AD-A058 788 AIR FORCE FLIGHT DYNAMICS LAB WRIGHT-PATTERSON AFB OHIO F/G 20/11
METHODS FOR DETERMINING MODAL PARAMETERS AND MASS, STIFFNESS AN--ETC(U)
JUN 78 C L KELLER

UNCLASSIFIED AFFDL-TR-78-59 NL

| OF |
AD
A058 788

END
DATE
FILMED
-11-78
DDC

LEVEL II 2

(14) AFFDL-TR-78-59

(10) Charles L./Keller

(6) METHODS FOR DETERMINING MODAL PARAMETERS AND MASS, STIFFNESS AND DAMPING MATRICES

Analysis and Optimization Branch
Structural Mechanics Division

(12) 77 p.

(11) Jun[illegible]78

TECHNICAL REPORT AFFDL-TR-78-59

(9) Interim Report. [illegible] Apr[illegible]77- [illegible] Mar[illegible]78

(16) 2304 (17) N1

AIR FORCE FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
Air Force Systems Command
Wright-Patterson Air Force Base, Ohio 45433

NOTICE

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

This report has been reviewed by the Information Office (IO) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication.

Charles L Keller

CHARLES L. KELLER
Project Engineer
Applied Mathematics Group
Analysis & Optimization Branch

Richard D Krobusek

RICHARD D. KROBUSEK, Major, USAF
Chief, Analysis & Optimization
Structural Mechanics Division

FOR THE COMMANDER

Howard L Farmer

HOWARD L. FARMER, Colonel, USAF
Chief, Structural Mechanics Division

Copies of this report should not be returned unless return is required by security considerations, contractural obligations, or notice on a specific document.

AIR FORCE/56780/21 August 1978 — 175

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFFDL-TR-78- 59	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) METHODS FOR DETERMINING MODAL PARAMETERS AND MASS, STIFFNESS AND DAMPING MATRICES		5. TYPE OF REPORT & PERIOD COVERED Interim Report April 77 - March 78 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Charles L. Keller		8. CONTRACT OR GRANT NUMBER(s) In-house research
9. PERFORMING ORGANIZATION NAME AND ADDRESS Applied Mathematics Group/FBRD Air Force Flight Dynamics Laboratory (AFSC) Wright-Patterson AFB, Ohio 45433		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS DoD Element 61102F 2304N102
11. CONTROLLING OFFICE NAME AND ADDRESS Analysis & Optimization Branch Structural Mechanics Division Air Force Flight Dynamics Laboratory Wright-Patterson Air Force Base, Ohio 45433		12. REPORT DATE June 1978 13. NUMBER OF PAGES 71
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Mass, Stiffness and Damping Matrices Modal Parameters System Identification Ground Vibration Testing Dynamic Structural Testing		

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The author describes, mathematically, methods for determining modal parameters and mass, damping and stiffness matrices from vibration tests. The emphasis is on methods which, at least in principle, are capable of determining the desired quantities from excitations at a single point. A method which determines the modal parameters and the mass, damping and stiffness matrices from the steady state response to sinusoidal excitations is proposed. A computer program of this method and the results of two numerical experiments are given. The first experiment is for the case of

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

simple characteristic values. In the second experiment, one characteristic value is of multiplicity two. The other methods examined usually determine the desired quantities from the transient response. The mathematical background for all the methods is given.

FOREWORD

This report describes work performed in the Air Force Flight Dynamics Laboratory under Project 2304N102, Mathematical Problems in Fluid Dynamics. This is an interim report on work carried out between April 1977 and March 1978.

The author thanks Dr. Karl G. Guderley, retired, for his many suggestions and stimulating discussions. The author thanks also Mr. Donald S. Clemm of the AFFDL Applied Mathematics Group for his programming assistance and write-up, Appendix D. Lastly, he wishes to acknowledge the cheerful assistance of secretaries Mrs. Rhonda Baab, Ms. Fran Thomas and Mrs. Lynn Curtis in the preparation of this report.

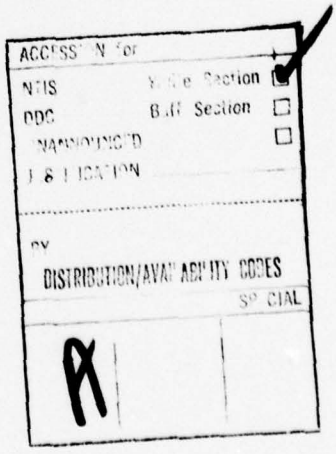

TABLE OF CONTENTS

LIST OF TABLES

SECTION I

INTRODUCTION

In recent years, the development of mini computers and certain algorithms have renewed interest that a practical way may be found to extract from measured vibration data the basic structural dynamic properties which govern the modal response. Such an ability could greatly enhance the usefulness of required ground vibration tests of new or modified aircraft for evaluation of aeroelastic, aeroservoelastic, dynamic loads and other dynamic phenomena directly related to aircraft safety.

As a consequence of these developments, a number of methods have been proposed by diverse groups, many of which are outside the airframe industry. The unfamiliarity of some of the methods, semantics problems, and proprietary considerations, have hindered understanding of the basics involved for evaluation, development or adaptation for more specific airframe use. This report describes the foundations of some of the more prominent methods on a common basis for initial comparisons and offers a method based on the steady-state sinusoidal response rather than the transient response used by the other methods.

We assume the structure is modeled by a system of second order differential equations with constant coefficients. In matrix and vector notation this system of equations can

be written as

$$M\ddot{x} + C\dot{x} + Kx = f$$

The matrices M, C and K are of order m and are called the mass, damping and stiffness matrices respectively. The elements of these matrices are real numbers. The forcing function or excitation $f = f(t)$ is a vector function of time with m components which may always be considered as known. The response $x = x(t)$ is also a vector function of time with m components which are determined by measurement. The components of the vector functions $f(t)$ and $x(t)$ may be complex valued.

If the vector function $u \exp(\lambda t)$ satisfies the homogeneous equation $M\ddot{x} + C\dot{x} + Kx = 0$ then the complex number λ is called a characteristic value and the vector u a characteristic or modal vector associated with λ. In this report we consider methods for determining the modal parameters, that is, the characteristic values λ or the natural frequencies and damping coefficients and modal vectors u. The natural frequencies (or resonant frequencies) and damping coefficients (or damping ratios) are readily obtained from the characteristic values and conversely.

The methods considered in this report have the capability of determining the vibration parameters and the matrices M, C and K from the measured responses x(t) to excitations f(t) which have only one (and always the same) component different

from zero. For most of these methods, only symmetery of the matrices M, C and K is required. It is not necessary to assume any relation (e.g. proportional damping) between M, C and K.

If the matrices M, C and K are not symmetric then, in general, one must excite the system at all stations. That is, one needs a linearly independent set of excitation functions f(t) and the corresponding responses x(t).

The excitation f(t) and the resulting response x(t) are experimental quantities determined by measurements. Accordingly these quantities are subject to error. For purposes of this report we assume ideal (error free) data. The various methods were examined to see if, at least in principle, the desired quantities could be obtained accurately by the method. The sensitivity of a method to errors in the data could be a decisive factor in the selection of a method. This aspect has not been treated because the author is not sufficiently familiar with the practical aspects of the experimentation procedures.

In Section II we propose a method for determining vibration parameters from the steady state response to harmonic excitations. The method is based upon Eq. (A36) of Appendix A. We show how to compute the characteristic values λ_k and the modal vectors u_k. We also show how to cope with the case of characteristic values of multiplicity greater than 1. For this case one must excite the system at a number of different stations separately. This number being the same as the multiplicity of the characteristic value.

The method of Section II is readily extended to deal with the case where the matrices M, C, and K are not symmetric. For this nonsymmetric case one must perform experiments (separately) at each station. The ability to treat the nonsymmetric case seems to have been overlooked for the most part.

In Section III we describe and discuss other methods for determining vibration parameters. We believe that we have described the major possible methods for determining vibration parameters from experiments at a single degree of freedom when no relations are assumed between the matrices M, C, and K. In Section IV we describe and give the results of numerical experiments with the method of Section II. We note possible areas for further investigation and present our conclusions.

The theoretical treatment may be carried out in either the time domain or the frequency domain. In general, the experimental data appears in the time domain. This time domain data is transformed to frequency domain data by numerically Fourier transforming the time data. However, if one measures the steady state response to harmonic excitations then, essentially, one obtains the experimental data directly in the frequency domain. Hence note that the method of Section II can also be used in methods which determine the frequency response function experimentally.

In Appendix A we give the mathematical background for the methods described in this report. In Appendix B we display the connection between the Fourier Integral, Fourier Series and trigonometric interpolating polynomials. Appendix B is background material for method 2 of Section III. Appendix C is background material for methods 4A and 4B of Section III. Finally in Appendix D we give instructions for the user and a program for the method of Section II.

SECTION II

VIBRATION PARAMETERS FROM THE STEADY STATE RESPONSE TO SINUSOIDAL EXCITATIONS

In this section we describe a procedure for determining the "complex frequencies" and associated complex mode shapes of a structure from its measured steady state response to sinusoidal excitations of frequencies close to resonance. In principle, only one "point" of the structure need be excited at appropriate frequencies, provided there are no multiple characteristic values. The case of no multiple characteristic values is treated first after which we show how to deal with multiple characteristic values.

Consider the following system of linear second order differential equations wtih constant coefficients

$$M\ddot{x} + C\dot{x} + Kx = f \tag{1}$$

In Eq. (1) M, C and K are real, square matrices of order m. The matrices M, C and K are referred to as the mass, damping and stiffness matrices respectively. The vectors $x = x(t)$ and $f = f(t)$ are of dimension m with time dependent components which may be complex valued.

In Appendix A it is shown that the steady state response to a harmonic excitation $f(t) = r \exp(i\omega t)$ of a system modeled by Eq. (1) is $y \exp(i\omega t)$ where

$$y = y(\omega, r) = \sum_{k=1}^{n} u_k v_k^T r/(i\omega - \lambda_k) \tag{2}$$

For $k = 1$ to $n = 2m$ the vector functions $x_k(t) = u_k \exp(\lambda_k t)$ are linearly independent and satisfy the homogeneous equation

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{3}$$

Similarly, the vectors $z_k(t) = v_k \exp(\lambda_k t)$ are linearly independent and satisfy the transposed homogeneous equation. The vectors u_k and v_k satisfy the normalization conditions expressed by Eq. (A17) of Appendix A. The complex numbers λ_k satisfy the characteristic equation

$$\det [\lambda^2 M + \lambda C + K] = 0 \tag{4}$$

The Eq. (2) is valid under very general circumstances. The only requirement for the validity of Eq. (2) is that for a root λ_k of multiplicity m_k the matrix $\lambda_k^2 M + \lambda_k C + K$ must be of rank $m - m_k$.

The complex numbers λ_k and the associated vectors u_k are the "complex frequencies" and mode shapes which we want to determine. The procedure to be described below extracts these quantities from Eq. (2). For the nonsymmetric case the vectors v_k are not scalar multiples of the corresponding u_k. For the determination of the v_k in the nonsymmetric case the system must be excited at each degree of freedom with frequencies close to the resonant frequencies.

In principle, the procedure described below is general. In the application of interest here the procedure is limited to systems for which the characteristic values λ_k lie close to the imaginary axis. More precisely the iteration procedure suggested below requires values ω_k satisfying the relation $| \lambda_k - i\omega_k | < | \lambda_j - i\omega_k |$ for all $\lambda_j \neq \lambda_k$.

For the case in which the matrices M, C and K are real symmetric matrices, and the roots λ_k of the determinantal equation, Eq. (4), are simple and well separated, the vectors v_k differ from the u_k by a numerical factor, at most. This case is considered now.

Let r_j denote the m dimensional unit vector whose $j^{\underline{th}}$ component is 1 and all others are zero. For k = 1 to m and ℓ = 1,2 let $y_{k\ell} \exp(i\omega_{k\ell}t)$ be the steady state response to $r_1 \exp(i\omega_{k\ell}t)$. Accordingly one makes two measurements for each characteristic value. The vectors $y_{k\ell}$ are computed from experimental data, that is, from the response to the sinusoidal excitation $r_1 \sin(\omega_{k\ell}t)$, see Appendix **A**, Eq. (A40). The values of $\omega_{k\ell}$ are chosen so that

$$\omega_{k1} < \text{Im}[\lambda_k] < \omega_{k2} \tag{5}$$

and ω_{k2} is much less than $\omega_{k+1,\,1}$. (The λ_k are indexed as described in Appendix A.) For definiteness we discuss the case where the excitation is in the first coordinate direction r_1. The procedure is the same regardless of the fixed coordinate direction excited.

