 0 \tag{4.21}$$

Hence, for $x \geq y > -1$, $x \geq 0$,

$$\frac{\Gamma(n+x+1)}{\Gamma(n+y+1)} < \frac{(n+x)^{n+x+\frac{1}{2}}}{(n+y)^{n+y+\frac{1}{2}}} \exp\left(y - x + \frac{1}{12(n+x)}\right), \quad n \geq 1$$

$$= (n+y)^{x-y}\left(1 + \frac{x-y}{n+y}\right)^{n+x+\frac{1}{2}} \exp\left(y - x + \frac{1}{12(n+x)}\right), \quad n \geq 1$$

$$< \left(\frac{1+x}{1+y}\right)^{x-y+\frac{1}{2}} (n+y)^{x-y} \exp\left(\frac{1}{12(1+x)}\right), \quad n \geq 1 \tag{4.22}$$

Therefore, from (4.18), for $n \geq 1$,

$$h_n^{(\alpha,\beta)} \binom{n+\alpha}{n} < \left\{ \frac{e^C}{\Gamma^2(1+\alpha)} \frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}} (n+\beta)^{\alpha} n^{\alpha} \left[\frac{(1+\alpha)(1+\alpha+\beta)}{1+\beta}\right]^{\alpha+\frac{1}{2}} \right\}^{\frac{1}{2}}$$

which proves (4.15) for $n \geq 1$ since $(\alpha+\beta+1)/2 \geq \max\{0, \beta\}$. The proof of 4.15 is completed by examination of the case $n = 0$.

Corollary 4.3 Let $p \geq 1$ be an integer. Let $\alpha \geq \beta > -1$ and $\alpha \geq 0$. *Then for all* $\pi_n \in P_n$, $\pi_n \neq 0$,

$$\frac{\|\pi_n\|_{2p}^{(\gamma,\beta)}}{\|\pi_n\|_2^{(\alpha,\beta)}} \leq \sqrt{h_n^{(\alpha,\beta)} \binom{n+\alpha}{n} \|S_n^{(\alpha,\beta)}\|_p^{(\gamma,\beta)}} \tag{4.23}$$

for all γ satisfying $0 \le \gamma \le \alpha$.

<u>Proof</u> Follows from (4.14) and Corollary 3.5 with $r = \infty$ and $s = 1$.

<u>Corollary 4.4</u> Let $p \ge 1$ be an integer. Let $\alpha \ge \beta > -1$ and $\alpha \ge 0$. Then for all $\pi_n \in P_n$, $\pi_n \ne 0$,

$$\frac{\|\pi_n\|_4^{(\alpha,\beta)}}{\|\pi_n\|_2^{(\alpha,\beta)}} \le \sqrt{\sqrt{n+1}\, h_n^{(\alpha,\beta)} \binom{n+\alpha}{n}} \tag{4.24}$$

<u>Proof</u> Use Corollary 4.3 with $\gamma = \alpha$ and the fact that

$$\|s_n^{(\alpha,\beta)}\|_2^{(\alpha,\beta)} = \sqrt{n+1}$$

<u>Theorem 4.2</u> Let $\alpha \ge \beta > -1$ and $\alpha \ge 0$. Then, for all $\pi_n \in P_n$, $\pi_n \ne 0$,

$$\|\pi_n\|_{2p}^{(\alpha,\beta)} < A\left(n + \frac{\alpha+\beta+3}{2}\right)^B \|\pi_n\|_2^{(\alpha,\beta)}, \quad p = 1, 2, \ldots \tag{4.25}$$

where

$$A = \sqrt{\alpha + \frac{3}{2}}\left(\frac{M}{\alpha + \frac{3}{2}}\right)^{1-\frac{1}{p}}$$

$$B = (1 + \alpha)(1 - \frac{1}{p})$$

and M is given by (4.16). Furthermore, for all $n \ge 0$,

$$\|\pi_n\|_\infty^{(\alpha,\beta)} \le \left(\frac{M}{\sqrt{2\alpha+2}}\right)\left(n + \frac{\alpha+\beta+3}{2}\right)^{1+\alpha} \|\pi_n\|_2^{(\alpha,\beta)} \tag{4.26}$$

and the exponent $1+\alpha$ in (4.26) cannot be replaced by a smaller number.

<u>Proof</u> The case $p = 1$ is trivial. For $p \geq 2$, use Corollary 3.7. In view of Lemma 4.2,

$$\|S_n^{(\alpha,\beta)}\|_\infty^{(\alpha,\beta)} = \sum_{k=0}^{n} h_k^{(\alpha,\beta)}\binom{k+\alpha}{k}$$

$$< M\sum_{k=0}^{n}(k+a)^{\alpha+\frac{1}{2}}$$

$$< M\int_0^{n+1}(k+a)^{\alpha+\frac{1}{2}}\,dk$$

$$< \frac{M}{\alpha+\frac{3}{2}}(n+a+1)^{\alpha+\frac{3}{2}}$$

where $a = (\alpha+\beta+1)/2$. Therefore, from (3.32),

$$\frac{\|\pi_n\|_{2p}^{(\alpha,\beta)}}{\|\pi_n\|_2^{(\alpha,\beta)}} \leq \left\{M(n+a)^{\alpha+\frac{1}{2}}\right\}^{\frac{1}{2}}\left\{\frac{M}{\alpha+\frac{3}{2}}(n+a+1)^{\alpha+\frac{3}{2}}\right\}^{\frac{1}{2}-\frac{1}{p}}\left\{\sqrt{n+1}\right\}^{\frac{1}{p}}$$

$$< \frac{M^{1-\frac{1}{p}}}{(\alpha+\frac{3}{2})^{\frac{1}{2}-\frac{1}{p}}}(n+a+1)^B$$

where

$$B = \frac{1}{2}(\alpha+\frac{1}{2}) + (\frac{1}{2}-\frac{1}{p})(\alpha+\frac{3}{2}) + \frac{1}{2p}$$

$$= (\alpha+1)(1-\frac{1}{p})$$

which proves (4.25). To prove (4.26), let

$$\pi_n(x) = \sum_{k=0}^{n} a_k h_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(x) \tag{4.27}$$

Then

$$\|\pi_n\|_\infty^{(\alpha,\beta)} = \left|a_0 h_0^{(\alpha,\beta)} P_0^{(\alpha,\beta)}(z) + \cdots + a_n h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(z)\right|,$$

some $z \in [-1,1]$

$$\leq \left\{\sum_{k=0}^{n} |a_k|^2\right\}^{\frac{1}{2}} \left\{\sum_{k=0}^{n} \left|h_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(z)\right|^2\right\}^{\frac{1}{2}}$$

$$\leq \|\pi_n\|_2^{(\alpha,\beta)} \left\{\sum_{k=0}^{n} \left[h_k^{(\alpha,\beta)} \binom{k+\alpha}{k}\right]^2\right\}^{\frac{1}{2}}$$

$$\leq M\|\pi_n\|_2^{(\alpha,\beta)} \left\{\sum_{k=0}^{n} \left(k + \frac{\alpha+\beta+1}{2}\right)^{2\alpha+1}\right\}^{\frac{1}{2}}, \text{ using (4.15)}$$

$$\leq M\|\pi_n\|_2^{(\alpha,\beta)} \left\{\left(n + \frac{\alpha+\beta+3}{2}\right)^{2\alpha+2} \Big/ (2\alpha + 2)\right\}^{\frac{1}{2}}$$

Simplifying the last inequality proves (4.26). Finally, the claim that $1 + \alpha$ cannot be replaced by any smaller exponent follows from examining the polynomial (4.27) with

$$a_k = h_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(1), \quad k = 0, 1, \ldots, n$$

Then proceeding as before, we see that

$$\|\pi_n\|_\infty^{(\alpha,\beta)} = \|\pi_n\|_2^{(\alpha,\beta)} \left\{\sum_{k=0}^{n} \left[h_k^{(\alpha,\beta)} \binom{k+\alpha}{k}\right]^2\right\}^{\frac{1}{2}} \tag{4.28}$$

Now, the proof of (4.22) can be altered to give a lower bound, which when applied to the quantity in brackets in (4.28) gives

$$h_k^{(\alpha,\beta)} \binom{k+\alpha}{k} > \tilde{M}(k-1)^{\alpha+\frac{1}{2}}, \quad k = 1, 2, \ldots$$

for some constant $\tilde{M}$ independent of k. Then

$$\left\{\sum_{k=0}^{n} \left[h_k^{(\alpha,\beta)} \binom{k+\alpha}{k}\right]^2\right\}^{\frac{1}{2}} > \tilde{M}\left\{\sum_{k=2}^{n} (k-1)^{2\alpha+1}\right\}^{\frac{1}{2}}$$

$$> \tilde{M}\left\{\int_{k=1}^{n} k^{2\alpha+1}\, dk\right\}^{\frac{1}{2}}$$

$$> \tilde{M}\{n^{2\alpha+2} - 1\}^{\frac{1}{2}} \geq \tilde{M}(n - 1)^{1+\alpha}, \; n \geq 1$$

This completes the proof.

We remark that Theorem 3.3 can be applied in the above situation for L^{ϕ}_{2p} norms. If this is done, the same result as (4.25) is obtained. Therefore, Corollary 3.7 is actually more general than Theorem 3.3. Similarly, it will be seen (Theorem 4.4) that Corollary 3.14 is more general than Theorem 3.6 in some situations.

<u>Theorem 4.3</u> Let $p \geq 1$ be an integer. Let $\alpha \geq \beta > -1$. For each integer $m \geq 1$, define the operator D^m on $\mathcal{P}_n$ by

$$D^m f(x) = \frac{d^m}{dx^m} f(x), \qquad f \in \mathcal{P}_n$$

Then, for all $\pi_n \in \mathcal{P}_n$, $\pi_n \neq 0$,

$$\frac{\|D^m \pi_n\|_{2p}^{(\alpha+m,\beta+m)}}{\|\pi_n\|_2^{(\alpha,\beta)}}$$

$$\leq \max_{0 \leq k \leq n} \sqrt{h_k^{(\alpha,\beta)} \|D^m P_k^{(\alpha,\beta)} D^m S_n^{(\alpha,\beta)}\|_p^{(\alpha+m,\beta+m)}} \tag{4.29}$$

<u>Proof</u> Szegö [31, Equation 4.21.7] shows that

$$D^m P_n^{(\alpha,\beta)}(x) = \frac{1}{2^m}(n + \alpha + \beta + 1)_m P_{n-m}^{(\alpha+m,\beta+m)}(x) \tag{4.30}$$

so that the functions $\{D^m P_0, D^m P_1, \cdots, D^m P_n\}$ satisfy the Nonnegativity Condition in $L^{\phi}_{2p}[-1,+1]$, where $\phi(x) = (1-x)^{\alpha+m}(1+x)^{\beta+m}$, by Lemma 4.1, since $\alpha+m \geq \beta+m > -1$ and $(\alpha+m) + (\beta+m) + 1 \geq 0$, for $m = 1, 2, 3, \ldots$, and for all $\alpha \geq \beta > -1$. Apply Theorem 3.5 directly.

<u>Theorem 4.4</u> Let $\alpha \geq \beta > -1$ and $\alpha \geq 0$. Then, for all $\pi_n \in \mathcal{P}_n$, $\pi_n \neq 0$,

$$\|\pi_n'\|_{2p}^{(\alpha+1,\beta+1)} < A(n+\alpha+\beta+2)^B \|\pi_n\|_2^{(\alpha,\beta)}, \quad p = 2, 3, \ldots \qquad (4.31)$$

where

$$A = \sqrt{\frac{\alpha + \frac{7}{2}}{3}} \left\{ \frac{\sqrt{3}\, M}{2(\alpha + 1)(\alpha + \frac{7}{2})} \right\}^{1-\frac{1}{p}}$$

$$B = (1 + \alpha)(1 - \frac{1}{p}) + 2 - \frac{1}{p}$$

and M is given by (4.16). Furthermore,

$$\|\pi_n'\|_\infty^{(\alpha+1,\beta+1)} \leq \frac{M}{2\sqrt{2}(\alpha+1)\sqrt{\alpha+3}} (n+\alpha+\beta+2)^{\alpha+3} \|\pi_n\|_2^{(\alpha,\beta)} \qquad (4.32)$$

and the exponent $3+\alpha$ in (4.32) cannot be replaced by a smaller number.

