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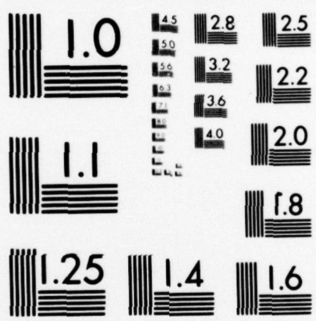

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LEVEL II

ON JACKKNIFING IN ESTIMATING THE FINITE END-POINTS OF A DISTRIBUTION*

by

Pranab Kumar Sen
University of North Carolina, Chapel Hill

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ABSTRACT

Sample extreme values are biased estimators of the end-points of a distribution, and hence, jackknifing is useful. However, the properties of jackknifing in such a case differ considerably from those in the regular case. These are studied here. Along with a modification of jackknifing, some applications are also considered.

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Key Words & Phrases: bias; extreme values; jackknifing; mean square; order of terminal contact; studentized form; Tukey-estimator of variance.

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1. INTRODUCTION

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d.r.v.) with a distribution function (df) F, defined on $(-\infty,\infty)$. It is assumed that F has a finite (unknown) lower end-point θ, that is

$$(1.1) \qquad -\infty < \theta = \sup\{x: F(x) = 0\} < \infty$$

and that $F(x)$ is continuous and monotonic in $x \in (\theta, \theta + \delta)$, for some $\delta > 0$. A natural estimator of θ is the sample minimum i.e.,

$$(1.2) \qquad \hat{\theta}_n = \min\{X_1, \ldots, X_n\} = X_{n,1} \qquad (n \geq 1) ,$$

where $X_{n,1} \leq \ldots \leq X_{n,n}$ stand for the ordered variables corresponding to $X_1, \ldots, X_n$; $n \geq 1$. $\hat{\theta}_n$ is a (strongly) consistent estimator of θ, but it is not an unbiased one; the nature of its bias depends on the order of terminal contact of F (at θ). It may therefore be appealing to use the jackknife estimator corresponding to $\hat{\theta}_n$.

Under quite general regularity conditions (viz., [1,2,4]), jackknifing meets three objectives: (a) *Bias reduction*. If θ_n^* be the jackknife estimator then $nE(\theta_n^* - \theta) \to 0$ as $n \to \infty$. (b) *Asymptotic normality*. If $n^{\frac{1}{2}}(\hat{\theta}_n - \theta)$ is asymptotically normal, then the same limit law holds for $n^{\frac{1}{2}}(\theta_n^* - \theta)$. (c) The Tukey estimator v_n^2 [defined by (2.5)] is a (strongly) consistent estimator of the variance of $n^{\frac{1}{2}}(\theta_n^* - \theta)$.

Since the asymptotic distributions of sample extrema are non-normal and, depending on the order of terminal contact, the bias of $\hat{\theta}_n$ is $O(n^{-a})$

for some $0 < a \le 1$, the effectiveness of jackknifing in regard to (a) and (b) remains to be examined carefully. Further, in this case, v_n^2 does not converge (stochastically). Along with the preliminary notions, expressions for θ_n^* and v_n^2 are considered in Section 2. The main results are studied in Section 3. Section 4 deals with a modification of jackknifing appropriate for the case of the bias of $O(n^{-a})$ for some $a < 1$. Some general remarks are made in the concluding section.

2. PRELIMINARY NOTIONS

We assume that for some non-negative integer m, $F(x)$ has continuous jth derivative $F^{(j)}(x)[= f^{(j-1)}(x)]$ for all $x \in (\theta, \theta+\delta)$, $\delta > 0$, $1 \le j \le m+1$. We denote the (right hand) derivatives at θ by $F_+^{(j)}(\theta) = f_+^{(j-1)}(\theta)$, $1 \le j \le m+1$ and $F_+^{(0)}(\theta) = 0$, $f_+^{(0)}(\theta) = f_+(\theta)$. Then, a *terminal contact of order* m is defined by

$$(2.1) \qquad F_+^{(j)}(\theta) = 0\ ,\quad 0 \le j \le m \quad \text{and} \quad 0 < f_+^{(m)}(\theta) < \infty\ .$$