Here now we describe the iterative procedure for determining the characteristic values λ_k and the associated vectors u_k and v_k. First we give the equations for computing initial values for λ_k, u_k and v_k. Then we give the equations for computing successive refinements of these quantities until preassigned tolerances are attained.

For j = 1 to m and ℓ = 1, 2, we have, on neglecting all but one term from Eq. (2),

$$y_{j\ell} = u_j v_j^T r_1/(i\omega_{j\ell} - \lambda_j) \tag{6}$$

as a first approximation. Set

$$\xi = \pm [y_{j1}^T y_{j1}/y_{j2}^T y_{j2}]^{1/2} = \frac{i\omega_{j2} - \lambda_j}{i\omega_{j1} - \lambda_j} \tag{7}$$

where the sign is chosen so that Im[ξ] > 0. (If one assumes a value for λ_j and computes ξ from Eq. (7) it is clear that this is the appropriate choice for the sign.) From Eq. (7) one obtains

$$\lambda_j = i(\xi\omega_{j1} - \omega_{j2})/(\xi - 1) \tag{8}$$

for j - 1 to m. Thus Eq. (8) gives a first approximation to λ_j. This first approximation of λ_j is needed for computing the first approximation of the vectors u_j and v_j. From Eq. (6) we obtain

$$y_{j1} - y_{j2} = u_j v_j^T \cdot r_1 i(\omega_{j2} - \omega_{j1})/(i\omega_{j1} - \lambda_j)(i\omega_{j2} - \lambda_j) \tag{9}$$

Now $v_j^T r_1 = v_{1j}$ is the first component of the vector v_j. We assume the vector u_j normalized so that its first component $u_{1j} = 1$. Then from Eq. (9) we have

$$v_{1j} = (y_{1j1} - y_{1j2})(i\omega_{j1} - \lambda_j)(i\omega_{j2} - \lambda_j)/i(\omega_{j2} - \omega_{j1}) \tag{10}$$

and the vector

$$u_j = (y_{j1} - y_{j2})(i\omega_{j1} - \lambda_j)(i\omega_{j2} - \lambda_j)/i(\omega_{j2} - \omega_{j1})v_{1j} \tag{11}$$

Hence v_j, as a scalar multiple of u_j, is given by

$$v_j = v_{1j} u_j \tag{12}$$

Thus in the first stage the computations performed are those indicated by Eqs. (7), (8), (10), (11) and (12).

Let λ_j^n, u_j^n, and v_j^n denote the new value which one is in the process of computing and let λ_k, u_k and v_k denote the present value of these quantities respectively. For j = 1 to m, set

$$\hat{y}_{j\ell} = y_{j\ell} - \sum_{\substack{k=1 \\ k\neq j}}^{n} u_k v_k^T r_1/(i\omega_{j1} - \lambda_k) \tag{13}$$

Compute

$$\xi = \pm [\hat{y}_{j1}^T \hat{y}_{j1}/\hat{y}_{j2}\hat{y}_{j2}]^{1/2} \tag{14}$$

where the sign is chosen as in Eq. (7), and

$$\lambda_j^n = i(\xi\omega_{j1} - \omega_{j2})/(\xi - 1) \tag{15}$$

Then

$$v_{1j}^n = \hat{y}_{1j1}(i\omega_{j1} - \lambda_j) \tag{16}$$

$$u_j^n = \hat{y}_{j1}(i\omega_{j1} - \lambda_j)/v_{1j} \tag{17}$$

and

$$v_j^n = v_{1j}u_j \tag{18}$$

where λ_j in Eqs. (16) and (17) is the λ_j^n computed in Eq. (15), v_{1j} in Eqs. (17) and (18) is the v_{1j}^n computed in Eq. (16), and in Eq. (18) u_j is the u_j^n computed in Eq. (17). At the same time one is computing λ_j^n, u_j^n and v_j^n one should record also

$$\lambda_{m+j}^n = \bar{\lambda}_j^n \tag{19}$$

$$u_{m+j}^n = \bar{u}_j^n \tag{20}$$

and

$$v_{m+j}^n = \bar{v}_j^n \tag{21}$$

These identifications should be made in the first stage also.

The computations in the refinement process, Eqs. (13) through (21), are repeated until either preassigned tolerance requirements are satisfied or the maximum number of passes through the refinement process is attained. In the latter event one is faced with the task of

determining why the tolerance requirements were not satisfied. If the tolerance requirements are satisfied, then the matrices M,C, and K are determined next.

Before discussing the computation of the matrices M,C, and K let us consider a modification of the above procedure which enables us to handle the case of multiple characteristic values. For definiteness suppose $\lambda_2 = \lambda_1$, $\lambda_4 = \lambda_3$ and the remaining characteristic values are simple. Let v_{jk} and u_{jk} denote the $j^{\underline{th}}$ component of the vectors v_k and u_k respectively. According to the discussion in Appendix A we may suppose

$$v_{21} = v_{12} = v_{43} = v_{34} = 0$$

$$u_{11} = u_{12} = u_{33} = u_{44} = 1 \tag{22}$$

As for the case of no repeated characteristic values the remaining u_k vectors are assumed normalized so that the first component has the value 1.

Let $y(\omega,r)\exp(i\omega t)$ denote the steady state response to the harmonic excitation $r\exp(i\omega t)$. The vector $y(\omega,r)$ is given by Eq. (2). Suppose we have obtained experimentally the vectors $y(\omega_{k\ell}, r_1)$ for $\ell = 1, 2$, see Eq. (15) for $k = 1, 5, 6, \cdots, m$ and also the vectors $y(\omega_{1\ell},r_2)$, $y(\omega_{3\ell},r_3)$ and $(y(\omega_{3\ell},r_4)$. From this data one obtains a first approximation to the characteristic values λ_k and the vectors u_k for $k = 1$ to m in exactly the same way as for the case of no repeated characteristic values.

Thus for first approximations compute as in Eq. (7)

$$\xi_j = \pm \left[y^T(\omega_{j1},r_k)y(\omega_{j1},r_k)/y^T(\omega_{j2},r_k)\cdot y(\omega_{j2},r_k)\right]^{1/2} \tag{23}$$

where $k = 1$ for $j = 1, 5, 6, \cdots, m$ and $k = 3$ for $j = 3$.

Then a first approximation to λ_j is given by

$$\lambda_j = i(\xi_j\omega_{j1} - \omega_{j2})/(\xi_j - 1) \tag{24}$$

for $j = 1, 3, 5, 6, \cdots, m$.

Next as in Eq. (9) we have

$$y(\omega_{j1},r_k)-y(\omega_{j2},r_k) = u_j v_j^T \cdot r_k i(\omega_{j2} - \omega_{j1})/(i\omega_{j1} - \lambda_j)(i\omega_{j2} - \lambda_j) \tag{25}$$

Let $y_p(\omega_{j\ell},r_k)$ denote the $p^{\underline{th}}$ component of the vector $y(\omega_{j\ell},r_k)$. Then from Eq. (25) we have

$$v_{kj} = [y_k(\omega_{j1},r_k)-y_k(\omega_{j2},r_k)](i\omega_{j1} - \lambda_j)(i\omega_{j2} - \lambda_j)/i(\omega_{j2} - \omega_{j1}) \tag{26}$$

where the value of k must be appropriately assigned depending on j, see below Eq. (27). Eq. (26) gives the $k^{\underline{th}}$ component of the vector v_j.

The modal vectors are obtained as before, Eq. (11). Thus

$$u_j = [y(\omega_{j1},r_k)-y(\omega_{j2},r_k)](i\omega_{j1} - \lambda_j)(i\omega_{j2} - \lambda_j)/i(\omega_{j2} - \omega_{j1})v_{kj} \tag{27}$$

where $k = 1$ for $j = 1, 5, 6, \cdots, m$, $k = 2$ for $j = 2$ and $k = j$ for $j = 3, 4$.

For values of j for which λ_j is not a multiple characteristic value the vectors v_j are given by

$$v_j = v_{1j}u_j \tag{28}$$

as before. For values of j for which λ_j is a multiple characteristic value the associated vectors v_j are linear combinations of the associated u_j vectors. Rather than use subscripted subscripts we chose to illustrate the procedure for determining the appropriate

linear combinations by example.

For the example being described the vectors v_1 and v_2 are linear combinations of the vectors u_1 and u_2. Thus

$$\left.\begin{aligned} v_1 &= a_{11}\, u_1 + a_{12}\, u_2 \\ v_2 &= a_{21}\, u_1 + a_{22}\, u_2 \end{aligned}\right\} \tag{29}$$

Now the vectors u_1 and u_2 are known from Eq. (27). The component v_{11} of v_1 and the component v_{22} of v_2 is known from Eq. (26). In addition, because of the special form that we were able to assume for the vectors v_1 and v_2 we have that the components $v_{21} = v_{12} = 0$ for the vectors v_1 and v_2 respectively. Utilizing this information we can extract a matrix equation from the system of vector equations, Eq. (29) for determining the coefficients a_{ij}. Thus we have

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & 0 \\ 0 & v_{22} \end{bmatrix} \tag{30}$$

After the coefficients a_{ij} are determined, the vectors v_1 and v_2 are completely specified (that is, the first approximation thereof) by Eq. (29).

In a similar fashion one has

$$\left.\begin{aligned} v_3 &= b_{11}\, u_3 + b_{12}\, u_4 \\ v_4 &= b_{21}\, u_3 + b_{22}\, u_4 \end{aligned}\right\} \tag{31}$$

From Eqs. (31) one obtains

$$\begin{bmatrix} u_{33} & u_{34} \\ u_{43} & u_{44} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} = \begin{bmatrix} v_{33} & 0 \\ 0 & v_{44} \end{bmatrix} \tag{32}$$

and hence the vectors v_3 and v_4 are completely determined also. From the above discussion it should be clear how to deal with any number of multiple characteristic values whatever their multiplicity.

The computations described by Eqs. (23) through (32) provide first approximations of the desired quantities. The refinement process for the case of multiple characteristic values is basically the same as for the case of simple characteristic values. We will abide by the same conventions as adopted for the case of all simple characteristic values see Eqs. (13) through (21).

Set as in Eq. (13)

$$\hat{y}(\omega_{j\ell},r_p) = y(\omega_{j\ell},r_p) - \sum_{\substack{k=1\\k\neq j}}^{n} u_k v_k^T r_p/(i\omega_{k\ell} - \lambda_k) \tag{33}$$

Then replace $y(\omega_{j\ell},r_k)$ for $\ell = 1, 2$ by $\hat{y}(\omega_{j\ell},r_k)$ in Eq. (23). With the value of ξ_j thus obtained compute

$$\lambda_j^n = i(\xi_j\omega_{j1} - \omega_{j2})/(\xi_j - 1) \tag{34}$$

Then

$$v_{kj}^n = [\hat{y}_1(\omega_{j1},r_k)](i\omega_{j1} - \lambda_j) \tag{35}$$

$$u_j^n = \hat{y}(\omega_{j1},r_k)/v_{kj} \tag{36}$$

and for values of j for which λ_j is not a repeated characteristic value

$$v_j^n = v_{1j}u_j \tag{37}$$

The v_j corresponding to a multiple characteristic value λ_j are computed as above, see Eqs. (29), (30), (31) and (32).

The refinement process described by Eqs. (33) thru (37) is repeated until the tolerance requirements are satisfied.

Once the characteristic values and the appropriately normalized characteristic vectors are known the system matrices M,C, and K are readily obtained. Set

$$P(\omega) = -\omega^2 M + i\omega C + K \tag{38}$$

and

$$Q(\omega) = \sum u_k v_k^T / (i\omega - \lambda_k) \tag{39}$$

We noted, Eq. (A43) of Appendix A, that

$$P(\omega) = Q^{-1}(\omega) \tag{40}$$

so that

$$P(\omega)Q(\omega) = I \tag{41}$$

Let the prime (′) denote differentiation with respect to ω. From Eq. (38) we have

$$\begin{aligned} K &= P(0) \\ iC &= P'(0) \\ -2M &= P''(0) \end{aligned} \tag{42}$$

From Eq. (41) we obtain

$$\begin{aligned} P(0) &= Q^{-1}(0) \\ P'(0) &= -Q^{-1}(0)Q'(0)Q^{-1}(0) \\ P''(0) &= -(2P'(0)Q'(0) + P(0)Q''(0))Q^{-1}(0) \end{aligned} \tag{43}$$

SECTION III
OTHER METHODS FOR DETERMINING THE VIBRATION PARAMETERS

In this section we describe and discuss other methods for determining the vibration parameters. We make no attempt nor claim to discuss all methods. Indeed in the discussion below other ways to achieve the same end will come to mind. Each variant of the methods described below may, of course, be considered another method.