<u>Proof</u> We will use Corollary 3.14. From (4.17) and (4.30), for $k \geq 1$,

$$\|h_k^{(\alpha,\beta)} P_k'^{(\alpha,\beta)}\|_\infty^{(\alpha+1,\beta+1)} = \frac{k+\alpha+\beta+1}{2} h_k^{(\alpha,\beta)} P_{k-1}^{(\alpha+1,\beta+1)}(1)$$

$$= \frac{k(k+\alpha+\beta+1)}{2(\alpha+1)} h_k^{(\alpha,\beta)} \binom{k+\alpha}{k} \qquad (4.33)$$

so that

$$M_\infty^{(\alpha+1,\beta+1)} = \max_{0 \leq k \leq n} \|h_k^{(\alpha,\beta)} P_k'^{(\alpha,\beta)}\|_\infty^{(\alpha+1,\beta+1)}$$

$$< \frac{M}{2(\alpha+1)} (n+\alpha+\beta+1)^{\alpha+\frac{5}{2}} \qquad (4.34)$$

from Lemma 4.2. Now, (4.33) implies also

$$\|S_n'^{(\alpha,\beta)}\|_\infty^{(\alpha+1,\beta+1)} < \frac{M}{2(\alpha+1)} \sum_{k=0}^{n} (k+\alpha+\beta+1)^{\alpha+\frac{5}{2}}$$

$$< \frac{M}{2(\alpha+1)(\alpha+\frac{7}{2})} (n+\alpha+\beta+2)^{\alpha+\frac{7}{2}} \qquad (4.35)$$

Next,

$$\|S_n'^{(\alpha,\beta)}\|_2^{(\alpha+1,\beta+1)}$$

$$= \left\| \sum_{k=0}^{n} h_k^{(\alpha,\beta)} P_k'^{(\alpha,\beta)} \right\|_2^{(\alpha+1,\beta+1)}$$

$$= \left\| \sum_{k=1}^{n} \frac{k+\alpha+\beta+1}{2} h_k^{(\alpha,\beta)} P_{k-1}^{(\alpha+1,\beta+1)} \right\|_2^{(\alpha+1,\beta+1)}$$

$$= \left\| \sum_{k=1}^{n} \frac{k+\alpha+\beta+1}{2} \frac{h_k^{(\alpha,\beta)}}{h_{k-1}^{(\alpha+1,\beta+1)}} h_{k-1}^{(\alpha+1,\beta+1)} \cdot P_{k-1}^{(\alpha+1,\beta+1)} \right\|_2^{(\alpha+1,\beta+1)}$$

$$= \left\{ \sum_{k=1}^{n} \left[\frac{k+\alpha+\beta+1}{2} \frac{h_k^{(\alpha,\beta)}}{h_{k-1}^{(\alpha+1,\beta+1)}} \right]^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ \sum_{k=1}^{n} \left(\frac{k+\alpha+\beta+1}{2} \right)^2 \left[\frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)\Gamma(k+1)}{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)} \cdot \frac{2^{\alpha+\beta+3}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+2)\Gamma(k)} \right] \right\}^{\frac{1}{2}}$$

$$= \left\{ \sum_{k=1}^{n} \left(\frac{k+\alpha+\beta+1}{2} \right)^2 \frac{4k}{k+\alpha+\beta+1} \right\}^{\frac{1}{2}}$$

$$\le \left\{ \sum_{k=1}^{n} (k+\alpha+\beta+1)^2 \right\}^{\frac{1}{2}}$$

$$< \frac{1}{\sqrt{3}} (n+\alpha+\beta+2)^{\frac{3}{2}} \tag{4.36}$$

Applying (4.34)-(4.36) in (3.51) proves (4.31). To prove (4.32), let π_n be as in (4.27). Then

$$\|\pi_n'\|_\infty = \left| \sum_{k=0}^{n} a_k h_k^{(\alpha,\beta)} P_k'^{(\alpha,\beta)}(z) \right|, \quad \text{some } z \in [-1,+1]$$

$$\leq \left\{ \sum_{k=0}^{n} |a_k|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{k=0}^{n} \left| h_k^{(\alpha,\beta)} P_k'^{(\alpha,\beta)}(z) \right|^2 \right\}^{\frac{1}{2}}$$

$$\leq \|\pi_n\|_2^{(\alpha,\beta)} \left\{ \sum_{k=0}^{n} \left[h_k^{(\alpha,\beta)} \frac{k(k+\alpha+\beta+1)}{2(\alpha+1)} \binom{k+\alpha}{k} \right]^2 \right\}^{\frac{1}{2}}$$

$$< \frac{M}{2(\alpha+1)} \|\pi_n\|_2^{(\alpha,\beta)} \left\{ \sum_{k=0}^{n} (n+\alpha+\beta+1)^{2\alpha+5} \right\}^{\frac{1}{2}}$$

$$< \frac{M}{2(\alpha+1)} \|\pi_n\|_2^{(\alpha,\beta)} \frac{(n+\alpha+\beta+2)^{\alpha+3}}{\sqrt{2\alpha+6}}$$

which proves (4.32). That the exponent $3+\alpha$ is best possible can be proved in the same manner as the analogous result in Theorem 4.2, by consideration of the polynomial π_n given by

$$a_k = h_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(1), \quad k = 0, 1, \ldots$$

in (4.27). This completes the proof.

B. Gegenbauer (Ultraspherical) Polynomials

Even more can be said for the Gegenbauer, or Ultraspherical, polynomials. These polynomials are defined (see Szegö [31, p. 80]) for $\nu > -\frac{1}{2}$, $\nu \neq 0$, by

$$P_n^{(\nu)}(x) = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu)} \frac{\Gamma(n+2\nu)}{\Gamma(n+\nu+\frac{1}{2})} P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x) \tag{4.37}$$

Vilenkin gives [33, p. 491] the expansion

$$P_n^{(\nu)}(x)P_m^{(\nu)}(x) = \sum_{k\in K} s(k,m,n;\nu)P_k^{(\nu)}(x) \tag{4.38}$$

where $K = \{|n-m|, |n-m| + 2, \ldots, n+m-2, n+m\}$, $2r = n+m+k$, and

$$s(k,m,n;\nu) = \frac{k+\nu}{r+\nu}\,\frac{\Gamma(r-n+\nu)\Gamma(r-m+\nu)\Gamma(r-k+\nu)\Gamma(k+1)\Gamma(r+2\nu)}{\Gamma(r-n+1)\Gamma(r-m+1)\Gamma(r-k+1)\Gamma^2(\nu)\Gamma(k+2\nu)} \tag{4.39}$$

Also, Szegö [31, Equations 4.10.27-8] states that Gegenbauer proved that

$$P_n^{(\nu)}(x) = \sum_{k=0}^{[n/2]} a(k,n;\mu,\nu)P_{n-2k}^{(\mu)}(x) \tag{4.40}$$

where $\nu > \mu > 0$ and

$$a(k,n;\mu,\nu) = \frac{(n-2k+\mu)\Gamma(\mu)\Gamma(k+\nu-\mu)\Gamma(n-k+\nu)}{k!\;\Gamma(\nu)\Gamma(\nu-\mu)\Gamma(n-k+\mu+1)} \tag{4.41}$$

Therefore,

$$a(k,n;\mu,\nu) > 0, \qquad \nu > \mu > 0 \tag{4.42}$$

Lemma 4.5 Let $\nu > 0$. The Gegenbauer polynomials

$$\{P_0^{(\nu)}(x), P_1^{(\nu)}(x), \ldots, P_n^{(\nu)}(x)\} \tag{4.43}$$

satisfy the Nonnegativity Condition in every space $L_{2p}^{\phi}[-1,+1]$, where $p = 1, 2, \ldots$, and

$$\phi(x) = (1 - x^2)^{\mu-\frac{1}{2}}, \qquad 0 < \mu \le \nu \tag{4.44}$$

Proof The nonnegativity of the coefficients in (4.38) shows that (4.43) satisfies the Nonnegativity Condition in $L_{2p}^{\omega}[-1,1]$, where $\omega(x) = (1 - x^2)^{\nu-\frac{1}{2}}$ for $\nu > 0$. The nonnegativity of the coefficients in the expansion (4.40) completes the proof.

Note that the expansions (4.40) and (4.6) are quite different in nature, so Lemma 4.5 gives new information, except in the case $\mu = \nu$.

Define, for $\nu > 0$,

$$h_n^{(\nu)} = \Gamma(\nu)\left\{\frac{2^{2\nu-1}(n+\nu)\Gamma(n+1)}{\pi\ \Gamma(n+2\nu)}\right\}^{\frac{1}{2}}, \ n \ge 0 \tag{4.45}$$

Then the functions $\{h_k^{(\nu)} P_k^{(\nu)}(x)\}$, $k = 0, 1, \ldots, n$ form an orthonormal set on $(-1,+1)$ with respect to $(1 - x^2)^{\nu-\frac{1}{2}}$ (see [31, Section 4.7]). Define

$$S_n^{(\nu)}(x) = \sum_{k=0}^{n} h_k^{(\nu)} P_k^{(\nu)}(x) \tag{4.46}$$

For all $\pi_n \in P_n$, we adopt the notation

$$\|\pi_n\|_p^{(\nu)} = \left\{\int_{-1}^{1} (1-x^2)^{\nu-\frac{1}{2}} |\pi_n(x)|^p \, dx\right\}^{\frac{1}{p}}, \ 1 \le p < \infty \tag{4.47}$$

where $\nu > 0$. Note that (4.47) is a special case of (4.11).

<u>Theorem 4.5</u> Let $\nu > 0$. Then, for all $0 \ne \pi_n \in P_n$,

$$\frac{\|\pi_n\|_{2p}^{(\mu)}}{\|\pi_n\|_2^{(\nu)}} \le \max_{0\le k\le n} \sqrt{h_k^{(\nu)} \|P_k^{(\nu)} S_n^{(\nu)}\|_p^{(\mu)}}, \ p = 1, 2, .. \tag{4.48}$$

for all $0 < \mu \le \nu$.

<u>Proof</u> Use Theorem 3.5 in light of Lemma 4.5.

<u>Corollary 4.6</u> For all $0 \ne \pi_n \in P_n$, and for $0 < \mu \le \nu$,

$$\frac{\|\pi_n\|_{2p}^{(\mu)}}{\|\ \ _n\|_2^{(\nu)}} \le \sqrt{h_n^{(\nu)} \binom{n+2\nu-1}{n} \|S_n^{(\nu)}\|_p^{(\mu)}} \tag{4.49}$$

<u>Proof</u> With $r = \infty$ and $s = 1$ in Corollary 3.12, we have

$$\frac{\|\pi_n\|_{2p}^{(\mu)}}{\|\pi_n\|_2^{(\nu)}} \le \max_{0\le k\le n} \sqrt{\|h_k^{(\nu)} P_k^{(\nu)}\|_\infty^{(\mu)} \|S_n^{(\nu)}\|_p^{(\mu)}} \tag{4.50}$$

Szegö [31, Equation 7.33.1] gives, for $\nu > 0$,

$$\max_{-1\le x\le +1} |P_k^{(\nu)}(x)| = P_k^{(\nu)}(1) = \binom{k+2\nu-1}{k}, \quad k \ge 0 \tag{4.51}$$

Therefore, considering (4.19),

$$\begin{aligned}
\|h_k^{(\nu)} P_k^{(\nu)}\|_\infty^{(\mu)} &= h_k^{(\nu)} \binom{k+2\nu-1}{k}, \qquad k \ge 0 \\
&= \frac{\Gamma(\nu)}{\Gamma(2\nu)} \left\{ \frac{2^{2\nu-1}}{\pi} \frac{(k+\nu)\Gamma(k+2\nu)}{\Gamma(k+1)} \right\}^{\frac{1}{2}} \\
&= \frac{\Gamma(\nu)}{\Gamma(2\nu)} \left\{ \frac{2^{2\nu-1}}{\pi} \frac{k+\nu}{k+2\nu} \frac{\Gamma(k+2\nu+1)}{\Gamma(k+1)} \right\}^{\frac{1}{2}} \\
&< \|h_{k+1}^{(\nu)} P_{k+1}^{(\nu)}\|_\infty^{(\mu)}, \quad k = 0, 1, 2, \ldots
\end{aligned} \tag{4.52}$$

This completes the proof.