Also, for the study of the bias, we assume that

$$(2.2) \qquad \nu_\alpha = \int_\theta^\infty |x|^\alpha dF(x) < \infty \quad \text{for some } \alpha > 0\ .$$

To define θ_n^*, we let for each i: $1 \le i \le n$,

$$(2.3) \quad \hat{\theta}_{n-1}^i = \min\{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\}\ ,\quad \hat{\theta}_{n,i} = n\hat{\theta}_n - (n-1)\hat{\theta}_{n-1}^i\ .$$

Then, $\hat{\theta}_{n-1}^i$ is equal to $X_{n,1}$ or $X_{n,2}$ according as X_i is $\ne$ or $= X_{n,1}$, $1 \le i \le n$. Also, (2.4) $$\theta^*_n = n^{-1}\sum_{i=1}^{n}\hat{\theta}_{n,i}$$

$$= X_{n,1} - n^{-1}(n-1)(X_{n,2} - X_{n,1}) \ , \quad n \geq 2 \ .$$

The Tukey estimator v_n^2, defined by

(2.5) $$v_n^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(\hat{\theta}_{n,i} - \theta^*_n)^2 = (n-1)\sum_{i=1}^{n}(\hat{\theta}^i_{n-1} - \theta^*_n)^2 \ ,$$

reduces in our case to

(2.6) $$v_n^2 = (X_{n,2} - X_{n,1})^2(n-1)(n^2+n-1)/n \quad (\sim \{n(X_{n,2} - X_{n,1})\}^2) \ .$$

For a terminal contact of order $m(\geq 0)$, we define

(2.7) $$b_{n,m} = \{nf_+^{(m)}(\theta)/(m+1)!\}^{1/(m+1)} \ , \quad a_m = 1/(m+1) \ .$$

Then, the limiting distribution of $b_{n,m}(\hat{\theta}_n - \theta)$ is known to be

(2.8) $$\Lambda_m(x) = \begin{cases} 0 \ , & x \leq 0 \ , \\ 1 - \exp(-x^{m+1}) \ , & x > 0 \ . \end{cases}$$

Also, by Theorem 3.1 of Sen (1961), as $n \to \infty$,

(2.9) $$b_{n,m}E(X_{n,r} - \theta) = \Gamma(r+a_m)/\Gamma(r) + o(1) \ , \quad \text{for every (fixed) } r(\geq 1) \ .$$

3. BASIC PROPERTIES OF JACKKNIFING

It follows from (2.4) that

(3.1) $$n(\theta^*_n - \theta) = n(X_{n,1} - \theta) - (n-1)(X_{n,2} - X_{n,1})$$

$$= (2n-1)(X_{n,1} - \theta) - (n-1)(X_{n,2} - \theta) \ .$$

Hence, from (2.9) and (3.1), we obtain that for a terminal contact of order m,

$$(3.2) \qquad b_{n,m}E(\theta_n^* - \theta) = (1-a_m)\Gamma(1+a_m) + o(1)$$
$$= (1-a_m)\{b_{n,m}E(\hat{\theta}_n - \theta)\} + o(1) \ .$$

For $m = 0$ i.e., $a_m = 1$, the right hand side (rhs) of (3.2) converges to 0, as $n \to \infty$, while for $m \geq 1$ (i.e., $a_m \leq \frac{1}{2}$), jackknifing leads to effectively $100(1-a_m)\%$ reduction in bias. Thus, the basic role of jackknifing is partially impaired for a terminal contact of order $m(\geq 1)$.

Theorem 1. *For a terminal contact of order* $m(\geq 0)$,

$$\Lambda_n^*(x) = \lim_{n\to\infty} P\{b_{n,m}(\theta_n^* - \theta) \leq x\}$$

$$(3.3) \qquad = \begin{cases} \int_0^\infty \exp\left\{-(2y^{a_m} - x)^{m+1}\right\}dy \ , & -\infty < x \leq 0 \ , \\ 1 - \exp(-x^{m+1}) + \int_{x^{m+1}}^\infty \exp\left\{-(2y^{a_m} - x)^{m+1}\right\}dy \ , & x > 0 \ , \end{cases}$$

where a_m *and* $b_{n,m}$ *are defined by* (2.7).

