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ADAPTIVE DETECTION OF RENEWAL PROCESSES.(U)

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ADAPTIVE DETECTION OF RENEWAL PROCESSES

A. Fogel and S.C. Schwartz

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Department of Electrical Engineering and Computer Science
Princeton University NJ
Princeton, New Jersey 08540

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ADAPTIVE DETECTION OF RENEWAL PROCESSES

A. Fogel and S.C. Schwartz
Department of Electrical Engineering and Computer Science
Princeton University
Princeton, NJ 08540

Abstract

In this paper, we consider the adaptive detection of renewal processes whose inter-arrival times are Gamma distributed. It is shown that the optimum detector exhibits a two-dimensional estimator-correlator structure for the two pertinent parameters. When the underlying statistics are partially known, the estimates appearing in the receiver cannot be implemented. Three suboptimum schemes with surprisingly good small sample performance are derived and compared.

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1. Introduction

An increasing number of communication systems process signals which can be modelled as point processes. These occur in various areas such as optical communications, nuclear medicine, and detection of seismic events. Oftentimes, the signals are assumed to be Poisson time-dependent processes and detection schemes under these assumptions have been investigated ([1]). However, many processes depart significantly from Poisson statistics; the measure of departure usually is taken as the hazard function ([2]) which is constant under the Poisson regime, but time-varying for other renewal processes.

A renewal process is by definition a point process in which the sequence of times between occurence of events consists of i.i.d. random variables. In this paper, we investigate the detection of renewal processes whose inter-arrival times are $\Gamma(\mu,k)$ distributed, i.e.

$$f(x|\mu,k) = \exp(-\frac{k}{\mu}x)x^{k-1}(\frac{k}{\mu})^{k}/\Gamma(k) \tag{1}$$

With two parameters, k and μ, the Gamma distribution is a good model for a variety of problems. It conveniently describes the Poisson regime for k = 1 and measures the departure from Poisson statistics through the parameters k([2]). In particular, characteristics of bunching are quite well described since

$$\left| \begin{array}{l} E(x) = \mu \\ Var(x) = \frac{\mu^2}{k} \end{array} \right.$$

so that

$$\frac{\sqrt{Var(x)}}{E(x)} = \frac{1}{\sqrt{k}} \tag{2}$$

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From (2), it can be seen that if k is greater than one, we have spreading of the observations (i.e. events are spaced regularly around the mean in time) whereas if k is less than one, events exhibit a bunching, or correlated, pattern.

We will investigate the following two hypotheses H_0 and H_1: under H_0, noise (dark current) is received and the process is Poisson with mean $1/\mu_0$; under H_1, the observed point process contains a random signal to be detected and the inter-arrival times are governed by (1). We will assume that the random signal under H_1 modulates the information bearing parameters μ and k, so that these are to be considered as random variables. Alternatively, one might consider μ as the information bearing parameter while k reflects the unknown dead time characteristic of a photomultiplier device.

In order to determine the structure of the optimum detector minimizing the average probability of error under a Bayesian criterion or maximizing the power for a fixed probability of false alarm, it is convenient to exploit the property that the Gamma distribution belongs to the exponential family. Indeed, let

$$\theta \triangleq (\theta_1, \theta_2)'$$

where

$$\theta_1 \triangleq -\frac{k}{\mu}$$

$$\theta_2 \triangleq k \tag{3}$$

and

$$h(x) \triangleq \frac{1}{x}$$

$$b(\theta) \triangleq \log\Gamma(\theta_2) - \theta_2 \log(-\theta_1)$$

Then, (1) can be written as

$$f(x|\theta) = h(x)\exp(\theta_1 x + \theta_2 \log x - b(\theta)) \tag{4}$$

which is the usual exponential form.

In the next section, we will extend some of the results of ([3]) to the two-dimensional exponential family and show that, independent of the bivariate prior density $\pi(\theta_1, \theta_2)$, the marginal density $f(x)$ is completely determined by the conditional mean estimates (CME) of θ_1 and θ_2. This resulting form for the marginal density leads to a general estimator-correlator structure for detectors based on likelihood ratios.

Since the optimum detector usually cannot be implemented because of insufficient a-priori knowledge of the statistics of μ and k, we will investigate the properties of some related suboptimum detectors. This is done in Section III. In particular, we will utilize a modified and a discrete maximum likelihood estimate ([4]) in forming suboptimum detectors. The simulations to be discussed illustrate the attractiveness of this suboptimum approach, especially in the important small sample case.

II. Detection of a Renewal Process with Gamma Inter-Arrival Times

A. Bayesian test

We suppose that under both hypotheses H_0 and H_1, n observations $(x_i, i=1,\ldots,n)$ independent and identically distributed (i.i.d.) are governed by (4): under H_0, $\theta = \theta_0 = (\theta_{10}, \theta_{20})'$ is a known vector, whereas under H_1, θ is a random vector with bivariate prior $\theta \sim \pi(\theta) = \pi(\theta_1, \theta_2)$. Moreover we assume that H_0 and H_1 occur with priors equal to p_0 and p_1 respectively. The detection problem admits a sufficient statistic ([5])

$$t = (t_1, t_2)' \tag{5}$$

where

$$t_1 = \sum_{i=1}^{n} x_i$$

$$t_2 = \sum_{i=1}^{n} \log x_i$$

so that H_0 and H_1 become equivalent to the following. Under both hypotheses

$$t \sim f(t|\theta) = \exp(\theta_1 t_1 + \theta_2 t_2 - nb(\theta) + P(t)) \tag{6}$$

where $\theta = \theta_0$ is a known vector under H_0 and under H_1, θ is a random vector with prior $\pi(\theta)$. Denoting the marginal of t under H_1 by $f(t)$, the optimum detector is the likelihood ratio

$$L(t) = \frac{f(t)}{f(t|\theta_0)} \mathop{\gtrless}_{H_0}^{H_1} \frac{p_0}{p_1} \tag{7}$$

Now, $f(t)$ is given by

$$f(t) = \iint f(t|\theta)\pi(\theta)d\theta \tag{8}$$

i.e.,

$$f(t) = \exp(B(t))\iint \exp(\theta_1 t_1 + \theta_2 t_2 - nb(\theta))\pi(\theta_1,\theta_2)d\theta_1 d\theta_2 \tag{9}$$

Take the partial derivative of $f(t)$ with respect to t_1 and t_2:

$$\frac{\partial \log f(t)}{\partial t_i} = \frac{\partial B(t)}{\partial t_i} + \frac{\iint \theta_i f(t|\theta)\pi(\theta)d\theta}{f(t)} \qquad i = 1,2 \tag{10}$$

Thus

$$\hat{\theta}_i(t) \triangleq E(\theta_i|t) = \frac{\partial \log f(t)}{\partial t_i} - \frac{\partial B(t)}{\partial t_i} \qquad i = 1,2 \tag{11}$$

Since

$$d\log f(t) = \frac{\partial \log f(t)}{\partial t_1} dt_1 + \frac{\partial \log f(t)}{\partial t_2} dt_2 \tag{12}$$

upon substituting (11) into (12), one obtains

$$d\log f(t) = \hat{\theta}_1(t)dt_1 + \hat{\theta}_2(t)dt_2 + dB(t) \tag{13}$$

(13) is a complete differential. Therefore if we integrate along a path such as represented in Fig. 1, we get

$$f(t) = K(\hat{\theta})\exp\left(\int_{t_0}^{t} \hat{\theta}_1(u)du_1 + \int_{t_0}^{t} \hat{\theta}_2(u)du_2 + B(t)\right) \tag{14}$$

where $K(\hat{\theta})$ is the normalizing constant and t_0 is chosen arbitrarily.