We remark that Corollary 4.6 is not a special case of Corollary 4.3.

From Szegö [31, Equation 4.7.14],

$$D^m P_n^{(\nu)}(x) = 2^m (\nu)_m P_{n-m}^{(\nu+m)}(x) \tag{4.53}$$

so that from Lemma 4.5, the polynomials

$$\{D^m P_0^{(\nu)}(x),\ D^m P_1^{(\nu)}(x),\ \ldots,\ D^m P_n^{(\nu)}(x)\}$$

satisfy the Nonnegativity Condition in all spaces $L_{2p}^{\phi}[-1,+1]$, where $\phi(t) = (1 - x^2)^\mu$, $0 < \mu \le \nu+m$. Therefore, we have a result which is much stronger than Theorem 4.3.

Theorem 4.6 Let $p \geq 1$ be an integer. Let $\nu > 0$. Define the operators D^m on P_n' as in Theorem 4.3. Then, for all $\pi_n \in P_n$, $\pi_n \neq 0$,

$$\frac{\|D^m \pi_n\|_{2p}^{(\mu)}}{\|\pi_n\|_2^{(\nu)}} \leq \max_{0 \leq k \leq n} \sqrt{h_k^{(\nu)} \|D^m P_k^{(\nu)} \cdot D^m S_n^{(\nu)}\|_p^{(\mu)}} \tag{4.54}$$

for all $0 < \mu \leq \nu+m$.

Proof Use Theorem 3.5 together with (4.53)

Theorem 4.7 Let $\nu \geq \frac{1}{2}$. Then for all $\pi_n \in P_n$, $\pi_n \neq 0$,

$$\|\pi_n'\|_{2p}^{(\nu)} < A(n+2\nu+1)^B \|\pi_n\|_2^{(\nu)}, \quad p = 2, 3, 4, \ldots \tag{4.55}$$

where

$$A = \sqrt{\nu+3}\left(\frac{Q}{\nu+3}\right)^{1-\frac{1}{p}}\left(\frac{8}{3}\sqrt{\frac{2}{5}}\right)^{\frac{1}{p}}$$

$$B = (\nu+\tfrac{1}{2})(1-\tfrac{1}{p}) + 2$$

and

$$Q = \frac{2^{-\frac{1}{4}}(\nu + \frac{1}{2})^{\nu-\frac{3}{4}} \exp(\frac{1}{12(1+2\nu)})}{\Gamma(\nu+\frac{1}{2})} \tag{4.56}$$

Furthermore,

$$\|\pi_n'\|_\infty^{(\nu)} \leq \frac{M}{(2\nu+1)\sqrt{2\nu+5}}(n+2\nu+1)^{\nu+\frac{5}{2}} \|\pi_n\|_2^{(\nu)} \tag{4.57}$$

where the constant M is given by setting $\alpha = \beta = \nu-\frac{1}{2}$ in (4.16), and the exponent $\nu+\frac{5}{2}$ in (4.57) cannot be replaced by a smaller number.

<u>Proof</u> We will use Corollary 3.14. We have

$$M_\infty^{(\nu)} = \max_{0\le k\le n} \|h_k^{(\nu)} P_k'^{(\nu)}\|_\infty^{(\nu)}$$

$$= \max_{1\le k\le n} 2\nu h_k^{(\nu)} P_{k-1}^{(\nu+1)}(1)$$

$$= \max_{1\le k\le n} \left\{ 2\nu\Gamma(\nu) \left[\frac{2^{2\nu-1}}{\pi} \frac{(k+\nu)\Gamma(k+1)}{\Gamma(k+2\nu)}\right]^{\frac{1}{2}} \binom{k+2\nu}{k-1} \right\}$$

$$= \max_{1\le k\le n} \left\{ \frac{2\nu\Gamma(\nu)}{\Gamma(2\nu+2)} \left[\frac{2^{2\nu-1}}{\pi} \frac{k(k+\nu)(k+2\nu)\Gamma(k+2\nu+1)}{\Gamma(k)}\right]^{\frac{1}{2}} \right\}$$

$$= \frac{2^{\nu+\frac{1}{2}}\Gamma(\nu+1)}{\sqrt{\pi}\;\Gamma(2\nu+2)} \left[\frac{n^2(n+\nu)(n+2\nu)\Gamma(n+2\nu+1)}{\Gamma(n+1)}\right]^{\frac{1}{2}} \tag{4.58}$$

From (4.22), we get

$$\frac{\Gamma(n+2\nu+1)}{\Gamma(n+1)} < (1+2\nu)^{2\nu+\frac{1}{2}} n^{2\nu} \exp\left(\frac{1}{12(1+2\nu)}\right), \ n \ge 1 \tag{4.59}$$

and so

$$M_\infty^{(\nu)} < Q(n+2\nu)^{\nu+2} \tag{4.60}$$

where, in simplifying Q, we have used the duplication formula for the Gamma function [1, Equation 6.1.18]

$$\Gamma(2z) = (2\pi)^{-\frac{1}{2}} 2^{2z-\frac{1}{2}} \Gamma(z)\Gamma(z+\tfrac{1}{2})$$

Now, (4.58) and (4.59) show that

$$\|S_n'^{(\nu)}\|_\infty^{(\nu)} = 2\nu \left\| \sum_{k=1}^{n} h_k^{(\nu)} P_{k-1}^{(\nu+1)} \right\|_\infty^{(\nu)}$$

$$= 2\nu \sum_{k=1}^{n} h_k^{(\nu)} P_{k-1}^{(\nu+1)}(1)$$

$$< Q \sum_{k=1}^{n} (k+2\nu)^{\nu+2}$$

$$< \frac{Q}{\nu+3}(n+2\nu+1)^{\nu+3} \tag{4.61}$$

Finally, recalling the expansion (4.40), we have

$$\|S_n'^{(\nu)}\|_2^{(\nu)} = 2\nu \Big\| \sum_{k=1}^{n} h_k^{(\nu)} P_{k-1}^{(\nu+1)} \Big\|_2^{(\nu)}$$

$$= 2 \Big\| \sum_{k=1}^{n} h_k^{(\nu)} \sum_{t=0}^{\left[\frac{k-1}{2}\right]} (k-2t+\nu-1) P_{k-2t-1}^{(\nu)} \Big\|_2^{(\nu)}$$

$$= 2 \Big\| \sum_{s=1}^{n} \Big\{ \sum_{k-2t=s} (k-2t+\nu-1) h_k^{(\nu)} \Big\} P_{s-1}^{(\nu)} \Big\|_2^{(\nu)}$$

$$= 2 \left[\sum_{s=1}^{n} \left\{ \frac{s+\nu-1}{h_{s-1}^{(\nu)}} \sum_{k-2t=s} h_k^{(\nu)} \right\}^2 \right]^{\frac{1}{2}} \tag{4.62}$$

What is needed is an estimate for the inner sum of (4.62) over $k-2t = s$, for each fixed $s \geq 1$. Let $N = [\frac{n-1}{2}]$.

$$\sum_{k-2t=s} h_k^{(\nu)} < \sum_{t=0}^{N} h_{s+2t}^{(\nu)}$$

$$= \frac{2^{\nu-\frac{1}{2}}\Gamma(\nu)}{\sqrt{\pi}} \sum_{t=0}^{N} \left\{ \frac{(s+2t+\nu)\Gamma(s+2t+1)}{\Gamma(s+2t+2\nu)} \right\}^{\frac{1}{2}}$$

$$= \frac{2^{\nu-\frac{1}{2}}\Gamma(\nu)}{\sqrt{\pi}} \sum_{t=0}^{N} \left\{ \frac{s+2t}{s+2t+2\nu-1} \cdots \frac{s+1}{s+2\nu} \frac{(s+1)}{\Gamma(s+2\nu)} (s+2t+\nu) \right\}^{\frac{1}{2}}$$

$$< \frac{2^{\nu-\frac{1}{2}}\Gamma(\nu)}{\sqrt{\pi}} \left\{ \frac{\Gamma(s+1)}{\Gamma(s+2\nu)} \right\}^{\frac{1}{2}} \sum_{t=0}^{N} (s+2t+\nu)^{\frac{1}{2}}$$

$$= \frac{2^{\nu-\frac{1}{2}}\Gamma(\nu)}{3\sqrt{\pi}} \left\{ \frac{\Gamma(s+1)}{\Gamma(s+2\nu)} \right\}^{\frac{1}{2}} (s+n+\nu+1)^{\frac{3}{2}} \tag{4.63}$$

Since

$$\left(\frac{s+\nu-1}{h_{s-1}^{(\nu)}} \right)^2 = \frac{\pi(s+\nu-1)\Gamma(s+2\nu-1)}{2^{2\nu-1}\Gamma^2(\nu)\Gamma(s)}$$

$$< \frac{\pi\ \Gamma(s+2\nu)}{2^{2\nu-1}\Gamma^2(\nu)\Gamma(s)} \tag{4.64}$$

(4.63) and (4.64) in (4.62) imply

$$\left\{\|S_n'^{(\nu)}\|_2^{(\nu)}\right\}^2 < 4 \sum_{s=1}^{n}\left\{\frac{\pi\,\Gamma(s+2\nu)}{2^{2\nu-1}\Gamma^2(\nu)\Gamma(s)}\right\}\left\{\frac{2^{2\nu-1}\Gamma^2(\nu)\Gamma(s+1)}{9\,\pi\,\Gamma(s+2\nu)}\right.$$

$$\left.\cdot\,(s+n+\nu+1)^3\right\}$$

$$< \frac{4}{9}\sum_{s=1}^{n}(s+n+\nu+1)^4$$

$$< \frac{4}{45}(2n+\nu+2)^5$$

so that

$$\|S_n'^{(\nu)}\|_2^{(\nu)} < \frac{8}{3}\sqrt{\frac{2}{5}}\left(n+\frac{\nu}{2}+1\right)^{\frac{5}{2}} \tag{4.65}$$

Finally, using (4.60), (4.61), and (4.65) in (3.51) concludes the proof of (4.55). A proof of (4.57) is unnecessary since the supremum norm is independent of ν and (4.32) holds for $\alpha = \beta = \nu-\frac{1}{2}$. This completes the proof.