Proof. Let $Z_n = b_{n,m}(\theta_n^* - \theta)$ and let

$$(3.4) \qquad Y_{n(1)} = nF(X_{n,1}) \quad \text{and} \quad Y_{n(2)} = n[F(X_{n,2}) - F(X_{n,1})] \ .$$

Then, by (2.1), (2.2), (2.7), (3.1) and (3.4) and proceeding as in the proof of Theorem 3.1 of Sen (1961), we obtain that

$$(3.5) \qquad E[Z_n - 2Y_{n(1)}^{a_m} + (Y_{n(1)} + Y_{n(2)})^{a_m}]^2 \to 0 \quad \text{as} \quad n \to \infty \ .$$

and hence, by the Chebychev inequality, we have

(3.6) $\Lambda_m^*(x) = \lim_{n\to\infty} P\left\{2Y_{n(1)}^{a_m} - (Y_{n(1)} + Y_{n(2)})^{a_m} \le x\right\}, \quad \forall -\infty < x < \infty.$

We may recall that $Y_{n(1)}$ and $Y_{n(2)}$ ar asymptotically independently distributed according to a common simple exponential law and they are non-negative rv's. For $x \le 0$, $\left[2Y_{n(1)}^{a_m} - (Y_{n(1)} + Y_{n(2)})^{a_m} \le x\right] \iff \left[Y_{n(2)} \ge (2Y_{n(1)}^{a_m} - x)^{m+1} - Y_{n(1)}\right]$ and the first equation in (3.3) follows directly by finding the conditional probability given $Y_{n(1)}$ and then itegrating it out over $Y_{n(1)}$. For $x > 0$, if $Y_{n(1)} \le x^{m+1}$, then $2Y_{n(1)}^{a_m} - (Y_{n(1)} + Y_{n(2)})^{a_m} \le Y_{n(1)}^{a_m} \le x$, while for $Y_{n(1)} > x^{m+1}$, as before we need $Y_{n(2)} \ge (2Y_{n(1)}^{a_m} - x)^{m+1} - Y_{n(1)}$, and hence, the last equation in (3.3) follows on parallel lines. Q.E.D.

For $m = 0$ (i.e., $a_m = 1$), Λ_0 in (2.8) is the simple exponential while Λ_0^* in (3.3) is the double exponential df. For $m \ge 0$, Λ_m and Λ_m^* are not the same df.

<u>Theorem 2</u>. *For a terminal contact of order* $m (\ge 0)$,

(3.7) $$\lim_{n\to\infty}\left\{E\left[b_{n,m}^2(\theta_n^* - \theta)^2\right]\right\} = \left\{1 - \frac{2a_m(1-a_m)}{1+a_m}\right\}\left[\lim_{n\to\infty}\left\{E\left[b_{n,m}^2(\hat{\theta}_n - \theta)^2\right]\right\}\right]$$

$$= \left(2a_m\Gamma 2a_m\{1 - 2a_m(1-a_m)/(1+a_m)\}\right).$$

<u>Proof</u>. Since $\hat{\theta}_n = X_{n,1}$, by an appeal to Theorem 3.1 of Sen (1961), we get that

(3.8) $$b_{n,m}^2 E(\hat{\theta}_n - \theta)^2 \to \Gamma 1+2a_m = 2a_m \Gamma 2a_m > 0.$$

Hence, to prove (3.7), by (3.5), it suffices to show that as $n \to \infty$, (3.9) $$E\left\{2Y_{n(1)}^{a_m} - (Y_{n(1)} + Y_{n(2)})^{a_m}\right\}^2 \to 2a_m\Gamma(2a_m)\{1 - 2a_m(1-a_m)/(1+a_m)\} .$$