The methods of primary interest to us have two distinctive features; First, they assume no relation whatsoever between the matrices M, C, and K. Secondly, these methods have the capability of determining (almost) all vibration parameters of interest from the data arising from exciting the system at a single degree of freedom, at least for the case where all the characteristics values are different from one another.

The first method which we describe requires that the system be excited at each degree of freedom individually. Hence, it is not a method of primary interest. However, given the problem of determining the matrices M, C, and K this first method seems to be an obvious approach to the solution. It shows, at least in principle, that the matrices M, C, and K can be determined from experimental data.

Let us excite the system of equations $M\ddot{x} + C\dot{x} + Kx = f$ at the kth coordinate with $\exp(i\omega_1 t)$. The steady state response will be $y_k \exp(i\omega_1 t)$ for k = 1 to m. (See the discussion and Eqs (A35) - (A40) of Appendix A . From this set of experiments one obtains the matrix equation

$$-\omega_1^2 M + K + i\omega_1 C = [y_1 \ldots y_m]_1^{-1} \tag{44}$$

The Eq. (44) is solvable for the matrix C. We have

$$C = (1/\omega_1)\,\mathrm{Im}[y_1 \cdots y_k]_1^{-1} \tag{45}$$

If this set of experiments is repeated with an $\omega_2 \neq \omega_1$ then we obtain a system of matrix equations

$$\begin{aligned} -\omega_1^2 M + K &= \mathrm{Re}[y_1 \cdots y_m]_1^{-1} \\ -\omega_2^2 M + K &= \mathrm{Re}[y_1 \cdots y_m]_2^{-1} \end{aligned} \tag{46}$$

which is readily solved for the matrices M and K.

Now let us note what is involved in this first method. The experimental requirement is a "shaker" set up for each degree of freedom of the system and an excitation at each degree of freedom of the system at frequencies ω_1 and ω_2.

The computations involved in this method are the determination of the columns of the matrices $[y_1 \cdots y_m]_1$, and $[y_1 \cdots y_m]_2$ (See Eq. (A40); computing the inverses of the complex matrices $[y_1 \cdots y_m]_1$, and $[y_1 \cdots y_m]_2$; obtaining the matrices M and K (Eq. 46) and the matrix C (Eq. (45)). In addition, if one needs the modal vectors, natural frequencies and damping ratios then one has to perform an eigenvalue and eigenvector analysis involving the matrices M, C, and K. Stated in another way one has to determine a linearly independent set of n functions of the form $u \exp(\lambda t)$ which satisfy the equation $M\ddot{x} + C\dot{x} + Kx = 0$.

In the determination of the matrices M, C, and K there are no simplifying assumptions. None of the operations or computations are performed approximately. Indeed even symmetry of the matrices M,C, and K is not needed or used.

The value of the components of the complex vectors y_k depend upon the frequency ω . As usual, let y_{jk} denote the jth component of the vector y_k. If one had a graph of $|y_{jk}|$ as a function of ω one would observe that $|y_{jk}|$ is small when ω is not close to a natural frequency of the system. This remark is evident from Eq (A36). Hence the inverses $[y_1 \cdots y_m]_1^{-1}$ and $[y_1 \cdots y_m]_2^{-1}$ obtained from experimental data may be very inaccurate for most choices of ω_1 and ω_2.

In the next method considered the frequency response function is determined experimentally as a function of frequency ω, [1-6]. This method is characterized by its use of techniques of Fourier Analysis. An analogous method based on the Laplace transform is also possible.

Let{F f(t)} denote the Fourier transform of the function f(t). We have $[\omega^2 M + i\omega C + K]F\{x(t)\} = F\{f(t)\}$or

$$F\{x(t)\} = [-\omega^2 M + i\omega C + K]^{-1} F\{f(t)\} \tag{47}$$

Now let $f_j(t)$ denote a vector valued function of t with all but the jth component identically zero. Also, let $x_j(t)$ denote the response to $f_j(t)$. Then the jth column of $[-\omega^2 M + i\omega C + K]^{-1}$ is given by

$$\text{jth column of } [-\omega^2 M + i\,\omega C + K]^{-1} = F\{x_j(t)\} \,/\, F\{f_j(t)\} \tag{48}$$

For j running through the values from 1 to m the matrix $[-\omega^2 M + i\omega C + K]^{-1}$ is completely specified as a function of ω , see Eq (A43).

If the matrices M, C, and K are symmetric and if the natural frequencies of the system are well separated then the vibration parameters can be determined from a single column of $[-\omega^2 M + i\omega C + K]^{-1}$ such as given

by Eq (48). For example, from Eq (A43) the first column of $[-\omega^2 M + i C\omega + K]^{-1}$ is the expression

$$\sum_{k=1}^{n} u_k v_{1k}/(i\omega-\lambda_k) \tag{49}$$

Thus the vectors $u_k v_{1k}$ and the complex numbers λ_k for k = 1 to n need to be determined so that the equality

$$\sum_{k=1}^{n} u_k v_{1k}/(i\omega-\lambda_k) = F\{x_1(t)\}/F\{f_1(t)\} \tag{50}$$

holds (at least in a curve fitting sense).

This second method is suited to an impulse type excitation but other forms of excitation may be used [4]. The experimental procedures for these first two methods differ. In this second method the transient part of the response is important in the analysis process, whereas, in the first method the steady state part was used. Note that the method of Section II uses the same kind of experimental data as the first method of this section.

For this second method we observe that there are two computational tasks. First, a column of the frequency response function must be computed, Eq (48), from the excitation and response data. Secondly, the modal vectors and characteristic values must be extracted from the computed frequency response function data. Let us examine these two tasks further.

For the first task the excitation data and the components of the vector response data are fitted by trigonometric interpolation polynomials to obtain the frequency response function data, Appendix B. If the system of differential equations $M\ddot{x} + C\dot{x} + Kx = f$ is of order m then one has up to m+1 fits by trigonometric polynomials, depending

upon the number of degrees of freedom of interest. For a complete analysis (determination of the matrices M, C, and K) all m+1 fits must be made. It is recognized that the frequency response function data is the result of certain approximations.

A number of ways for extracting the characteristic values and vectors from the frequency response function data are noted in [3,4]. The problem of course is the determination of the quantities $u_k v_{1k}$ and λ_k so that the expression (49) fits the frequency response function data. Present practice is to least squares fit the frequency response function data by the expression (49). The "normal" equations, resulting from the direct approach to a least squares fit of the expression (49) to the frequency response function data, are not solvable exactly. Thus the modal vectors, natural frequencies and damping ratios are determined approximately. The graph of the expression (49), after the quantities $u_k v_{1k}$ and λ_k have been determined, when compared with the graph of the frequency response function data, gives a visual check of the fitting process.

It is clear from the above discussion that there is considerable data manipulation and processing associated with this second method. The desired vibration parameters are obtained by numerical (approximate) methods. On the other hand this second method may not be as sensitive to errors in the data as some of the other methods.

In method 2 (just considered) the vibration parameters are obtained from the measured (experimentally determined) frequency response function. In method 3 to be considered next, the vibration parameters are obtained from the measured impulse response function. This method

is based on Eq (A31) when expressed as Eq (A45), Appendix A.

When the forcing function f(t) has only one component, for example, the first component different from zero then Eq (A45) simplifies to

$$x(t) = \int_o^t I_1(t-\tau) f_1(\tau) d\tau \tag{51}$$

Set $t_k = \tau_k = k\Delta\tau$. Observe that $I_1(t_j - \tau_k) = I_1(\tau_{j-k})$ for $k \leq j$. If the integral in Eq. (51) is evaluated by the trapezoidal rule we have

$$x(t_j) = (I_1(t_j)\ f_1(0)/2 + \sum_{k=1}^{j} I_1(\tau_{j-k})\ f_1(\tau_k))\Delta\tau \tag{52}$$

Note that for j=1 Eq (52) is solvable for $I_1(\tau_1)$. Now that $I_1(\tau_1)$ is known, then for j=2 Eq (52) is solvable for $I_1(\tau_2)$ and so on. In this way the impulse response function $I_1(t)$ is determined from experimental data.

The impulse response function data is used to determine the vibration parameters. From Appendix A, Eq (A34) we have

$$I_1(t) = \sum_{k=1}^{n} u_k v_{1k} \exp(\lambda_k t) \tag{53}$$

The quantities $u_k v_{1k}$ and λ_k need to be determined so that the function defined by Eq (53) fits the impulse response function data obtained from Eq(52).

We will now describe briefly a way to obtain the vibration parameters from the impulse response function data. This procedure is given also in [7, pp 270-280]. A component of $I_1(t)$ is of the same form as $I_1(t)$ as given by Eq (53). That is, the jth component of the vector

valued function $I_1(t)$ is scalar function of the form

$$g(t) = \sum_{k=1}^{n} a_k(\exp(\lambda_k t) \tag{54}$$

Briefly stated, the problem is to determine the complex numbers λ_k and the coefficients a_k from the data $t_j = t+jh$ and $g_j = g(t_j)$ for $j = 0,1,2$ and so on.

The problem is solved in two stages. In the first stage the λ_k's are determined and in the second stage the coefficients a_k. First, we determine the coefficients C_k of a difference equation

$$g(t) + C_1 g(t+h) +\ldots+ C_n g(t+nh) = 0 \tag{55}$$

which the functions exp $(\lambda_k t)$ (and hence any linear combination of the exp $(\lambda_k t)$ also) will satisfy for a fixed value of h and all values of t. Using the data we obtain the system of equations

$$C_1 g_k + C_2 g_{k+1} +\ldots+ C_n g_{k+n-1} = g_{k-1} \tag{56}$$

for k=1, ... , n which is solved for the C_j s.

Now set

$$p = \exp(\lambda h)$$

then

$$\exp(\lambda jh) = p^j$$

We see that $g(t) = \exp(\lambda t)$ will satisfy Eq. (55) if p is a root of the polynomial equation

$$C_n p^n +\ldots+ C_1 p+1 = 0 \tag{57}$$

Then

$$\lambda h = \log_e p \tag{58}$$

Thus the λ_k are determined by the n roots p_k of Eq. (57).

The n roots p_k of Eq. (57) are used also in the determination of the coefficients a_k. From Eq. (54) we obtain the system of equations

$$a_1 p_1^k + \ldots + a_n p_n^k = g_k \tag{59}$$

for k = 0,1, ..., n-1. This system of equations determines the coefficient a_k. A method for computing the values of the a_k's from Eqs. (59) is given in [7, p 274].

Let $\tilde{p}_k$ denote an n-dimensional vector whose components are 1, p_k, ..., p_k^{n-1} respectively, where p_k is a root of Eq. (57). The coefficient matrix for the C_k's, Eq. (56) can be written in the dyadic form

$$\sum_{k=1}^{n} a_k p_k \exp(\lambda_k t)\, \tilde{p}_k \tilde{p}_k^T \tag{60}$$

It is clear from this expression that the coefficient matrix for the C_k's is nonsingular provided the roots p_k of Eq. (57) are all different. This expression may also be helpful in examining the coefficient matrix of the C_k's for ill conditioning.

The fact that the characteristic values λ_k are all different does not guarantee that the coefficient matrix for the C_k's is nonsingular. For example, if the characteristic values λ_j and λ_k and the sample interval h satisfy the condition

$$h(\lambda_k - \lambda_j) = 2\pi i \tag{61}$$

then

$$p_j = \exp(\lambda_j h) = \exp(\lambda_k h) = p_k \tag{62}$$

and the coefficient matrix, by Eq. (60), is singular.

Let us review briefly the computations involved in this method. First, there are the numerical integrations in the determination of the impulse response function data. Secondly, the linear system given by Eq. (56) must be solved for the difference equation coefficients C_j. Errors in the coefficient matrix and in the right hand side of Eq. (56) will affect the C_j's. Thirdly, the roots of Eq. (57) must be determined. Lastly, the linear system given by Eq. (59) must be solved for the amplitudes a_k for each degree of freedom of interest.

We will refer to the next two methods as method 4A and 4B respectively, [8, 9, 10]. In these methods the initial (or original) data is a solution x(t) of the homogeneous equation $M\ddot{x} + Cx + Kx = 0$. The vector function x(t) is a linear combination of the functions $u_k \exp(\lambda_k t)$. Thus x(t) is of the same form as the impulse response function.