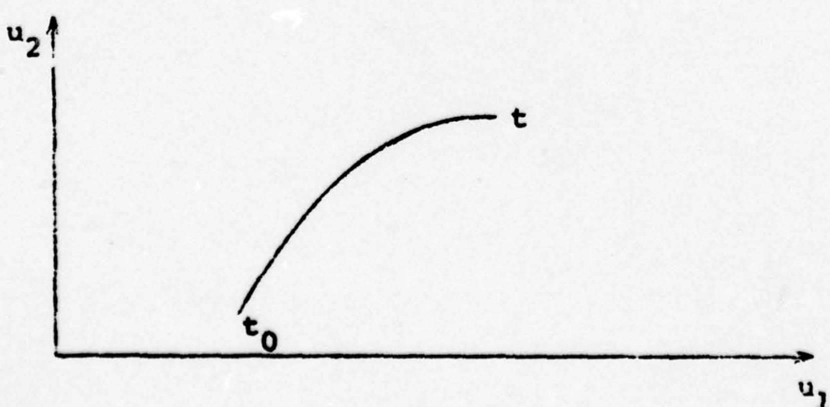


Fig. 1

Let

$$r(\hat{\theta}) \triangleq \int_{t_0}^{t} \hat{\theta}_1(u)du_1 + \int_{t_0}^{t} \hat{\theta}_2(u)du_2 - \theta_{10}t_1 - \theta_{20}t_2 \tag{15}$$

After substitution of (6) and (14) into (7), one can write the likelihood-ratio as

$$L(t) = K(\hat{\theta})\exp(nb(\theta_0))\exp(r(\hat{\theta})) \tag{16}$$

Following [6], the constant appearing in (16) can be written in a more convenient way. We multiply (16) by $f(t|\theta_0)$ and integrate with respect to t. Since f(t) integrates to 1, we obtain

$$[K(\hat{\theta})\exp(nb(\theta_0))]^{-1} = E_{H_0}(\exp(r(\hat{\theta}))) \tag{17}$$

Substituting (17) into (16) and taking logarithms, the log-likelihood ratio becomes

$$\ell(\hat{\theta}) = r(\hat{\theta}) - E_{H_0}(\exp(r(\hat{\theta}))) \tag{18}$$

This is compared to the threshold $\ln(p_0/p_1)$ for an optimum Bayes test. As seen from (18) and (15), the optimum receiver is completely determined by the CME's of θ_1 and θ_2 and displays an estimator-correlator structure.

B. Neyman-Pearson test

The Neyman-Pearson test is easily derivable from (18), i.e.

$$r(\hat{\theta}) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma(\hat{\theta}) \tag{19}$$

where $\gamma(\hat{\theta})$ is chosen so that the probability of false alarm is set equal to a level α.

The above results constitute a canonical detector structure for renewal processes with gamma inter-arrival times. One must, of course, specify the prior distribution $\pi(\theta_1, \theta_2)$. When this distribution is not known, Eqs. (18) and (19) suggest replacing the CME's by other estimates which are good approximations to it and which require less prior information. This is the subject of the next section.

Finally, it should be clear that the results of this section can easily be extended to the n-dimensional exponential family.

III. Adaptive Detection of Renewal Processes

Let

$$\theta_{10} = -1/\mu_0 \;, \quad \theta_{20} = 1$$
$$\theta_1 = -k/\mu \;, \quad \theta_2 = k \tag{20}$$

Then, the optimum Bayesian and Neyman-Pearson tests are given by (18) and (19). As suggested above, these tests are often not used because of insufficient prior knowledge or because the CME's of θ_1 and θ_2 are simply difficult to implement. Consequently, it is natural to investigate the properties of suboptimum detectors obtained by substituting suboptimum estimators for the CME's in (18) or (19). In ([4]), it is shown that good approximations to CME's can be derived from modifications of the maximum likelihood estimate (MLE). As one might expect, the resulting detector performance is close to the optimum. What is surprising is that this is true even for very small samples (n=3 or 4). We now derive three detection schemes based on the MLE and modifications of it. This is done in increasing order of assumed prior knowledge. The first is the MLE which assumes no prior knowledge on μ or k. The truncated MLE assumes that the range of μ and k is known. Finally, the discrete MLE assumes further that the parameter k can only take on one of a finite number of values.

A. MLE Detector

We first have to calculate the MLE's of μ and k or, equivalently, of θ_1 and θ_2. From (3) and (6) the maximum likelihood equations have the form

$$-\frac{\tilde{\theta}_2}{\tilde{\theta}_1} = \tilde{\mu} = t_1/n$$
$$\psi(\tilde{\theta}_2) - \log(-\tilde{\theta}_1) = t_2/n \tag{21}$$

where $\tilde{\mu}$, $\tilde{\theta}_1$, $\tilde{\theta}_2$ denote the MLE's of the corresponding parameters and ψ is the derivative of the Gamma function. The solution to (21) is not immediate and does not lend itself to analytic integration. However, if one assumes that k (i.e. θ_2) is sufficiently large so that Stirling's formula ([7]) can be used, we have

$$\psi(\tilde{\theta}_2) \simeq \log \tilde{\theta}_2 - \frac{1}{2\tilde{\theta}_2}$$

and

$$-\frac{\tilde{\theta}_2}{\tilde{\theta}_1} = \frac{t_1}{n}$$

$$\tilde{\theta}_2 = \tilde{k} = \frac{1}{2}\left(\log\left(\frac{t_1}{n}\right) - \frac{t_2}{n}\right)^{-1} \tag{22}$$

For later use, we make the following observations:

1) $\tilde{\theta}_1$ and $\tilde{\theta}_2$ can now be integrated.

2) $\tilde{k}$ is a reasonable estimate since it is always positive, a property which stems from the fact that the arithmetic mean is larger than the geometric mean. We now have to calculate the integrals

$$I(\tilde{\theta}_1) \triangleq \int_{t_0}^{t} \tilde{\theta}_1(u)\,du_1 \tag{23}$$

and

$$I(\tilde{\theta}_2) \triangleq \int_{t_0}^{t} \tilde{\theta}_2(u)\,du_2 \tag{24}$$

where the integrations should be performed along a convenient path. In Appendix A, these integrations are carried out, the final result being:

$$I(\tilde{\theta}_1) + I(\tilde{\theta}_2) = -\frac{n}{2}\log\left[\log\left(\frac{t_1}{n}\right) - \frac{t_2}{n}\right] \tag{25}$$

Note again that in (25), the sign of the argument raises no problem since it is positive. We then obtain the MLE Bayesian detector by substituting (25) into (19). We also have to calculate the quantity

$$K' \triangleq E_{H_0} \exp\left(\frac{t_1}{\mu_0} - t_2 - \frac{n}{2} \log\left[\log \frac{t_1}{n} - \frac{t_2}{n}\right]\right) \tag{26}$$

Rewrite (26) as:

$$K' = E_{H_0} \frac{\exp\left[n\left(\log \frac{t_1}{n} - \frac{t_2}{n}\right)\right]}{\left[\log \frac{t_1}{n} - \frac{t_2}{n}\right]^{n/2}} \exp\left[- n \log \frac{t_1}{n} + \frac{t_1}{\mu_0}\right] \tag{27}$$

By a theorem due to Pitman ([8], page 217), t_1 and $\log \frac{t_1}{n} - \frac{t_2}{n}$ are independent. This property permits the factoring of the expectation in (27) and since t_1 is Gamma distributed under H_0, we have

$$E_{H_0} \exp\left(- n \log \frac{t_1}{n} + \frac{t_1}{\mu_0}\right) = \frac{1}{\Gamma(n)} \int_0^\infty \left(\frac{t_1}{n}\right)^{-n} \exp\left(\frac{t_1}{\mu_0}\right)$$

$$\times \exp\left(- \frac{t_1}{\mu_0}\right) t_1^{\,n-1} \left(\frac{1}{\mu_0}\right)^n dt_1 \tag{28}$$

$$= \frac{1}{\Gamma(n)} \frac{1}{\mu_0^{\,n}} \frac{1}{n^n} \int_0^\infty t_1^{-1} dt_1 = \infty$$

so that

$$K' = \infty \tag{29}$$

Hence, the MLE yields an undefined Bayesian detector, a phenomenon encountered for other classes of problems ([9], Sec. V.B, and [10], Sec. 3.4 of Chap. 2).