Towards this, we may note that $E\left[Y_{n(1)}^{2a_m}\right] \to \Gamma(1+2a_m) = 2a_m\Gamma(2a_m)$, $E(Y_{n(1)} + Y_{n(2)})^{2a_n}$ $\to \Gamma(2+2a_m) = 2a_m(1+2a_m)\Gamma(2a_m)$ while $E\left\{Y_{n(1)}^{a_m}(Y_{n(1)} + Y_{n(2)})^{a_m}\right\} = E\left\{\left[E(Y_{n(1)}^{a_m} | Y_{n(1)} + Y_{n(2)})\right](Y_{n(1)} + Y_{n(2)})^{a_m}\right\} \sim E\left\{(Y_{n(1)} + Y_{n(2)})^{2a_m+1}/(a_m+1)\right\} \to \left\{\Gamma(2+2a_m)\right\}/(1+a_m) =$ $\left\{2a_m(1+2a_m)\Gamma(2a_m)\right\}/(a_m+1)$, and hence, (3.9) follows by some standard steps. Q.E.D.

For $m = 0$ (i.e., $a_m = 1$), the second factor on the rhs of (3.7) is equal to 1, so that both $\hat{\theta}_n$ and θ_n^* have the same asymptotic variance, though their df's are not the same. For $m \geq 1$ (i.e., $a_m \leq \frac{1}{2}$), $2a_m(1-a_m)/(1+a_m) > 0$ and is bounded from above by 1/3. Thus, from (3.2) and (3.7) we have that jackknifing reduces both the asymptotic bias and the asymptotic mean square to a fractional extent. This characteristic is different from the regular case where there is a complete reduction of asymptotic bias but no reduction of the asymptotic mean square.

From (2.6), (2.7) and (3.4), it follows that for a terminal contact of order $m(\geq 0)$,

(3.10) $$\left|n^{-1}b_{n,m}v_n - \left\{(Y_{n(1)} + Y_{n(2)})^{a_m} - Y_{n(1)}^{a_m}\right\}\right| \xrightarrow{p} 0 , \quad \text{as } n \to \infty .$$

Since $(Y_{n(1)} + Y_{n(2)})^{a_m} - Y_{n(1)}^{a_m} \to \left\{(Y_1 + Y_2)^{a_m} - Y_1^{a_m}\right\}$, where Y_1 and Y_2 are i.i.d.r.v. having the simple exponential df on $[0,\infty)$, $n^{-1}b_{n,m}$ either converges to a positive constant (when $m = 0$) or goes to 0 (when $m \geq 1$), it follows that either (for $m = 0$) v_n has a non-degenerate asymptotic df or (for $m \geq 1$) it goes to $+\infty$, in probability as $n \to \infty$. This characteristic is also different from the regular case where $v_n \overset{p}{\to} a$ constant, as $n \to \infty$. Nevertheless, for the studentized form, we have for a terminal contact of order $m(\geq 0)$,

$$T_n = n(\theta_n^* - \theta)/v_n = b_{n,m}(X_{n,1} - \theta)/b_{n,m}(X_{n,2} - X_{n,1}) - (n-1)/n$$

(3.11)
$$+ o_p(1) \overset{\mathcal{D}}{\to} Y_1^{a_m}/\left\{(Y_1 + Y_2)^{a_m} - Y_1^{a_m}\right\} - 1 ,$$

so that noting that $Y^* = Y_2/Y_1$ has the Fisher's variance-ratio distribution with degrees of freedom (2,2), we have from (3.11) that

(3.12)
$$[1 + (1 + T_n)^{-1}]^{m+1} - 1 \overset{\mathcal{D}}{\to} Y^* = Y_2/Y_1 .$$

For $m = 0$, we have a simplified form

(3.13)
$$T_n + 1 \overset{\mathcal{D}}{\to} Y_1/Y_2 \overset{\mathcal{D}}{=} Y^* .$$

Both (3.12) and (3.13) have important statistical applications.