It is clear that the procedures described in method 3 could be used to determine the characteristic values and associated characteristic vectors from the function x(t). The function x(t) is original experimental data whereas the impulse response function is computed from experimental data. Thus the numerical integration step of method 3 used to obtain the impulse response function could be eliminated in the case

where x(t) is a free response of the structure. However, the system matrices M, C, and K are not computable from the results so obtained (see remarks which follow Eq (A43) of Appendix A).

The equations and details of computations in the methods 4A and 4B are given in Appendix C. Here we only need to note what is involved in these two methods and contrast and compare them with the other methods whenever possible.

The first step in Methods 4A and 4B is the determination of the "system" equations. The second step is the determination of the characteristic values and vectors from the system equations obtained in the first step. All the previous methods, except for the very first of this section, first determined the characteristic values and a particular normalization of the characteristic vectors so that the system equations can be determined from these quantities.

In method 4A the system equations determined are the equations of an equivalent first order system, Eqs (C4) and (C5) of Appendix C. Thus, the matrices M, C, and K are not obtained in this method. A number of numerical integrations, see Eqs. (C20) and (C21), are needed to produce the data required in the determination of the system equations, see Eq (C9).

The method 4B is free of the numerical integrations required in method 4A. This freedom is achieved by determining a system of difference equations which has the same solutions as the differential equations (see the discussion following Eq (C22) of Appendix C). The matrices M, C, and K are not obtained in method 4B.

After the system equations are obtained an eigenvalue and

eigenvector analysis are performed to obtain the characteristic values and vectors. One should note that the matrices involved in the computations in methods 4A and 4B are twice as large as those in the other methods.

In the preceding pages of this section we have described methods 2, 3, 4A and 4B. These methods essentially satisfy our self imposed requirements. That is, the matrices M, C, and K are not assumed linearly connected and the vibration parameters can be determined from the response to the excitation at a single point. In the discussion of these methods one acquires some awareness of computational requirements. However, the previous discussion was primarily concerned with the foundations of the methods. Here now we want to make quantitative estimates of some computational costs associated with two of the methods.

The Fast Fourier Transform Algorithm provides a significant reduction in the cost of the computations indicated by Eq. (48). Thus an operations count for the process which gives the value of a component of $F\{x_j(t)\}$ at N stations is of the order of $N \log_2 N$. In method 3 the determination of the value of a component of $I_1(t)$ at N stations is the same as solving a triangular system of linear equations. Hence, the operations count is of the order of $N(N+1)/2$.

The method of Section II and the methods 2 and 3 can determine the natural frequencies, damping ratios and corresponding mode shapes at specified degrees of freedom of interest. That is, (we haven't emphasized this point) one doesn't have to compute these vibration parameters for all m degrees of freedom. However, the system matrices M, C, and K can be determined by the method given above only if the

components of the appropriately normalized characteristic vectors are known for all m degrees of freedom.

In methods 4A and 4B "equivalent" system matrices of order n = 2m are determined first. Thus, one must select m degrees of freedom and collect and use the response data at all of these degrees of freedom. The selection of the data collection stations and the sampling frequency may influence the calculation of the characteristic values and associated characteristic vectors. There are similar difficulties associated with the other methods.

SECTION IV
RESULTS AND CONCLUSIONS

In this section we describe the numerical experiments used in testing the method of Section II and give the results of these experiments. We note again some of the observations made in the previous sections.

Sample problems to test the method of Section II are readily constructed. One uses Eq. (A36) of Appendix A for this purpose. Thus, in Eq. (A36) we assigned appropriate values to λ_k and u_k for k=1 to m. This in turn fixes the values of λ_k, u_k and v_k for k=1 to n=2m. Then for a given vector r and values of ω Eq (A36) gives vectors y. We used the vectors y, obtained in this manner, in the method of Section II to compute the λ_k, u_k and v_k.

For a system with m=4 the computed values of λ_k, u_k and v_k agreed with the assigned values to about 8 decimal places after 10 or so iterations. In Appendix D we display the computed values and assigned values in one case where the characteristic values are all different and in a second case for a characteristic value of multiplicity 2. These results show that the method of Section II is computationally feasible and capable of high accuracy.

The methods discussed in Section III are also capable of high accuracy, at least in principle. However, as is well known, the excessive cost of high accuracy frequently results in compromise. Limitations on computational capabilities also limit the attainable accuracy.

The method of Section II requires the steady state response

to a sinusoidal excitation. Method 2 computes the frequency response function from impulse or random excitation. Method 3 computes an impulse response function also from an impulse-like excitation. Methods 4A and 4B are based upon the free response. Thus, the way in which a structure is excited influences the choice of method for computing vibration parameters. Alternately, first choosing a method for computing vibrations parameters determines how the structure is excited.

From these remarks it is clear that the various methods complement one another. We have noted that the method of Section II assumes the structure is lightly damped. If this is not the case then methods 3 or 4A or 4B are possibly better suited to determining the vibration parameters. On the other hand if one wants the matrices M, C, and K then a method other than method 4A or 4B should be used.

In all the methods there is some stage which involves the determination of the characteristic values λ_k and the modal vectors u_k. In the method of Section II, the numbers λ_k and the vectors u_k are determined directly from the experimental data. In the other methods certain intermediate steps are necessary before calculating the λ_k and u_k. Thus in the methods 4A and 4B one first determines the matrices of a system equivalent (in the sense noted earlier) to the system given by the matrices M, C and K. The λ_k and u_k are then obtained as the solution to the generalized algebraic eigenvalue problem defined by the matrices of the "equivalent" system.

In method 3 one first determines an impulse response function which we denoted by $I_j(t)$. The function $I_j(t)$ is defined as the

solution of a Volterra integral equation of the first kind, Eq. (51). By transforming to the frequency domain, the integral equation problem of method 3 is reduced to the algebraic problem of method 2. See [11]. The impulse response function (or its transform, that is, the frequency response function, is assumed to have a specific form, Eq. (53) or Eq. (50)). After either the impulse response or the frequency response function is known the identification of parameters λ_k and u_k begins.

Our purpose in this report was to present (Section II) a method for determining the characteristic values, modal vectors and the mass, damping and stiffness matrices M, C, and K from the steady state response to sinusoidal excitations at frequencies close to resonance. The mathematical background for this method (Appendix A) with a few additions is also the background for other methods which we described and discussed in Section III. Thus, we were able to note difficulties (e.g., an integration being replaced by a numerical process) with each method.

The effects of these difficulties need to be investigated further. These effects could be determined by analysis but, possibly, more readily by numerical experiments. Thus numerical experiment could determine the computer resources required by each method to obtain the solution to a prescribed accuracy. Secondly, the sensitivity of each method to error in original data could be estimated also by numerical experiment.

APPENDIX A

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In this section we give the mathematical background (notation and assumptions) for procedures for determining mass, stiffness and damping properties of a structure. The mathematics required is a knowledge of the nature and properties of the solutions to a system of second order linear differential equations (Lagrange's equation) with constant real coefficients. The determining of the structural parameters, mathematically speaking, is of the nature of an inverse problem. Specifically, we are concerned with determining the differential operator (those constant coefficients). Such a system of differential equations can be characterized by the solutions of the homogeneous equation and the transposed homogeneous equation. The roots of its characteristic equation and the associated characteristic vectors are required for one such characterization.

Below we develop expressions for solutions of a system of second order linear differential equations with constant, real coefficients. These expressions are obtained from expressions for the solutions to an equivalent first order system of equations. The results obtained are found in texts which discuss systems of ordinary differential equations in some detail, however not always in the explicit form given below. For additional details concerning systems of differential equations see references [12, 13, 14]

Let

$$[MD^2 + CD + K]\, x(t) = f(t) \qquad \text{(A1)}$$

where D indicates differentiation with respect to t, denote a system of second order differential equations. The mass, damping and stiffness matrices, M, C and K, are square matrices of order m with real entries. The vectors $x = x(t)$ and $f = f(t)$ are of dimension m with time dependent components which may be complex valued. The matrices M, C and K are usually symmetric but this condition is not essential for the development given here.

It is assumed that the homogeneous equation

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{A2}$$

has $n = 2m$ linearly independent solutions. The associated characteristic equation is

$$\det [M\lambda^2 + C\lambda + K] = 0 \tag{A3}$$

The left hand side of Eq. (A3) is a polynomial in λ with real coefficients. This polynomial is of actual degree n. Since the coefficients of the characteristic equation are real, the complex conjugate of each complex root of Eq. (A3) is also a root.

If λ is a simple root of the characteristic equation then there is a solution of the homogeneous equation of the form

$$x(t) = u \exp(\lambda t) \tag{A4}$$

where u is a nontrivial constant vector. The vector u satisfies the condition

$$[M\lambda^2 + C\lambda + K]\, u = 0 \tag{A5}$$

and is unique except for a non-zero multiplicative factor. If λ is a root of multiplicity $k > 1$ we assume the matrix $M\lambda^2 + C\lambda + K$ is of rank $m - k$. In this event there is a linear manifold of vectors of dimension k and every vector u in this linear manifold satisfies Eq. (A5).

Accordingly for every such vector u, $x(t) = u \exp(\lambda t)$ is a solution of the homogeneous equation.

The roots of the characteristic equation are oftentimes referred to as characteristic values or eigenvalues. The vectors u which satisfy Eq. (A5), will be referred to as the modal or characteristic vectors or eigenvectors corresponding to the characteristic value λ.

The characteristic equation

$$\det [M^T\lambda^2 + C^T\lambda + K^T] = 0 \tag{A6}$$

of the transposed homogeneous equation

$$M^T\ddot{x} + C^T\dot{x} + K^Tx = 0 \tag{A7}$$

has exactly the same roots as Eq. (A3). For a given λ the matrices $M\lambda^2 + C\lambda + K$ and $M^T\lambda^2 + C^T\lambda + K^T$ have the same rank. Consequently, the dimensions of the solution sets of Eqs. (A2) and (A7), corresponding to a characteristic value λ, are the same. If the matrices M, C and K are symmetric then the solution sets are identical. In either event there are n characteristic values λ_k and associated constant vectors u_k and v_k such that for k = 1 to n

$$x_k(t) = u_k \exp(\lambda_k t)$$

and

$$z_k(t) = v_k \exp(\lambda_k t)$$

are linearly independent solutions of Eqs. (A2) and (A7) respectively.

Set $x_1(t) = x(t)$ and $x_2(t) = \dot{x}_1(t)$. Then the system of first order equations

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & -I \\ K & C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix} \tag{A8}$$

is equivalent to the system given by Eq. (A1). If the matrices M, C, and K are symmetrical one can obtain an equivalent first order system which is also symmetrical [15]. Express Eq. (A8) symbolically as

$$A\dot{y} + By = \tilde{f} \tag{A9}$$

Basic or fundamental solutions of the homogeneous equation

$$A\dot{y} + By = 0 \tag{A10}$$

are of the form $\tilde{u} \exp(\lambda t)$, where $\tilde{u}$ is a vector of dimension n. It is clear that a linearly independent set of modal vectors is given by

$$\tilde{u}_k = \begin{bmatrix} u_k \\ \lambda_k u_k \end{bmatrix} \tag{A11}$$

for k = 1 to n, where λ_k and u_k are the characteristic values and associated modal vectors of a set of linearly independent solutions of Eq. (A2). The characteristic values λ_k satisfy the characteristic equation

$$\det [\lambda A + B] = 0 \tag{A12}$$

and the associated modal vectors $\tilde{u}_k$ satisfy the condition

$$[\lambda_k A + B]\tilde{u}_k = 0 \tag{A13}$$

For the transposed homogeneous first order system

$$A^T\dot{y} + B^T y = 0 \tag{A14}$$

the modal vectors $\tilde{v}_k$, for k = 1 to n, are given by

$$\tilde{v}_k = \begin{bmatrix} -(1/\lambda_k)K^T v_k \\ v_k \end{bmatrix} \tag{A15}$$

In Eq. (A15) λ_k and v_k denote the characteristic values and modal vectors of a set of linearly independent solutions of Eq. (A7). The stiffness matrix K is not required to obtain the v_k from $\tilde{v}_k$. A modal vector $\tilde{v}_k$ corresponding to the characteristic value λ_k satisfies the condition

$$\lambda_k \tilde{v}_k^T A + \tilde{v}_k^T B = 0 \tag{A16}$$

From Eqs. (A13) and (A16) it follows that if $j \neq k$ and $\lambda_j \neq \lambda_k$, then the associated modal vectors $\tilde{v}_j$ and $\tilde{u}_j$ satisfy the orthogonality conditions

$$\tilde{v}_j^T A \tilde{u}_k = 0 \text{ and } \tilde{v}_j^T B \tilde{u}_k = 0 \tag{A17}$$

