AD-A058 810 ROYAL AIRCRAFT ESTABLISHMENT FARNBOROUGH (ENGLAND) F/G 8/5
EVALUATION OF 14TH-ORDER HARMONICS IN THE GEOPOTENTIAL.(U)
FEB 78 D G KING-HELE, D M WALKER, R H GOODING

UNCLASSIFIED RAE-TR-78015 DRIC-BR-62474 NL

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ROYAL AIRCRAFT ESTABLISHMENT

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Technical Report 78015

February 1978

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UDC 521.6 : 517.564.4

ROYAL AIRCRAFT ESTABLISHMENT

(9) Technical Report 78015 (14) RAE-TR-78015

Received for printing 1 Feb[redacted]78

(11)

(6) EVALUATION OF 14th-ORDER HARMONICS IN THE GEOPOTENTIAL (12) 40p.

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(18) DRIC (19) BR-62474

SUMMARY

The Earth's gravitational potential is now usually expressed in terms of a double series of tesseral harmonics with several hundred terms, up to order and degree at least 20. The harmonics of order 14 can be evaluated by analysing changes in satellite orbits which experience 14th-order resonance, when the track over the Earth repeats after 14 revolutions.

In this Report we describe our first evaluation of individual 14th-order coefficients in the geopotential from analysis of the variations in inclination and eccentricity of satellite orbits passing through 14th-order resonance under the action of air drag. Using results from eleven satellites, we find the following values for normalized coefficients of harmonics of order 14 and degree ℓ, $\bar{C}_{\ell,14}$ and $\bar{S}_{\ell,14}$, for $\ell = 14, 15 \ldots 22$:

ℓ	$10^9\bar{C}_{\ell,14}$	$10^9\bar{S}_{\ell,14}$
14	-38.5 ± 2.9	-7.8 ± 2.2
15	4.5 ± 1.1	-23.8 ± 0.3
16	-22.3 ± 3.6	-36.0 ± 3.8
17	-15.0 ± 2.6	16.8 ± 1.2
18	-24.0 ± 4.9	-3.2 ± 3.7
19	-1.6 ± 2.8	-7.6 ± 1.0
20	8.8 ± 5.8	-15.4 ± 4.6
21	18.2 ± 3.6	-10.6 ± 1.9
22	-14.5 ± 8.1	9.9 ± 6.4

These values provide a test of the accuracy of the 14th-order coefficients in comprehensive geoid models. Detailed comparisons with three recent models are made, showing good agreement on some coefficients and discrepancies on others.

Departmental Reference: Space 545

310450

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1 INTRODUCTION

One of the finest achievements in the Earth sciences in the past few years has been the accurate determination of the shape of the geoid, obtained by assuming that the geopotential can be expanded in a double series of tesseral harmonics, then analysing hundreds of thousands of satellite observations and solving the equations to obtain values for hundreds of the harmonic coefficients. Examples of such comprehensive geoid models already published include the Goddard Earth Model 8 (GEM 8)[1], the Smithsonian Standard Earth IV.3 (SSE IV.3)[2] and the European GRIM 2 model[3]. The accuracy of the solutions has been improving rapidly as more laser ranging data are included: the most recent Goddard Earth Model[4], GEM 10, includes 598 geopotential coefficients and is complete to degree and order 21; GEM 10 utilizes 840000 observations including 200000 laser ranges (many on Geos 3 and Lageos), and gives a geoid which may be accurate to about 1 metre over most of the world.

Though the overall geoid is well determined in these comprehensive solutions, most of the values of individual geopotential coefficients of degree higher than about 12 are poorly determined: in GEM 7, Wagner[5] estimated that the errors exceed 30% for degrees higher than 12. The values of coefficients of a particular order can, however, be accurately determined by analysing the orbits of satellites which experience resonance of that order, as their orbits slowly contract under the influence of air drag. For example, 15th-order resonance occurs when the Earth spins once relative to the orbital plane while the satellite completes 15 revolutions. The satellite then crosses the equator at intervals of 24° in longitude, and its track over the Earth repeats after 15 revolutions. The 15th-order harmonics in the geopotential have maxima every 24° in longitude, and therefore have the same effect on the satellite on every revolution. Consequently the perturbations which they produce tend to build up, and if near-resonance continues for several months, the resulting changes in the orbital parameters (particularly the orbital inclination) are large enough to be analysed, and to give an accurate value for a 'lumped harmonic' appropriate for that orbit - the lumped harmonic being a linear function of the individual harmonics of order 15 and degree 15, 17, 19 .... (if the inclination is being analysed). By obtaining such values from orbits at differing inclinations, the values of the individual coefficients can be determined. The 15th-order resonance is particularly fruitful, because it occurs when the average height of a satellite over the Earth is about 500 km, and numerous satellites are 'dragged through' 15th-order resonance.
015 Three years ago we analysed the changes at 15th-order resonance for 13 satellites,

obtaining[6,7] values for coefficients of order 15 and degree 15, 17, 19 ....33 (from variations in inclination) and coefficients of degree 16, 18, 20 and 22 (from variations in eccentricity). These sets of values for 15th-order harmonic coefficients were better than the conflicting sets in the comprehensive geoid models existing at that time, and have subsequently been incorporated in GRIM 2.

The success of this analysis, which also yielded tentative values for 30th-order harmonics, encouraged the idea of studying other resonances. We looked at the feasibility of utilizing 'half resonances', such as 29:2 (when the satellite makes 29 revolutions while the Earth spins twice), and in 1976 Walker[8] succeeded in analysing 29:2 resonance for Ariel 1, and obtained values for lumped 29th-order harmonics; while Hiller and King-Hele[9] analysed 31:2 resonance for Proton 4, though with less accurate results, because of the high drag.

The other obvious resonances to study are 14th and 16th order. The latter is difficult because 16th-order resonance occurs when the average height of the satellite is near 200 km, when the drag is usually severe, and the resonance does not last long enough to build up a measurable perturbation. For 14th-order resonance the average height is 800 km, and the problem is lack of drag: the satellites linger on the brink of resonance for many years before experiencing the effects fully. After waiting for two years, we now have four satellites which give good results at 14th-order resonance and, by combining these with three analysed by Wagner[5] and results from four high-drag satellites, we have evaluated 14th-order coefficients of degree 14, 15, 16 ....22, with tentative values for higher degrees.

2 THEORETICAL EQUATIONS FOR ORBITAL CHANGES NEAR 14TH-ORDER RESONANCE

If we accept that a double infinite series of tesseral harmonics is the best representation of the geopotential,[10] its longitude-dependent part can be written in normalized form[11] at an exterior point (r,θ,λ) as

$$\frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\left(\frac{R}{r}\right)^{\ell} P_{\ell}^{m}(\cos\theta)\left\{\bar{C}_{\ell m}\cos m\lambda+\bar{S}_{\ell m}\sin m\lambda\right\}N_{\ell m}\quad , \qquad (1)$$

where r is the distance from the Earth's centre, θ is co-latitude, λ is longitude (positive to the east), μ is the gravitational constant for the Earth (398601 km^3/s^2), R is the Earth's equatorial radius (6378.1 km), $P_{\ell}^{m}(\cos\theta)$ is the associated Legendre function of order m and degree ℓ, and $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$ are the normalized tesseral harmonic coefficients, of which 015 those of order $m = 14$ particularly concern us here. The normalizing factor $N_{\ell m}$ is given by[11]

$$N_{\ell m}^2 = \frac{2(2\ell + 1)(\ell - m)!}{(\ell + m)!} . \qquad (2)$$

The rate of change of inclination i caused by a relevant pair of coefficients, $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$, near $\beta:\alpha$ resonance may be written[12]

$$\frac{di}{dt} = \frac{n}{\sin i}\left(\frac{R}{a}\right)^{\ell} \bar{F}_{\ell mp} G_{\ell pq} (k \cos i - m) \mathcal{R}\left[j^{\ell - m + 1}(\bar{C}_{\ell m} - j\bar{S}_{\ell m}) \exp\{j(\gamma\Phi - q\omega)\}\right] , \qquad \ldots\ldots (3)$$

where $\bar{F}_{\ell mp}$ is Allan's normalized inclination function[13], $G_{\ell pq}$ is a function of eccentricity e for which explicit forms have been derived by Gooding[12], $\mathcal{R}$ denotes 'real part of' and $j = \sqrt{-1}$. The resonance angle Φ is defined by the equation

$$\Phi = \alpha(\omega + M) + \beta(\Omega - \nu) , \qquad (4)$$

where ω is the argument of perigee, M the mean anomaly, Ω the right ascension of the node and ν the sidereal angle. The indices γ, q, k and p in equation (3) are integers, with γ taking the values 1, 2, 3 .... and q the values 0, ±1, ±2, ...; the equations linking ℓ, m, k and p are[12]: $m = \gamma\beta$; $k = \gamma\alpha - q$; $2p = \ell - k$.

Here $\beta = 14$ and $\alpha = 1$, and the m-suffix of a relevant $(\bar{C}_{\ell m}, \bar{S}_{\ell m})$ pair is given uniquely by the choice of γ. The values of ℓ to be taken must be such that $\ell \geqslant m$ and $(\ell - k)$ is even. The successive coefficients which arise (for given γ and q) may usefully be gathered together in a lumped form and written as[12]

$$\bar{C}_m^{q,k} = \sum_{\ell} Q_\ell^{q,k} \bar{C}_{\ell m} , \qquad \bar{S}_m^{q,k} = \sum_{\ell} Q_\ell^{q,k} \bar{S}_{\ell m} , \qquad (5)$$

where ℓ increases in steps of 2 from its minimum permissible value ℓ_0, and the Q_ℓ are constant coefficients with $Q_{\ell 0} = 1$.

For the 14:1 resonance the most important terms in equation (3) are likely to be those with $\gamma = 1$, because $\gamma = 2$ gives $m = \gamma\beta = 28$, and the 28th-order
015 harmonics are expected to have a much smaller effect. Of the terms with $\gamma = 1$, those with $q = 0$, 1 and -1 are likely to be the most important, since terms with $q = \pm 2$ have an extra e factor, and $e < 0.08$ for the satellites analysed here. So, for the present, we concentrate on the terms with

$(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$. With $\gamma = 1$, we have $m = 14$ and $k = 1 - q$, so that the three pairs of values of (γ,q) above are associated with the following three pairs of values of $[q,k]$: $[0,1]$, $[1,0]$ and $[-1,2]$. With these assumptions, equation (3) may be rewritten for 14th-order resonance as[12-14]

$$\frac{di}{dt} = \frac{n}{\sin i}\left(\frac{R}{a}\right)^{14}\left[\frac{R}{a}(14 - \cos i)\bar{F}_{15,14,7}\left\{\bar{S}_{14}^{0,1}\sin\Phi + \bar{C}_{14}^{0,1}\cos\Phi\right\}\right.$$
$$+ \frac{15e}{2}(14)\bar{F}_{14,14,7}\left\{\bar{C}_{14}^{1,0}\sin(\Phi - \omega) - \bar{S}_{14}^{1,0}\cos(\Phi - \omega)\right\}$$
$$+ \frac{11e}{2}(14 - 2\cos i)\bar{F}_{14,14,6}\left\{\bar{C}_{14}^{-1,2}\sin(\Phi + \omega) - \bar{S}_{14}^{-1,2}\cos(\Phi + \omega)\right\}$$
$$\left. + \text{ terms in } e^{|q|}\,{\cos \atop \sin}(\gamma\Phi - q\omega)\right] . \qquad (6)$$

The three pairs of lumped coefficients $\bar{C}_m^{q,k}$ and $\bar{S}_m^{q,k}$ appearing in equation (6) may be written in terms of the individual geopotential coefficients $(\bar{C}_{\ell m},\bar{S}_{\ell m})$ as indicated in equations (5). Explicitly, with the $Q_\ell^{q,k}$ expressed in terms of the $\bar{F}$ functions, we have[12-14]

$$\bar{C}_{14}^{0,1} = \bar{C}_{15,14} - \frac{\bar{F}_{17,14,8}}{\bar{F}_{15,14,7}}\left(\frac{R}{a}\right)^2\bar{C}_{17,14} + \frac{\bar{F}_{19,14,9}}{\bar{F}_{15,14,7}}\left(\frac{R}{a}\right)^4\bar{C}_{19,14} - \ldots. \qquad (7)$$

$$\bar{C}_{14}^{1,0} = \bar{C}_{14,14} - \frac{17\bar{F}_{16,14,8}}{15\bar{F}_{14,14,7}}\left(\frac{R}{a}\right)^2\bar{C}_{16,14} + \frac{19\bar{F}_{18,14,9}}{15\bar{F}_{14,14,7}}\left(\frac{R}{a}\right)^4\bar{C}_{18,14} - \ldots. \qquad (8)$$

$$\bar{C}_{14}^{-1,2} = \bar{C}_{14,14} - \frac{13\bar{F}_{16,14,7}}{11\bar{F}_{14,14,6}}\left(\frac{R}{a}\right)^2\bar{C}_{16,14} + \frac{15\bar{F}_{18,14,8}}{11\bar{F}_{14,14,6}}\left(\frac{R}{a}\right)^4\bar{C}_{18,14} - \ldots. \qquad (9)$$

and similarly for S, on replacing C by S throughout.