In contrast, the Neyman-Pearson test is well defined and derived by substituting (25) into (19). Dividing by n, we get

$$\ell(\tilde{\theta}) = - \frac{1}{2} \log\left[\log \frac{t_1}{n} - \frac{t_2}{n}\right] + \frac{1}{\mu_0} \frac{t_1}{n} - \frac{t_2}{n} \underset{H_0}{\overset{H_1}{\gtrless}} v(\tilde{\theta}) \tag{28}$$

B. Truncated MLE Detector

Since the optimum detector is determined by CME's, one might expect that by modifying the MLE for some given partial a-priori knowledge of the parameters, the resulting estimates will be closer to the CME's, and the associated detector will exhibit a performance which is closer to that of the optimum. This will indeed be the case. In this sub-section, we assume prior knowledge of the dynamic range of μ and k, i.e., the boundaries are known:

$$\mu \in [\mu_\ell, \mu_u]$$

$$k \in [k_\ell, k_u]$$

We consider the following estimates:

$$\bar{\mu} = \frac{t_1}{n} \text{ if } 0 \le \frac{t_1}{n} \le \mu_u$$
$$\bar{\mu} = \mu_u \text{ if } \frac{t_1}{n} > \mu_u \tag{29}$$

and

$$\bar{k} = \tilde{k} \quad \text{if } k_\ell \le \tilde{k} \le k_u$$
$$\bar{k} = k_u \quad \text{if } \tilde{k} > k_u \tag{30}$$
$$\bar{k} = k_\ell \quad \text{if } \tilde{k} < k_u$$

The associated estimates $\bar{\theta}_1$ and $\bar{\theta}_2$ are given by

$$\bar{\theta}_1 = -\frac{\bar{k}}{\bar{\mu}} \quad , \quad \bar{\theta}_2 = \bar{k} \tag{31}$$

In Appendix B, we calculate the integrals $I(\bar{\theta}_1)$ and $I(\bar{\theta}_2)$ defined as in (23), (24). With the following definitions

$$[f(x)]^+ \begin{cases} = f(x) & \text{if } f(x) \geq 0 \\ = 0 & \text{if } f(x) < 0 \end{cases}$$

$$c(x) \begin{cases} = 1 & \text{if } x \geq 0 \\ = 0 & \text{if } x < 0 \end{cases}$$

$$y_1 = \frac{t_1}{n}, \quad y_2 = \frac{t_2}{n}$$

$$a = \log \frac{t_1}{n} - \frac{t_2}{n}$$

$$\nu_\ell = \log \frac{t_1}{n} - \frac{1}{2k_\ell}$$

$$\nu_u = \log \frac{t_1}{n} - \frac{1}{2k_u}$$

we show

$$\begin{aligned} \frac{1}{n}(I(\bar{\theta}_1) + I(\bar{\theta}_2)) = & - k_\ell(\log y_1 - 1) - k_\ell(-\nu_\ell)^+ \times c(y_1 - 1) \\ & + \frac{1}{2}\, c(y_1-1)\{[\log(2k_\ell \log y_1)]\, c(-\nu_\ell) - [\log(2k_u)\log y_1]\, c(-\nu_u)\} \\ & + k_\ell(\log y_1 - \log \mu_u)^+ - \frac{k_\ell}{\mu_u}(y_1 - \mu_u)^+ + k_\ell y_2 \qquad (32) \\ & - k_\ell(y_2 - \nu_\ell)^+ - \frac{1}{2}[\log(2k_\ell a)] \times c(y_2 - \nu_\ell) \\ & + \frac{1}{2}[\log(2k_\ell a)]\, c(y_2 - \nu_u) + k_u(y_2 - \nu_u)^+ \end{aligned}$$

Equation (32) is obviously a complicated expression. However, it is easily implementable on a computer using the built-in positive difference function or on special purpose hardware using limiters. We consider only the associated N-P detector which is obtained by substituting (32) into (19).

C. Discrete MLE Detector

Here, we assume the a-priori knowledge of the dynamic range of μ and also suppose that k can only take on an integer value drawn from a finite set, i.e.

$$\mu \in [\mu_\ell, \mu_u]$$

$$k \in \{k_\ell, k_\ell + 1, \ldots, k_u\}$$

we form the following estimates[1]

$$\begin{aligned} \hat{\mu} &= \frac{t_1}{n} \text{ if } \mu_\ell \le \frac{t_1}{n} \le \mu_u \\ \hat{\mu} &= \mu_u \text{ if } \frac{t_1}{n} > \mu_u \\ \hat{\mu} &= \mu_\ell \text{ if } \frac{t_1}{n} < \mu_\ell \end{aligned} \tag{33}$$

and

$$\begin{aligned} \hat{k} &= k_\ell \quad \text{if } \tilde{k} \le k_\ell + \tfrac{1}{2} \\ \hat{k} &= k_\ell + i \text{ if } k_\ell + i - \tfrac{1}{2} < \tilde{k} \le k_\ell + i + \tfrac{1}{2} \\ &\quad \text{for } i = 1, \ldots, k_u - k_\ell - 1 \\ \hat{k} &= k_u \quad \text{if } \tilde{k} > k_u - \tfrac{1}{2} \end{aligned} \tag{34}$$

while

$$\begin{aligned} \hat{\theta}_1 &= -\frac{\hat{k}}{\hat{\mu}} \\ \hat{\theta}_2 &= \hat{k} \end{aligned} \tag{35}$$

---

[1] The notation here should not be confused with the CME notation.

The resulting N-P detector obtained by substituting $I(\hat{\theta}_1)$ and $I(\hat{\theta}_2)$ into (19), is derived in Appendix C. We introduce the following notation:

$$y_1 = \frac{t_1}{n}, \quad y_2 = \frac{t_2}{n}$$

$$\nu_i = \log y_1 - \frac{1}{2(k_\ell+i)-1}, \quad i = 1,\ldots,k_u - k_\ell$$

$$s_i = \exp[y_2 + \frac{1}{2(k_\ell+i)-1}], \quad i = 1,\ldots,k_u - k_\ell$$

We then have

1) If $y_1 > 1$

$$\frac{1}{n}(I(\hat{\theta}_1) + I(\hat{\theta}_2)) = -k_\ell(\log y_1 - 1) + k_\ell(\log y_1 - \log\mu_u)^+ + \sum_{i=1}^{k_u-k_\ell}(-\nu_i)^+ + k_\ell y_2 + \sum_{i=1}^{k_u-k_\ell}(y_2 - \nu_i)^+ \tag{36}$$

2) If $\mu_\ell < y_1 < 1$

$$\frac{1}{n}(I(\hat{\theta}_1) + I(\hat{\theta}_2)) = -k_\ell(\log y_1 - 1) + \sum_{i=1}^{k_u-k_\ell}(y_2 - \nu_i)^+ + k_\ell y_2 \tag{37}$$

3) If $y_1 < \mu_\ell$

$$\frac{1}{n}(I(\hat{\theta}_1) + I(\hat{\theta}_2)) = -k_\ell(\log\mu_\ell - 1) + \sum_{i=1}^{k_u-k_\ell}(y_2 - \log\mu_\ell + \frac{1}{2(k_\ell+i)-1})^+ - \sum_{i=1}^{k_u-k_\ell}(\frac{s_i}{\mu_\ell} - 1)^+ + \frac{1}{\mu_\ell}\sum_{i=1}^{k_u-k_\ell}(s_i - y_1)^+ + \frac{k_\ell}{\mu_\ell}(\mu_\ell - y_1) + k_\ell y_2 \tag{38}$$

As commented on in Section III.B, this receiver is also not that difficult to implement.

D. Simulation Results

Simulations have been performed for the Neyman-Pearson tests associated with $\tilde{\theta}$, $\bar{\theta}$ and $\hat{\theta}$, and are denoted respectively by DET.1, DET.2 and DET.3. Under H_0, the observations are exponentially distributed with mean $1/\mu_0$. Under H_1, they are $\Gamma(k/\mu,k)$ distributed, k is uniformly distributed on the integers $\{k_\ell, k_\ell+1, \ldots k_u\}$ and μ is independent of k and uniform on $[\mu_\ell, \mu_u]$. For this example, the optimum test which we designate DET.4, can be obtained directly from the likelihood ratio calculated in Appendix D. It should be noted that although available in this example, this detector cannot be set into an estimator-correlator structure and, as indicated in Table 3 below, the computing time required for its implementation is much larger than that of any of the tests previously described.