Even though λ_k takes on the same value for several consecutive values of the index k, the associated model vectors $\tilde{u}_k$ and $\tilde{v}_k$ can be determined so that the orthogonality conditions Eq. (A17) still hold for $j \neq k$. The vectors $\tilde{u}_k$ and $\tilde{v}_k$ may be normalized so that in addition to the orthogonality conditions they satisfy also the conditions

$$\tilde{v}_K^T A \tilde{u}_k = 1 \text{ and } \tilde{v}_k^T B \tilde{u}_k = -\lambda_k \tag{A18}$$

Write

$$U = [\tilde{u}_1 \cdots \tilde{u}_n] \text{ and } V = [\tilde{v}_1 \cdots \tilde{v}_n] \tag{A19}$$

Here U denotes a matrix whose kth column is the vector $\tilde{u}_k$. Similarly, the kth column of the matrix V is the vector $\tilde{v}_k$. Let Λ denote a diagonal matrix with the characteristic values λ_k in the diagonal positions. The relations given in Eqs. (A17) and (A18) can be expressed in matrix form also, namely

$$V^T A U = I \text{ and } V^T B U = -\Lambda \tag{A20}$$

Note that the $\tilde{v}$ vectors corresponding to a characteristic value λ of multiplicity greater than 1 may be assumed to have a special form, at least if the coordinate vectors are appropriately ordered. For definiteness, suppose λ_1 is of multiplicity 2. Let $\tilde{v}_1$ and $\tilde{v}_2$ be linearly independent modal vectors corresponding to λ_1. Then any linear combination of $\tilde{v}_1$ and $\tilde{v}_2$ is also a modal vector

corresponding to λ_1. From this fact it follows that there are linearly independent vectors $\tilde{v}_1$ and $\tilde{v}_2$ corresponding to λ_1 such that, see Eq. (A15), the first component of v_1 is not zero, the second component is zero, the first component of v_2 is zero, and the second component is not zero. Corresponding to these vectors $\tilde{v}_1$ and $\tilde{v}_2$ of special form there are vectors $\tilde{u}_1$ and $\tilde{u}_2$ which satisfy the orthogonality and normality conditions, Eqs. (A17) and (A18).

Thus whatever the degree of multiplicity of λ greater than 1, we may assume that there is a linearly independent set of $\tilde{v}$ vectors, the v portion of which has the special form just described for some consecutive set of components. Moreover, corresponding to these $\tilde{v}$ vectors of special form there are $\tilde{u}$ vectors which satisfy the orthogonality conditions Eq. (A17). The normality conditions can be met by appropriately adjusting (there are infinitely many ways) the magnitudes of the $\tilde{u}$ and $\tilde{v}$ vectors.

Let $\tilde{\Lambda} = \tilde{\Lambda}(t)$ denote the diagonal matrix of order n for which the kth diagonal element is $\exp(\lambda_k t)$. Set (see Eq. (A19))

$$\tilde{U}(t) = U\tilde{\Lambda} \tag{A21}$$

Then

$$\dot{\tilde{U}}(t) = U\tilde{\Lambda}\Lambda = U\tilde{\Lambda}\Lambda \tag{A22}$$

and

$$\dot{\tilde{A}}\tilde{U} + B\tilde{U} = [AU\Lambda + BU]\tilde{\Lambda} = 0 \tag{A23}$$

We now seek a particular solution y(t) of Eq. (A9) where we suppose

$$y(t) = \tilde{U}(t)z(t) \tag{A24}$$

and the vector function z(t) is to be determined. Substitute $\dot{y}(t)$ and y(t), as determined from Eq. (A24), into Eq. (A9) and use the results,

Eq. (A23) to obtain

$$AU\,\tilde{\Lambda}\,\dot{z}(t) = \tilde{f}(t) \tag{A25}$$

Multiplying Eq. (A25) by $\tilde{\Lambda}^{-1}(t)V^T$, we obtain

$$z(t) = \int_0^t \tilde{\Lambda}^{-1}(\tau)v^T\tilde{f}(\tau)\,d\tau \tag{A26}$$

Hence

$$y(t) = \int_0^t U\,\tilde{\Lambda}(t-\tau)V^T\tilde{f}(\tau)\,d\tau \tag{A27}$$

Partition the matrices U, $\tilde{\Lambda}(t-\tau)$ and V^T into (m x m)-submatrices

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad \tilde{\Lambda}(t-\tau) = \begin{bmatrix} \tilde{\Lambda}_{11} & 0 \\ 0 & \tilde{\Lambda}_{22} \end{bmatrix} \tag{A28}$$

and

$$V^T = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

This partitioning is in conformity with the partitioning of the vectors $\tilde{u}$, $\tilde{v}$ and y, Eqs. (A11), (A15) and (A8). One readily obtains from Eq. (A27)

$$x_1(t) = \int_0^t [U_{11}\tilde{\Lambda}_{11}(t-\tau)V_{12} + U_{12}\tilde{\Lambda}_{22}(t-\tau)V_{22}]f(\tau)\,d\tau \tag{A29}$$

It is clear that the term $U_{11}\tilde{\Lambda}_{11}(t-\tau)V_{12}f(\tau)$ represents

$$[\exp\lambda_1(t-\tau)u_1 \cdots \exp\lambda_m(t-\tau)u_m] \begin{bmatrix} v_1^T\,\tilde{f}(\tau) \\ \vdots \\ v_m^T\,\tilde{f}(\tau) \end{bmatrix} \tag{A30}$$

$$= [\exp\lambda_1(t-\tau)u_1v_1^T \cdots \exp\lambda_m(t-\tau)u_mv_m^T]\,f(\tau)$$

A similar expression is obtained for the term $U_{12}\tilde{\Lambda}_{22}(t-\tau)V_{22}f(\tau)$.

Thus we obtain, in dyadic form, the expression for $x_1(t)$

$$x_0(t) = \sum_{k=1}^{n} \int_0^t u_k v_k^T \exp \lambda_k(t - \tau) f(\tau)\, d\tau \qquad \text{(A31)}$$

Note that the subscript on x has changed from 1 to 0. The use of the subscript 1 is discontinued because we no longer care to emphasize the relation to Eq. (A8). The subscript 0 is used because $x_0(t)$ as given by Eq. (A31) is the particular solution which satisfies Eq. (A1) with zero initial conditions.

Take $f(t) = h(t)r_j$ where $h(t) = 0$ for $t < 0$, $h(t) = 1$ for $t \geq 0$ and r_j denotes the jth coordinate vector. The jth component of r_j is 1 and all other components are zero. Also, let v_{jk} denote the jth component of the vector v_k. From Eq. (A31) the step response $H_j(t)$ due to a unit step excitation at the jth station is

$$H_j(t) = \int_0^t \sum_{k=1}^{n} u_k v_{jk} \exp \lambda_k(t - \tau)\, d\tau \qquad \text{(A32)}$$

That is

$$H_j(t) = -\sum_{k=1}^{n} (i/\lambda_k)\, u_k v_{jk} + \sum_{k=1}^{n} (1/\lambda_k)\, u_k v_{jk} \exp (\lambda_k t) \qquad \text{(A33)}$$

The impulse response $I_j(t)$ is obtained from the step response $H_j(t)$ by differentiation. Thus differentiating Eq. (A33) gives

$$I_j(t) = \sum_{k=1}^{n} u_k v_{jk} \exp (\lambda_k t) \qquad \text{(A34)}$$

The response $x_\omega(t)$ to the harmonic excitation r exp (iωt) is

$$x_\omega(t) = -\sum_{k=1}^{n} [u_k v_k^T \cdot r \exp(\lambda_k t)] + \sum_{k=1}^{n} [u_k v_k^T \cdot r \exp(i\omega t)]/(i\omega - \lambda_k) \tag{A35}$$

In Eq. (A35) the response $x_\omega(t)$ is expressed as the "sum" of two sums. The first sum is a linear combination of solutions of the homogeneous equation, Eq. (A2), called the complementary function. The second sum is a particular solution of Eq. (A1) with f(t) = r exp (iωt). If the characteristic values λ_k are complex numbers with Re $[\lambda_k] < 0$ then the complementary function goes to zero as t becomes large. The complementary function is a transient and the particular solution is called the steady state solution.

Whether the complementary function is transient or not, set

$$y = [\sum_{1}^{n} u_k v_k^T/(i\omega - \lambda_k)] \cdot r \tag{A36}$$

Then y exp(iωt) is a particular solution of Eq. (A1) when f(t) = r exp(iωt). For the time being we will refer to y exp(iωt) as the steady state solution. Many procedures for determining values for structural parameters are based on Eq. (A36). It follows that

$$[-\omega^2 M + K + i\omega C]\, y = r \tag{A37}$$

and from this equation that

$$[-\omega^2 M + K - i\omega C]\, \bar{y} = r \tag{A38}$$

Then (by linearity)

$$[MD^2 + CD + K]\, [y \exp(i\omega t) - \bar{y} \exp(-i\omega t)]/2i = r \sin \omega t \tag{A39}$$

That is, $[y \exp(i\omega t) - \bar{y} \exp(-i\omega t)]/2i$ is a particular response to the sinusoidal excitation $r \sin\omega t$.

This response to the sinusoidal excitation can be rewritten as

$$[y \exp(i\omega t) - \bar{y} \exp(-i\omega t)]/2i = [(y - \bar{y})/2i] \cos \omega t + [(y + \bar{y})/2] \sin \omega t$$
$$= \text{Im } [y] \cos \omega t + \text{Re } [y] \sin \omega t \qquad \text{(A40)}$$

From this equation it is clear that if the steady state response to a sinusoidal excitation $r \sin \omega t$ is known then the steady state response to the harmonic excitation $r \exp(i\omega t)$ is known and conversely.

Let a_j denote the $j^{\underline{th}}$ component of Re [y] and b_j the $j^{\underline{th}}$ component of Im [y]. Then in the usual fashion we have

$$a_j \sin \omega t + b_j \cos \omega t = [a_j^2 + b_j^2]^{1/2} \sin (\omega t + \theta_j) \qquad \text{(A41)}$$

when

$$\tan \theta_j = b_j/a_j \qquad \text{(A42)}$$

Eq. (A41) provides an alternate expression for the steady state response to a sinusoidal excitation.

The Eqs. (A36) and (A37) contain an important relationship, namely

$$[-\omega^2 M + K + i\omega C]^{-1} = \sum_1^n u_k v_k^T/(i\omega - \lambda_k) \qquad \text{(A43)}$$

The right hand side of Eq. (A43) is the frequency response function. If the modal vectors u_k and v_k and the characteristic values λ_k are known, then Eq. (A43) can be used to determine the matrices M, C and K. We must remember that Eq. (A43) is not true for arbitrary modal vectors u_k and v_k. The Eq. (A43) was obtained by assuming the vectors u_k and v_k appropriately normalized.

Let $f_j(t)$ denote the $j^{\underline{th}}$ component of the vector f(t) and as before,

v_{jk} the j<u>th</u> component the vector v_k. Then

$$v_k^T f(t) = \sum_{j=1}^{m} v_{jk} f_j(t) \tag{A44}$$

Using this result we can rewrite Eq. (A31) as

$$x_0(t) = \int_0^t \sum_{k=1}^{n} u_k \left(\sum_{j=1}^{m} v_{jk} f_j(\tau) \right) \exp \lambda_k (t - \tau) d\tau$$

$$= \int_0^t \left[\sum_{j=1}^{m} \left(\sum_{k=1}^{n} u_k v_{jk} \exp \lambda_k (t - \tau) \right) f_j(\tau) \right] d\tau$$

That is, we obtain

$$x_0(t) = \int_0^t \left(\sum_{j=1}^{m} I_j(t - \tau) \cdot f_j(\tau) \right) d\tau \tag{A45}$$

The Eq. (A45) expresses the response $x_0(t)$ as a sum over j of the convolution of the j<u>th</u> impulse response with the j<u>th</u> component of the excitation.

APPENDIX B

THE FOURIER INTEGRAL, FOURIER SERIES AND TRIGONOMETRIC INTERPOLATING POLYNOMIALS

In this section we exhibit the relation between the Fourier series representation, a trigonometric interpolating polynomial representation and the Fourier Transform of a real valued function f(t). The results of this section may be found or ferreted out of standard references such as [7, 15, 16]. They are included here for completeness and ready reference. Let us assume that the function f(t) satisfies whatever conditions needed so that any indicated operations on or representations of f(t) are valid.