The rate of change of eccentricity e caused by the (ℓ,m) harmonic near $\beta:\alpha$ resonance can be written[12]

$$\frac{de}{dt} = n\left(\frac{R}{a}\right)^{\ell}\bar{F}_{\ell mp}G_{\ell pq}\left\{\frac{q - \frac{1}{2}(k + q)e^2}{e}\right\}\mathcal{R}\left[j^{\ell-m+1}(\bar{C}_{\ell m} - jS_{\ell m})\exp j(\gamma\Phi - q\omega)\right] . \qquad \ldots\ldots (10)$$

The most important terms in (10) are likely to be those with $(\gamma,q) = (1,1)$ and $(1,-1)$, but for consistency with equation (6) we also give explicitly the terms with $(\gamma,q) = (1,0)$. Equation (10) may then be written for 14th-order resonance as

$$\begin{aligned}\frac{de}{dt} = \frac{n}{2}\left(\frac{R}{a}\right)^{14}\Bigg[& e\left(\frac{R}{a}\right)\bar{F}_{15,14,7}\left(\bar{S}^{0,1}_{14}\sin\Phi + \bar{C}^{0,1}_{14}\cos\Phi\right) \\ & - 15\bar{F}_{14,14,7}\left\{\bar{C}^{-1,0}_{14}\sin(\Phi-\omega) - \bar{S}^{-1,0}_{14}\cos(\Phi-\omega)\right\} \\ & + 11\bar{F}_{14,14,6}\left\{\bar{C}^{-1,2}_{14}\sin(\Phi+\omega) - \bar{S}^{-1,2}_{14}\cos(\Phi+\omega)\right\} \\ & + \text{terms in}\left[e^{|q|-1}\left\{q - \tfrac{1}{2}(k+q)e^2\right\}\begin{matrix}\cos\\ \sin\end{matrix}(\gamma\Phi - q\omega)\right]\Bigg] \end{aligned} \qquad (11)$$

where the $\bar{C}^{q,k}_{14}$ and $\bar{S}^{q,k}_{14}$ are given by equations (7) to (9).

3 METHOD

For a number of years we have been receiving US Navy orbital data on selected low-drag satellites in near-circular orbits close to 14th-order resonance. With the data now accumulated on two of these satellites, 1965-16G and 1971-120B, accurate fitting of the 14th-order theoretical equations is possible. Useful results are also derived from the US Navy data on 1965-81A, a satellite of higher eccentricity, which passed through resonance in 1970. We first fit the inclination and eccentricity data separately, using the computer program THROE[15], and then make a combined fitting of inclination and eccentricity, using the SIMRES program[12]. Our fourth satellite is Ariel 2 (1964-15A), for which Gooding has analysed the variations in inclination and eccentricity after 14th-order resonance[12].

In determining the coefficients of 14th order and *odd* degree (degree 15, 17, 19 ...), we have utilized data from six other satellites. Of these, the most important are three low-drag satellites accurately analysed by Wagner[5]: 1963-26A; 1961-15G; and 1971-120A before resonance. The other three are high-drag satellites, two analysed by Klokočnik[16] and one by Hiller[17], for which the results are inevitably of poorer accuracy, but still useful in filling gaps in the range of inclinations.

In determining coefficients of 14th order and *even* degree (degree 14, 16, 18 ....), the four primary satellites provide the main data base; but results[18] from the high-drag satellite 1971-106A are also included, and some subsidiary results from Wagner's fittings of 1963-26A and 1961-15G are used.

4 THE INDIVIDUAL SATELLITES

4.1 The four primary orbits

4.1.1 Surcal 2, 1965-16G

Surcal 2 was one of eight small satellites launched on 9 March 1965 into near-circular orbits having inclinations near 70° and heights near 920 km. Of the eight satellites, 1965-16G has decayed most rapidly: its period decreased from 103.5 minutes initially to 101.8 minutes in July 1977, and it passed through 14th-order resonance on 13 November 1976.

Values of inclination from US Navy orbits from January 1976 to July 1977 are plotted in Fig 1, after removal of (1) lunisolar and zonal harmonic perturbations, using the computer program PROD[19] with 1-day integration steps; and (2) the effects of atmospheric rotation and the precession of the pole, using THROE. The unbroken line in Fig 1 shows the theoretical curve fitted by THROE with $(\gamma,q) = (1,0)$, while the broken line shows the fitting of inclination and eccentricity together using SIMRES, with $(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$. The close agreement of the two curves is extremely satisfactory; the rms scatter of the points about the curve, 0.0012°, is slightly smaller than the expected accuracy of the US Navy orbits. The values of the lumped coefficients for $(\gamma,q) = (1,0)$ from the THROE run are virtually the same as from SIMRES. Each THROE run includes a calculation of the measure of fit ε (where ε^2 is the sum of squares of weighted residuals, divided by the number of degrees of freedom). In the SIMRES fitting, i and e were weighted on the basis of the values of ε obtained in the contributing THROE runs. In Fig 1 the value of $\dot{\Phi}$ increases from -2.9 deg/day at the start to +2.8 deg/day at the end.

The values of eccentricity from US Navy orbits are plotted in Fig 2 after removal of zonal harmonic perturbations. The unbroken line shows the curve fitted by THROE with $(\gamma,q) = (1,1)$ and $(1,-1)$, the broken line shows the fitting of i and e together by SIMRES, with $(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$, and the dotted line shows the THROE solution with $(\gamma,q) = (1,0)$ $(1,\pm1)$ and $(2,\pm1)$. The agreement between the first two curves is good, but the points exhibit a curious unfitted oscillation with an amplitude of order 0.0002 and a period of about 75 days. This cannot be due to lunisolar perturbations, which have been calculated and amount to only 0.00001; and there is very little improvement in the fitting on taking into account the terms $(\gamma,q) = (1,\pm2)$ or $(0,3)$. But the addition of the $(\gamma,q) = (2,\pm1)$ terms does give a better fit: the measure of fit ε decreases from 1.10 to 0.94. In the SIMRES solution the rms scatter of the 015 points about the curve is 0.00012, which is rather greater than the expected accuracy of the US Navy values.

Because of these deficiencies in fitting, we looked with some suspicion on the values of the lumped coefficients obtained from SIMRES for the $(\gamma,q) = (1,\pm1)$ terms. This suspicion proved well founded: in the solutions for individual coefficients of even degree, the lumped harmonics did not conform at all well with those obtained from the other satellites, even when three of the four standard deviations were increased by a factor of 3. We therefore looked again at the THROE runs for e and found that the run with $(\gamma,q) = (1,0)$, $(1,\pm1)$ and $(2,\pm1)$ gave values in better conformity with the other satellites. The values of lumped coefficients for $q = \pm 1$ from this run were substituted in the equations (without any modifications in sd) and gave better solutions for the individual coefficients: although only two out of 14 equations were altered, the standard deviations of the solutions were reduced by 15% on average. Our aim is to obtain the best values of the individual coefficients, so this *ex post facto* alteration was welcome. This fitting is presumably better because the $q = \pm 2$ $(m = 28)$ terms are important for this satellite, and should not be allowed to contaminate the $q = \pm 1$ $(m = 14)$ coefficients which we are trying to evaluate.

Tables 1 to 3 record the values of the lumped harmonics as used in the solutions for individual coefficients, for 1965-16G and all the other satellites. The values for 1965-16G in Table 1 are from the SIMRES fitting, the values in Tables 2 and 3 from the THROE fitting of e, as already mentioned. These Tables also give the corresponding values of the $Q_{\ell}^{q,k}$ coefficients defined in equation (5), and similar data for all the other satellites used.

4.1.2 <u>1971-120B (Meteor 10 rocket?)</u>

The Meteor 10 satellite and its rocket were launched on 29 December 1971 into a near-circular orbit at a height near 860 km, at an inclination of 81.25°. In April 1976 three fragments appeared in orbit, and the object 1971-120B, which had previously been approaching 14th-order resonance, reappeared in an orbit which had just passed resonance. Some difference of opinion exists about the identity of 1971-120B, but this is irrelevant in our analysis.

Fig 3 shows the weekly US Navy values of inclination between April 1976 and August 1977, after removal of the usual perturbations (lunisolar, odd harmonic, etc). The unbroken line indicates the theoretical curve for $(\gamma,q) = (1,0)$ fitted by THROE. The rms scatter of the points about the curve is 0.0021. The
015 SIMRES fitting of i and e together, with equal weighting on the basis of the

Table 1

Values of $\bar{C}^{0,1}_{14}$ and $\bar{S}^{0,1}_{14}$ from the $(\gamma,q) = (1,0)$ terms in the final fittings, as used in the solutions, with $Q^{0,1}_{\ell}$ values

Satellite	i (deg)	Q_{15}	Q_{17}	Q_{19}	Q_{21}	Q_{23}	Q_{25}	Q_{27}	Q_{29}	Q_{31}	$10^9\bar{C}^{0,1}_{14}$	$10^9\bar{S}^{0,1}_{14}$	$\bar{F}_{15,14,7}$	$10^9\bar{F}_{15,14,7}\bar{C}^{0,1}_{14}$	$10^9\bar{F}_{15,14,7}\bar{S}^{0,1}_{14}$
1973-22A	48.4	1.0	-2.996	3.930	-2.026	-0.611	1.348	0.046	-0.740	0.069	-63 ± 52	-115 ± 56	0.07340	-4.6 ± 3.8	-8.4 ± 4.1
1963-26A	49.7	1.0	-2.918	3.559	-1.313	-1.334	1.178	0.591	-0.801	-0.345	18 ± 18	-82 ± 11	0.09128	1.6 ± 1.6	-7.5 ± 1.0
1964-15A	51.2	1.0	-2.464	2.280	-0.174	-1.070	0.245	0.561	-0.117	-0.321	75 ± 30	-98 ± 23	0.1226	9.2 ± 3.7	-12.0 ± 2.8
1961-15G	66.8	1.0	-0.251	-0.465	-0.186	0.088	0.170	0.103	0.002	-0.056	5.5 ± 1.0	-22.5 ± 0.5	0.4900	2.7 ± 0.5	-11.0 ± 0.2
1971-18B	69.9	1.0	0.093	-0.302	-0.293	-0.124	0.028	0.094	0.084	0.038	32 ± 22	-2 ± 20	0.5292	17.0 ± 11.5	-1.3 ± 10.5
1965-16G	70.1	1.0	0.113	-0.288	-0.293	-0.134	0.017	0.088	0.084	0.043	-2.1 ± 0.3	-16.6 ± 0.9	0.5303	-1.1 ± 0.2	-8.8 ± 0.5
1973-82A	74.0	1.0	0.475	0.062	-0.149	-0.196	-0.151	-0.077	-0.012	0.029	-46 ± 30	-52 ± 55	0.5127	-24 ± 15	-27 ± 28
1971-120A	81.2	1.0	0.926	0.743	0.544	0.365	0.221	0.113	0.039	-0.007	1.4 ± 2.4	-20.0 ± 1.2	0.2722	0.4 ± 0.3	-5.4 ± 0.3
1971-120B	81.2	1.0	0.925	0.741	0.542	0.364	0.219	0.112	0.039	-0.007	-3.7 ± 2.8	-19.1 ± 1.2	0.2722	-1.0 ± 0.4	-5.2 ± 0.3
(1965-82	32.0	1.0	-5.798	18.638	-41.240	68.274	-87.406	86.287	-62.033	24.986	6800 ± 2700	-3300 ± 1500	0.00123	8.4 ± 3.3	-4.1 ± 1.8)

Table 2

Values of $\bar{C}_{14}^{-1,0}$ and $\bar{S}_{14}^{-1,0}$ from the $(\gamma,q) = (1,1)$ terms in the final fittings, as used in the solutions of equations (14), with values of $Q_{\ell}^{1,0}$