We simulated hypotheses H_0 and H_1 1000 times (m=1000) and calculated the empirical distributions of the four tests under both hypotheses. To determine the various thresholds for a significance level α, we used the following non-parametric method discussed by Davis and Andreadakis [11], and which can also be found in ([12]). Let r(1), r(2),...,r(m) be the order statistics of any of the tests investigated under H_0. The $(1-\alpha)_{th}$ quantile $q_{1-\alpha}$ is such that

$$Pr_{H_0}\{r > q_{1-\alpha}\} = \alpha$$

Consider the event

$$E = \{r(j) > q_{1-\alpha}\}$$

E occurs if at least (m-j+1) values of r are greater that $q_{1-\alpha}$, corresponding to the probability of having at least (m-j+1) successes in m Bernouilli trials with α being the probability of a success. Hence

$$\Pr\{E\} = I_{\alpha}(m-j+1, j)$$

where $I_{\alpha}(a,b)$ is the incomplete Beta function. In this case, it can be approximated by

$$N\left(\frac{j-1-m(1-\alpha)}{\sqrt{m\alpha(1-\alpha)}}\ ,\ 1\right)$$

where

$$N(a,1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}}\, dx$$

Consequently, for m=1000, if we choose j=963 (or j=918), there is a 96.4% probability that the false alarm is less than 5% (or 10%) when the thresholds are taken to be r(963) and r(918), respectively.

The results, summarized in the following tables, illustrate some significant differences in the small sample case for various values of the parameters. In general, DET.3 is superior to DET.2 which in turn, performs better than DET.1. DET.3 is quite frequently much better than DET.1 and very close to the optimum. In the large sample case (n greater than 10), as one might expect, the detectors have similar power.

Number of Samples	μ_0	μ_ℓ	μ_u	k_ℓ	k_u	Power of DET.1	Power of DET.2	Power of DET.3	Power of DET.4 (optimum)
3	.3	.5	3	1	3	.731	.773	.862	.876
4	.5	1	3	1	3	.689	.784	.818	.858
4	3	1	5	5	9	.354	.443	.525	.435
4	1	.5	3	4	7	.433	.484	.615	.681
4	1	.5	3	2	5	.346	.324	.460	.523
10	1	.5	3	2	5	.708	.797	.760	.832

Table 1. $\alpha = 5\%$

Number of Samples	μ_0	μ_ℓ	μ_u	k_ℓ	k_u	Power of DET.1	Power of DET.2	Power of DET.3	Power of DET.4 (optimum)
3	.3	.5	3	1	3	.816	.824	.911	.915
4	.5	1	3	1	3	.760	.832	.887	.901
4	3	1	5	5	9	.549	.699	.728	.760
4	1	.5	3	4	7	.635	.681	.746	.799
4	1	.5	3	2	5	.495	.501	.577	.660
10	1	.5	3	2	5	.800	.879	.848	.893

Table 2. $\alpha = 10\%$

DET.1	DET.2	DET.3	DET.4	
112	112	115	495	seconds

Table 3.

Approximate Sum of Computing Time for the First and Two Last Rows of Table 1.

IV. Conclusion

In this paper, we investigated the detection of renewal processes whose inter-arrival times are Gamma distributed. We first developed the structure of the optimum Bayesian and Neyman-Pearson tests for the two-dimensional exponential family. The main characteristic of these detectors is that they fall into the category of estimator-correlators since they are determined by integrals of the CME's of the two pertinent parameters. One implication of this structure is the implementation of suboptimum tests by substituting various estimates for the CME's.

We then applied the estimator-correlator property to the case of Gamma distributed observations and investigated three related tests. The first detector, DET.1, is based on the MLE, and as previously observed, the Bayesian test is undefined whereas the Neyman-Pearson version seems to perform quite well even for a small number of samples. The second test, DET.2, is based on the truncated MLE which assumes knowledge of the dynamic range (boundary) of the parameters. Finally, we investigated the properties of DET.3, the test based on the discrete MLE of k assuming that k can only assume a value on a finite set of integers. This test is well-suited for the situation where observations are taken at the output of a photomultiplier with a fixed dead-time characteristic. DET.3 and DET.2 outperformed DET.1 and often in the small sample case, the performance of DET.3 is markedly superior to that of DET.1 and very close to the optimum. Consequently, based on these preliminary simulations, DET.3 is a fairly adequate test for the detection of a large class of renewal processes.

Appendix A

We integrate the MLE's $\tilde{\theta}_1$ and $\tilde{\theta}_2$ which are solutions to (22). They are rewritten as:

$$\tilde{\theta}_1 = - \frac{n}{2t_1} \left(\log \frac{t_1}{n} - \frac{t_2}{n}\right)^{-1} \tag{1.A}$$

$$\tilde{\theta}_2 = \frac{1}{2} \left(\log \frac{t_1}{n} - \frac{t_2}{n}\right)^{-1} \tag{2.A}$$

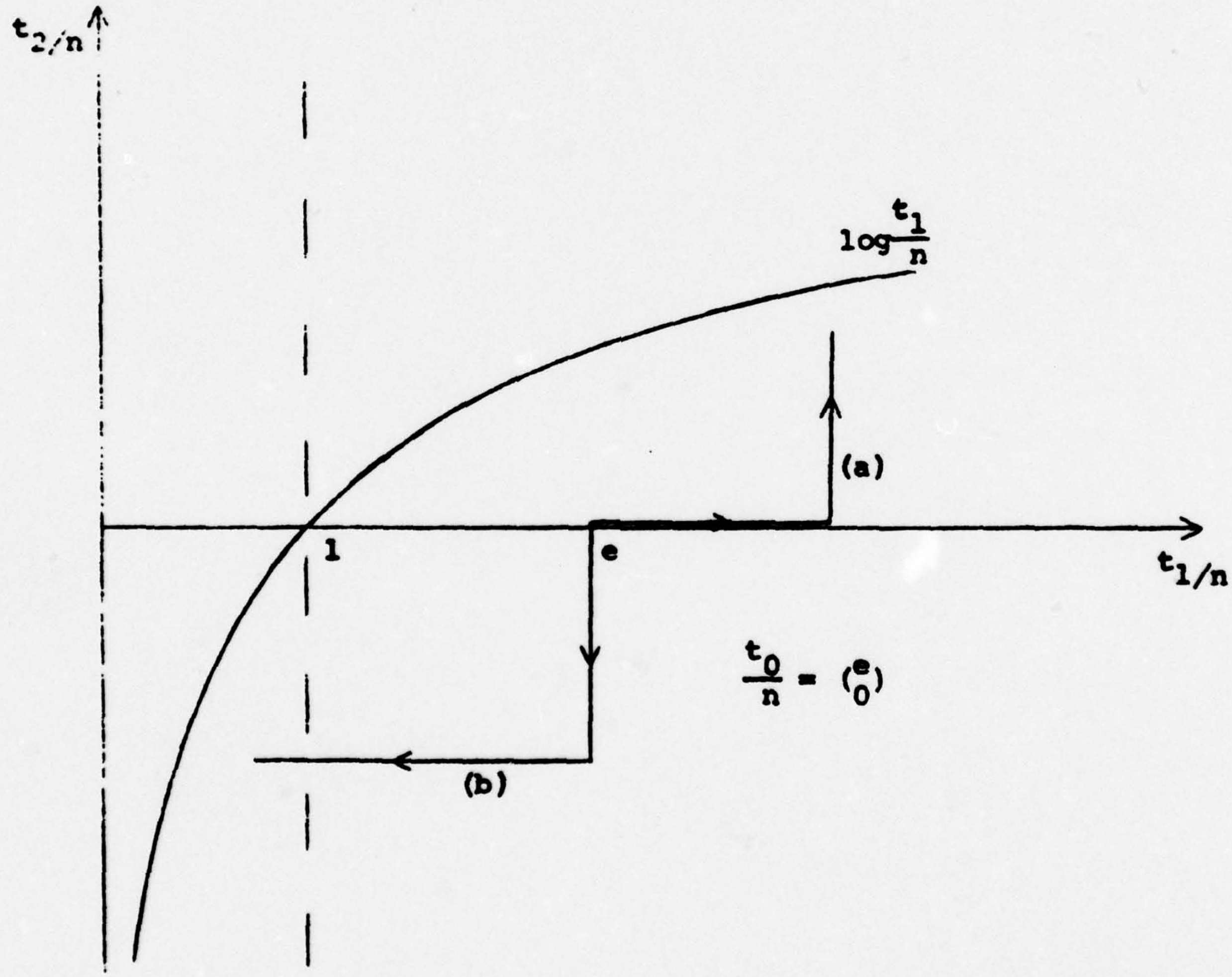


Fig. A

Since

$$\log \frac{t_1}{n} - \frac{t_2}{n} \geq 0$$

the admissible points $\frac{t}{n}$ are located below the curve

$$y = \log \frac{t_1}{n}$$

and therefore one should be careful in choosing the path of integration.