If f(t) is periodic of period τ then f(t) is representable as a trigonometric series.

$$f(t) = a_0/2 + \sum_1^\infty (a_n \cos n\omega t + b_n \sin n\omega t) \tag{B1}$$

where $\omega = 2\pi/\tau$. The Fourier coefficients are given by the equations

$$a_n = (2/\tau) \int_0^\tau f(t) \cos n\,\omega\, dt$$

$$b_n = (2/\tau) \int_0^\tau f(t) \sin n\,\omega\, dt \tag{B2}$$

Substituting the complex form

$$\cos n\omega t = [\exp(in\omega t) + \exp(-in\omega t)]/2 \tag{B3}$$

and

$$\sin n\omega t = [\exp(in\omega t) - \exp(-in\omega t)]/2i \tag{B4}$$

in Eq. (B1) and collecting like terms in exp (inωt) and exp (-inωt) one obtains

$$f(t) = a_0/2 + (1/2) \sum_{n=1}^{\infty} [(a_n - ib_n) \exp (in\omega t) + (a_n + ib_n) \exp (-in\omega t)] \tag{B5}$$

Next, setting

$$\begin{aligned} C_0 &= a_0/2 \\ C_n &= (a_n - ib_n)/2 \\ C_{-n} &= (a_n + ib_n)/2 \end{aligned} \tag{B6}$$

(note $C_{-n} = \bar{C}_n$) one obtains

$$f(t) = \sum_{-\infty}^{\infty} C_k \exp (ik\omega t) \tag{B7}$$

the Fourier series representation of f(t) in complex form. The complex Fourier coefficients C_k are given by

$$C_k = \frac{1}{\tau} \int_0^{\tau} f(t) \exp (-in\omega t)\, dt \tag{B8}$$

A function $\hat{f}(t)$ defined by the equation

$$\hat{f}(t) = \sum_{k=1}^{N} \hat{C}_k \exp (i\omega kt) = \sum_{k=1}^{N} \hat{C}_k \exp (2\pi ikt/\tau) \tag{B9}$$

is a trigonometric interpolation polynomial in complex form. The complex coefficients $\hat{C}_k$ are determined by the function values assigned to $\hat{f}(t)$ at the points $t = j\tau/N$ for $j = 1$ to N. First, we obtain a rule for computing the coefficients $\hat{C}_k$. Next, if f(t) is a periodic function of period τ and if $\hat{f}(t)$ is the trigonometric interpolating polynomial satisfying the conditions

$$\hat{f}(j\tau/N) = f(j\tau/N) \tag{B10}$$

for j = 1 to N, we obtain the relation between the coefficients $\hat{C}_k$ of the interpolating polynomial and C_k of the Fourier series.

The desired results are obtained from readily established properties of the N, $N^{\underline{th}}$ roots of unity. The principal $N^{\underline{th}}$ root of unity is

$$r_1 = \exp(2\pi i/N) \tag{B11}$$

Then

$$r_1^N = 1 \tag{B12}$$

and

$$\begin{aligned} r_1^N - 1 &= 0 \\ &= (r_1 - 1)(r_1^{N-1} + r_1^{N-2} + \cdots + r_1 + 1) \end{aligned} \tag{B13}$$

Since for $N \neq 1$, $r_1 \neq 1$ it follows that

$$r_1^{N-1} + r_1^{N-2} + \cdots + r_1 + 1 = 0 \tag{B14}$$

Set

$$r_k = r_1^k = \exp(2\pi ik/N). \tag{B15}$$

for k = 1 to N (or k = 0 to N - 1, as convenient). Then

$$r_k^N = 1$$

and it follows

$$r_k^{N-1} + r_k^{N-2} + \cdots + r_k + 1 = 0 \tag{B16}$$

provided $k \neq N$ (or 0) and for $k = N$ (or 0)

$$r_N^{N-1} + r_N^{N-2} + \cdots + r_N + 1 = N \tag{B17}$$

From Eqs. (B15) and (B14) it follows that

$$r_1 + r_2 + \cdots + r_N = 0 \tag{B18}$$

and from Eq. (15)

$$r_k^j = r_j^k \tag{B19}$$

Note also that

$$\begin{aligned} \bar{r}_k &= \exp(-2\pi ik/N) = \exp(2\pi iN/N) \cdot \exp(-2\pi ik/N) \\ &= r_{N-k} \end{aligned} \tag{B20}$$

Consider the matrix

$$M = \begin{bmatrix} r_1^1 & r_1^2 & r_1^3 & \cdots & r_1^N \\ r_2^1 & r_2^2 & r_2^3 & \cdots & r_2^N \\ & & \vdots & & \\ r_N^1 & r_N^2 & r_N^3 & \cdots & r_N^N \end{bmatrix} \tag{B21}$$

Since $r_k^j = r_j^k$ the matrix M is symmetric. The matrix product

$$\bar{M} \cdot M = [s_{j\ell}] \tag{B22}$$

where

$$s_{j\ell} = \sum_{k=1}^{N} \bar{r}_j^k \cdot r_k^\ell \tag{B23}$$

1) If $\ell = j$ then

$$s_{jj} = \sum_{k=1}^{N} \exp(-2\pi ijk/N) \exp(2\pi ijk/N) = N \tag{B24}$$

2) If $\ell \neq j$ then

$$s_{j\ell} = \sum_{k=1}^{N} r_{N-j}^{k} \cdot r_{\ell}^{k} = \sum_{k=1}^{N} r_{N+\ell-j}^{k} = 0 \tag{B25}$$

since $r_{N+\ell-j} \neq 1$. It follows from Eqns. (B24) and (B25) that

$$\bar{M} M = M \bar{M} = N I \tag{B26}$$

From Eq. (B9) for $t = j\tau/N$ we have

$$\hat{f}(j\tau/N) = \sum_{k=1}^{N} \hat{C}_k r_j^k \tag{B27}$$

for $j = 1$ to N. This system of equations can be expressed in vector form

$$\begin{bmatrix} \hat{f}(\tau/N) \\ \hat{f}(2\tau/N) \\ \vdots \\ \hat{f}(\tau) \end{bmatrix} = M \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \\ \vdots \\ \hat{C}_N \end{bmatrix}$$

It follows from Eq. (B26) that

$$\begin{bmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{bmatrix} = (1/N)\bar{M} \begin{bmatrix} \hat{f}(\tau/N) \\ \vdots \\ \hat{f}(\tau) \end{bmatrix} \tag{B29}$$

Eqns. (B28) and (B29) exhibit the relation between the coefficients $\hat{C}_k$ and the values of the function $\hat{f}(t)$ at the points $t = j\tau/N$ for $j = 1$ to N. Thus if we have any (real valued) function $f(t)$ which is well defined at the points $j\tau/N$ for $j = 1$ to N and if we set $\hat{f}(j\tau/N) = f(j\tau/N)$ then Eq. (B29) determines the coefficients $\hat{C}_k$ and Eq. (B9) defines a periodic function of period τ which satisfies the conditions $\hat{f}(j\tau/N) = f(j\tau/N)$ for $j = 1$ to N.

From Eq. (B29) we see that the coefficients $\hat{C}_k$ are obtained as the product of a vector multiplied by a matrix. If one defines an operation as a multiplication of two (complex) numbers followed by an addition, then the product of an N-vector by a matrix of order N, in general, requires N^2 operations. An algorithm is developed in [17] which, in principle, determines the C_k's in less than $2\ N \log_2 N$ operations. Implementations of this algorithm are referred to as Fast Fourier Transformers.

Let $f(t)$ denote a periodic function of period τ and let $\hat{f}(t)$ be the trigonometric polynomial which interpolates $f(t)$ at the points $t = j\tau/N$ for $j = 1$ to N. For $t = j\tau/N$, for $j = 1$ to N we have by Eq. (B7)

$$\begin{aligned} f(j\tau/N) &= \sum_{-\infty}^{\infty} C_k \exp(2\pi ikj/N) \\ &= \sum_{-\infty}^{\infty} C_k [\exp 2\pi ik/N]^j \end{aligned} \tag{B30}$$

Now we can express the integer k, $-\infty < k < \infty$, as

$$k = \ell \cdot N + k_1 \tag{B31}$$

where $0 \leq k_1 \leq N - 1$. Hence

$$\exp 2\pi ik/N = \exp 2\pi ik_1/N \tag{B32}$$

It follows that

$$\sum_{-\infty}^{\infty} C_k [\exp 2\pi ik/N]^j = \sum_{-\infty}^{\infty} C_{\ell N} + [\exp 2\pi i/N]^j \sum_{\infty}^{\infty} C_{\ell N+1} + \cdots + [\exp 2\pi i(N-1)/N]^j \sum_{-\infty}^{\infty} C_{\ell N + N-1} \tag{B33}$$

and then from Eqs. (B27) and (B33) that for k = 1 to N - 1

$$\hat{C}_k = \sum_{-\infty}^{\infty} C_{\ell N+k} \tag{B34}$$

and

$$\hat{C}_N = \sum_{-\infty}^{\infty} C_{\ell N}$$

Let f(t) be defined only for $0 \leq t \leq \tau$. Here now we will exhibit the relation between the Fourier coefficients of some periodic extensions of f(t) and particular values of the Fourier transform of f(t). First, however, let us describe what we mean when f(t) is extended periodically "in the standard way". It is clear that for any value of t, $-\infty < t < \infty$ we can write

$$t = a\tau + t_1 \tag{B35}$$

and $0 \leq t_1 < \tau$. Then we set

$$f(t) = f(t_1) \tag{B36}$$

The complex Fourier coefficients of f(t) are given by Eq. (B8)

$$\tau C_k = \int_0^{\tau} f(t) \exp(-ik\omega t)\, dt \tag{B37}$$

(It is instructive to plot, or at least, draw some vertical line

segments representing $\tau|C_k|$ at points $k\omega$ on a horizontal axis for $k = 0, \pm 1, \pm 2, \cdots$.

Suppose we define

$$f_1(t) = f(t) \text{ for } 0 \leq t \leq \tau$$
$$f_1(t) = 0 \quad \text{for } \tau < t \leq 2\tau \tag{B38}$$

and then extend $f_1(t)$ periodically in the standard way. The complex Fourier Coefficients for $f_1(t)$ are

$$2\tau\, C_k^{(1)} = \int_0^{\tau} f(t) \exp(-ik\omega t/2)\, dt \tag{B39}$$

Observe that

$$2\tau\, C_{2k}^{(1)} = \tau C_k \tag{B40}$$

for $k = 0, \pm 1, \pm 2, \cdots$. We note that doubling the interval from τ to 2τ has halved the distance between successive plots of the values $2\tau|C_k^{(1)}|$ as compared to successive plots of the values $\tau|C_k|$. Repeating this process, that is, increase the interval from 2τ to 4τ define

$$f_2(t) = f(t) \text{ for } 0 \leq t \leq \tau$$
$$= 0 \quad \text{for } \tau < t \leq 4\tau \tag{B41}$$

and extend $f_2(t)$ periodically in the standard way then the distance between successive plots of the values $4\tau|C_k^{(2)}|$ is half the distance between successive plots of the values $2\tau|C_k^{(1)}|$. Thus we can make the number of complex Fourier Coefficients which plot in some fixed interval about the origin as large as we please by choosing the interval

$\tau < t \leq b$ sufficiently large.

If now we define the Fourier Transform of f(t) by the equation

$$\phi(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_{0}^{\tau} f(t)\, e^{-i\omega t}\, dt \tag{B42}$$

then it is clear that

$$\phi(2\pi k/\tau) = \tau C_k \tag{B43}$$

for $n = 0, \pm 1, \pm 2, \cdots$ and that

$$\phi(\pi k/\tau) = 2\tau C_k^{(1)} \tag{B44}$$

for $n = 0, \pm 1, \pm 2, \cdots$, and so on.

The Fourier series representation and Fourier transform of a function f(t) are not readily available for use in general because of the intergration process required for the determination of the Fourier coefficient and the Fourier transform. However, in view of the relations exhibited above between the Fourier coefficients, certain values of the Fourier transform and the coefficients of the trigonometric interpolating polynomial it is clear that if the trigonometric polynomial interpolates f(t) (or an appropriate periodic extension of f(t)) at sufficiently many points, the coefficients of the trigonometric polynomial will be good approximations of the Fourier coefficients and, when properly scaled, of these certain values of the Fourier transform of f(t).

APPENDIX C
SYSTEMS OF LINEAR DIFFERENCE EQUATIONS

In this section we review the material which is basic for the two methods for determining certain structural parameters given in [8,9] and [10] respectively.