Satellite	i (deg)	Q_{14}	Q_{16}	Q_{18}	Q_{20}	Q_{22}	Q_{24}	Q_{26}	Q_{28}	$10^9\bar{C}_{14}^{-1,0}$	$10^9\bar{S}_{14}^{-1,0}$	$\bar{F}_{14,14,7}$	$10^9\bar{F}_{14,14,7}\bar{C}_{14}^{-1,0}$	$10^9\bar{F}_{14,14,7}\bar{S}_{14}^{-1,0}$
1963-26A	49.7	1.0	-7.381	18.736	-22.411	8.705	8.458	-8.538	-3.120	-493 ± 314	538 ± 314	0.01398	-6.9 ± 4.4	7.5 ± 4.4
1964-15A	51.2	1.0	-6.761	15.095	-14.281	1.078	8.115	-2.493	-4.799	-483 ± 128	490 ± 133	0.02053	-9.9 ± 2.6	10.1 ± 2.7
1971-106A	65.7	1.0	-2.601	0.379	1.409	0.568	-0.439	-0.686	-0.303	17 ± 59	-137 ± 175	0.1690	2.9 ± 9.9	-23 ± 30
1961-15G	66.8	1.0	-2.351	0.026	1.181	0.727	-0.146	-0.564	-0.414	12 ± 14	70 ± 24	0.1886	2.3 ± 2.6	13.2 ± 4.5
1965-16G	70.1	1.0	-1.595	-0.669	0.352	0.673	0.436	0.038	-0.227	17 ± 16	68 ± 51	0.2598	4.4 ± 4.2	17.8 ± 13.2
1971-120B	81.2	1.0	0.172	-0.146	-0.272	-0.292	-0.256	-0.195	-0.128	-34.9 ± 4.7	-12.2 ± 4.0	0.5234	-18.3 ± 2.5	-6.4 ± 2.1
1965-81A	87.4	1.0	0.569	0.387	0.270	0.188	0.129	0.086	0.055	-60.0 ± 6.6	-35.4 ± 11.4	0.6075	-36.4 ± 4.0	-21.5 ± 6.9

Table 3

Values of $\bar{C}_{14}^{-1,2}$ and $\bar{S}_{14}^{-1,2}$ from the $(\gamma,q) = (1,-1)$ terms in the final fittings, as used in the solutions of equations (15), with values of $Q_{\ell}^{-1,2}$

Satellite	i (deg)	Q_{14}	Q_{16}	Q_{18}	Q_{20}	Q_{22}	Q_{24}	Q_{26}	Q_{28}	$10^9\bar{C}_{14}^{-1,2}$	$10^9\bar{S}_{14}^{-1,2}$	$\bar{F}_{14,14,6}$	$10^9\bar{F}_{14,14,6}\bar{C}_{14}^{-1,2}$	$10^9\bar{F}_{14,14,6}\bar{S}_{14}^{-1,2}$
1963-26A	49.7	1.0	-4.869	7.254	-2.939	-3.131	2.589	1.691	-1.740	-29 ± 87	-145 ± 320	0.05695	-1.7 ± 5.0	-8.3 ± 18.1
1964-15A	51.2	1.0	-4.340	5.254	-0.604	-3.034	0.625	1.963	-0.216	-48 ± 71	100 ± 70	0.07669	-3.7 ± 5.4	7.7 ± 5.4
1971-106A	65.7	1.0	-1.008	-0.812	-0.041	0.458	0.451	0.143	-0.149	-186 ± 160	66 ± 42	0.3542	-66 ± 57	23 ± 15
1961-15G	66.8	1.0	-0.826	-0.818	-0.208	0.304	0.432	0.246	-0.021	-41 ± 24	30 ± 12	0.3802	-15.6 ± 9.2	11.4 ± 4.6
1965-16G	70.1	1.0	-0.299	-0.637	-0.495	-0.168	0.116	0.245	0.219	-38 ± 25	-11 ± 43	0.4625	-17 ± 12	-5 ± 20
1971-120B	81.2	1.0	0.623	0.444	0.307	0.193	0.102	0.032	-0.017	-81.4 ± 10.4	-34.9 ± 4.8	0.6224	-50.7 ± 6.5	-21.7 ± 3.0
1965-81A	87.4	1.0	0.512	0.333	0.237	0.179	0.140	0.114	0.094	-52.3 ± 8.0	12.1 ± 34.4	0.5831	-30.5 ± 4.6	7.1 ± 20.0

THROE runs, is shown as a broken line in the few regions where it differs perceptibly from the unbroken line. The value of $\dot{\Phi}$ increases from 0.34 deg/day initially to 0.58 deg/day at the end. The fitting of inclination, and the agreement between the two curves, is excellent. The values for the lumped coefficients obtained from SIMRES for the $(\gamma,q) = (1,0)$ terms are

$$10^9 \bar{C}_{14}^{0,1} = -3.7 \pm 1.4 \quad \text{and} \quad 10^9 \bar{S}_{14}^{0,1} = -19.1 \pm 1.2 \ .$$

Wagner[5] obtained values of these lumped harmonics for 1971-120A before resonance. His values are 1.4 ± 1.2 and -20.0 ± 1.2 . The agreement is excellent, considering that different 'wings' of the resonance are being analysed, and that neither analysis covers the exact resonance. (In the final solution the sd was doubled on both values of $\bar{C}_{14}^{0,1}$, to avoid their clashing with each other.)

The values of eccentricity for 1971-120B, after removal of odd zonal harmonic perturbations, are plotted in Fig 4, and the unbroken line shows the theoretical curve for $(\gamma,q) = (1,1)$ and $(1,-1)$ as fitted by THROE. The SIMRES fitting of i and e together gives an almost identical result. Fig 4 shows that the fitting is extraordinarily good, the rms scatter of the points about the curve being 0.000035, which is smaller than the likely errors in the values.

4.1.3 OGO 2, 1965-81A

Launched on 14 October 1965, OGO 2 entered an eccentric orbit with perigee height 420 km and apogee 1520 km, at an inclination of 87.4°. Because of its low perigee, 1965-81A suffered much greater drag than the two satellites previously discussed: the period, initially 104.4 minutes, steadily decreased under the influence of air drag, and the orbit passed through 14th-order resonance on 17 June 1970.

In equation (6) for di/dt , the first term in curly brackets (for $q = 0$) is usually dominant. But 1965-81A happens to be near the inclination (86.2°) where $\bar{F}_{15,14,7}$ is zero; also the eccentricity is quite large, 0.064, and consequently the $q = \pm 1$ terms are larger. Numerically, we find

$$\left\{\frac{R}{a}(14 - \cos i)\bar{F}_{15,14,7}\right\} \Big/ \left\{\frac{15e}{2}(14)\bar{F}_{14,14,7}\right\} = 0.2 \ .$$

Also it happens that at this inclination $\bar{C}^{-1,0}_{14}$ and $\bar{S}^{-1,0}_{14}$ are about five times larger numerically than $\bar{C}^{0,1}_{14}$ and $\bar{S}^{0,1}_{14}$. So we might expect that the $(\gamma,q) = (1,0)$ terms in (6) would be negligible. This expectation is fulfilled. Fitting i with $(\gamma,q) = (1,0)$ alone gives poor results; a much better fitting is obtained with $(1,1)$ and $(1,-1)$; and the use of all three pairs gives no advantage.

So for 1965-81A, both the inclination and eccentricity are best fitted by using the terms $(\gamma,q) = (1,1)$ and $(1,-1)$ only. Both orbital parameters therefore provide values of the *same* lumped coefficients, and this satellite provides a severe test of their compatibility. Table 4 gives values of these four lumped coefficients obtained from the fitting of i alone by THROE, e alone by THROE and i and e by SIMRES. The agreement is as good as can be expected from a fitting of four coefficients to only 26 data points.

Figs 5 and 6 show the observational values and the fitted curves. The fittings of i in Fig 5 are quite satisfactory, but Fig 6 gives the impression that the fitting of e alone is rather too oscillatory, and that the SIMRES fitting is not oscillatory enough. This is in conformity with Table 4, where all the values from the SIMRES solution are numerically smaller than those obtained from fitting e alone. All-in-all, the fittings and the lumped coefficients are good, considering the small number of observational values. In Figs 5 and 6 the values of $\dot{\Phi}$ run from -12.4 deg/day at the beginning to $+3.9$ deg/day at the end. The rms scatter of the values about the unbroken curves in Figs 5 and 6 is 0.0012° for i and 0.00007 for e; for the broken curves the corresponding values are 0.0013° and 0.00012.

For 1965-81A the possibility arises that the terms with $(\gamma,q) = (1,\pm 2)$ may be important. Their inclusion was fruitless with i, but did improve the fitting for e; however, too many constants (nine in all) were being determined from too few points (26), so it was no surprise to find that nearly all the lumped coefficients were undetermined and apparently much too large. To assess the effects of the relevant lumped coefficients, $(\bar{C},\bar{S})^{2,-1}_{14}$ and $(\bar{C},\bar{S})^{-2,3}_{14}$, their values were computed from the values of the individual coefficients of degree 15, 17, 19 and 21 (Table 5), and these values were used with THROE to calculate the effect of the $(\gamma,q) = (1,\pm 2)$ terms on e. The effect was found to be very small, the maximum change in e during the run being 0.000012. So the $(\gamma,q) = (1,\pm 2)$ terms were ignored, and we adopted the SIMRES solution for $(\gamma,q) = (1,\pm 1)$ given in Table 4. In the final solution the sd of $\bar{S}^{-1,2}_{14}$ was doubled, but the other
015 values fitted well.

Table 4

Values of lumped coefficients obtained from 1965-81A by fitting i alone, e alone, or i and e together

	From i alone THROE	From e alone THROE	From e and i SIMRES
$10^9\bar{C}_{14}^{1,0}$	-55 ± 7	-152 ± 18	-60 ± 7
$\bar{S}_{14}^{1,0}$	-32 ± 9	-92 ± 26	-35 ± 11
$\bar{C}_{14}^{-1,2}$	-43 ± 7	-63 ± 18	-52 ± 8
$\bar{S}_{14}^{-1,2}$	9 ± 17	294 ± 12	12 ± 17

4.1.4 Ariel 2, 1964-15A

Ariel 2, the second Anglo-US satellite, was launched on 27 March 1964 into an orbit with inclination 51.64° with period 101.3 minutes. Initially, the heights of perigee and apogee were 290 and 1360 km respectively, and the eccentricity was 0.075. Using Minitrack observations, Gooding[20] computed the orbit at 210 epochs during the first year after launch, and has now[12] determined values of lumped 14th-order harmonics by analysing the variations in inclination and eccentricity; the orbit was already post-resonant at launch. Because of the relatively high eccentricity and the multitude of values, the best fitting is obtained with eight pairs of values of (γ,q) - the basic trio and five other pairs. The values of inclination and eccentricity, and the curves fitted by SIMRES, are shown in Figs 7 and 8. Although the fittings do not look so convincing as those in Figs 1 to 6, there are far more values available for Ariel 2, and the variations in the orbital parameters extend much further away from resonance, because of the high eccentricity, as also happens for example[21] with Vanguard 3.

For Ariel 2, the Q_{ℓ} coefficients do not fall off so rapidly as for the three previous satellites (see Tables 1 to 3), and the neglect of harmonics of degree greater than 23 may be a source of error. The effect of these higher harmonics was calculated using the values in Tables 5 and 7, and the only sd needing to be increased for this reason was that of $\bar{C}_{14}^{0,1}$, which was increased from 23 to 30 × 10^{-9} . The numerical values of the lumped harmonics as used in the solutions are given in Tables 1 to 3.

4.2 The orbits analysed by Wagner

In a thorough assessment of the accuracy of the Goddard Earth Models, Wagner[5] has analysed the 14th-order resonant variations in inclination (but not eccentricity) of three satellites relevant to our work. The first of these three orbits is that of 1963-26A, which passed through 14th-order resonance in March 1968, with inclination 49.7° and eccentricity 0.057. The second orbit is that of 1961-15G, a fragment from the exploded rocket of Transit 4A; this fragment passed through 14th-order resonance in September 1971, with inclination 66.8° and eccentricity 0.017. The third satellite is 1971-120A, Meteor 10, which Wagner analysed for the years 1972-1975, some years before it was due to reach resonance. As already mentioned, Wagner's values are close to those we have obtained from 1971-120B after resonance, and the two satellites, which are in nearly identical orbits, should together define a reliable mean value.

Wagner's values for the $q = 0$ coefficients from these three satellites are given in Table 1, the standard deviation for $\bar{C}_{14}^{0,1}$ for 1971-120A being doubled, as mentioned in section 4.1.2.

A fourth orbit analysed by Wagner, or rather a set of orbits of several fragments from the 1965-82 launch at inclination 32°, has been used to strengthen our 8-coefficient solution for odd-degree coefficients. For 1965-82 the values of the Q_ℓ are largest for $\ell = 23$ to 29, and the geopotential coefficients of these degrees will make a dominant contribution to the lumped coefficient. So 1965-82 is unsuitable unless at least eight coefficients are being evaluated. Numerical data appear at the end of Table 1.

For 1963-26A and 1961-15G, Wagner also obtains values for the $q = \pm 1$ coefficients. These are given in Tables 2 and 3, but the standard deviations obtained by Wagner[5] have been doubled, because his values are derived from the variation in inclination only, and are not combined inclination-eccentricity fittings, as our $q = \pm 1$ values are.

4.3 The four high-drag orbits

Klokočnik[16] has analysed the 14th-order resonant variations in inclination (but not eccentricity) for two satellites with perigee heights near 200 km, namely Intercosmos 9, 1973-22A, inclination 48.4°, and Intercosmos 10, 1973-82A, inclination 74°. Because of the high drag, the values of the lumped harmonics are rather inaccurate, and in the solution it was necessary to increase the sd
015 of 1973-22A by a factor of 2 and the sd of 1973-82A by a factor of 5.