1) Case 1: $\frac{t_1}{n} > 1$

We integrate along path (a). Then

$$I(\tilde{\theta}_1) = \int_{t_0}^{t} \tilde{\theta}_1(u)du_1 = -\frac{1}{2}\int_{t_0}^{t} \frac{n}{u_1(\log \frac{u_1}{n} - \frac{t_2}{n})} du_1 \tag{3.A}$$

Along the part of the path for which the integral does not vanish, we have

$$t_2 = t_{20} = 0$$

Thus

$$I(\tilde{\theta}_1) = -\frac{n}{2}\int_{e}^{t_1/n} \frac{d\sigma}{\sigma\log\sigma} = -\frac{n}{2}\int_{1}^{\log\frac{t_1}{n}} \frac{ds}{s}$$

$$I(\tilde{\theta}_1) = -\frac{n}{2} \log\log \frac{t_1}{n} \tag{4.A}$$

Now

$$I(\tilde{\theta}_2) = \int_{t_0}^{t} \tilde{\theta}_2(u)du_2 = \frac{1}{2}\int_{t_0}^{t} \frac{du_2}{\log \frac{u_1}{n} - \frac{u_2}{n}} \tag{5.A}$$

Along the part of the path where the integral does not vanish, we have

$$u_1 = t_1$$

Thus

$$I(\tilde{\theta}_2) = \frac{1}{2}\int_0^{t_2} \frac{du_2}{\log\frac{t_1}{n} - \frac{u_2}{n}} = -\frac{n}{2}\log\left|\frac{\log\frac{t_1}{n} - \frac{t_2}{n}}{\log\frac{t_1}{n}}\right| \tag{6.A}$$

2) Case 2: $\frac{t_1}{n} < 1$

Here, we integrate along path (b). We have

$$I(\tilde{\theta}_1) = -\frac{1}{2}\int_{ne}^{t_1} \frac{n}{u_1(\log\frac{u_1}{n} - \frac{t_2}{n})}\, du_1$$

and using the same changes of variables as those leading to (4.A) one obtains:

$$I(\tilde{\theta}_1) = -\frac{n}{2}\log\left|\frac{\log\frac{t_1}{n} - \frac{t_2}{n}}{1 - \frac{t_2}{n}}\right| \tag{7.A}$$

Similarly,

$$I(\tilde{\theta}_2) = \frac{1}{2}\int_0^{t_2} \frac{du_2}{1 - \frac{u_2}{n}} = -\frac{n}{2}\log(1 - \frac{t_2}{n}) \tag{8.A}$$

Finally, from (4.A), (6.A), (7.A) and (8.A), we obtain in both cases

$$I(\tilde{\theta}_1) + I(\tilde{\theta}_2) = -\frac{n}{2}\log\left[\log\frac{t_1}{n} - \frac{t_2}{n}\right] \tag{9.A}$$

Appendix B

Here, we integrate the estimates $\bar{\theta}_1$ and $\bar{\theta}_2$ given in (31). As in Appendix A, the admissible region of values that $\frac{t}{n}$ can take on is located below the curve $y = \log \frac{t_1}{n}$. Several cases have to be investigated which will be referred to in Fig. B below.

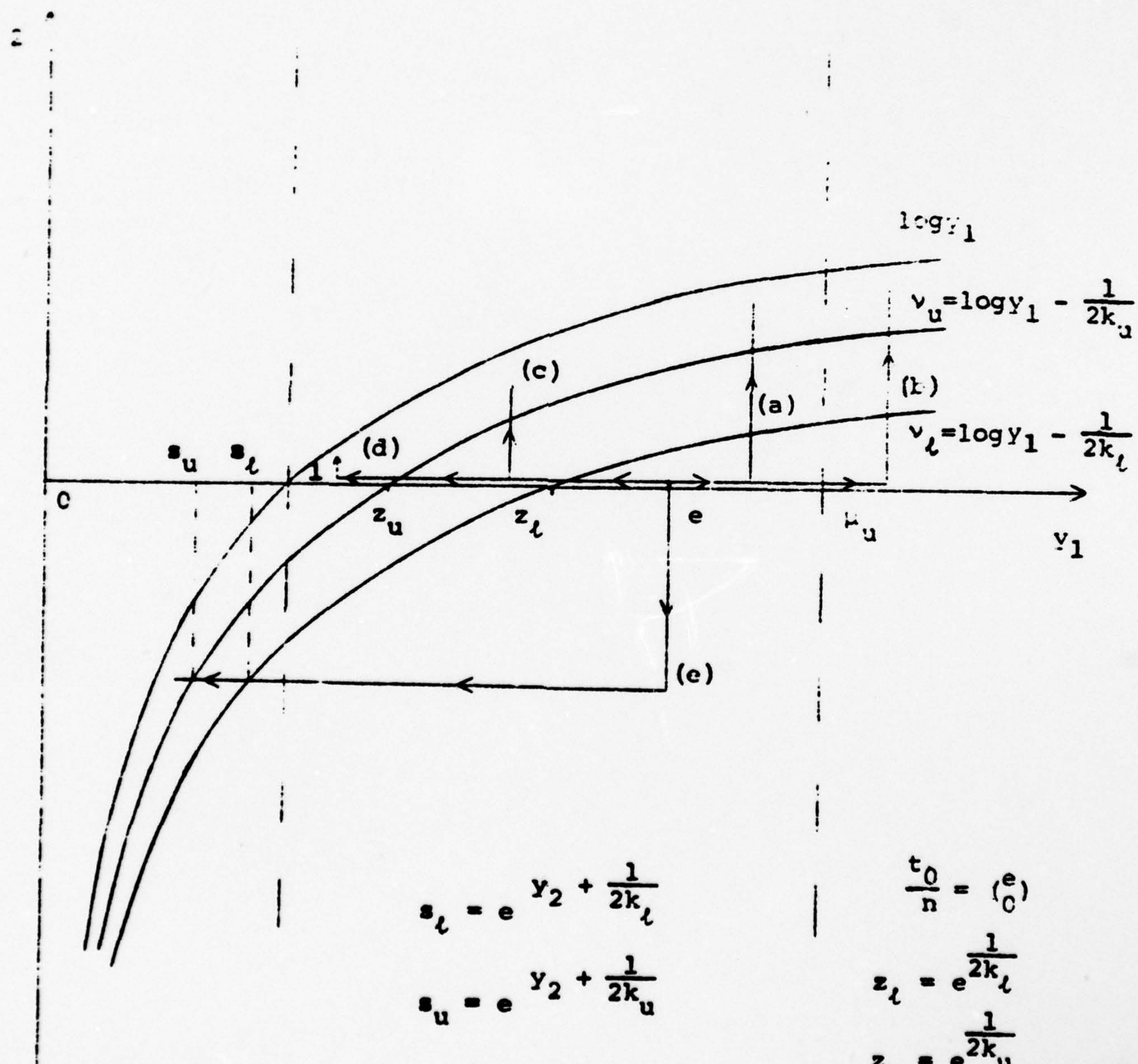


Fig. B

We assume that $\mu_u > e$, $k_\ell > \frac{1}{2}$ and make use of the notation introduced for (32) and in Fig. B. It is easily verified that

$$\bar{k} = k_\ell \text{ iff } y_2 < \nu_\ell$$
$$\bar{k} = \tilde{k} \text{ iff } \nu_\ell \le y_2 \le \nu_u \qquad (1.B)$$
$$\bar{k} = k_u \text{ iff } y_2 > \nu_u$$