Theorem 4.7 and Theorem 4.4 give an interesting comparison. We have

$$\|\pi_n'\|_p^{(\nu+1)} \le \|\pi_n'\|_p^{(\nu)}, \quad \nu \ge \frac{1}{2},\ p \ge 1 \tag{4.66}$$

since $0 < (1-x^2)^{\nu+1-\frac{1}{2}} = (1-x^2)^{\nu+\frac{1}{2}} \le (1-x^2)^{\nu-\frac{1}{2}}$ provided $\nu \ge \frac{1}{2}$. Now, defining for $\nu \ge \frac{1}{2}$,

$$U_{n,p}^{(\nu)} = \max_{0\ne\pi_n\in P_n}\left\{\frac{\|\pi_n'\|_p^{(\nu+1)}}{\|\pi_n\|_p^{(\nu)}}\right\},\ p \ge 1 \tag{4.67}$$

and

$$V_{n,p}^{(\nu)} = \max_{0\ne\pi_n\in P_n}\left\{\frac{\|\pi_n'\|_p^{(\nu)}}{\|\pi_n\|_2^{(\nu)}}\right\},\quad p \ge 1 \tag{4.68}$$

we see that (4.66) implies that

$$U_{n,p}^{(\nu)} \le V_{n,p}^{(\nu)}, \quad \nu \ge \frac{1}{2},\ p \ge 1 \tag{4.69}$$

But much more than (4.69) can be said. Theorem **4.4** gives

$$U_{n,2p}^{(\nu)} < A_1(n+2\nu+1)^{2+(\nu+\frac{1}{2})(1-\frac{1}{p})-\frac{1}{p}}, \nu \ge \frac{1}{2}, n \ge 0,\ p = 2,3,4,\ldots$$

while Theorem 4.7 gives

$$V_{n,2p}^{(\nu)} < A_2(n+2\nu+1)^{2+(\nu+\frac{1}{2})(1-\frac{1}{p})}, \quad \nu \ge \frac{1}{2},\ n \ge 0,\ p = 2,3,4,\ldots$$

where the constants A_1 and A_2 can be taken independent of both n and p.

C. Laguerre Polynomials

Let $L_n^{(\alpha)}(x)$ be the n-th degree generalized Laguerre polynomial, $\alpha > -1$, as defined by Szegö [31, Chapter V]. These polynomials satisfy the orthogonality relation

$$\int_0^\infty L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) x^\alpha e^{-x}\, dx = \frac{\delta_{n,m}}{\{g_n^{(\alpha)}\}^2} \tag{4.70}$$

where

$$g_n^{(\alpha)} = \left\{\Gamma(1+\alpha)\binom{n+\alpha}{n}\right\}^{-\frac{1}{2}} \tag{4.71}$$

Let

$$S_n^{(\alpha)} = \sum_{k=0}^{n} (-1)^k g_k^{(\alpha)} L_k^{(\alpha)}(x) \tag{4.72}$$

Define the norms, for $\pi_n \in P_n$,

$$\|\pi_n\|_p^{(\alpha)} = \left\{\int_0^\infty |\pi_n(x)|^p x^\alpha e^{-x}\, dx\right\}^{\frac{1}{p}}, \ p \ge 1 \tag{4.73}$$

Note that $\|\pi_n\|_\infty^{(\alpha)} = \infty$ for all nonconstant $\pi_n \in P_n$. Also, the notation (4.72) and (4.73) should not be confused with earlier (identical) notation for the Gegenbauer polynomials.

Szegö [31, Problem 94] states that

$$(-1)^{k+m+n} \int_0^\infty L_k^{(\alpha)}(x) L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^\alpha e^{-x} dx \geq 0 \qquad (4.74)$$

for all $k,m,n = 0, 1, \ldots$. Since the expansion

$$(-1)^{n+m} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) = \sum_{k=0}^{n+m} C(k,m,n;\alpha)(-1)^k L_k^{(\alpha)}(x) \qquad (4.75)$$

certainly exists, it is easy to see that (4.74) implies

$$C(k,m,n;\alpha) \geq 0 \qquad (4.76)$$

for all $k,m,n = 0, 1, \ldots$. Watson [34] gives an explicit expression involving a hypergeometric function, but a simple form seems unavailable. Even for $\alpha = 0$, the formula for the coefficients seems rather complicated (see Gillis and Weiss [39]).

Szegö [31, Equation 5.1.14] gives

$$\frac{d}{dx}\left[(-1)^n L_n^{(\alpha)}(x)\right] = (-1)^{n-1} L_{n-1}^{(\alpha+1)} \qquad (4.77)$$

and Bailey [5] attributes to Erdelyi the relation

$$x^m (-1)^n L_n^{(\alpha+m)}(x) = \frac{\Gamma(1+\alpha+m+n)}{n!} \sum_{k=0}^{m} \frac{(m)_k (n+k)!}{k!\,\Gamma(1+\alpha+n+k)} \cdot (-1)^{n+k} L_{n+k}^{(\alpha)}(x) \qquad (4.78)$$

It is a simple matter to see that the appropriate Nonnegativity Conditions are satisfied in order to prove the next result.

<u>Theorem 4.8</u> Let $p \geq 1$ be an integer. Let $\alpha > -1$. Then the equations (4.70)-(4.78) imply that for all $\pi_n \in P_n$, $\pi_n \neq 0$,

$$\frac{\|\pi_n\|_{2p}^{(\alpha)}}{\|\pi_n\|_2^{(\alpha)}} \leq \sqrt{\max_{0\leq k\leq n} g_k^{(\alpha)} \|L_k^{(\alpha)} S_n^{(\alpha)}\|_p^{(\alpha)}} \tag{4.79}$$

$$\frac{\|\pi_n'\|_{2p}^{(\alpha+1)}}{\|\pi_n\|_2^{(\alpha)}} \leq \sqrt{\max_{1\leq k\leq n} g_k^{(\alpha)} \|L_k'^{(\alpha)} S_n'^{(\alpha)}\|_p^{(\alpha+1)}} \tag{4.80}$$

$$\frac{\|x\pi_n'\|_{2p}^{(\alpha)}}{\|\pi_n\|_2^{(\alpha)}} \leq \sqrt{\max_{0\leq k\leq n} g_k^{(\alpha)} \|L_k'^{(\alpha)} S_n'^{(\alpha)}\|_p^{(\alpha+2p)}} \tag{4.81}$$

<u>*Proof*</u> *Use Theorem 3.5.*

Further information seems difficult to extract from Theorem 4.8 primarily because the polynomials are not essentially bounded with respect to $x^\alpha e^{-x}$, so that Corollaries 3.7 and 3.14, as well as Theorems 3.3 and 3.6, are not applicable. Since estimates for the higher norms of $L_n^{(\alpha)}(x)$ do not seem to be available, the utility of Theorem 4.8 seems limited.

Turán [32] proves that for $n \geq 1$

$$\max_{0\neq\pi_n\in P_n} \left\{\frac{\|\pi_n'\|_2^{(0)}}{\|\pi_n\|_2^{(0)}}\right\} = \frac{1}{2\sin(\frac{\pi}{4n+2})} \tag{4.82}$$

where the norms in (4.82) are, of course, given by (4.73) and the maximum in (4.82) is attained only for nonzero constant multiples of

$$\pi_n(x) = \sum_{k=1}^{n} \sin\left(\frac{k\pi}{2n+1}\right) L_k^{(0)}(x)$$

We know of no other results related to (4.79) through (4.81).

D. Hermite Polynomials

Let $H_n(x)$ be the n-th degree Hermite polynomial, as defined by Szegö [31, Chapter V]. The Hermite polynomials satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\,dx = \frac{\delta_{n,m}}{\{h_n\}^2} \tag{4.83}$$

where

$$h_n = \{\sqrt{\pi}\ 2^n\ n!\}^{-\frac{1}{2}} \tag{4.84}$$

Let

$$S_n(x) = \sum_{k=0}^{n} h_k H_k(x) \tag{4.85}$$

Lebedev [21, p. 96] gives the expansion

$$H_n(x) H_m(x) = \sum_{k=0}^{\min(n,m)} 2^k k! \binom{m}{k}\binom{n}{k} H_{n+m-2k}(x) \tag{4.86}$$

and from Szegö [31, Equation 5.5.10],

$$\frac{d}{dx} H_n(x) = 2nH_{n-1}(x) \tag{4.87}$$

Define for all $\pi_n \in P_n$,

$$\|\pi_n\|_p = \left\{\int_{-\infty}^{\infty} e^{-x^2} |\pi_n(x)|^p\, dx\right\}^{\frac{1}{p}},\ p \geq 1 \tag{4.88}$$

<u>Theorem 4.9</u> Let $p \geq 1$ be an integer. Then, for all $\pi_n \in P_n$, $\pi_n \neq 0$,

$$\frac{\|\pi_n\|_{2p}}{\|\pi_n\|_2} \le \max_{0\le k\le n} \sqrt{h_k \|H_k S_n\|_p} \tag{4.89}$$

$$\frac{\|\pi_n'\|_{2p}}{\|\pi_n\|_2} \le \max_{1\le k\le n} \sqrt{2kh_k \|H_{k-1} S_n'\|_p} \tag{4.90}$$

where the norms are defined by (4.88).

Proof The appropriate Nonnegativity Conditions are satisfied because of (4.86) and (4.87). Use Theorems 3.2 and 3.5.

The remarks concerning the limited utility of Theorem 4.8 apply here as well.

The only bound in the literature related to (4.89) or (4.90) seems to be one mentioned by Turan [32], who states that E. Schmidt [35] proves that

$$\max_{0\ne\pi_n\in P_n} \left\{\frac{\|\pi_n'\|_2}{\|\pi_n\|_2}\right\} = \sqrt{2n} \tag{4.91}$$

where the norms are, of course, given by (4.88). The maximum (4.91) is attained only for nonzero constant multiples of $H_n(x)$.

E. Remarks

In conclusion, there are two results of a general nature which can be useful in guaranteeing that the Nonnegativity Condition is satisfied. Let $p_0(x)$, $p_1(x)$, ..., be any sequence of orthogonal polynomials normalized so that $p_n(x) = x^n + \ldots$. Askey [3] proves that if

$$p_1(x)p_n(x) = p_{n+1}(x) + a_n p_n(x) + b_n p_{n-1}(x) \tag{4.92}$$

holds for $n = 1, 2, \ldots$, and if $a_n \geq 0$, $b_n > 0$, and $a_{n+1} \geq a_n$, $b_{n+1} \geq b_n$, then

$$p_n(x)p_m(x) = \sum_{k=|n-m|}^{n+m} A_k p_k(x), \quad A_k \geq 0 \tag{4.93}$$

holds for all $n,m = 0, 1, \ldots$. Askey applies this result successfully to Laguerre, Hermite, Charlier, and Meixner polynomials.

In another direction, let $\omega(x)$ be a positive function on $(0,\infty)$ such that $\int_0^\infty x^n \omega(x)dx$ exists for $n = 0, 1, \ldots$. Let $\{p_n(x)\}$ be the orthonormal polynomials with respect to $\omega(x)$ standardized by $p_n(0) > 0$. This can be done since all the zeros of $p_n(x)$ are interior to $[0,\infty)$. Let $\{p_n^{(\alpha)}(x)\}$ be the polynomials orthonormal with respect to $x^\alpha \omega(x)$ and standardized by $p_n^{(\alpha)}(0) > 0$, where $\alpha \geq 1$ is a fixed integer. Then Askey [4] shows that

$$p_n^{(\alpha)}(x) = \sum_{k=0}^{n} \alpha_k p_k(x), \quad \alpha_k > 0 \tag{4.94}$$

for all $n = 0, 1, \ldots$. Askey conjectures that (4.94) holds for any $\alpha > 0$.

Chapter V

REPRESENTATION THEOREM

A. Permutable Operators on $\otimes^p V$

In previous chapters, we have defined the operators $M_{n,p}$ in Lemma 2.1, $M^{(1)}_{n,p}$ in equation (2.73), $L^{\phi}_{n,p}$ in Lemma 3.2, and $E^{\phi}_{n,p}$ in Lemma 3.9. In this chapter, algebraic properties common to all these operators are studied and a Representation Theorem is proved (Theorem 5.8).

Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let V be the vector space $\overset{n}{\times} F = F^n$ equipped with the inner product

$$(x,y)_V = \bar{y}^T x, \qquad x,y \in V \tag{5.1}$$

The earlier definition of $\Gamma = \Gamma_{n,p}$ in (2.15) is slightly altered in that the common index set $\{0, 1, 2, \ldots, n\}$ is, in this chapter, replaced by $\{1, 2, \ldots, n\}$. Therefore, Γ has n^p elements. Lexicographic ordering is still defined here by (2.16). Let $\{e_1, \ldots, e_n\}$ be the usual basis for V; that is, e_k has all zero components except the k-th component which is 1. Define the Kronecker product

$$e^{\otimes}_{\alpha} = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \in \overset{p}{\times} F^n \cong F^{n^p}$$

to be that vector with all zero components except the $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$ component which is 1. Let $x_k = \langle x^k_1, \ldots, x^k_n \rangle^T \in V$, $k = 1, \ldots, p$. Define their Kronecker product

$$x_1 \otimes \cdots \otimes x_p \in F^{n^p}$$

by

$$x_1 \otimes \cdots \otimes x_p = \sum_{\alpha=(\alpha_1,\ldots,\alpha_p)\in\Gamma} (x^1_{\alpha_1} \cdots x^p_{\alpha_p}) e^{\otimes}_{\alpha} \tag{5.2}$$

Define

$$\overset{p}{\otimes} V = \mathrm{Span}\{e^{\otimes}_{\alpha} : \alpha \in \Gamma_{n,p}\} \tag{5.3}$$

Any element of $\otimes^p V$ of the form (5.2) is said to be decomposable. Any element of $\otimes^p V$ which cannot be written in the form (5.2) for some vectors x_1, ..., x_p in V is indecomposable. If $u \in \otimes^p V$ and $v \in \otimes^p V$, define the inner product by

$$(u,v) = \bar{v}^T u \tag{5.4}$$

It is not hard to see that if $u = x_1 \otimes \cdots \otimes x_p$ and $v = y_1 \otimes \cdots \otimes y_p$, then

$$(u,v) = (x_1,y_1)_V \cdots (x_p,y_p)_V$$

$$= \prod_{k=1}^{p} \bar{y}_k^T x_k \tag{5.5}$$

Let S_p be the group of permutations on the integers {1, 2, ..., p}. For each $\sigma \in S_p$, define the permutation operator $P(\sigma)$ on $\otimes^p V$ via

$$P(\sigma) x_1 \otimes \cdots \otimes x_p = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(p)} \tag{5.6}$$

for all decomposable elements $x_1 \otimes \cdots \otimes x_p$ in $\otimes^p V$. Since the basis elements $\{e^{\otimes}_{\alpha} : \alpha \in \Gamma\}$ of $\otimes^p V$ are decomposable, $P(\sigma)$ can be linearly extended to all $\otimes^p V$. That this extension of (5.6) yields a unique and well defined linear operator on $\otimes^p V$ is proved by Marcus [23, Chapter 1]. Thus, for all $\sigma \in S_p$, $P(\sigma) \in \mathcal{L}(\otimes^p V)$, the space of all linear operators on $\otimes^p V$.

Let $T \in \mathcal{L}(V)$, the space of linear operators on V. Then define $\otimes^p T \in \mathcal{L}(\otimes^p V)$ via

$$\otimes^p T\, x_1 \otimes \cdots \otimes x_p = (Tx_1) \otimes \cdots \otimes (Tx_p) \in \otimes^p V \quad (5.7)$$

for all decomposable elements in $\otimes^p V$. Since $\{e_\alpha^\otimes : \alpha \in \Gamma\}$ are decomposable, $\otimes^p T$ can be linearly extended to all $\otimes^p V$. That this extension yields an unambiguous operator on all $\otimes^p V$ is shown by Marcus [23, Chapter 2]. Let the matrix of T with respect to $\{e_1, \ldots, e_n\}$ be $[a_{i,j}]$ and let the matrix of $\otimes^p T$ *with respect to* $\{e_\alpha^\otimes, \alpha \in \Gamma\}$ be $[A_{\alpha,\beta}]$. Then Marcus shows that

$$A_{\alpha,\beta} = \prod_{k=1}^{p} a_{\alpha_k,\beta_k} \quad (5.8)$$

for all $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$, $\beta = (\beta_1, \ldots, \beta_p) \in \Gamma$. Note that (5.8) merely states that the matrix of the Kronecker product operator $\otimes^p T$ is just the Kronecker product of the matrix of T with itself p times. (For a definition of Kronecker product of matrices, see Chapter II.)

Marcus [23, p. 245] defines $B \in \mathcal{L}(\otimes^p V)$ to be a bisymmetric operator if and only if

$$BP(\sigma) = P(\sigma)B, \quad \forall \sigma \in S_p \quad (5.9)$$

Let $\mathcal{B}_p$ denote the totality of all bisymmetric operators on $\otimes^p V$. The matrix of $B \in \mathcal{B}_p$ is denoted

$$B = [b_{\alpha,\beta}], \qquad \alpha,\beta \in \Gamma \tag{5.10}$$

Marcus [23, Theorem 2.6] proves that $B \in \mathcal{B}_p$ if and only if, for every $\sigma \in S_p$,

$$b_{\alpha,\beta} = b_{\alpha\sigma,\beta\sigma}, \qquad \alpha,\beta \in \Gamma \tag{5.11}$$

where $\alpha\sigma$ and $\beta\sigma$ are defined by

$$\begin{aligned} \alpha\sigma &= (\alpha_1,\cdots,\alpha_p)\sigma = (\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(p)}) \in \Gamma \\ \beta\sigma &= (\beta_1,\cdots,\beta_p)\sigma = (\beta_{\sigma(1)},\cdots,\beta_{\sigma(p)}) \in \Gamma \end{aligned} \tag{5.12}$$

A much deeper result is

Theorem 5.1 A linear operator B on $\otimes^p V$ is bisymmetric if and only if B is a linear combination of Kronecker product operators $\otimes^p T$, $T \in \mathcal{L}(V)$. In other words,

$$\mathcal{B}_p = \mathrm{Span}\{\otimes^p T : T \in \mathcal{L}(V)\} \tag{5.13}$$

Furthermore, in (5.13), T can be taken to be nonsingular.

Proof See Marcus [23, Theorem 2.7] and the remarks on page 249].

Up to now, we have presented only known results. We will now define and study an apparently new subspace of the bisymmetric operators $\mathcal{B}_p$. We define an operator $L \in \mathcal{L}(\otimes^p V)$ to be a permutable operator if and only if

$$LP(\sigma) = P(\sigma)L = L, \qquad \forall \sigma \in S_p \tag{5.14}$$

Let $\mathcal{E}_p$ be the space of all permutable operators on $\otimes^p V$.

Theorem 5.2 The symmetrizer S on $\otimes^p V$ is an element of $\mathcal{E}_p$.

Proof The symmetrizer $S \in \mathcal{L}(\otimes^p V)$ is defined by (see Marcus [23, Theorem 2.6])

$$S = \frac{1}{p!} \sum_{\gamma \in S_p} P(\gamma) \tag{5.15}$$

Let $\sigma \in S_p$. Then

$$\begin{aligned} SP(\sigma) &= \left(\frac{1}{p!} \sum_{\gamma \in S_p} P(\gamma)\right) P(\sigma) \\ &= \frac{1}{p!} \sum_{\gamma \in S_p} P(\gamma\sigma) \end{aligned} \tag{5.16}$$

where the last equation follows from Marcus [23, p. 72]. Since, for fixed $\sigma \in S_p$, $\{\gamma\sigma : \gamma \in S_p\} = S_p$, we see from (5.16) that

$$SP(\sigma) = S$$

Similarly, $P(\sigma)S = S$ and this concludes the proof.

The next theorem is the analog of equation (5.11).

Theorem 5.3 A linear operator L on $\otimes^p V$ is permutable if and only if, for every $\sigma, \gamma \in S_p$,

$$a_{\alpha,\beta} = a_{\alpha\sigma,\beta\gamma}, \qquad \alpha, \beta \in \Gamma \tag{5.17}$$

where $[a_{\alpha,\beta}]$ is the matrix of L with respect to the basis $\{e_\alpha^\otimes, \alpha \in \Gamma\}$.

Proof We have

$$Le_{\beta}^{\otimes} = \sum_{\alpha\in\Gamma} a_{\alpha,\beta} e_{\alpha}^{\otimes} \tag{5.18}$$

so that

$$\begin{aligned} P(\sigma)Le_{\beta}^{\otimes} &= \sum_{\alpha\in\Gamma} a_{\alpha,\beta}\left(P(\sigma)e_{\alpha}^{\otimes}\right) \\ &= \sum_{\alpha\in\Gamma} a_{\alpha,\beta}\, e_{\alpha\sigma^{-1}}^{\otimes} \\ &= \sum_{\alpha\in\Gamma} a_{\alpha\sigma,\beta}\, e_{\alpha}^{\otimes} \end{aligned} \tag{5.19}$$

Similarly, (5.18) gives

$$\begin{aligned} LP(\gamma)e_{\beta}^{\otimes} &= L\, e_{\beta\gamma^{-1}}^{\otimes} \\ &= \sum_{\alpha\in\Gamma} a_{\alpha,\beta\gamma^{-1}}\, e_{\alpha}^{\otimes} \end{aligned} \tag{5.20}$$

Since $LP(\gamma) = L = P(\sigma)L$, equating (5.19) and (5.20) gives, for all $\sigma,\ \gamma \in S_p$,

$$a_{\alpha\sigma,\beta} = a_{\alpha,\beta\gamma^{-1}}, \qquad \alpha,\beta \in \Gamma \tag{5.21}$$

In (5.21) replace β by $\beta\gamma$ to get

$$a_{\alpha\sigma,\beta\gamma} = a_{\alpha,\beta}$$

and this completes the proof.

Marcus [23, p. 132] defines the completely symmetric space $V^{(p)}$ to be the range of the symmetrizer S defined by (5.15) and shows that

$$\dim V^{(p)} = \binom{n+p-1}{p} \tag{5.22}$$

Furthermore, S is idempotent; that is, $S^2 = S$. Therefore, $u \in \mathrm{Range}(S) = V^{(p)}$ if and only if $Su = u$, if and only if $P(\sigma)u = u$ for all $\sigma \in S_p$. This last statement follows from the fact that $P(\sigma)S = S$ for all $\sigma \in S_p$. The next result was pointed out to the author by Herbert Robinson in a private conversation.

Theorem 5.4 $L \in \mathcal{L}(\otimes^p V)$ is a permutable operator if and only if

$$L(\mathrm{Null}(S)) = \{0\} \tag{5.23}$$

and

$$L(\mathrm{Range}(S)) \subset V^{(p)} \tag{5.24}$$

Proof First, Let L be permutable. Let $u \in \otimes^p V$. Then

$$\begin{aligned} S(Lu) &= \frac{1}{p!} \sum_{\sigma \in S_p} (P(\sigma)L)u \\ &= \frac{1}{p!}\left(\sum_{\sigma \in S_p} L\right) u \\ &= Lu \end{aligned}$$

and so $Lu \in \mathrm{Range}(S) = V^{(p)}$ which proves (5.24). Let $u \in \mathrm{Null}(S)$. Then $Lu = (LS)u = L(Su) = 0$ which proves (5.23). Conversely, suppose (5.23) and (5.24) hold. Let $\omega \in \otimes^p V$. Since S is hermitian, there exists $u \in \mathrm{Range}(S) = V^{(p)}$ and $v \in \mathrm{Null}(S)$ such that $\omega = u + v$. Then for all $\sigma \in S_p$, $P(\sigma)u = u$ and $P(\sigma)v \in \mathrm{Null}(S)$, since $SP(\sigma)v = P(\sigma)(Sv) = 0$. Now

$$\begin{aligned} LP(\sigma)\omega &= L(P(\sigma)u) + L(P(\sigma)v) \\ &= Lu + 0 \\ &= L(u + v) \\ &= L\omega \end{aligned}$$

Similarly, since $Lu \in V^{(p)}$ by (5.24),

$$
\begin{aligned}
P(\sigma)L\omega &= P(\sigma)(Lu) + P(\sigma)(Lv) \\
&= Lu + P(\sigma)(0) \\
&= Lu \\
&= L(u + v) \\
&= L\omega
\end{aligned}
$$

Hence $L = P(\sigma)L = LP(\sigma)$ and so L is permutable.