Hiller[17] has recently analysed the 14th-order resonant variation of inclination and eccentricity for China 2 rocket, 1971-18B, at inclination 69.9°, but again the orbit did not remain near resonance for long enough to allow an accurate analysis, and it was necessary to increase the sd of $\bar{C}^{0,1}_{14}$ by a factor of 2.

Despite their limitations, the results from these three orbits helped to fill gaps in the coverage of inclination, in the odd-degree solutions. The values of the lumped coefficients from the three orbits are given in Table 1.

The fourth high-drag orbit used is that of Cosmos 462, 1971-106A, with perigee height 240 km, which has been analysed by Walker[18]. This orbit has some similarities with 1965-81A. The $q = 0$ terms are found to have little effect, and the analysis yields values of the $q = \pm 1$ coefficients, which are given in Tables 2 and 3. In the analysis of 1971-106A, the perturbations in i and e due to $q = \pm 2$ terms were calculated, using lumped coefficients evaluated from the solutions of Table 5, and the values of i and e were then cleared of these perturbations. This is a refinement which should always be used in resonance analyses if reliable lumped harmonics can be pre-calculated.

4.4 Data not used

Many satellites move in orbits which are close to 14th-order resonance but do not experience exact resonance. These orbits in 'shallow resonance' (as it is often called) have been extensively studied over the years and utilized in the comprehensive geoid models. Also Reigber and Balmino[22] have obtained solutions for the individual coefficients of 14th order up to degree 30, from analysis of a number of shallow resonant orbits. Their results have been incorporated in the GRIM 2 model; so we thought it best not to mix their values with ours, but to obtain an independent solution for comparison.

5 THE SOLUTIONS

5.1 Solutions for odd-degree coefficients (ℓ = 15, 17, 19 ...)

The fittings of the $(\gamma, q) = (1,0)$ terms give us nine values of the lumped coefficients with $[q,k] = [0,1]$, listed in Table 1. Inserting these values in equations (5), we obtain nine equations for 14th-order C-coefficients of odd degree,

$$\bar{C}_{15,14} + Q^{0,1}_{17}\bar{C}_{17,14} + Q^{0,1}_{19}\bar{C}_{19,14} + \ldots = \bar{C}^{0,1}_{14} \qquad (12)$$

with nine similar equations for the coefficients $\bar{S}^{0,1}_{14}$. The numerical values of the Q_ℓ terms and lumped coefficients for each satellite are given in the first nine rows of Table 1.

We follow the method which proved successful in solving for 15th-order coefficients. To the nine equations of the form (12) we add constraint equations of the form

$$\bar{C}_{\ell,14} = 0 \pm 10^{-5}/\ell^2 \tag{13}$$

and then solve by least squares for 3, 4, 5 ....r coefficients from 12, 13, 14 .... 14 ....(9 + r) equations. The value of r is chosen empirically after examination of the solutions, to give what is believed to be the best set of values for the individual coefficients. The quantity $10^{-5}/\ell^2$ derives[11] from "Kaula's rule of thumb" for the order of magnitude of a coefficient of degree ℓ. Now that individual coefficients are being evaluated with better accuracy, it is becoming apparent that, at least for $12 < \ell < 30$, a better average value for the coefficients would be about $0.7 \times 10^{-5}/\ell^2$, with $10^{-5}/\ell^2$ itself regarded rather as an upper limit. We found that the standard deviations of our solutions were slightly improved by making this 30% reduction in the right-hand side of equation (13); but the values of the coefficients were little altered, so it seemed better to allow the more relaxed constraint, as given in (13).

In the course of the solutions of equations (12) and (13), some values of lumped harmonics were found to be persistently ill-fitting. We tried doubling their assumed standard deviations, and if the solution was significantly improved, the doubled sd was used*. The changes made have already been mentioned; the values given in Table 1 incorporate the increased sd and are those used in the solutions.

When the nine equations itemized in Table 1, together with the constraint equations (13), were solved by least squares for 3, 4, 5 ....9 coefficients, the values of ε, the measure of fit, were 2.13, 1.08, 1.07, 1.07, 1.06, 1.06, 1.06 for the C-coefficients, and 1.23, 0.59, 0.59, 0.58, 0.58, 0.58, 0.58, for the S-coefficients. (As before, ε^2 is defined as the sum of the squares of the weighted residuals divided by the number of degrees of freedom, which here is always 9.) Since ε fails to decrease when the number of coefficients evaluated goes beyond 4, the 4-coefficient solutions, for ℓ = 15, 17, 19 and 21, are recommended as best, and are underlined in Table 5.

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* Purists sometimes query the legitimacy of this procedure. Our aim is to derive the best possible values of the geopotential coefficients: this aim is not helped by giving too high a weight to inaccurate data. The inaccurate values
015 could be omitted altogether; but this would imply a judgment that zero weighting is optimal, which is unlikely to be true unless the data are entirely spurious. Our procedure is an attempt to optimize the weighting empirically.

The 8-coefficient solutions are also given in Table 5, however, for two reasons: (a) to show how the values of the first four coefficients are affected by inclusion of higher harmonics; and (b) to provide a comparison with GEM 10 and GRIM 2, which go to degree 29. When solving for eight coefficients, we include a further equation, that from the 1965-82 satellites at 32° inclination, given in the last row of Table 1.

Table 5

Solutions for geopotential harmonics of order 14 and odd degree

	$10^9 \bar{C}_{\ell,14}$					$10^9 \bar{S}_{\ell,14}$				
ℓ	4-coeff	8-coeff	GEM 10	GRIM 2	SSE IV	4-coeff	8-coeff	GEM 10	GRIM 2	SSE IV
15	4.5 ± 1.1	5.1 ± 3.8	3.9	2.2	4.8	-23.8 ± 0.3	-24.2 ± 1.8	-24.6	-24.3	-33.8
17	-15.0 ± 2.6	-17.4 ± 6.0	-15.9	-26.3	-13.7	16.8 ± 1.2	17.7 ± 2.7	10.9	9.2	13.2
19	-1.6 ± 2.8	0.9 ± 4.2	-5.8	-7.6	-3.2	-7.6 ± 1.0	-7.9 ± 1.9	-12.6	-12.1	-4.4
21	18.2 ± 3.6	15.9 ± 12.4	19.7	0	9.5	-10.6 ± 1.9	-11.5 ± 5.7	10.3	-7.1	14.0
23		8 ± 10	.8	9	17		0 ± 5	-5	20	-21
25		-16 ± 12	-23	17	-22		1 ± 6	17	-2	21
27		14 ± 13	23	-22	17		2 ± 7	6	15	-5
29		-2 ± 12	-11	17			-2 ± 6	10	15	

Table 6

Weighted residuals for each satellite in the 4-coefficient solution of Table 5

Satellite	Residuals for $\bar{C}^{0,1}_{14}$	$\bar{S}^{0,1}_{14}$
1973-22A	-1.33	-0.58
1963-26A	-0.04	0.35
1964-15A	1.34	-0.76
1961-15G	-0.09	-0.00
1971-18B	1.55	0.72
1965-16G	0.02	-0.01
1973-82A	-1.35	-0.68
1971-120A	0.90	-0.37
1971-120B	-1.05	0.36

The weighted residual, that is

(observational value minus computed value) ÷ (standard deviation),

for each satellite in the 4-coefficient solution is given in Table 6. The residuals for the 8-coefficient solution are very similar. It is obvious from Table 6 that the formal standard deviations of the solutions for the C-coefficients could be reduced by omitting the three high-drag satellites. When the equations are solved with these three satellites omitted, it is found that the values of the C-coefficients remain within ½ sd of those in Table 5, but the standard deviations are reduced by 20% on average. Although this solution is formally more accurate, it is probably less reliable because of the wider gaps in the coverage of inclination, so we prefer the solution given in Table 5.

Equation (6) shows that a truer measure of the strength of the resonance effects is given not by the lumped coefficients alone but by these coefficients multiplied by $\bar{F}_{15,14,7}$, and values of the lumped coefficients multiplied in this way are given in the last two columns of Table 1 (which also shows that the most accurate results are those from 1961-15G, 1965-16G and 1971-120).

The values of the lumped coefficients multiplied by $\bar{F}_{15,14,7}$ are also plotted in Fig 9, with curves showing the values given by the 4-coefficient solutions. It is seen that the solutions are entirely satisfactory, in the sense that there are no strong oscillations in the gaps between the data points. The points indicated by the triangles in Fig 9 are at an inclination of 86.2°, where $\bar{F}_{15,14,7} = 0$: a satellite in a circular orbit at this inclination would suffer no perturbation in inclination as it passes 14th-order resonance (except possibly from 28th-order coefficients).

5.2 Solutions for even-degree coefficients (ℓ = 14, 16, 18 ...)

For the orbits of the four basic satellites and 1971-106A, the $(\gamma,q) = (1,\pm1)$ terms in the fittings of the variations in eccentricity and inclination have given two equations each for the 14th-order coefficients of even degree, of the form

$$\bar{C}_{14,14} + Q_{16}^{1,0}\bar{C}_{16,14} + Q_{18}^{1,0}\bar{C}_{18,14} + \dots = \bar{C}_{14}^{1,0} \qquad (14)$$

$$\bar{C}_{14,14} + Q_{16}^{-1,2}\bar{C}_{16,14} + Q_{18}^{-1,2}\bar{C}_{18,14} + \dots = \bar{C}_{14}^{-1,2} \qquad (15)$$

and similarly for S , on replacing C by S throughout. The numerical values of the Q_ℓ and the lumped harmonics on the right-hand side, for each of these satellites and for the two satellites of Wagner, are given in Table 2, for equation (14), and Table 3, for equation (15). To these 14 equations for $\bar{C}_{\ell,14}$ (and 14 for $\bar{S}_{\ell,14}$) we add the constraint equations (13), and solve for r coefficients from (14 + r) equations, with r = 3, 4, ...9 .

With 14 equations to be fitted simultaneously, there is little scope for the values of the coefficients to adjust themselves to fit non-conforming lumped values. So it was no surprise to find a number of obstinately ill-fitting lumped coefficients, for which it was advantageous to increase the sd . They were as follows. The sd of $\bar{C}_{14}^{-1,2}$ was doubled for 1961-15G and 1971-120B, and multiplied by 10 for 1971-106A. The sd of $\bar{S}_{14}^{1,0}$ was multiplied by 5 for 1971-106A. The sd of $\bar{S}_{14}^{-1,2}$ was doubled for 1963-26A and 1965-81A. The values given in Tables 2 and 3 are those used in the solutions, after these increases in sd have been made.

The 14 equations of type (14) or (15), itemized in Tables 2 and 3, plus the constraint equations (13), were solved by least squares for 3, 4 ....9 coefficients. The values of ε , the measure of fit, were 1.03, 1.01, 0.91, 0.89, 0.89, 0.89, 0.89 for the C-coefficients and 0.92, 0.74, 0.69, 0.68, 0.68, 0.68, 0.68 for the S-coefficients. For both C and S , therefore, ε decreases significantly - by 10% and 7% respectively - on going from the 4- to the 5-coefficient solution, but does not decrease significantly thereafter. So the 5-coefficient solutions, for ℓ = 14, 16, 18, 20 and 22, are recommended as the best, and are given underlined in Table 7. The 9-coefficient solution is also given in Table 7, however, to show how little the solution is affected by including higher-degree coefficients, and to provide a comparison with GRIM 2.

Equations (6) and (11) show that a better measure of the effect of the lumped coefficients is given by $\bar{F}_{14,14,7}\bar{C}_{14}^{1,0}$ and $\bar{F}_{14,14,6}\bar{C}_{14}^{-1,2}$ than by the lumped coefficients alone. The values of these quantities (S as well as C) are listed in the last two columns of Tables 2 and 3, and the values are plotted in Figs 10 and 11, with the curves given by the 5-coefficient solution. From Fig 11 alone it may seem at first sight that the curves are making rather a poor job of fitting the points; but the fit is of course simultaneous for the C (or S) values in both Fig 10 and Fig 11, so that the form of the curve is a compromise between their conflicting claims.