1. <u>Case 1:</u> $y_1 > 1$

a) $z_\ell = e^{\frac{1}{2k_\ell}} < y_1 \le \mu_u$

Integrate along the path (a):

$$I(\bar{\theta}_1)/n = -\int_e^{y_1} \frac{k_\ell}{u_1} du_1 = -k_\ell(\log y_1 - 1) \qquad (2.B)$$

We now integrate $\bar{\theta}_2$ and several subcases have to be considered.

i) If $y_2 > \nu_u$, then

$$I(\bar{\theta}_2)/n = \int_0^{\nu_\ell} k_\ell du_2 + \frac{1}{2}\int_{\nu_\ell}^{\nu_u} \frac{du_2}{\log y_1 - u_2} + \int_{\nu_u}^{y_2} k_u du_2$$

or

$$I(\bar{\theta}_2)/n = k_\ell\nu_\ell - \frac{1}{2}\log\left(\frac{k_\ell}{k_u}\right) + k_u(y_2 - \nu_u) \qquad (3.B)$$

ii) If $\nu_\ell \le y_2 \le \nu_u$

$$I(\bar{\theta}_2)/n = k_\ell\nu_\ell - \frac{1}{2}\log(2k_\ell a) \qquad (4.B)$$

iii) If $y_2 < \nu_\ell$, then

$$I(\bar{\theta}_2)/n = k_\ell y_2 \qquad (5.B)$$

Then, (3.B - 5.B) can be rewritten in a single formula, i.e.,

$$I(\bar{\theta}_2)/n = k_\ell y_2 - k_\ell(y_2-\nu_\ell)^+ - \frac{1}{2}\log(2k_\ell a)c(y_2-\nu_\ell)$$
$$+ \frac{1}{2}\log(2k_u a)c(y_2-\nu_u) + k_u(y_2 - \nu_u)^+ \qquad (6.B)$$

b) If $y_1 > \mu_u$, integrate along path (b):

$$I(\bar{\theta}_1)/n = -\int_e^{\mu_u} \frac{k_\ell}{u_1}\,du_1 + \int_{\mu_u}^{y_1} -\frac{k_\ell}{\mu_u}\,du_1$$

or

$$I(\bar{\theta}_1)/n = -k_\ell(\log\mu_u - 1) - \frac{k_\ell}{\mu_u}(y_1-\mu_u) \qquad (7.B)$$

In this case, $I(\bar{\theta}_2)$ is again given by (6.B).

c) If $z_u \le y_1 \le z_\ell$, then integrate along path (c),

$$I(\bar{\theta}_1)/n = -\int_e^{z_\ell} \frac{k_\ell}{u_1}\,du_1 - \frac{1}{2}\int_{z_\ell}^{y_1} \frac{du_1}{u_1\log u_1}$$

or

$$I(\bar{\theta}_1)/n = -k_\ell\left(\frac{1}{2k_\ell} - 1\right) - \frac{1}{2}\log(2k_\ell\log y_1) \qquad (8.B)$$

Again, $I(\bar{\theta}_2)$ is given in (6.B).

d) If $1 < y_1 < z_u$, we integrate along path (d), so that

$$I(\bar{\theta}_1)/n = -k_\ell\left(\frac{1}{2k_\ell} - 1\right) - \frac{1}{2}\log\frac{k_\ell}{k_u} - k_u\left(\log y_1 - \frac{1}{2k_u}\right) \qquad (9.B)$$

and again, $I(\bar{\theta}_2)$ is given by (6.B). We can include (2.B, 7.B, 8.B, 9.B) within a single formula, i.e.,

$$I(\bar{\theta}_1)/n = -k_\ell(\log y_1 - 1) - k_\ell(-\nu_\ell)^+ + \frac{1}{2}\log(2k_\ell\log y_1)c(-\nu_\ell)$$
$$- \frac{1}{2}\log(2k_u\log y_1)c(-\nu_u) + k_\ell\left(\log\frac{y_1}{\mu_u}\right)^+ - \frac{k_\ell}{\mu_u}(y_1-\mu_u)^+ \qquad (10.B)$$

2. Case 2: $y_1 < 1$ (Path (e))

a) If $y_2 < \nu_\ell$

$$I(\bar{\theta}_1)/n = -k_\ell(\log y_1 - 1) \tag{11.B}$$

b) If $\nu_\ell \le y_2 \le \nu_u$,

$$I(\bar{\theta}_1)/n = -k_\ell(\log s_\ell - 1) - \frac{1}{2}\int_{s_\ell}^{y_1} \frac{du_1}{u_1(\log u_1 - y_2)}$$

or

$$I(\bar{\theta}_1)/n = -k_\ell(\log s_\ell - 1) - \frac{1}{2}\log(2k_\ell a) \tag{12.B}$$

c) If $y_2 > \nu_u$,

$$I(\bar{\theta}_1)/n = -k_\ell(\log s_\ell - 1) - \frac{1}{2}\log\frac{k_\ell}{k_u} - k_u(\nu_u - y_2) \tag{13.B}$$

Eqs. (11.B - 13.B) may be summarized as

$$\begin{aligned} I(\bar{\theta}_1)/n = &- k_\ell(\log y_1 - 1) - k_\ell(y_2-\nu_\ell)^+ - \frac{1}{2}\log(2k_\ell a)(y_2-\nu_\ell)^+ \\ &+ \frac{1}{2}\log(2k_u a)(y_2-\nu_u)^+ + k_u(y_2-\nu_u)^+ \end{aligned} \tag{14.B}$$

For a), b), c), of case 2, we have

$$I(\bar{\theta}_2)/n = k_\ell y_2 \tag{15.B}$$

From (6.B), (10.B), (14.B) and (15.B), one obtains Eq. (32).

Appendix C

In this appendix, the discrete MLE detector is derived. The estimates $\hat{\mu}$, $\hat{k}$, $\hat{\theta}_1$ and $\hat{\theta}_2$ are given in (33), (34) and (35) and have various forms according to the position of t in the plane. The paths of integration are represented in Fig. C1 and Fig. C2 below.