For the first method given in [8,9] the system of m second order equations

$$M\ddot{x} + C\dot{x} = Kx = f \tag{C1}$$

is exchanged for the system of n = 2m first order equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -M^{-1}Kx_1 - M^{-1}Cx_2 + M^{-1}f \end{aligned} \tag{C2}$$

This system may be written as

$$\dot{y} = Ay + \tilde{f} \tag{C3}$$

where the matrix A in partitioned form is given by

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \tag{C4}$$

Suppose we know a nontrivial solution y(t) of the homogeneous equation

$$\dot{y} = Ay \tag{C5}$$

For n different values of time t_k, k = 1 to n set

$$y_k = y(t_k) \tag{C6}$$

and

$$\dot{y}_k = dy/dt|_{t = t_k} \tag{C7}$$

From Eq. (C5) we obtain the matrix equation

$$[\dot{y}_1 \cdots \dot{y}_n] = A[y_1 \cdots y_n] \tag{C8}$$

It follows that if the matrix $[y_1 \cdots y_n]$ is nonsingular then

$$A = [\dot{y}_1 \cdots \dot{y}_n]\ [y_1 \cdots y_n]^{-1} \tag{C9}$$

We shall see that the homogeneous solution $y(t)$ must be a linear combination of the n linearly independent solutions defined by the modal vectors and characteristic values, Appendix A. Thus we may write

$$y(t) = \tilde{u}_1 \exp(\lambda_1 t) + \cdots + \tilde{u}_n \exp(\lambda_n t) \tag{C10}$$

where here $\tilde{u}_k$ denotes either the $k^{\underline{th}}$ modal vector or the zero vector. For brevity let us write also

$$z_{jk} = \exp(\lambda_j t_k) \tag{C11}$$

Then the matrix $[y_1 \cdots y_n]$ may be expressed as the product of two matrices

$$[y_1 \cdots y_n] = [\tilde{u}_1\ \tilde{u}_2 \cdots \tilde{u}_n] \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} \tag{C12}$$

It is clear from Eq. (C12) that if for any values of $k, \tilde{u}_k$ is the zero vector then the matrix $[y_1 \cdots y_n]$ is singular.

The next task is the determination of functions $y(t)$ and $\dot{y}(t)$ which satisfy the condition $\dot{y} = Ay$ when the given information is the second derivative $\ddot{x}(t)$ of a function $x(t)$ which satisfies $M\ddot{x} + Cx + Kx = 0$. Let us set

$$\ddot{x}(t) = \hat{x}(t) + \hat{x}_0 \tag{C13}$$

Hence by Eq. (C2) we have

$$\dot{x}_2(t) = \hat{x}(t) + \hat{x}_0 \tag{C14}$$

$$\dot{x}_1(t) = x_2(t) = \int_0^t [\hat{x}(\tau) + \hat{x}_0]\, d\tau + C_1 \tag{C15}$$

and

$$x_1(t) = \int_0^t d\tau \int_0^\tau [\hat{x}(s) + \hat{x}_0]\, ds + C_1 t + C_2 \tag{C16}$$

The quantities $x_1(t)$, $\dot{x}_1(t)$, $x_2(t)$ and $\dot{x}_2(t)$ defined by Eqs. (C14), (C15) and (C16) satisfy the equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{C17}$$

Now since Eq. (C17) holds in particular for $t = 0$, the Eq. (C17) simplifies to

$$\begin{bmatrix} \int_0^t [\hat{x}(\tau) + \hat{x}_0]\, d\tau \\ \hat{x}(t) \end{bmatrix} = A \begin{bmatrix} \int_0^t d\tau \int_0^\tau [\hat{x}(s) + \hat{x}_0]\, ds + C_1 t \\ \int_0^t [\hat{x}(\tau) + \hat{x}_0]\, d\tau \end{bmatrix} \tag{C18}$$

All the quantities in Eq. (C18) are known except the integration constant C_1. Replace t by αt in Eq. (C18). One then has

$$\begin{bmatrix} \int_0^{\alpha t} [\hat{x}(\tau) + \hat{x}_0]\, d\tau \\ \hat{x}(\alpha t) \end{bmatrix} = A \begin{bmatrix} \int_0^{\alpha t} d\tau \int_0^{\tau} [\hat{x}(s) + \hat{x}_0]\, ds + C_1 \alpha t \\ \int_0^{\alpha t} [\hat{x}(\tau) + \hat{x}_0]\, d\tau \end{bmatrix} \tag{C19}$$

Finally multiply Eq. (C18) by α and substract from Eq. (C19). One obtains an equation which contains the matrix A but no other unknown quantities.

Thus from the above discussion we obtain for the vector functions $\dot{y}(t)$ and $y(t)$

$$\dot{y}(t) = \begin{bmatrix} \int_0^{\alpha t} [\hat{x}(\tau) + \hat{x}_0]\, d\tau - \alpha \int_0^{t} [\hat{x}(\tau) + \hat{x}_0]\, d\tau \\ \hat{x}(\alpha t) - \hat{x}(t) \end{bmatrix} \tag{C20}$$

and

$$y(t) = \begin{bmatrix} \int_0^{\alpha t} d\tau \int_0^{\tau} [\hat{x}(s) + \hat{x}_0]\, ds - \alpha \int_0^{t} d\tau \int_0^{\tau} [\hat{x}(s) + \hat{x}_0]\, ds \\ \int_0^{\alpha t} [\hat{x}(\tau) + x_0]\, d\tau - \alpha \int_0^{t} [\hat{x}(\tau) + \hat{x}_0]\, d\tau \end{bmatrix} \tag{C21}$$

The quantities $y(t)$ and $\dot{y}(t)$ are computed from the measured quantity $\ddot{x}(t)$. They are obtained by a numerical integration process which has been made somewhat more laborius by the necessary elimination of an integration constant. Convenient values for α are readily apparent. Let us summarize the discussion of this section up to this point. We noted above that the homogeneous system of equations ($f \equiv 0$), Eq. (C1)

has a set of linearly independent solutions. Next given a linear combination of all these linearly independent solutions, the matrix A of an equivalent first order system, Eq. (C5), can be determined from the known values of this solution of the homogeneous system Eq. (C1).

The second method, given in [10], is similar in principle to the first. However, it avoids the numerical integration, Eqs. (C20) and (C21) required by the first method. The procedure which enables one to bypass the numerical integration is described in [7, pp. 272-280].

This second method is based on the fact that corresponding to the homogeneous differential equation, Eq. (C1), there is a companion difference equation (of second differences) having the same solution set. Then, as above, given a linear combination of all the members of a linearly independent set of solutions of the homogeneous equation ($f \equiv 0$), Eq. (C1), one can determine an equivalent equation of first differences. Let us explain in detail the remark just made.

Consider a system of equations of second differences

$$x(t + 2h) + Bx(t + h) + Cx(t) = 0 \tag{C22}$$

where B and C denote matrices of order m and x(t) is a vector valued function of t of dimension m. It is clear that

$$x(t) = u \exp(\lambda t) \tag{C23}$$

satisfies Eq. (C22) if

$$\eta = \exp(\lambda h)$$

satisfies the characteristic equation

$$\det[\eta^2 I + B\eta + C] = 0 \tag{C24}$$

and u is a corresponding characteristic vector satisfying the condition

$$[\eta^2 I + B\eta + C]u = 0 \tag{C25}$$

Set

$$\begin{aligned} x_1(t + h) - x_2(t) &= 0 \\ x_2(t + h) + Cx_1(t) + Bx_2(t) &= 0 \end{aligned} \tag{C26}$$

The system of equations, Eq. (C26) can be written as

$$\begin{bmatrix} x_1(t + h) \\ x_2(t + h) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ C & B \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 0 \tag{C27}$$

a system of equations of first differences which is equivalent to Eq. (C22).

Let

$$x(t) = \sum_{k=1}^{n} u_k \exp(\lambda_k t) \tag{C28}$$

n = 2m, be a sum of linearly independent functions which satisfy the conditions Eqs. (C24) and (C25). If the values of x(t) are known at times t_k and $t_k + h$ for k = 1 to N then we have the matrix equation

$$\begin{bmatrix} x_1(t_1 + h) & \cdots & x_1(t_n + h) \\ x_2(t_1 + h) & \cdots & x_2(t_n + h) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ C & B \end{bmatrix} \begin{bmatrix} x_1(t_1) & \cdots & x_1(t_n) \\ x_2(t_1) & \cdots & x_2(t_n) \end{bmatrix} = 0 \tag{C29}$$

Eq. (C29) is solvable for the matrix of the system (C27) if the matrix

$$\begin{bmatrix} x_1(t) & \cdots & x_1(t_n) \\ x_2(t) & \cdots & x_2(t_n) \end{bmatrix}$$

is nonsingular

From this discussion it is clear that given a function x(t) as defined by Eq. (C28) one can determine a system of equations of first differences which the n-dimensional vector valued function

$$y(t) = \begin{bmatrix} x(t) \\ x(t + h) \end{bmatrix} \tag{C30}$$

satisfies. Thus we determine the matrices

$$\Phi = \begin{bmatrix} x(t_1) & \cdots & x(t_n) \\ x(t_1 + h) & \cdots & x(t_n + h) \end{bmatrix} \tag{C31}$$

and

$$\tilde{\Phi} = \begin{bmatrix} x(t_1 + h) & \cdots & x(t_n+h) \\ x(t_1 + 2h) & \cdots & x(t_n + 2h) \end{bmatrix} \tag{C32}$$

Then, if Φ is nonsingular, y(t) defined by Eq. (C30) satisfies the system of equations of first differences

$$y(t + k) - \tilde{\Phi}\, \Phi^{-1}\, y(t) = 0 \tag{C33}$$

APPENDIX D

COMPUTER PROGRAM

We now describe the organization of, and the calling procedure for invoking the use of, a collection of subprograms, written in FORTRAN to implement the method described in Section II. It accepts, as input, the frequencies ω_{k1} and ω_{k2}, the vectors Y_{k1} and Y_{k2} (obtained from experimental data), together with various other data, and produces, as output, the characteristic values, λ_k, the modal vectors, U_k and V_k, and the mass, damping, and stiffness matrices, M, C, and K. A section on program testing is also included.

ORGANIZATION - Subroutine AAS, called by the user, allocates virtual storage in the work array, W, and calls subprogram BBS. BBS monitors and controls the computations and calls into use the following subprograms:

EIGEN - computes an approximation to a characteristic value, λ_k, given the associated frequencies ω_{k1}, ω_{k2}, and the vectors Y_{k1} and Y_{k2} (or $\hat{Y}_{k1}$ and $\hat{Y}_{k2}$). See Eqs. (7), (8), (14) and (15), Section II.

VUS - computes an approximation to a modal vector U_k and the first component of the associated vector V_k, given the frequencies ω_{k1}, and ω_{k2}, the characteristic value, λ_k, and the vectors Y_{k1} and Y_{k2} (or $\hat{Y}_{k1}$ and $\hat{Y}_{k2}$). See Eqs. (10),(11),(16), and (17), Section II.

YVECS - computes $\hat{Y}_{k1}$ and $\hat{Y}_{k2}$, given the vectors Y_{k1} and Y_{k2}, the frequencies ω_{k1}, ω_{k2}, and the current approximations to the characteristic value, λ_k, and the modal vectors U_k and V_k. See Eq. (13), Section II.

RECVR - computes the system matrices M, C, and K, given the characteristic values, λ_k, and the modal vectors U_k and V_k. See Eqs. (38)-(43), Section II.

INV, MTMT, LU3 - used by RECVR to perform various matrix operations.

Together, these subprograms utilize 10 cells of temporary work space in blank COMMON, in addition to the work array, W.

CALLING PROCEDURE - The calling program, written by the user, will contain the equivalent of the following FORTRAN statements:

```
EXTERNAL DATA
COMPLEX LAMBDA(M),U(M,M),V(M,M)
REAL EM(M,M),C(M,M),KA(M,M),W(NW)
CALL AAS (M,DATA,TOL,IMAX,LAMBDA,U,V,EM,C,KA,I,W)
```

where:

M is an integer input variable, indicating an m by m system of second order differential equations.

DATA is the name of a user written subroutine subprogram, called by BBS to obtain certain input data. The form of this is described in a later section. The name "DATA" is symbolic, of course, and can be any legal subroutine name that does not conflict with ones in use in the collection. (See list under ORGANIZATION)

TOL is a real input variable, the tolerance used to control program convergence. Let λ_k^i be the ith iterate of λ_k. The process is said to converge, no further iterations being performed, if $|\lambda_k^i - \lambda_k^{i-1}| < TOL$, for all k, k=1,2,...,m. Some care should be exercised here to

ensure that TOL is not unreasonably small.

IMAX is an integer input variable, the maximum number of iterations allowed. $15 \leq IMAX \leq 30$ would probably be reasonable.