Table 7

Solutions for geopotential harmonics of order 14 and even degree

	$10^9\bar{C}_{\ell,14}$					$10^9\bar{S}_{\ell,14}$				
ℓ	5-coeff	9-coeff	GEM 10	GRIM 2	SSE IV	5-coeff	9-coeff	GEM 10	GRIM 2	SSE IV
14	-38.5 ± 2.9	-38.9 ± 3.1	-51.2	-69.3	-56.3	-7.8 ± 2.2	-8.1 ± 2.4	-5.4	-1.2	-3.2
16	-22.3 ± 3.6	-22.6 ± 4.2	-18.8	-29.8	-20.1	-36.0 ± 3.8	-35.3 ± 4.0	-37.9	-40.6	-33.3
18	-24.0 ± 4.9	-21.6 ± 5.8	-8.0	-10.1	0.6	-3.2 ± 3.7	-4.4 ± 4.2	-10.1	-23.8	-27.2
20	8.8 ± 5.8	7.4 ± 6.7	13.1	8.0	17.3	-15.4 ± 4.6	-13.8 ± 5.3	-10.6	-44.5	-26.6
22	-14.5 ± 8.1	-11.2 ± 9.9	9.7	-23.7	-0.8	9.9 ± 6.4	7.2 ± 8.0	6.2	-1.2	6.6
24		-8 ± 12	-18	18	-44		5 ± 9	4	5	-28
26		0 ± 11	7	25			-3 ± 8	1	-9	
28		1 ± 11	-7	-19			-2 ± 8	-11	-9	
30		0 ± 10		11			0 ± 7		7	

Table 8

Weighted residuals for each satellite in the 5-coefficient solution of Table 7

Satellite	Residuals for $\bar{C}^{-1,0}_{14}$	$\bar{S}^{-1,0}_{14}$	$\bar{C}^{-1,2}_{14}$	$\bar{S}^{-1,2}_{14}$
1963-26A	0.49	-0.29	0.64	-0.95
1964-15A	-0.72	0.54	-0.26	-0.15
1971-106A	0.04	-1.17	-1.14	0.71
1961-15G	-0.07	0.18	-1.43	-0.07
1965-16G	0.68	0.30	-0.76	-0.51
1971-120B	0.45	0.01	-1.75	-0.08
1965-81A	0.13	-0.31	0.77	1.20

6 COMPARISONS

6.1 Odd-degree coefficients

Comparison between the 4-coefficient and the 8-coefficient solutions in
015 Table 5 shows that the solutions are extremely stable. The values in the 8-coefficient solution differ from the corresponding values in the 4-coefficient solution by less than the standard deviation of the latter (except for $\bar{S}_{15,14}$, where the sd is unusually small). This stability is a very welcome feature of the solutions.

Comparison of the 4- and 8-coefficient solutions with GEM 10 shows good agreement among the C coefficients (within 1.5 sd); but our S coefficients, which have very low standard deviations, do not agree so well with GEM 10 (except for ℓ = 15, where the difference is only 3%).

Comparison with GRIM 2 shows poor agreement with C coefficients and better agreement than GEM 10 for the S coefficients.

Comparison with SSE IV.3 shows quite good agreement with our solutions for the C coefficients, but considerable divergences for the S coefficients.

Our value and those from the three comprehensive geoid solutions are largely independent, but not entirely so, since Wagner's results have some influence on GEM 10, as well as on our values.

The GEM 8 model agrees with our values just about as well as GEM 10. However, no detailed comparisons are made, because we feel that the strongly oscillatory high-degree terms in GEM 8 for the C coefficients (the values are -27, $+26$ and -41×10^{-9} for ℓ = 25, 27 and 29) are more likely to be an artefact of the solution than realistic values. This oscillatory tendency may also affect GEM 10 and SSE IV.3, though to a lesser extent.

Although the three comprehensive models do not always agree either with each other or with our values, they agree much better with our values than happened with the 15th-order coefficients, no doubt because several shallow resonant 14th-order orbits are included in the comprehensive solutions. The agreement is best for ℓ = 15, 17 and 19, with all four solutions indicating:

a low positive value (around 5*) for $\bar{C}_{15,14}$ and a value close to -24 for $\bar{S}_{15,14}$;

a value near -15 for $\bar{C}_{17,14}$, and near $+15$ for $\bar{S}_{17,14}$;

for $\bar{C}_{19,14}$, the verdict is 'small and probably negative', and for $\bar{S}_{19,14}$ 'negative and larger' (near -10).

For higher values of ℓ the agreement is not so good, and the near-unanimity over $\bar{C}_{25,14}$ and $\bar{C}_{27,14}$ may be a chance effect.

The comparison may perhaps fairly be summarized by saying that our solutions and those from the comprehensive models are recognizably similar, but further work on both methods is needed to achieve better agreement.

6.2 Even-degree coefficients

Comparison of our 5-coefficient and 9-coefficient solutions in Table 7 shows that, as for the odd-degree solutions, the values are extremely stable. The values in the 9-coefficient solution differ from the corresponding values in the 5-coefficient solution by less than half the standard deviation of the latter. The chief imperfection is the rather large standard deviations, which are two or three times greater than for the odd-degree coefficients. This reflects the fact that the most easily measured orbital change at resonance, that in the inclination, usually depends primarily on the odd-degree coefficients.

Comparison of our 9-coefficient solutions with the comprehensive models shows considerable differences in the C terms for $\ell = 14$ and 18. For $\ell = 14$ the values* from GEM 10, GRIM 2 and SSE IV.3 are between -50 and -70, whereas our solution gives -39 ± 3. For $\ell = 16$, the three comprehensive models differ, but their average is -23, in agreement with our values. For $\ell = 18$ our value is -22 ± 6, while the three comprehensive models give values between -10 and +1. For the S coefficients the agreement is much better, and GEM 10 agrees with our values to within 1.4 sd for all values of ℓ (up to 28). For $\ell = 14$ our 5-coefficient S solutions give -8 ± 2, while the other three range between -5 and -1. For $\ell = 16$ we have -36 ± 4, while the other three range between -41 and -33. For $\ell = 18$ the agreement is not so good. For $\ell = 20$ the GRIM 2 value of -44.5 seems unlikely, but the other two are fairly close to ours. For $\ell = 22$ all three are within 2 sd of ours.

For even-degree coefficients, our values and those from the comprehensive models are almost completely independent, since Wagner's limited results for $(\gamma, q) = (1, \pm 1)$ have very little influence on our solutions.

The conclusions from Table 7 are (a) that the agreement is already good for $\bar{S}_{\ell,14}$, but there are discrepancies for $\bar{C}_{\ell,14}$, and (b) that the standard deviations of our solutions need to be improved by analysis of further low-drag orbits.

7 CONCLUSIONS

We have for the first time determined the values of the coefficients of individual 14th-order harmonics in the geopotential from analysis of changes in the inclination and eccentricity of satellites with orbits which passed through 14th-order resonance. Results from the eleven orbits are used. Although values up to degree 30 are obtained, the recommended set of values runs from degree 14
015 to 22 only, and is as follows:

* In this paragraph we drop the factor 10^{-9}.

ℓ	$10^9\bar{C}_{\ell,14}$	$10^9\bar{S}_{\ell,14}$
14	-38.5 ± 2.9	-7.8 ± 2.2
15	4.5 ± 1.1	-23.8 ± 0.3
16	-22.3 ± 3.6	-36.0 ± 3.8
17	-15.0 ± 2.6	16.8 ± 1.2
18	-24.0 ± 4.9	-3.2 ± 3.7
19	-1.6 ± 2.8	-7.6 ± 1.0
20	8.8 ± 5.8	-15.4 ± 4.6
21	18.2 ± 3.6	-10.6 ± 1.9
22	-14.5 ± 8.1	9.9 ± 6.4

Tables 5 and 7 show that these values are not appreciably disturbed when higher-degree coefficients are included in the evaluations. The odd-degree coefficients, derived primarily from changes in orbital inclination, are more accurate than the even-degree coefficients, which are determined primarily from changes in eccentricity.

Values obtained in this way from resonance effects provide an independent test of the accuracy of the comprehensive geoid models derived in recent years. Detailed comparisons with three of these models, GEM 10, GRIM 2 and SSE IV.3, show that their values of 14th-order harmonic coefficients are generally similar to ours: see Tables 5 and 7. The best agreement - within 1½ sd for the even-degree S coefficients and the odd-degree C coefficients - is with GEM 10. However, there are also significant discrepancies, and further work on both methods is needed to achieve better agreement.

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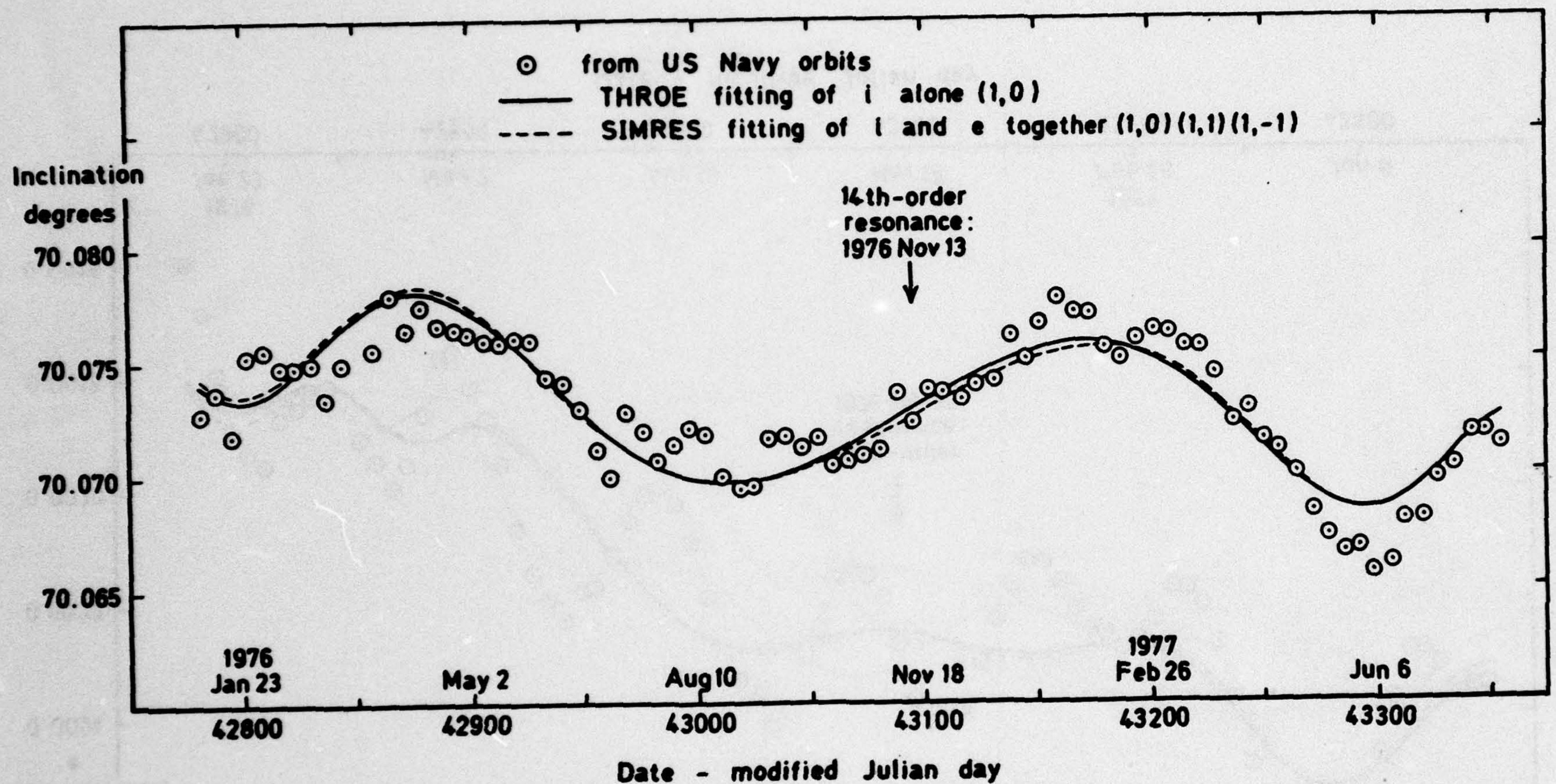