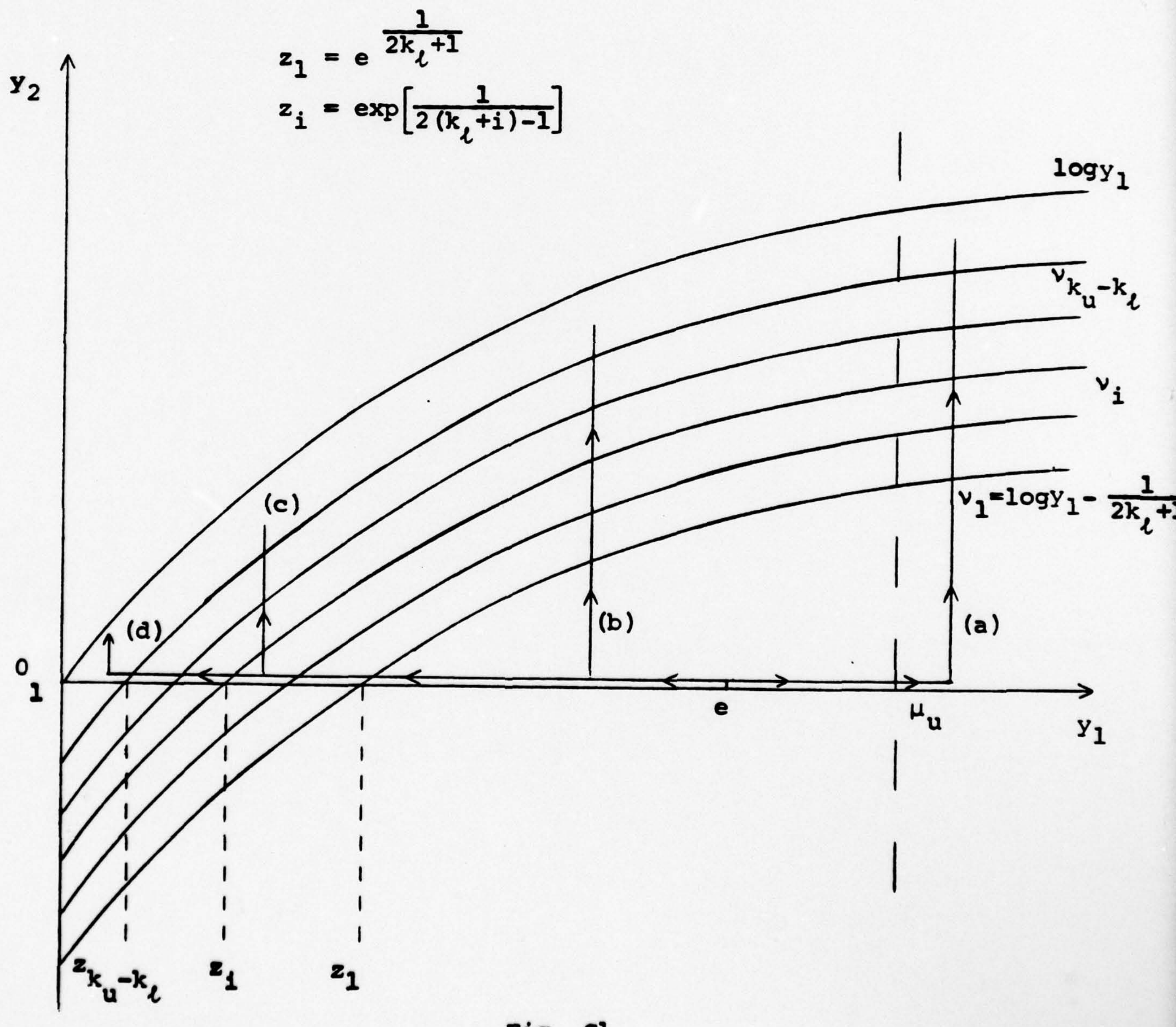
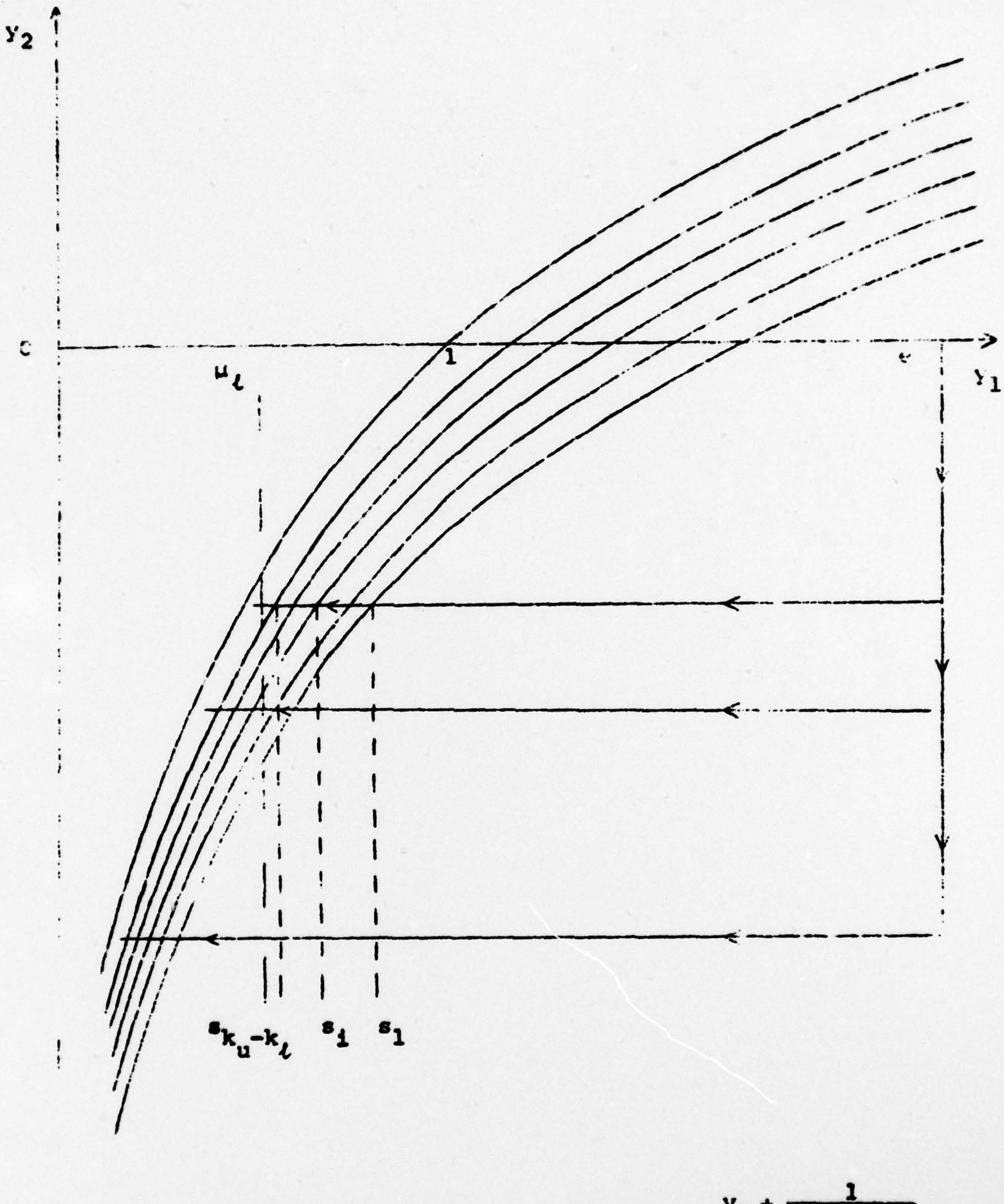


Fig. C1



$$s_i = e^{\,y_2 + \frac{1}{2(k_\ell+i)-1}}$$

Fig. C2

We assume that

$$\mu_\ell < 1 \quad , \quad \mu_u > e$$

$$\text{and } k_\ell \geq 1$$

With the notation introduced below (35), it is readily verified that (34) is equivalent to:

$$\begin{aligned} \hat{k} &= k_\ell \quad \text{iff } y_2 \leq \nu_1 \\ \hat{k} &= k_\ell + i \text{ iff } \nu_i < y_2 \leq \nu_{i+1} \ , \ i = 1,\ldots,k_u - k_\ell - 1 \\ \hat{k} &= k_u \quad \text{iff } \nu_{k_u - k_\ell} < y_2 \end{aligned} \tag{1.C}$$

1. Case 1: If $y_1 > 1$, then consider Fig. C.1 and integrate along the appropriate path.

a) If $y_1 \geq z_1$, $I(\hat{\theta}_1)$ is given in (2.B) and (7.B).

i) If $y_2 \leq \nu_1$,

$$I(\hat{\theta}_2) = \int_0^{t_2} k_\ell du_2 = nk_\ell y_2 \tag{2.C}$$

ii) If $\nu_1 < y_2 \leq \nu_2$,

$$I(\hat{\theta}_2)/n = k_\ell \nu_1 + (k_\ell + 1)(y_2 - \nu_1) = k_\ell y_2 + (y_2 - \nu_1) \tag{3.C}$$

iii) If $\nu_1 \leq y_2 \leq \nu_3$,

$$I(\hat{\theta}_2)/n = k_\ell y_2 + (y_2 - \nu_1) + (y_2 - \nu_2) \tag{4.C}$$

One can summarize (2.C - 4.C) and the other subcases as

$$I(\hat{\theta}_2)/n = k_\ell y_2 + \sum_{i=1}^{k_u - k_\ell} (y_2 - \nu_i)^+ \tag{5.C}$$

b) If $z_2 < y_1 \leq z_1$, then

$$I(\hat{\theta}_1)/n = -k_\ell\left(\frac{1}{2k_\ell+1} - 1\right) - \int_{z_1}^{y_1} \frac{k_\ell+1}{u_1}\, du_1$$

or

$$I(\hat{\theta}_1)/n = -k_\ell(\log y_1 - 1) - v_1 \tag{6.C}$$

$I(\hat{\theta}_2)$ is still given by (5.C).

c) If $z_3 < y_1 \le z_2$,

$$I(\hat{\theta}_1)/n = -k_\ell(\log y_1 - 1) - v_1 - v_2 \tag{7.C}$$

Cases a), b), c), and all other subsequent cases for $y_1 > 1$, can be rewritten as

$$I(\hat{\theta}_1)/n = -k_\ell(\log y_1 - 1) + k_\ell\left(\log \frac{y_1}{\mu_u}\right)^+ + \sum_{i=1}^{k_u - k_\ell} (-v_i)^+ \tag{8.C}$$

while $I(\hat{\theta}_2)$ is given by (5.C).