Corollary 5.1 $L \in E_p$ implies that the rank of L is less than or equal to

$$\binom{n+p-1}{p}$$

Proof Range $(L) \subset V^{(p)}$ and $\dim V^{(p)}$ is $\binom{n+p-1}{p}$.

Corollary 5.2 If $T \in \mathcal{L}(V)$ is nonsingular, $p \geq 2$ and $n \geq 2$, then $\otimes^p T \notin E_p$.

Proof Marcus [23, p. 54-63] shows that

$$\operatorname{rank}[\otimes^p T] = (\operatorname{rank} T)^p = n^p$$

Since

$$n^p > \binom{n+p-1}{p}, \text{ for } p \geq 2 \text{ and } n \geq 2$$

the rank of $\otimes^p T$ is too large to allow $\otimes^p T \in E_p$.

Corollary 5.3 For $p \geq 1$,

$$E_p = \{MS \mid M \in \mathcal{L}(V^{(p)})\} \tag{5.25}$$

Proof Let $M \in \mathcal{L}(V^{(p)})$. Since $MS(\mathrm{Null}(S)) = \{0\}$ and $MS(\mathrm{Range}(S)) = M(V^{(p)}) \subset V^{(p)}$, we have, by Theorem 5.4,

$$E_p \supset \{MS \mid M \in \mathcal{L}(V^{(p)})\}$$

Now, to prove the reverse inclusion, let $L \in E_p$. Then $SL = LS = L$, so that $L\omega = LS\omega$ for all $\omega \in \otimes^p V$. Since S is the projection of $\otimes^p V$ onto $V^{(p)}$, we can always write

$$L\omega = MS\omega \in V^{(p)}$$

where M is the restriction of L to $V^{(p)}$. Thus, $M \in \mathcal{L}(V^{(p)})$. This concludes the proof.

The significance of this last corollary is twofold. First, it shows that the permutable operators are essentially general linear operators on $V^{(p)}$. Second, because of this general nature, not much can be said about the eigenstructure of permutable operators. Despite this, however, we do have the following theorem.

Theorem 5.5 If L is a hermitian permutable operator on $\otimes^p V$, then L has at most n ($= \dim V$) decomposable orthogonal eigenvectors in $\otimes^p V$ with nonzero eigenvalues.

Proof Let $u = x \otimes \cdots \otimes x_p \neq 0$ be such that $Lu = \lambda u$, $\lambda \neq 0$. We claim that $u_k = x_k \otimes \cdots \otimes x_k$ satisfies

$$Lu_k = \lambda u_k, \qquad k = 1, \ldots, p \tag{5.26}$$

Since Range $(L) \subset V^{(p)}$, $x_1 \otimes \cdots \otimes x_p \in V^{(p)}$, so that for all $\sigma \in S_p$,

$$\begin{aligned} x_1 \otimes \cdots \otimes x_p &= P(\sigma^{-1}) x_1 \otimes \cdots \otimes x_p \\ &= x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)} \end{aligned}$$

Marcus [23, Theorem 2.3] implies that

$$x_k = c_k x_{\sigma(k)}, \qquad k = 1, \ldots, p \tag{5.27}$$

with

$$\prod_{k=1}^{p} c_k \neq 0$$

Clearly (5.27) implies

$$u_k = x_k \otimes \cdots \otimes x_k = \hat{c}_k \, x_1 \otimes \cdots \otimes x_p, \qquad k = 1, \ldots, p \tag{5.28}$$

for some $\hat{c}_k \neq 0$. From (5.28) follows (5.26). Now, let $v = y_1 \otimes \cdots \otimes y_p \neq 0$ be such that $Lv = \Lambda v$, $\Lambda \neq 0$. As before, we have

$$v_k = y_k \otimes \cdots \otimes y_k = \tilde{c}_k y_1 \otimes \cdots \otimes y_p, \ \tilde{c}_k \neq 0$$

and

$$Lv_k = \Lambda v_k, \qquad k = 1, \ldots, p$$

Since L is hermitian, u and v may be taken orthogonal; that is, with $C = (\hat{c}_1 \tilde{c}_1)^{-1}$,

$$\begin{aligned} 0 = (u,v) &= C(x_1 \otimes \cdots \otimes x_1, \ y_1 \otimes \cdots \otimes y_1) \\ &= C \prod_{k=1}^{p} (x_1, y_1)_V \\ &= C\{(x_1, y_1)_V\}^p \end{aligned}$$

so that $x_1 \perp y_1$ in V. Therefore, any set of decomposable eigenvectors of L are such that the "first" vectors in any Kronecker product representation of these eigenvectors are pairwise orthogonal. Since dim V = n, there can be at most n such eigenvectors which do not map to 0 under L. This completes the proof.

There exist "natural" candidates for the decomposable eigenvectors for a hermitian permutable operator L on $\otimes^p V$. Let

$$\lambda_n' = \max_{x \in V} \frac{(x \otimes \cdots \otimes x, Lx \otimes \cdots \otimes x)}{(x \otimes \cdots \otimes x, x \otimes \cdots \otimes x)}$$

Clearly, λ_n' is well defined. Let $x_n \in V$ be any vector for which the maximum λ_n' is attained. Let

$$\lambda_{n-1}' = \max_{\substack{x \in V \\ x \perp x_n}} \frac{(x \otimes \cdots \otimes x, Lx \otimes \cdots \otimes x)}{(x \otimes \cdots \otimes x, x \otimes \cdots \otimes x)}$$

and let $x_{n-1} \in V$ be any vector for which this maximum is attained. Continuing in this fashion, one generates the sequence of real numbers

$$\lambda_1' \le \lambda_2' \le \cdots \le \lambda_n'$$

and the sequence $x_k \otimes \cdots \otimes x_k \in \otimes^p V$, $k = 1, \ldots, n$. Are the elements $x_k \otimes \cdots \otimes x_k$ the decomposable eigenvectors of L with corresponding eigenvalues λ_k'? The answer is no, since L need not have any decomposable eigenvectors. On the other hand, we conjecture that if λ is an eigenvalue of L with a corresponding decomposable eigenvector, then $\lambda \in \{\lambda_1', \lambda_2', \ldots, \lambda_n'\}$.

Since a permutable operator $L \in \mathcal{L}(\otimes^p V)$ is also bisymmetric, L has the representation of Theorem 5.1. Specifically, there exists a smallest integer $N \ge 1$, and constants

$c_1, \ldots, c_N$ in F, and linear operators $A_1, \ldots, A_N$ in $\mathcal{L}(V)$ such that

$$L = \sum_{k=1}^{N} c_k \underbrace{A_k \otimes \cdots \otimes A_k}_{p \text{ factors}} \tag{5.29}$$

where $\det A_k = \pm 1$, $k = 1, \ldots, N$.

What more can be said of the matrices $A_1, \ldots, A_N$ in (5.29)? One question is whether or not the matrices $A_1, \ldots, A_N$ in (5.29) are necessarily hermitian if L is hermitian. The example

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_1 \otimes A_1 + \frac{1}{2} A_2 \otimes A_2 - \frac{1}{2} A_3 \otimes A_3$$

where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

gives the answer, since A_3 is not hermitian, provided only that N = 3 is the smallest number of matrices possible in this case. The proof that two matrices do not suffice is quite easy. However, we do not know that the representation (5.29) is, in any sense, unique. Therefore, it is still conceivable that there exist for this example three matrices which are hermitian and represent L.

<u>Theorem 5.6</u> Let L be a permutable operator on $\otimes^p V$, where $p = n = \dim V \geq 2$. Then any representation of L in the form (5.29) has the property

$$\sum_{k=1}^{N} c_k \det A_k = 0$$

<u>Proof</u> Let $[a_{ij}^{(k)}]$ be the $n \times n$ matrix of A_k, $k = 1, \ldots, N$. Throughout this proof, let $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Gamma$ and $\beta = (\beta_1, \ldots, \beta_p) \in \Gamma$. Since the $n^p \times n^p$ matrix of $A_k \otimes \cdots \otimes A_k$ is, from (5.8),

$$\left[\prod_{t=1}^{p} a_{\alpha_t,\beta_t}^{(k)}\right]_{\alpha,\beta \in \Gamma}$$

we have the matrix of L given by

$$\left[\sum_{k=1}^{N} c_k \prod_{t=1}^{p} a_{\alpha_t,\beta_t}^{(k)}\right]_{\alpha,\beta \in \Gamma}$$

By Theorem 5.3, for all $\sigma, \gamma \in S_p$,

$$\sum_{k=1}^{N}\left(c_k \prod_{t=1}^{p} a_{\alpha_t,\beta_t}^{(k)}\right) = \sum_{k=1}^{N}\left(c_k \prod_{t=1}^{p} a_{\alpha_{\gamma(t)},\beta_{\sigma(t)}}^{(k)}\right)$$

so that

$$\sum_{k=1}^{N} c_k\left(\prod_{t=1}^{p} a_{\alpha_t,\beta_t}^{(k)} - \prod_{t=1}^{p} a_{\alpha_{\gamma(t)},\beta_{\sigma(t)}}^{(k)}\right) = 0 \tag{5.30}$$

Since $p = n$, we can put $\alpha_1 = 1, \alpha_2 = 2, \ldots, \alpha_p = n$, and for $\gamma(1) = 1, \ldots, \gamma(p) = n$, (5.30) becomes

$$\sum_{k=1}^{N} c_k\left(\prod_{t=1}^{p} a_{t,\beta_t}^{(k)} - \prod_{t=1}^{p} a_{t,\beta_{\sigma(t)}}^{(k)}\right) = 0 \tag{5.31}$$

Let σ be any odd permutation in S_p. Now, whenever the p-tuple $(\beta_1,\ldots,\beta_p)$ is an even permutation of the first p integers, $(\beta_{\sigma(1)},\ldots,\beta_{\sigma(p)})$ is an odd permutation. Therefore, summing (5.31) over all those coordinates $(\beta_1,\ldots,\beta_p)$ which are even permutations of the first p integers,

$$0 = \sum_{k=1}^{N} c_k \left\{ \sum_{\beta \text{ even}} \left[\prod_{t=1}^{p} a^{(k)}_{t,\beta_t} - \prod_{t=1}^{p} a^{(k)}_{t,\beta_{\sigma(t)}} \right] \right\}$$

$$= \sum_{k=1}^{N} c_k \det A_k \tag{5.32}$$

where (5.32) follows by definition of the determinant of A_k and the fact that as β runs over all even permutations, $\beta\sigma$ must run over all odd permutations. This concludes the proof.

Theorem 5.6 can be extended to the case $1 < p \leq n$.

<u>Theorem 5.7</u> Let L be a permutable operator on $\otimes^p V$, where $1 < p \leq n = \dim V$. Then any representation of L in the form (5.29) has the property

$$\sum_{k=1}^{N} c_k \det A_k^{(p)} = 0 \tag{5.33}$$

where the matrices $A_1^{(p)}, \ldots, A_N^{(p)}$ are any $p \times p$ submatrices of $A_1, \ldots, A_N$, respectively, formed by elimination of the same $n - p$ rows and $n - p$ columns from each of the matrices $A_1, \ldots, A_N$.

<u>Proof</u> Let $r_1, \ldots, r_p$ and $s_1, \ldots, s_p$ be the row and column indices, respectively, retained in the construction

of a particular set of $A_1^{(p)}, \ldots, A_N^{(p)}$. Equation (5.30) is still valid. Specialize (5.30) by taking $\alpha_1 = r_1, \ldots, \alpha_p = r_p$ and $\gamma(1) = 1, \ldots, \gamma(p) = p$, so that (5.30) becomes

$$\sum_{k=1}^{N} c_k \left(\prod_{t=1}^{p} a_{r_t,\beta_t}^{(k)} - \prod_{t=1}^{p} a_{r_t,\beta_{\sigma(t)}}^{(k)} \right) = 0 \qquad (5.34)$$

Now let σ be any odd permutation in S_p. Whenever the p-tuple $(\beta_1, \ldots, \beta_p)$ is an even permutation of the first p integers, $(\beta_{\sigma(1)}, \ldots, \beta_{\sigma(p)})$ must be an odd permutation of the first p integers. Therefore, summing (5.34) over those coordinates $(\beta_1, \ldots, \beta_p)$ which are even permutations yields (5.33) and completes the proof.