Fig 1 1965-16G: variation of inclination at 14th-order resonance

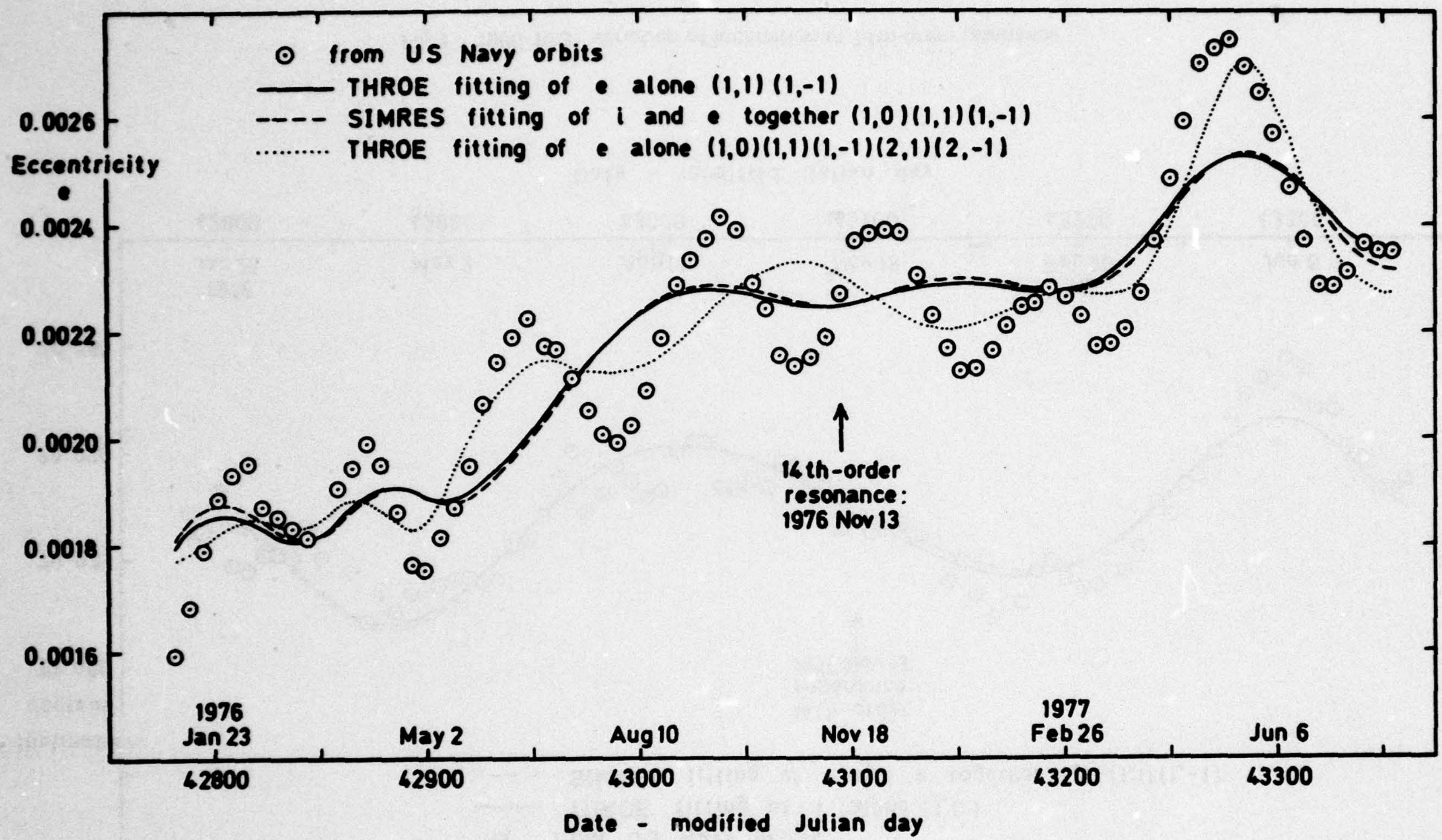

Fig 2 1965-16G: variation of e at 14th-order resonance

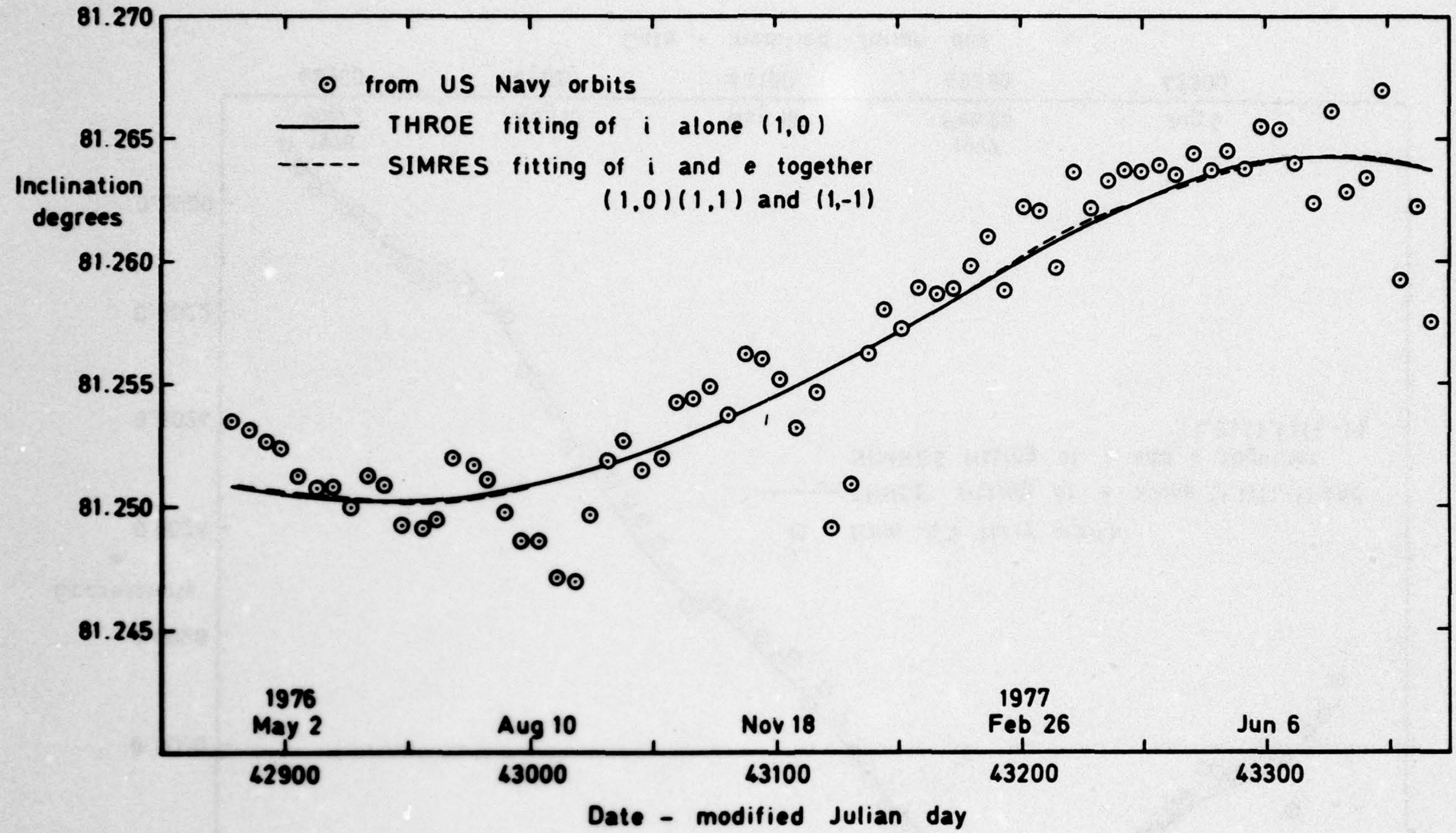

Fig 3 1971-120B: variation of inclination near 14th-order resonance

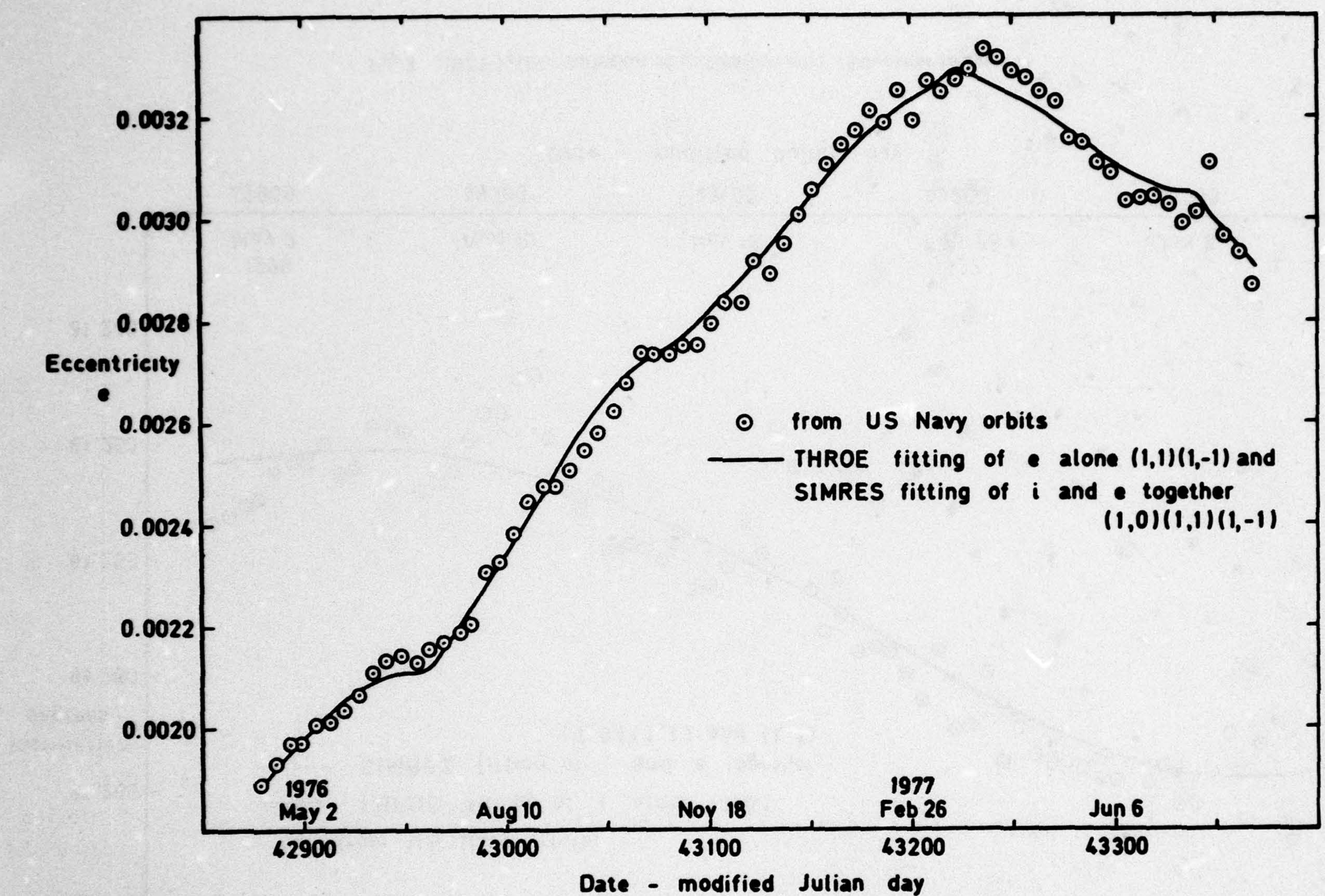

Fig 4 1971-120B: variation of e near 14th-order resonance

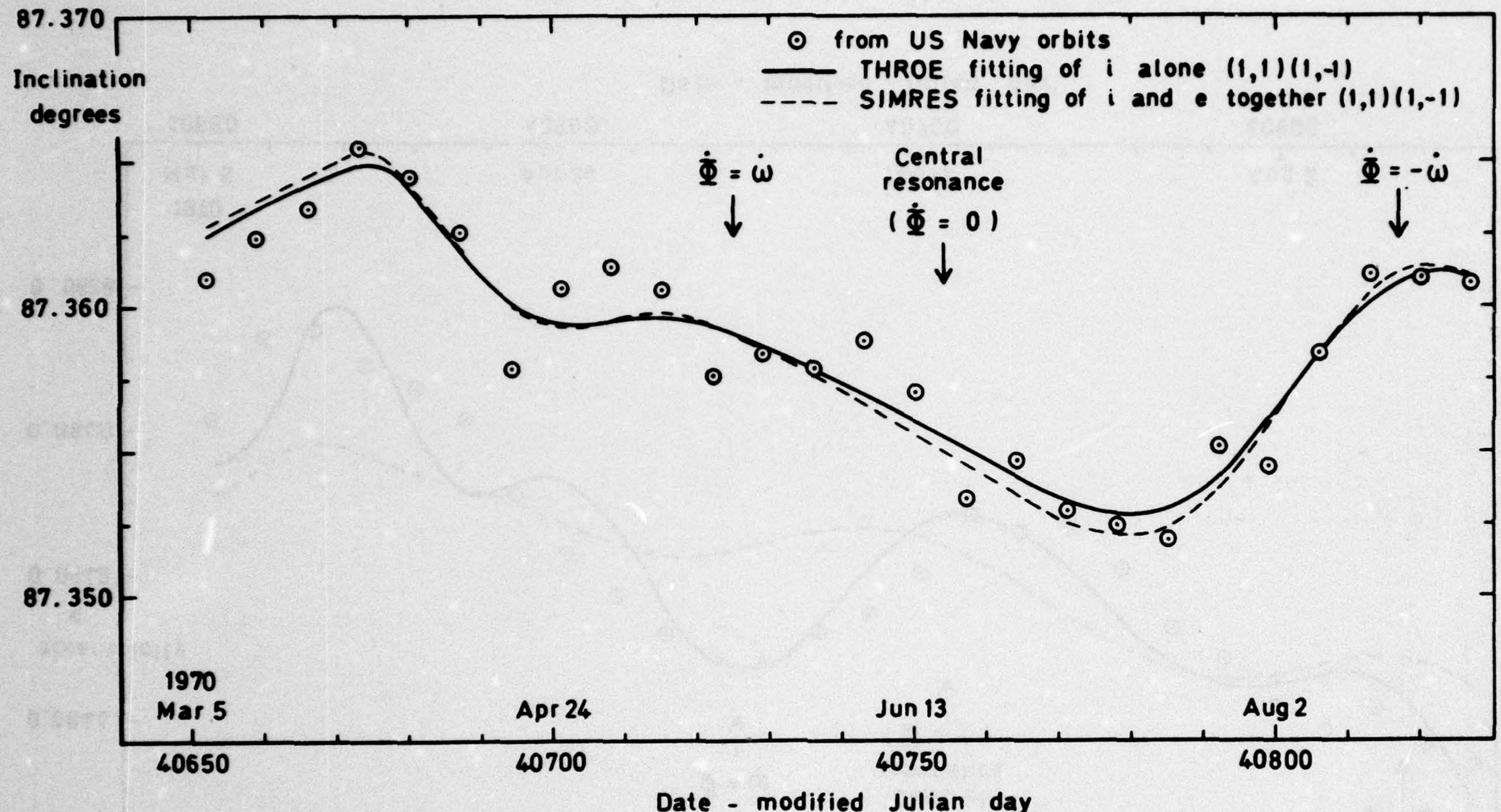

Fig 5 1965-81A: variation of inclination at 14th-order resonance

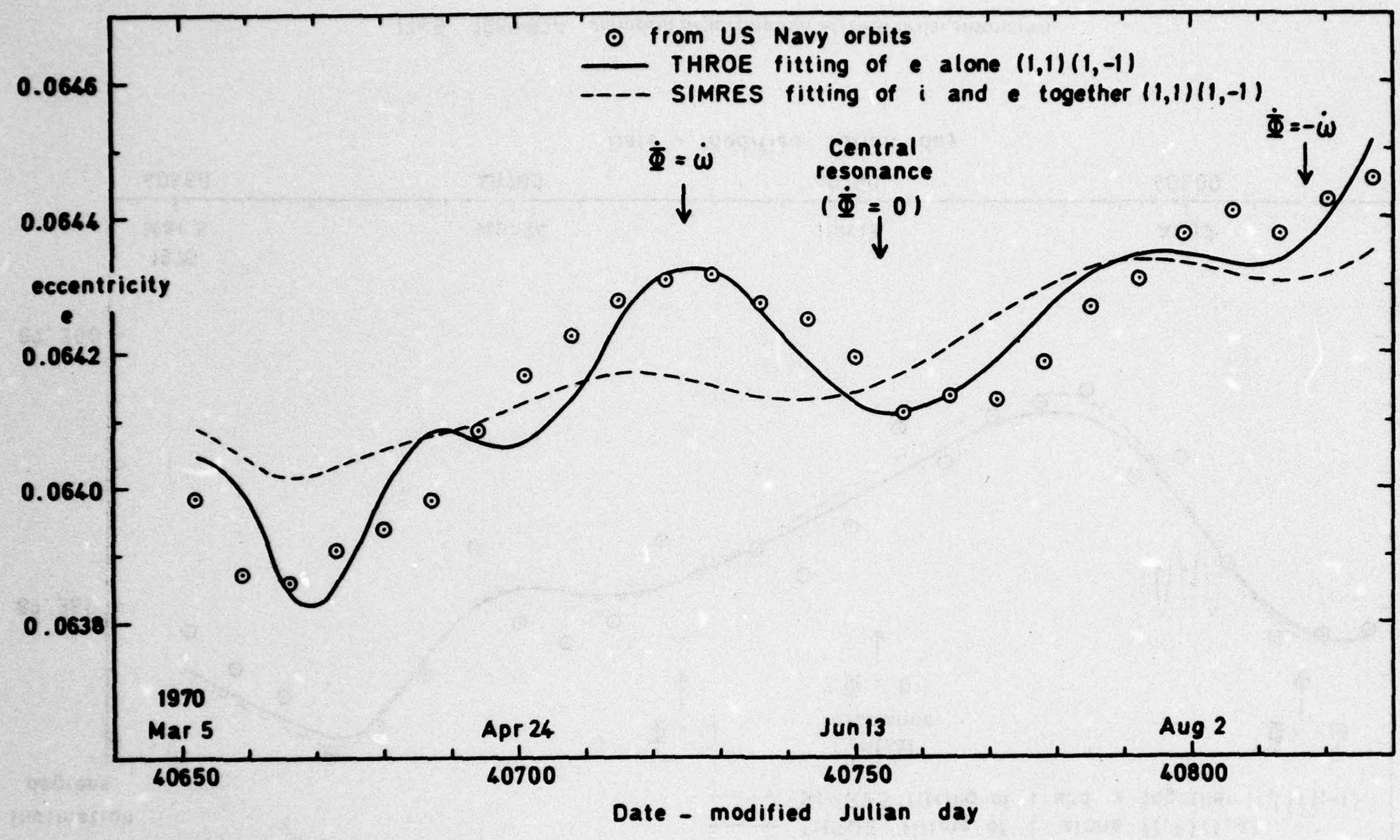

Fig 6 1965-81A: variation of eccentricity at 14th-order resonance

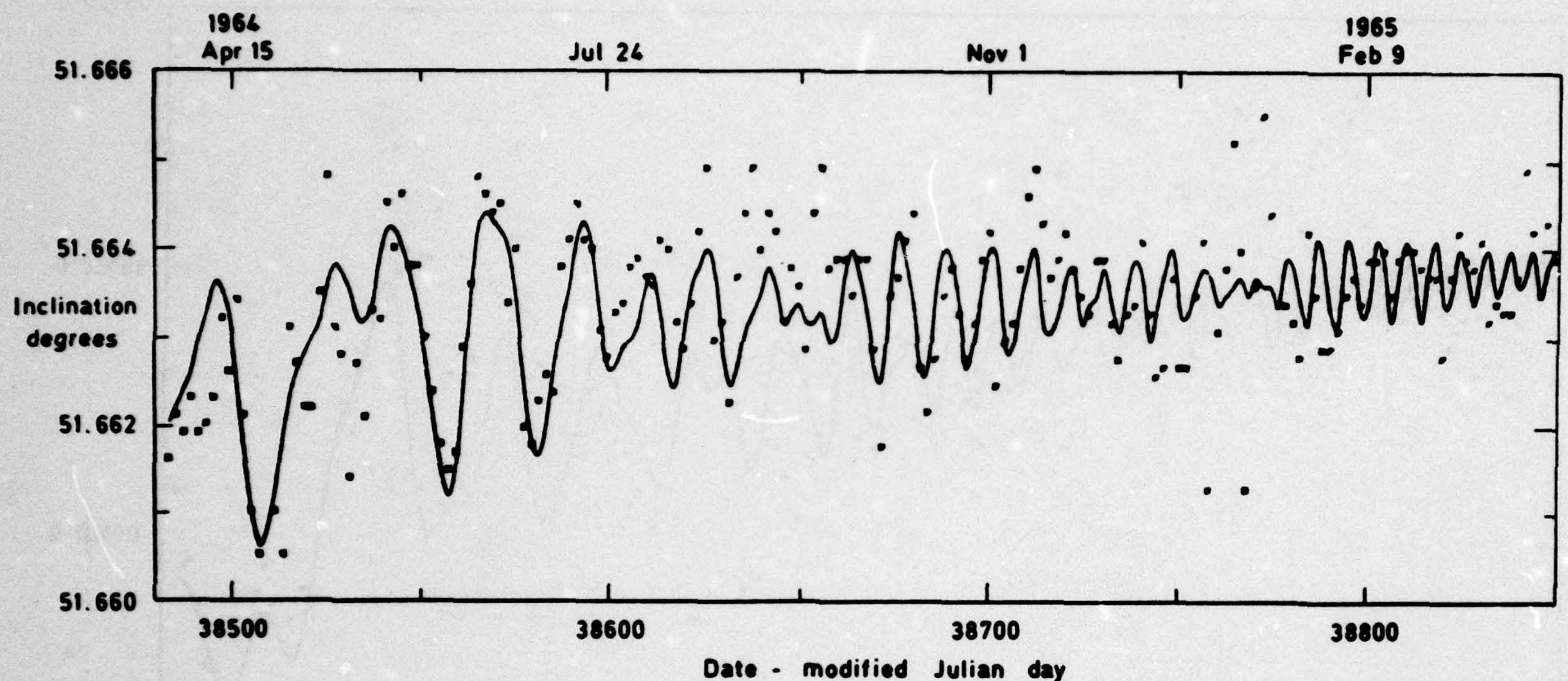

Fig 7 1964-15A: variation of inclination after resonance

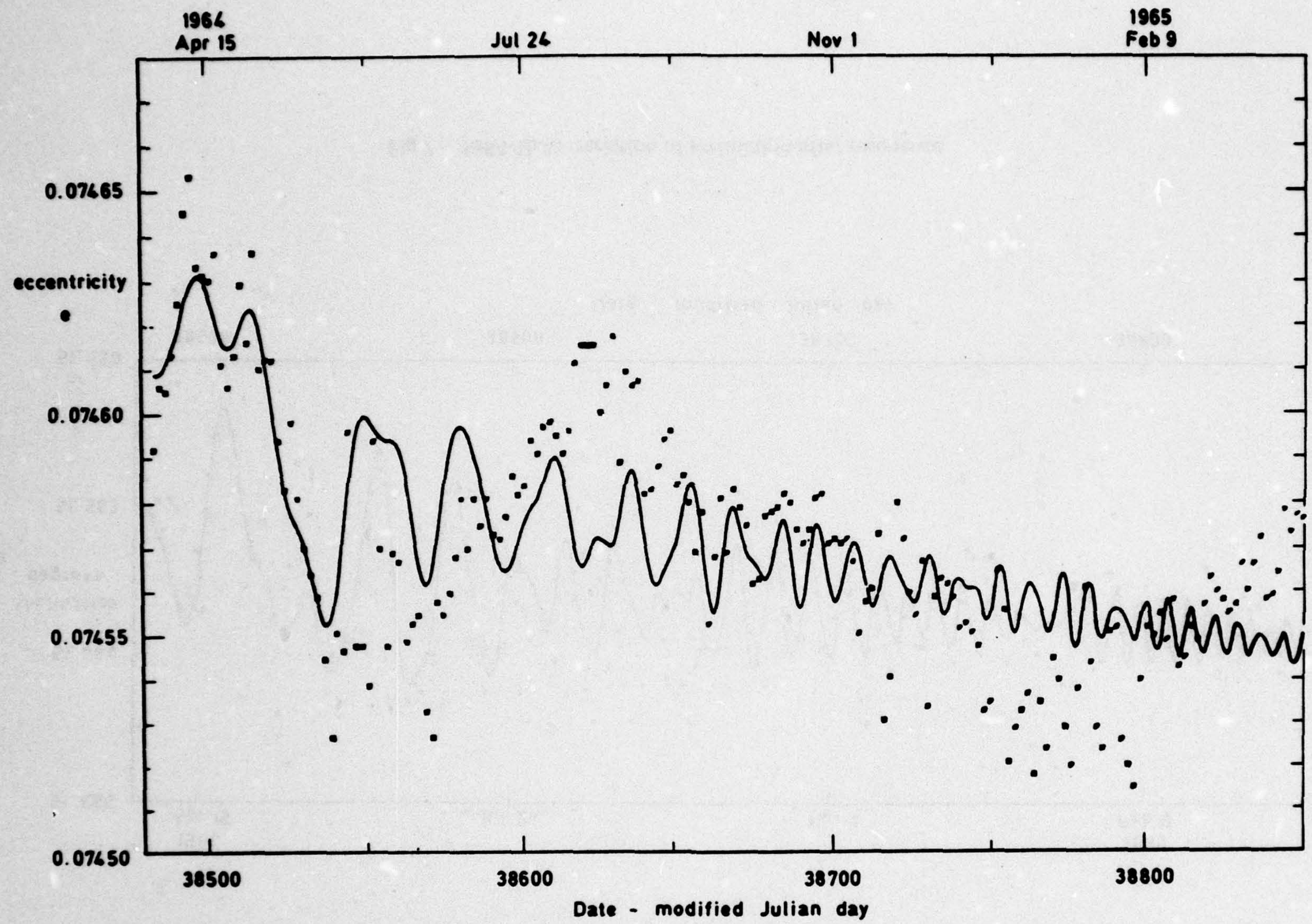

Fig 8 1964-15A: variation of eccentricity after resonance

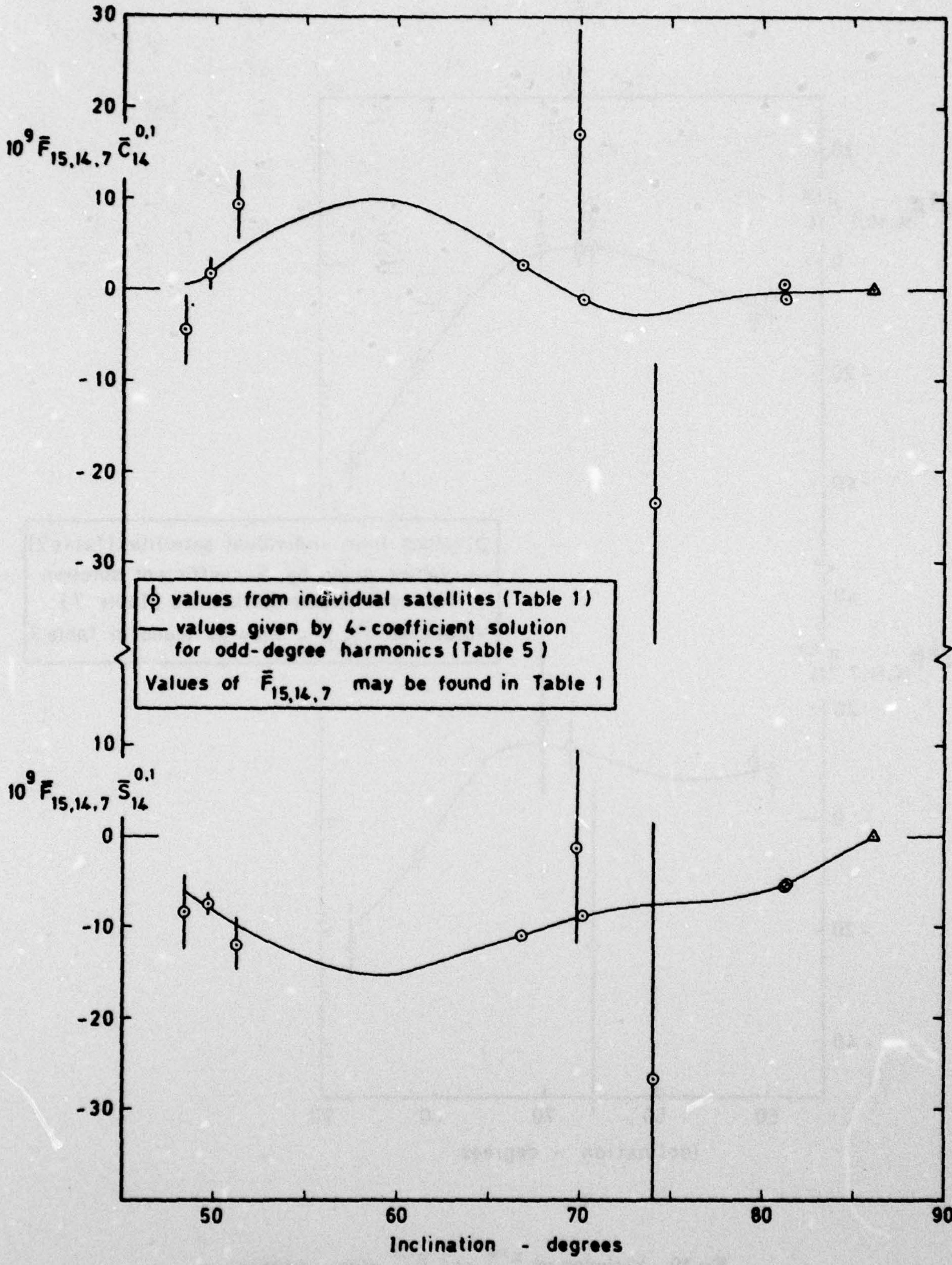

Fig 9 Variation of $\bar{C}_{14}^{0,1}$ and $\bar{S}_{14}^{0,1}$ with inclination

Fig 10

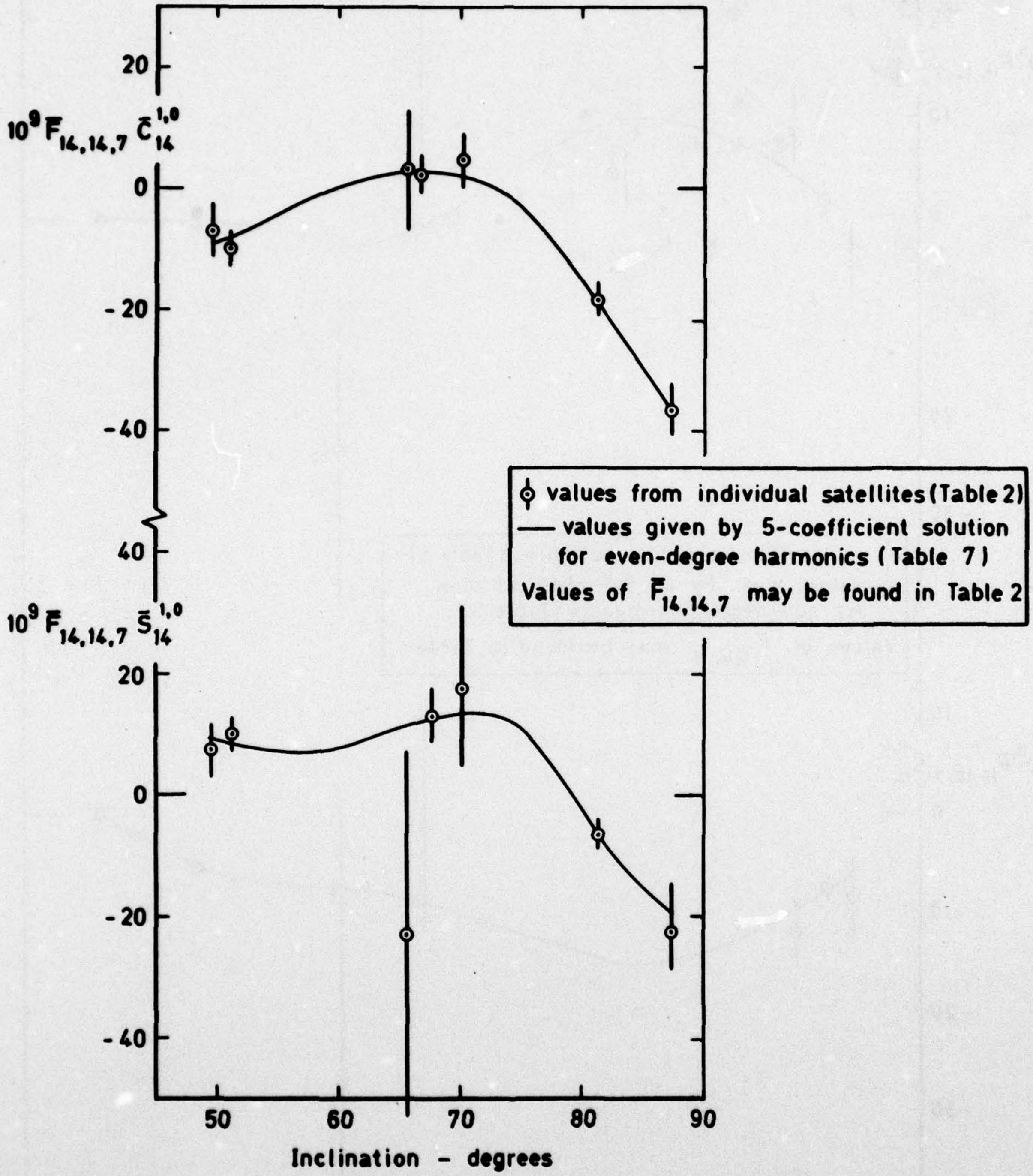

Fig 10 Variation of $\bar{C}_{14}^{1,0}$ and $\bar{S}_{14}^{1,0}$ with inclination

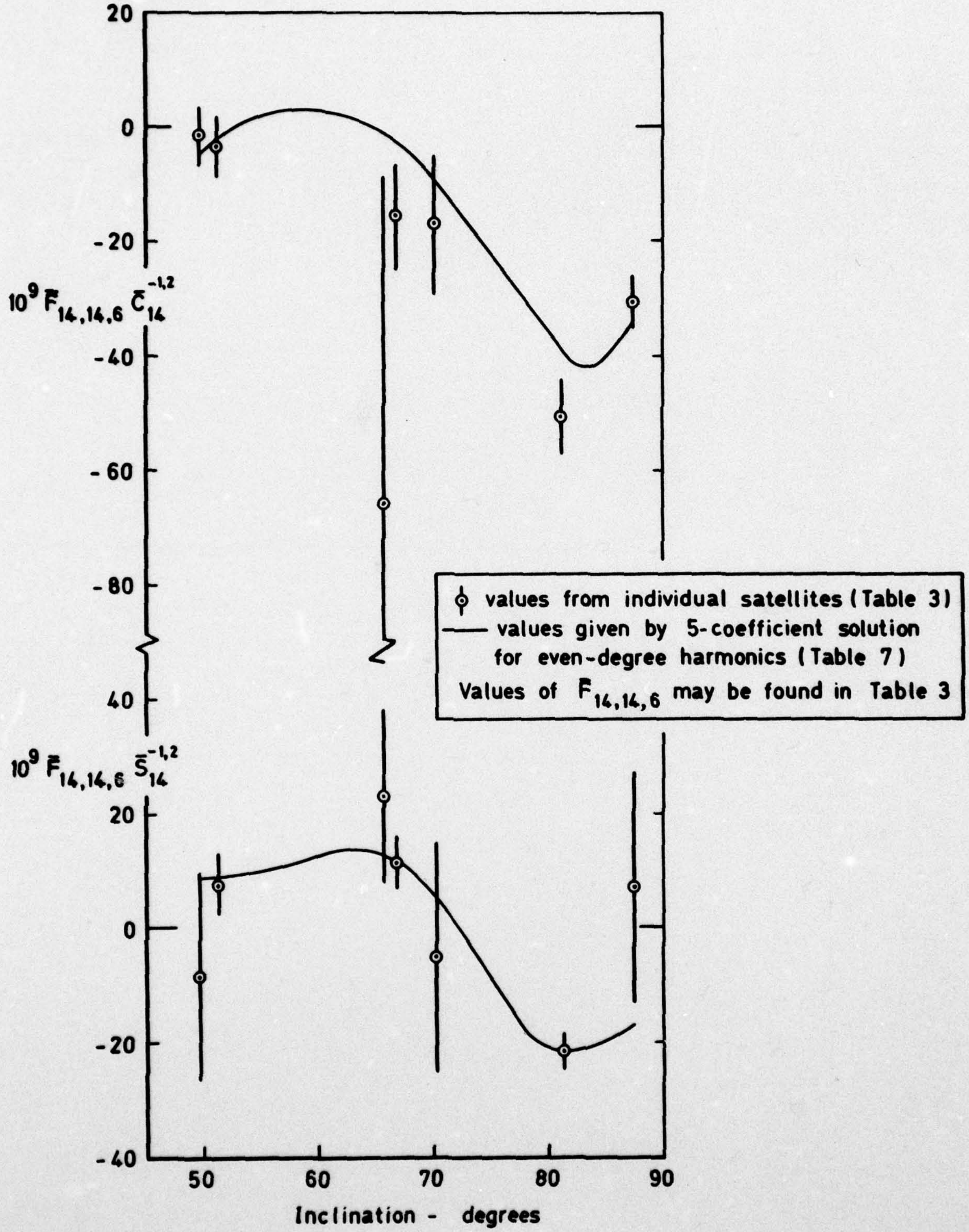