2. <u>Case 2</u>: If $\mu_\ell < y_1 < 1$, then consider Fig. C.2 and integrate along the appropriate paths.

a) If $y_1 > s_1$,

$$I(\hat{\theta}_1)/n = -k_\ell(\log y_1 - 1) \tag{9.C}$$

b) If $s_2 < y_1 \le s_1$

$$I(\hat{\theta}_1)/n = -k_\ell(\log y_1 - 1) + (y_2 - v_1) \tag{10.C}$$

and in general, for $\mu_\ell < y_1 < 1$, we get

$$I(\hat{\theta}_1)/n = -k_\ell(\log y_1 - 1) + \sum_{i=1}^{k_u - k_\ell} (y_2 - v_i)^+ \tag{11.C}$$

3. <u>Case 3</u>: If $y_1 \le \mu_\ell$, we have to consider the various values that μ_ℓ can assume.

a) If $\mu_\ell > s_1$, we investigate the following subcases.

i) If $y_1 > s_1$, then similarly as before

$$I(\hat{\theta}_1)/n = -k_\ell(\log\mu_\ell - 1) + \frac{k_\ell}{\mu_\ell}(\mu_\ell - y_1) \tag{12.C}$$

ii) If $s_2 < y_1 \le s_1$,

$$I(\hat{\theta}_1)/n = -k_\ell(\log\mu_\ell - 1) + \frac{1}{\mu_\ell}(s_1 - y_1) + \frac{k_\ell}{\mu_\ell}(\mu_\ell - y_1) \tag{13.C}$$

(12.C), (13.C) and the subsequent cases can be written as

$$I(\hat{\theta}_1)/n = -k_\ell(\log\mu_\ell - 1) + \frac{1}{\mu_\ell}\sum_{i=1}^{k_u - k_\ell}(s_i - y_1)^+ + \frac{k_\ell}{\mu_\ell}(\mu_\ell - y_1) \tag{14.C}$$

b) If $s_2 < \mu_\ell \le s_1$, again several sub-cases have to be investigated. If $s_2 < y_1$, we have using (10.C)

$$I(\hat{\theta}_1)/n = -k_\ell(\log\mu_\ell - 1) + (y_2 - \log\mu_\ell + \frac{1}{2k_\ell + 1}) - \frac{k_\ell + 1}{\mu_\ell}(y_1 - \mu_\ell) \tag{15.C}$$

and in general for $s_{i+1} \le y_1 \le s_i$, $i = 2,\ldots,k_u - k_\ell$, we get

$$\begin{aligned} I(\hat{\theta}_1)/n = &-k_\ell(\log\mu_\ell - 1) + (y_2 - \log\mu_\ell + \frac{1}{2k_\ell + 1}) \\ &+ \frac{1}{\mu_\ell}\sum_{i=2}^{k_u - k_\ell}(s_i - y_1)^+ + \frac{k_\ell + 1}{\mu_\ell}(\mu_\ell - y_1) \end{aligned} \tag{16.C}$$

We can regroup (14.C) and (16.C) as

$$\begin{aligned} I(\hat{\theta}_1)/n = &-k_\ell(\log\mu_\ell - 1) + (y_2 - \log\mu_\ell + \frac{1}{2k_\ell + 1})^+ \\ &- \frac{1}{\mu_\ell}(s_1 - \mu_\ell)^+ + \frac{1}{\mu_\ell}\sum_{i=1}^{k_u - k_\ell}(s_i - y_1)^+ + \frac{k_\ell}{\mu_\ell}(\mu_\ell - y_1) \end{aligned} \tag{17.C}$$

In fact, (17.C) can be generalized to the cases $s_{i+1} < \mu_\ell \le s_i$, $i = 2,\ldots,k_u-k_\ell$ as follows:

$$I(\hat{\theta}_1)/n = -k_\ell(\log\mu_\ell - 1) + \sum_{i=1}^{k_u-k_\ell}\left(y_2 - \log\mu_\ell + \frac{1}{2(k_\ell+i)-1}\right)^+$$
$$- \frac{1}{\mu_\ell}\left[\sum_{i=1}^{k_u-k_\ell}(s_i-\mu_\ell)^+ - (s_i-y_1)^+\right] + \frac{k_\ell}{\mu_\ell}(\mu_\ell - y_1) \qquad (18.C)$$

For $y_1 < 1$, we always have

$$I(\hat{\theta}_2) = nk_\ell y_2 \qquad (19.C)$$

Finally, Eqs. (5.C), (8.C), (11.C), (18.C) and (19.C) yield the results stated in (36), (37), and (38).

Appendix D

We derive the optimum detector DET.4. The marginal of t under H_1 which we denote by f(t), can be written as:

$$f(t) = \iint f(t|\mu,k)\pi(k)\pi(\mu)\,dk\,d\mu \qquad (1.D)$$

where $\pi(k)$ and $\pi(\mu)$ are the priors on k and μ respectively. Consequently, we have

$$f(t) = \frac{1}{(k_u-k_\ell+1)(\mu_u-\mu_\ell)} \sum_{k=k_\ell}^{k_u} \int_{\mu_\ell}^{\mu_u} \left(\frac{k}{\mu}\right)^{nk} \frac{1}{\Gamma^n(k)} \exp\left(-\frac{k}{\mu}t_1+kt_2+B(t)\right)d\mu \qquad (2.D)$$

Let

$$J = \int_{\mu_\ell}^{\mu_u} \exp\left(-\frac{k}{\mu}t_1\right)\frac{1}{\mu^{nk}}\,d\mu \qquad (3.D)$$

We integrate by parts:

$$J = \left[\exp\left(-\frac{k}{\mu}t_1\right)\frac{1}{t_1 k\mu^{nk-2}}\right]_{\mu_\ell}^{\mu_u} + \frac{nk-2}{kt_1}\int_{\mu_\ell}^{\mu_u} \exp\left(-\frac{k}{\mu}t_1\right)\frac{1}{\mu^{nk-1}}\,d\mu \qquad (4.D)$$

Iterating the integrations by parts and denoting

$$z = \frac{kt_1}{\mu}$$

one gets

$$J = \frac{(nk-2)!}{(kt_1)^{nk-1}}\left[e^{-z}\sum_{i=0}^{nk-2}\frac{z^i}{i!}\right]_{\frac{kt_1}{\mu_\ell}}^{\frac{kt_1}{\mu_u}} \qquad (5.D)$$

Hence, substituting (5.D) into (2.D) and using (6), the likelihood-ratio which we denote by L(t), is equal to:

$$L(t) \triangleq \frac{f(t)}{f(t|\theta_0)} = C \exp\frac{t_1}{\mu_0} \sum_{k=k_\ell}^{k_u} \frac{k(nk-2)!}{t_1^{nk-1}\Gamma^n(k)} \exp[(k-1)t_2]$$

$$\times \left[e^{-z} \sum_{i=0}^{nk-2} \frac{z^i}{i!} \right]_{\frac{kt_1}{\mu_\ell}}^{\frac{kt_1}{\mu_u}} \tag{6.D}$$

where

$$C = \frac{\mu_0^{\,n}}{(k_u-k_\ell+1)(\mu_u-\mu_\ell)}$$

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