B. The Representation Theorem for L_{2p} Norms

The algebraic properties of the preceding section yield a representation theorem for L_{2p} norms. The representation is given here in the context of Chapter III, but it is easily generalized to arbitrary finite dimensional spaces of measurable functions on which an L_{2p} norm can be defined.

Theorem 5.8 (Representation Theorem) Let $\omega(t)$ be a Lebesgue measurable function defined on the real interval (a,b), $-\infty \le a < b \le \infty$, such that

$$0 < \int_a^b \omega(t)dt < \infty$$

Let P_n be a real subspace of $L_{2p}^{\omega}[a,b]$, for some integer $p \ge 1$, and let $\{h_0, h_1, \ldots, h_n\}$ be a basis for P_n. Then there exists an integer $N \ge 1$, and nonzero real constants

$c_1, \ldots, c_N$, and $(n+1) \times (n+1)$ real matrices $A_1, \ldots, A_N$ satisfying

$$\det A_k = \pm 1, \qquad k = 1, \ldots, N \tag{5.35}$$

such that, for all $\pi_n(t) = a_0h_0(t) + a_1h_1(t) + \cdots + a_nh_n(t) \in P_n$,

$$\|\pi_n\|_{2p}^{\omega} \equiv \left\{c_1\left(x^TA_1x\right)^p + \cdots + c_N\left(x^TA_Nx\right)^p\right\}^{\frac{1}{2p}} \tag{5.36}$$

where $x = \langle a_0\ a_1 \cdots a_n\rangle^T \in \mathbb{R}^{n+1}$ and the norm in (5.36) is given by (3.2). Furthermore, if $1 < p \le n+1$, then

$$\sum_{k=1}^{N} c_k \det A_k^{(p)} = 0 \tag{5.37}$$

where $A_1^{(p)}, \ldots, A_N^{(p)}$ are any $p \times p$ submatrices of $A_1, \ldots, A_N$, respectively, formed by elimination of the same $n-p+1$ rows and $n-p+1$ columns from each of the matrices $A_1, \ldots, A_N$.

<u>Proof</u> First, we show that the operator L on $\otimes^p V$, $V = \mathbb{R}^{n+1}$, defined via the matrix

$$\left[(h_{\beta_1} \cdots h_{\beta_p}, h_{\alpha_1} \cdots h_{\alpha_p})_\omega\right]_{\alpha,\beta \in \Gamma}$$

is a permutable operator. Let $\sigma \in S_p$ and $\gamma \in S_p$. Since

$$(h_{\beta_1} \cdots h_{\beta_p}, h_{\alpha_1} \cdots h_{\alpha_p})_\omega = (h_{\beta_{\sigma(1)}} \cdots h_{\beta_{\sigma(p)}}, h_{\alpha_{\gamma(1)}} \cdots h_{\alpha_{\gamma(p)}})_\omega$$

Theorem 5.3 shows that L is permutable. From Theorem 5.1 and the representation (5.29) and equation (3.12),

$$\left(\|\pi_n\|_{2p}^{\omega}\right)^{2p} = (x \otimes \cdots \otimes x)^T \,(L\, x \otimes \cdots \otimes x)$$

$$= (x \otimes \cdots \otimes x)^T \left(\sum_{k=1}^{N} c_k\, A_k \otimes \cdots \otimes A_k\right)$$

$$\cdot\; x \otimes \cdots \otimes x$$

$$= \sum_{k=1}^{N} c_k (x \otimes \cdots \otimes x)^T$$

$$\cdot\; \{(A_k x) \otimes \cdots \otimes (A_k x)\}$$

$$= \sum_{k=1}^{N} c_k (x^T A_k x) \cdots (x^T A_k x)$$

$$= \sum_{k=1}^{N} c_k (x^T A_k x)^p$$

Using Theorem 5.7 completes the proof.

Remark Theorem 5.8 is stated for the real case, but it could just as easily have been stated for the complex case instead.

Theorem 5.8 raises an interesting question. Do there exist representations of the form (5.36) satisfying (5.35) but not (5.37)? The operator L defined in the proof of Theorem 5.8 gives rise to a representation (5.36) which necessarily must satisfy (5.37). So the question may be recast in the following manner. Does a representation (5.36) necessarily lead to a representation of the operator L? The answer is no. Let $x = \langle a\ b\rangle^T \in \mathbb{R}^2$. Then

$$\left\{\int_0^1 (a + bt)^4 dt\right\}^{\frac{1}{4}} = \{x \otimes x^T L\, x \otimes x\}^{\frac{1}{4}}$$

where

$$L = \begin{bmatrix} 1 & 1/2 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/4 & 1/5 \end{bmatrix}$$

(We note that L resembles the Hilbert matrix.) Clearly L is permutable, and a computation shows that

$$x \otimes x^T L\, x \otimes x = \sum_{k=1}^{5} c_k (x^T A_k x)^2$$

where

$$c_1, \ldots, c_5 = \frac{1}{4}, -\frac{1}{40}, -\frac{1}{40}, \frac{31}{40}, -\frac{87}{240}$$

$$A_1, \ldots, A_5 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since

$$\sum_{k=1}^{5} c_k \det A_k = -\frac{201}{240} \neq 0$$

we know that, by Theorem 5.6,

$$L \neq \sum_{k=1}^{5} c_k A_k \otimes A_k$$

It would be interesting to know what the correct (i.e., smallest) value of N is here. By direct example, it can be shown that $N \leq 7$, but whether or not 7 matrices are required is an open question.

Do there exist $L \in \mathcal{L}(\otimes^p V)$ such that

$$(x_1 \otimes \cdots \otimes x_p, L\, x_1 \otimes \cdots \otimes x_p) > 0 \tag{5.38}$$

for all $x_1 \otimes \cdots \otimes x_p \in \otimes^p V$ and yet L is not positive definite on $\otimes^p V$? The answer is yes for $p \geq 2$ and $n \geq 1$. In fact, L can be taken to be a permutable operator as well. Examples are the operators L defined in the proof of Theorem 5.8, which satisfy (5.38) and cannot be positive definite because the rank of L must be less than $(n+1)^p$ for $p \geq 2$ and $n \geq 1$, by Corollary 5.1. (Incidentally, L is positive semidefinite by Theorem 3.3.)

C. Open Approximation Questions in $\otimes^p V$

We end this chapter with two conjectures and some open questions. Let L be a hermitian permutable operator on $\otimes^p V$. Then the Rayleigh quotient

$$\max_{x_1 \otimes \cdots \otimes x_p} \frac{(x_1 \otimes \cdots \otimes x_p, L\, x_1 \otimes \cdots \otimes x_p)}{(x_1 \otimes \cdots \otimes x_p, x_1 \otimes \cdots \otimes x_p)} \tag{5.39}$$

is certainly bounded above by the spectral radius of L. We conjecture that (5.39) can be computed by the Quadratic Relaxation Algorithm of Chapter VI, modified slightly in equations (6.5) through (6.9) to accommodate the more general form (5.39). We also conjecture that the Rayleigh quotient

$$\max_{x_1 \otimes \cdots \otimes x_p} \frac{(x_1 \otimes \cdots \otimes x_p, L\, x_1 \otimes \cdots \otimes x_p)}{(x_1 \otimes \cdots \otimes x_p, M\, x_1 \otimes \cdots \otimes x_p)} \tag{5.40}$$

can also be computed by a Quadratic Relaxation Algorithm patterned after the one in Chapter VI, where both L and M in (5.40) are permutable operators on $\otimes^p V$ with M satisfying the condition (5.38).

Finally, we ask the following approximation questions. Let $v_0 \in \otimes^p V$ be any element of $\otimes^p V$. What can be said about

$$\varepsilon_0 = \min_{x_1 \otimes \cdots \otimes x_p \in \otimes^p V} \|v_0 - x_1 \otimes \cdots \otimes x_p\| \tag{5.41}$$

where the norm in (5.41) is defined via the inner product (5.4)? Clearly, if v_0 is decomposable, then $\varepsilon_0 = 0$. In a somewhat different vein, we can ask, "How dense is the set of decomposable elements in the unit sphere of $\otimes^p V$?" Specifically, what can be said about

$$\varepsilon = \max_{\substack{v_0 \in \otimes^p V \\ \|v_0\| \le 1}} \ \min_{\substack{x_1 \otimes \cdots \otimes x_p \in \otimes^p V \\ \|x_1 \otimes \cdots \otimes x_p\| \le 1}} \|v_0 - x_1 \otimes \cdots \otimes x_p\| \tag{5.42}$$

and is ε different from

$$\varepsilon' = \max_{\substack{v_0 \in \otimes^p V \\ \|v_0\| \le 1}} \{\varepsilon_0\} \quad ? \tag{5.43}$$

These questions seem to be difficult.

Chapter VI

QUADRATIC RELAXATION ALGORITHM

A. The Algorithm

We propose, without proof of convergence, an algorithm for the computation of $\pi_n^* \in P_n$ such that

$$R_{n,2p} = \max_{0 \neq \pi_n \in P_n} \frac{\|D\pi_n\|_{2p}^{\phi}}{\|\pi_n\|_2^{\omega}} \tag{6.1}$$

$$= \frac{\|D\pi_n^*\|_{2p}^{\phi}}{\|\pi_n^*\|_2^{\omega}}, \qquad p = 1, 2, 3, \ldots \tag{6.2}$$

where D is a linear transformation as specified below. The algorithm is presented in the context of Chapter III, but without assuming the Nonnegativity Condition. It can be adapted in an obvious manner to a more general setting in abstract measure spaces. Alternatively, it can be adapted easily to more general algebraic settings as mentioned at the end of Chapter V.

Let P_n be a subspace of $L_2^{\omega}[a,b] \cap L_{2p}^{\phi}[c,d]$, for some integer $p \geq 1$, where $\omega(x) > 0$ and $\phi(x) > 0$ a.e. on the intervals (a,b) and (c,d), respectively, and satisfy the conditions (3.1). Let $D: P_n \to L_{2p}^{\phi}[c,d]$ be an arbitrary linear transformation on P_n. The norms in (6.1) above are defined by (3.2). We will be keeping n fixed throughout this discussion, so we modify our notation to allow $\pi, \pi_1, \ldots, \pi_p$ to be arbitrary functions in P_n.

Lemma 6.1 For n = 0, 1, 2, ... and p = 1, 2, 3, ...,

$$(R_{n,2p})^{2p} = \max_{\substack{0\neq\pi_j\in P_n\\ j=1,2,\dots,p}} \left\{ \frac{\int_c^d \prod_{j=1}^{p} |D\pi_j(x)|^2 \phi(x)\,dx}{\prod_{j=1}^{p} \int_a^b |\pi_j(x)|^2 \omega(x)\,dx} \right\} \tag{6.3}$$

Proof We have

$$\max_{0\neq\pi\in P_n} \frac{\int_c^d |D\pi(x)|^{2p} \phi(x)\,dx}{\left[\int_a^b |\pi(x)|^2 \omega(x)\,dx\right]^p}$$

$$\leq \max_{\substack{0\neq\pi_j\in P_n\\ j=1,\dots,p}} \frac{\int_c^d \prod_{j=1}^{p} |D\pi_j(x)|^2 \phi(x)\,dx}{\prod_{j=1}^{p} \int_a^b |\pi_j(x)|^2 \omega(x)\,dx} \tag{6.4.1}$$

$$\leq \max_{\substack{0\neq\pi_j\in P_n\\ j=1,\dots,p}} \prod_{j=1}^{p} \left[\frac{\left(\int_c^d |D\pi_j(x)|^{2p} \phi(x)\,dx\right)^{\frac{1}{p}}}{\int_a^b |\pi_j(x)|^2 \omega(x)\,dx} \right] \tag{6.4.2}$$

$$\leq \prod_{j=1}^{p} \max_{0\neq\pi_j\in P_n} \left[\frac{\left(\int_c^d |D\pi_j(x)|^{2p} \phi(x)\,dx\right)^{\frac{1}{p}}}{\int_a^b |\pi_j(x)|^2 \omega(x)\,dx} \right] \tag{6.4.3}$$

$$= \max_{0\neq\pi\in P_n} \frac{\int_c^d |D\pi(x)|^{2p} \phi(x)\,dx}{\left[\int_a^b |\pi(x)|^2 \omega(x)\,dx\right]^p}$$

where Lemma 3.1 was used in (6.4.2). Hence the inequalities (6.4.1), (6.4.2), and (6.4.3) are in fact equalities and this concludes the proof.