Fig 11 Variation of $\bar{C}_{14}^{-1,2}$ and $\bar{S}_{14}^{-1,2}$ with inclination

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1. DRIC Reference (to be added by DRIC)	2. Originator's Reference	3. Agency Reference	4. Report Security Classification/Marking
	RAE TR 78015	N/A	UNCLASSIFIED

5. DRIC Code for Originator	6. Originator (Corporate Author) Name and Location
850100	Royal Aircraft Establishment, Farnborough, Hants, UK
5a. Sponsoring Agency's Code	**6a. Sponsoring Agency (Contract Authority) Name and Location**
N/A	N/A

7. Title
Evaluation of 14th-order harmonics in the geopotential

7a. (For Translations) Title in Foreign Language

7b. (For Conference Papers) Title, Place and Date of Conference

8. Author 1. Surname, Initials	9a. Author 2	9b. Authors 3, 4	10. Date	Pages	Refs.
King-Hele, D.G.	Walker, Doreen M.C.	Gooding, R.H.	February 1978	38	22

11. Contract Number	12. Period	13. Project	14. Other Reference Nos.
N/A	N/A		Space 545

15. Distribution statement
(a) Controlled by – MOD(PE) D. Space
(b) Special limitations (if any) –

16. Descriptors (Keywords) (Descriptors marked * are selected from TEST)

Gravitational field. Geopotential. Satellite orbits. Resonance. Orbital analysis.

17. Abstract

The Earth's gravitational potential is now usually expressed in terms of a double series of tesseral harmonics with several hundred terms, up to order and degree at least 20. The harmonics of order 14 can be evaluated by analysing changes in satellite orbits which experience 14th-order resonance, when the track over the Earth repeats after 14 revolutions.

In this Report ~~we~~ describe ~~our~~ the first evaluation of individual 14th-order coefficients in the geopotential from analysis of the variations in inclination and eccentricity of satellite orbits passing through 14th-order resonance under the action of air drag. ~~Using~~ results from eleven satellites, ~~we find~~ these values for eighteen normalized coefficients of harmonics of order 14 and degree 14, 15 ....22. These values provide a test of the accuracy of the 14th-order coefficients in comprehensive geoid models. Detailed comparisons with three recent models are made, showing good agreement on some coefficients and discrepancies on others.

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