4. A MODIFICATION OF θ_n^*

We have observed in (3.2) that for $m \geq 1$, $b_{n,m}E(\theta_n^* - \theta)$ does not converge to 0 as $n \to \infty$. Let C_n be the sigma-field generated by $X_{n,1}, \ldots, X_{n,n}$ and by X_{n+j}, $j \geq 1$ (so that C_n is non-increasing in n). Then, in the regular case, [cf. (2.11) of Sen (1977)], we have

(4.1)
$$\theta_n^* - \hat{\theta}_n = (n-1)E\{(\hat{\theta}_n - \hat{\theta}_{n-1})|C_n\} .$$

In our case, for $m \geq 1$, $nb_{n,m}E(\hat{\theta}_n - \hat{\theta}_{n-1}) = -a_m\Gamma(1+a_m) + o(1)$, where as $b_{n,m}E(\hat{\theta}_n - \theta) = \Gamma(1+a_m) + o(1)$, and thereby, we get the resulting bias in (3.2). To eliminate the, we may consider the modified estimator

$$\theta^{**}_{n,m} = \hat{\theta}_n + \frac{1}{a_m} E\{(\hat{\theta}_n - \hat{\theta}_{n-1}) | C_n\} \tag{4.2}$$

$$= X_{n,1} - (m+1)n^{-1}(n-1)(X_{n,2} - X_{n,1}) \,.$$

In that case, we have

$$b_{n,m}E(\theta^{**}_{n,m} - \theta) \to 0 \quad \text{as} \quad n \to \infty \,. \tag{4.3}$$

Also, following the same line as in the proof of Theorem 1, we obtain that

$$\Lambda^{**}_m(x) = \lim_{n\to\infty} P\{b_{n,m}(\theta^{**}_{n,m} - \theta) \leq x\}$$

$$= \begin{cases} \displaystyle\int_0^\infty \exp\left\{-\left[\{(m+2)y^{a_m} - x\}/(m+1)\right]^{m+1}\right\} dy, & -\infty < x \leq 0 \,, \\ \displaystyle 1 - \exp\{-x^{m+1}\} + \int_{x^{m+1}}^\infty \exp\left\{-\left[\{(m+2)y^{a_m} - x\}/(m+1)\right]^{m+1}\right\} dy, & 0 < x < \infty \,. \end{cases} \tag{4.4}$$

Also, following the line of proof of Theorem 2, we have

$$\lim_{n\to\infty} E\left\{b^2_{n,m}(\theta^{**}_{n,m} - \theta)^2\right\} = (2a_m\Gamma(2a_m))\left\{1 - \frac{2a_m}{1+a_m}(m+1)[(m+1)a_m - 1]\right\} = 2a_m\Gamma(2a_m)$$

$$= \lim_{n\to\infty} E\left\{b^2_{n,m}(\hat{\theta}_n - \theta)^2\right\} \geq \lim_{n\to\infty} E\left\{b^2_{n,m}(\theta^*_n - \theta)^2\right\} \,. \tag{4.5}$$

Thus, whereas $\theta^{**}_{n,m}$ eliminates bias to the desired extent, it fails to reduce the mean square. In this sense, it is similar to the case of θ^*_n in the regular case. [Though Λ^{**}_m and Λ_m are not the same.]

Finally, for the studentized case, in (3.11)-(3.13), the only changes we need to made is to replace T_n by $T_n + m$; the rest remains the same.

5. SOME REMARKS

We have so far considered the case of the lower end-point. The case of the upper end-point (if finite) follows on parallel lines. Secondly, in practical applications, when the form of F is not specified but the order of terminal contact is assumed to be known [viz., $m = 0$ when F is U-shaped or inverted J-shaped, etc.], the studentized form in (3.11)-(3.13) may most conveniently be used to provide a jackknife test for a null hypothesis $H_0: \theta = \theta_0$ (specified) or a confidence interval for the unknown θ. For a symmetric df with both end-points finite, jackknifing of the extreme mid-range (for estimating or testing for the location of the df) can be made — the jackknife estimator corresponding to the smallest and the largest order statistic are also asymptotically independent.

REFERENCES

[1] Arvesen, J.M. (1969). Jackknifing U-statistics. *Ann. Math. Statist.* 40, 2076-2100.

[2] Schucany, W.R., Gray, H.L. and Owen, D.M. (1971). Bias reduction in estimation. *Jour. Amer. Statist. Assoc.* 66, 524-533.

[3] Sen, P.K. (1961). A note on the large sample behaviour of extreme sample values from distributions with finite end-points. *Calcutta Statist. Assoc. Bull.* 10, 106-115.

[4] Sen, P.K. (1977). Some invariance principles relating to jackknifing and their role in sequential analysis. *Ann. Statist.* 5, 316-329.

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