The Quadratic Relaxation Algorithm is based on Lemma 6.1.

Algorithm (Quadratic Relaxation)

(1) Let $\pi_1^{(0)}, \pi_2^{(0)}, \ldots, \pi_p^{(0)}$ be any given nonzero functions in P_n, and define

$$T^{(0)} = \frac{\int_c^d \prod_{j=1}^{p} |D\pi_j^{(0)}(x)|^2 \phi(x)\,dx}{\prod_{j=1}^{p} \int_a^b |\pi_j^{(0)}(x)|^2 \omega(x)\,dx} \tag{6.5}$$

Set $k = 0$ and $r = 1$.

(2) Given $\pi_1^{(k)}, \pi_2^{(k)}, \ldots, \pi_p^{(k)}$ in P_n, and $1 \le r \le p$, define

$$T^{(k+1)} = \max_{0 \neq \pi \in P_n} \left\{ \frac{\int_c^d |D\pi(x)|^2\, W^{(k)}(x)\,dx}{M^{(k)} \int_a^b |\pi(x)|^2 \omega(x)\,dx} \right\} \tag{6.6}$$

where

$$W^{(k)}(x) = \prod_{\substack{j=1 \\ j \neq r}}^{p} |D\pi_j^{(k)}(x)|^2 \phi(x) \tag{6.7}$$

$$M^{(k)} = \prod_{\substack{j=1 \\ j \neq r}}^{p} \int_a^b |\pi_j^{(k)}(x)|^2 \omega(x)\,dx \tag{6.8}$$

Let $\tilde{\pi}$ be any nonzero polynomial for which the ratio in (6.6) attains its maximum. Define

$$\pi_j^{(k+1)} = \begin{cases} \tilde{\pi}, & \text{if } j = r \\ \pi_j^{(k)}, & \text{if } j \neq r \end{cases} \tag{6.9}$$

(3) Increase k by 1. Replace r by

$$r - \left[\frac{r}{p}\right]p + 1$$

where [ ] denotes the greatest integer function.

(4) Go to step (2).

The sequence $T^{(k)}$, $k = 0, 1, 2, \ldots$, generated by the algorithm certainly has a limit since

$$T^{(0)} \le T^{(1)} \le T^{(2)} \le \cdots \le R_{n,2p}^{2p} \tag{6.10}$$

which follows directly from (6.6) and (6.3). Also, for each $j = 1, 2, \ldots, p$, the normalized sequence

$$\left\{\frac{\pi_j^{(k)}}{\|\pi_j^{(k)}\|_2^{\omega}}, \quad k = 0,1,2,\ldots\right\} \tag{6.11}$$

must have at least one limit point. Let S_j be the set of limit points of (6.11), and define

$$\mathcal{S}_j = \{\varepsilon \ell \mid \ell \in S_j \text{ and } |\varepsilon| = 1\}$$

Then there exists

$$\hat{\pi} \in \mathcal{S}_1 \cap \mathcal{S}_2 \cap \cdots \cap \mathcal{S}_p \subset \mathcal{P}_n \tag{6.12}$$

The proof of (6.12) is an immediate consequence of Lemma 3.1 and the definition of $\hat{\pi}$ in (6.9). In essence, (6.12) states that each of the p sequences of polynomials defined by (6.11) has a subsequence which converges to $\hat{\pi}$. Unfortunately, this is not enough to assert that $\hat{\pi}$ is an extremal polynomial for $R_{n,2p}$.

<u>Conjecture</u> If $\hat{\pi} \in S_1 \cap \cdots \cap S_p$, then

$$R_{n,2p} = \frac{\|D\hat{\pi}\|_{2p}^{\phi}}{\|\hat{\pi}\|_2^{\omega}} = \lim_{k\to\infty} T^{(k)} \tag{6.13}$$

If the extremal polynomial π^* for $R_{n,2p}$ is unique up to constant multiples, then we further conjecture that

$$\lim_{k\to\infty} \frac{\pi_j^{(k)}}{\|\pi_j^{(k)}\|_2^{\omega}} = \hat{\pi} = \pi^*,\ j = 1,\ 2,\ \ldots,\ p \tag{6.14}$$

B. <u>Computational Considerations</u>

Before proceeding to an example, some remarks on the solution of (6.6) are in order. Let

$$\pi(x) = \sum_{r=0}^{n} a_r h_r(x)$$

and let $a = \langle a_0\ a_1\ \cdots\ a_n\rangle^T \in \mathbb{C}^{n+1}$. Then we have

$$T^{(k+1)} = \frac{1}{M^{(k)}} \max_{a\in\mathbb{C}^{n+1}} \left\{\frac{\bar{a}^T A a}{\bar{a}^T B a}\right\} \tag{6.15}$$

where A and B are hermitian matrices of dimension (n+1) × (n+1), and B is positive definite. Explicitly, letting $A = [a_{ij}]$ and $B = [b_{ij}]$, we have

$$a_{ij} = \int_c^d Dh_j(x)\ \overline{Dh_i(x)}\ W^{(k)}(x)dx \tag{6.16}$$

$$b_{ij} = \int_a^b h_j(x)\ \overline{h_i(x)}\ \omega(x)dx \tag{6.17}$$

Since $W^{(k)}(x)$ is a known function, by definition of the algorithm, the matrices A and B can be found explicitly.

It is well known (see, e.g., [13]) that the ratio of hermitian forms (6.15) is maximized by the largest eigenvalue of the eigenproblem

$$Aa = \lambda Ba \tag{6.18}$$

and that all the eigenvalues of (6.18) are nonnegative. Let λ_{max} be the largest eigenvalue of (6.18). Then it is also well known [13] that

$$\frac{\bar{a}^T Aa}{\bar{a}^T Ba} = \lambda_{max} \tag{6.19}$$

if and only if $a \neq 0$ lies in the eigenspace of λ_{max}. Thus we can find the coefficients of $\tilde{\pi}$ in (6.9) by computing any vector ($\neq 0$) in the eigenspace of the largest eigenvalue of (6.18).

The eigenproblem (6.18) is equivalent to the eigenproblem

$$B^{-1}Aa = \lambda a \tag{6.20}$$

Numerically, however, solving (6.20) leads to annoying difficulties. Although A and B are both hermitian, the product $B^{-1}A$ is not, in general, hermitian. Therefore, to solve (6.20) on a computer, one has to use a computer program for solving the eigenproblem of a general complex matrix. Numerical roundoff in the computation of the product $B^{-1}A$ then yields computed eigenvalues which are not strictly real. To avoid this difficulty, it is better to solve the eigenproblem by another method. Martin and Wilkinson [25] give an efficient method for solving (6.18) when A and B are real symmetric matrices. It is easy to see how to modify their method to adapt it to the case A and B hermitian.

Incidentally, it can be shown [13] that if A and B are real, then the eigenvectors can be taken to be real as well. Therefore, if the Conjecture is true, then there exist extremal polynomials of $R_{n,2p}^{(k)}$ defined by (1.6) having real coefficients.

C. Example

We apply the Quadratic Relaxation Algorithm to the maximum problem

$$R = \max_{a_0,a_1,a_2 \in \mathbb{R}} \frac{\left\{\int_1^{3/2} |a_0 + a_1x + a_2x^2|^6 dx\right\}^{\frac{1}{6}}}{\left\{\int_{-1}^{1} |a_0 + a_1x + a_2x^2|^2 dx\right\}^{1/2}} \tag{6.21}$$

Define, for $a = (a_0, a_1, a_2) \in \mathbb{R}^3$,

$$F(x;a) = |a_0 + a_1x + a_2x^2|^2 \tag{6.22}$$

Recalling Lemma 6.1, we have

$$R = \max_{a,b,c \in \mathbb{R}^3} \left\{\frac{\int_1^{3/2} F(x;a)F(x;b)F(x;c)dx}{\int_{-1}^{1} F(x;a)dx \int_{-1}^{1} F(x;b)dx \int_{-1}^{1} F(x;c)dx}\right\}^{\frac{1}{6}} \tag{6.23}$$

The Quadratic Relaxation Algorithm is easy to apply. At each step we compute the matrices in (6.15) via (6.16) and (6.17). Although these integrals can be computed explicitly, we prefer to approximate them numerically by the trapezoidal rule. Specifically, the integral (6.16) is replaced by

$$a_{ij} = \frac{\frac{1}{2}}{N-1} \sum_{r=1}^{N}{}' \, Dh_j(x_r)\overline{Dh_i(x_r)}W^{(k)}(x_r) \tag{6.24}$$

where N = 500 and $+1 = x_1, \ldots, x_N = 3/2$ are equispaced in [1, 3/2]. Similarly, the integral (6.17) is replaced by

$$b_{ij} = \frac{2}{N-1} \sum_{r=1}^{N}{}' \, h_j(y_r)\overline{h_i(y_r)}\,\omega(x_r) \tag{6.25}$$

where N = 500 and $-1 = y_1, \ldots, y_N = +1$ are equispaced in [-1,+1]. The prime on the summation signs in (6.24) and (6.25) means that the first and last terms are taken with weight 1/2. In this example, of course, the operator D in (6.24) is the identity operator and the basis functions $\{h_0, h_1, h_2\}$ are $\{1, x, x^2\}$.

Starting with the vectors

$a^{(0)} = (-3, -2, -1)$

$b^{(0)} = (-2, -1, 0)$

$c^{(0)} = (-1, 0, 1)$

we get, for the first three steps in the Quadratic Relaxation Algorithm,

$a^{(1)} = (-.26238420, +.21498490, +.94071037)$

$b^{(1)} = b^{(0)}$

$c^{(1)} = c^{(0)}$

$T^{(1)} = 5.2856113$

$a^{(2)} = a^{(1)}$

$b^{(2)} = (-.26378420, +.21190785, +.94106987)$

$c^{(2)} = c^{(1)}$

$$T^{(2)} = 75.203450$$

$$a^{(3)} = a^{(2)}$$
$$b^{(3)} = b^{(2)}$$
$$c^{(3)} = (-.26120187, +.21748675, +.94046430)$$
$$T^{(3)} = 1601.0732$$

We used the convergence criterion

$$0 \le T^{(k+1)} - T^{(k)} < 10^{-6}$$

and found that the algorithm converged in the eighth step to

$$a^{(8)} = (-.26118456, +.21752387, +.94046053)$$
$$\left.\begin{array}{l} b^{(8)} = a^{(8)} \\ c^{(8)} = a^{(8)} \end{array}\right\} \text{to within 8 significant digits}$$
$$T^{(8)} = 1601.3516$$

If the algorithm has converged to an extremal polynomial, then we have

$$R = 3.4204332 = (1601.3516)^{\frac{1}{6}}$$

and is attained by the extremal polynomial

$$\pi^*(x) = .94046053x^2 + .21752387x - .26118456$$

The algorithm converged to the polynomial π^* for every set of initial vectors $a^{(0)}$, $b^{(0)}$, and $c^{(0)}$ that was tried. All computations were performed on a Univac 1108 in single precision which gives 8 or 9 significant decimal digits, although the summations (6.24) and (6.25) to compute the matrices of the eigenproblems (6.15) were accumulated in double precision which gives 18 or 19 significant decimal digits.

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