LAMBDA is a complex output array, of dimension m, containing the characteristic values, λ_k, k=1,2,..,m.

U,V are complex, m by m, output arrays, containing the modal vectors, U_k and V_k, respectively. U(I,K) will contain the I-th component of vector U_k, and similarly for V(I,K) and V_k.

EM,C,KA are real, m by m, output arrays, containing the mass damping, and stiffness matrices, M, C, and K, respectively.

I is an integer output variable, containing the number of iterations required for convergence. If convergence is not achieved, I is set to IMAX +1.

W is an array used by the subprograms as temporary work space. It must be dimensioned at least $4m^2+6m$.

DESCRIPTION OF SUBROUTINE DATA - this subprogram, written by the user, is called by BBS to provide it with the following data:

1. the frequencies ω_{k1} and ω_{k2}, k=1,2,...,m.
2. the vectors Y_{k1} and Y_{k2}, k=1,2,...,m; and of length m.
   These are obtained from experimental data.

This subprogram should contain the equivalent of the following FORTRAN statements:

```
SUBROUTINE DATA (M,OMEGA,Y1,Y2)
REAL OMEGA(N)
COMPLEX Y1(M,M),Y2(M,M)
```

where:

M is an integer input variable, the same M as provided in the initial call to AAS.

OMEGA is a real output variable, of dimension N=2M, which will contain ω_{k1} and ω_{k2}, as follows:

OMEGA(1) will contain ω_{11}

OMEGA(2) will contain ω_{21}

. .
. .
. .

OMEGA(M) will contain ω_{m1}

OMEGA(M+1) will contain ω_{12}

. .
. .
. .

OMEGA(N) will contain ω_{m2}

Y1,Y2 are complex, m by m, output arrays, which will contain the vectors Y_{k1} and Y_{k2}, respectively. Y1(I,K) will contain the I-th component of vector Y_{k1}, I,k=1,2...,m; and similarly for Y2(I,K) and Y_{k2}.

PROGRAM TESTING - The results of two test cases with m=4 are given in Tables I and II. For these tests the values of λ_k, U_k and V_k were specified. Then given the input frequencies, ω_{k1} and ω_{k2}, we computed the vectors Y_{k1} and Y_{k2}, using Eq. (2) of Section II. Using a tolerance of 10^{-8}, convergence was achieved in 11 iterations. The following tables display the agreement achieved between the specified and computed values of λ_k and U_k. Similar agreement was achieved for V_k, obviously.

In the first test case the λ_k are all different. In the second case $\lambda_1=\lambda_2$. The remaining λ_k and the U_k and V_k are the same as in the first case. The specified values for the λ_k and U_k are given in the tables. For the V_k we took $V_1 = i\sqrt{2}\cdot 10^{-3}U_1$, $V_2 = i\sqrt{3}\cdot 10^{-3}U_2$, $V_3 = i\sqrt{2}\cdot 10^{-3}U_3$ and $V_4 = i\sqrt{3}\cdot 10^{-3}U_4$. Note in Table 2 that the computed values of U_1 and U_2 are linear combinations of the specified values of U_1 and U_2.

TABLE I

CHARACTERISTIC VALUES ALL DIFFERENT

Data	Exact	Computed
λ_1	-0.2 + i	-.20000000 + 1.0000000 i
λ_2	-0.05 + 2i	-.050000000 + 2.0000000 i
λ_3	-0.1 + 3i	-.10000000 + 3.0000000 i
λ_4	-0.175 + 4i	-.17500000 + 3.9999999 i
U_1	1	1.
	2/3	.66666668 + .841 X 10^{-8} i
	1	1.0000000 + .228 X 10^{-8} i
	0	-.173 X 10^{-8} - .154 X 10^{-8} i
U_2	1	1.
	-3/2	-1.5000000 - .390 X 10^{-9} i
	0	.148 X 10^{-7} + .677 X 10^{-8} i
	1	1.0000001 + .177 X 10^{-8} i
U_3	1	1.
	2/3	.66666665 - .313 X 10^{-8} i
	-13/9	-1.4444444 - .214 X 10^{-8} i
	0	.137 X 10^{-7} + .123 X 10^{-7} i
U_4	1	1.
	-1/2	-.50000002 + .364 X 10^{-8} i
	0	.142 X 10^{-8} - .355 X 10^{-8} i
	-5/2	-2.4999999 + .108 X 10^{-7} i

TABLE II

A CHARACTERISTIC VALUE OF MULTIPLICITY 2

Data	Exact	Computed
λ_1	-0.2 + i	-.20000000 + 1.0000000 i
λ_2	-0.2 + i	-.20000000 + 1.0000000 i
λ_3	-0.1 + 3i	-.10000000 + 3.0000000 i
λ_4	-0.175 + 4i	-.17500000 + 4.0000000 i
U_1	1	1.
	2/3	-.52610556 - .704 X 10^{-12} i
	1	.44948974 + .123 X 10^{-11} i
	0	.55051026 + .285 X 10^{-12} i
U_2	1	1.
	-3/2	-2.7340926 + .343 X 10^{-10} i
	0	-.56958119 - .253 X 10^{-10} i
	1	1.5695812 - .232 X 10^{-11} i
U_3	1	1.
	2/3	.66666667 - .253 X 10^{-9} i
	-13/9	-1.4444444 + .401 X 10^{-9} i
	0	-.840 X 10^{-10} + .117 X 10^{-9} i
U_4	1	1.
	-1/2	-.50000000 - .374 X 10^{-11} i
	0	-.292 X 10^{-10} + .631 X 10^{-10} i
	-5/2	-2.5000000 + .427 X 10^{-9} i

```
      SUBROUTINE AAS (M,DATA,TOL,IMAX,LAMBDA,U,V,EM,C,KA,I,W)
C
C          THIS SUBROUTINE ALLOCATES VIRTUAL STORAGE IN THE WORK
C     ARRAY, W, DEPENDING ON THE PARAMETER, M, AND CALLS THE CORE
C     SUBROUTINE BBS.
C
      EXTERNAL DATA
      DIMENSION W(1)
      K=2*M
      IY1=1+K
      IY2=IY1+K
      IY=IY2+K
      K=K*M
      IYT=IY+K
      K=M*M
      IT3=IY
      IT4=IT3+K
      IT5=IT4+K
      CALL BBS (M,DATA,TOL,IMAX,LAMBDA,U,V,EM,C,KA,I,W(1)
     1  ,W(IY1),W(IY2),W(IY),W(IYT),W(IT3),W(IT4),W(IT5))
      RETURN
      END
```

```
      SUBROUTINE BBS (M,DATA,TOL,IMAX,LAMBDA,U,V,EM,C,KA,LL
     1   ,OMEGA,Y1,Y2,Y,YT,T3,T4,T5)
C
C          THIS SUBROUTINE MONITORS AND CONTROLS ALL CALCULATIONS,
C     INCLUDING THE READING OF THE INPUT DATA VIA THE USER WRITTEN
C     SUBROUTINE 'DATA'.
C
      EXTERNAL DATA
      DIMENSION OMEGA(1)
      COMPLEX T,LAMBDA(M),Y1(M),Y2(M),V(M,M),U(M,M),Y(M,M),YT(M,M)
C
C          READ INPUT DATA: OMEGA'S AND RESPONSE VECTORS Y
C
      CALL DATA (M,OMEGA,Y,YT)
C
C          FIRST APPROXIMATION
C
      DO 20 K=1,M
      OM1=OMEGA(K)
      OM2=OMEGA(K+M)
C
C          OBTAIN EIGENVALUE, LAMBDA(K)
C
      CALL EIGEN (Y(1,K),YT(1,K),OM1,OM2,LAMBDA(K),M)
C
C          OBTAIN EIGENVECTORS, U AND V
C
      CALL VUS (Y(1,K),YT(1,K),LAMBDA(K),OM1,OM2,V(1,K),U(1,K),M)
 20   CONTINUE
C
C          REFINEMENT PROCEDURE
C
      DO 50 L=1,IMAX
      KN=0
C
      DO 40 K=1,M
      OM1=OMEGA(K)
      OM2=OMEGA(K+M)
C
C          OBTAIN NEW Y VECTORS, Y1 AND Y2
C
      CALL YVECS (K,OMEGA,Y,YT,LAMBDA,V,U,Y1,Y2,M)
C
C          OBTAIN NEW EIGENVALUE, T = LAMBDA(K)
C
      CALL EIGEN (Y1,Y2,OM1,OM2,T,M)
C
C          OBTAIN NEW EIGENVECTORS
C
      CALL VUS (Y1,Y2,T,OM1,OM2,V(1,K),U(1,K),M)
```

```
C
C          TEST FOR CONVERGENCE AND STORE NEW EIGENVALUE
C
      IF (CABS(LAMBDA(K)-T) .GE. TOL) KN=KN+1
 40   LAMBDA(K)=T
C
C          IF CONVERGENCE WAS MET, GO TO 60
C
      IF (KN .EQ. 0) GO TO 60
 50   CONTINUE
C
C          CONVERGENCE FAILED
C
      LL=IMAX+1
      GO TO 70
C
C          SET ITERATION COUNT AND COMPUTE THE
C     REMAINING ELEMENTS OF VECTOR V
C
 60   LL=L
 70   DO 80 K=1,M
      DO 80 J=2,M
 80   V(J,K)=V(1,K)*U(J,K)
C
C          RECOVER MATRICES M, C, K
C
      CALL RECVR (U,V,LAMBDA,M,EM,KA,C,T3,T4,T5)
      RETURN
      END


      SUBROUTINE EIGEN (Y1,Y2,OM1,OM2,T,M)
C
C          THIS SUBROUTINE COMPUTES AN APPROXIMATION TO LAMBDA(K),
C     GIVEN THE FREQUENCIES OMEGA(K1) AND OMEGA(K2), AND THE
C     CURRENT APPROXIMATION TO THE VECTORS Y(K1) AND Y(K2).
C
      COMPLEX Y1(1),Y2(1),T,T1,T2
      COMMON T1,T2
      T1=(0.,0.)
      T2=(0.,0.)
      DO 10 I=1,M
      T1=T1+Y1(I)**2
 10   T2=T2+Y2(I)**2
      T=CSQRT(T1/T2)
      IF (AIMAG(T) .LT. 0.) T=-T
      T1=CMPLX(0.,OM1)
      T2=CMPLX(0.,OM2)
      T=(T*T1-T2)/(T-1.)
      RETURN
      END
```

```
      SUBROUTINE VUS (Y,YT,LAMBDA,OM1,OM2,V,U,M)
C
C          THIS SUBROUTINE COMPUTES VECTOR U(K) AND THE 1-ST
C     ELEMENT OF VECTOR V(K), GIVEN THE CURRENT APPROXIMATION
C     TO THE CHARACTERISTIC VALUE LAMBDA(K), THE FREQUENCIES
C     OMEGA(K1) AND OMEGA(K2), AND THE CURRENT APPROXIMATION
C     TO THE VECTORS Y(K1) AND Y(K2).
C
      COMPLEX Y(1),YT(1),LAMBDA,V,U(1),D1
      COMMON D1
      D1=YT(1)-Y(1)
      V=D1/(1./(CMPLX(0.,OM2)-LAMBDA)-1./(CMPLX(0.,OM1)-LAMBDA))
      U(1)=(1.,0.)
      DO 10 L=2,M
      U(L)=(YT(L)-Y(L))/D1
   10 CONTINUE
      RETURN
      END



      SUBROUTINE YVECS (K,OMEGA,Y,YT,LAMBDA,V,U,Y1,Y2,M)
C
C          THIS SUBROUTINE COMPUTES AN APPROXIMATION TO THE
C     VECTORS Y(K1) AND Y(K2), GIVEN THE FREQUENCIES OMEGA(K1)
C     AND OMEGA(K2), THE RESPONSE VECTORS Y, AND THE CURRENT
C     VALUES OF ALL LAMBDA'S, V'S, AND U'S.
C
      DIMENSION OMEGA(1)
      COMPLEX Y(M,M),YT(M,M),LAMBDA(1),V(M,M),U(M,M),Y1(1),Y2(1),
     1  T1,T2,S1,S2,T
      COMMON T1,T2,S1,S2,T
      T1=CMPLX(0.,OMEGA(K))
      T2=CMPLX(0.,OMEGA(K+M))
      DO 30 I=1,M
      S1=Y(I,K)
      S2=YT(I,K)
      DO 20 L=1,M
      T=V(1,L)*U(I,L)
      IF (L .EQ. K) GO TO 10
      S1=S1-T/(T1-(LAMBDA(L)))
      S2=S2-T/(T2-(LAMBDA(L)))
   10 S1=S1-(CONJG(T))/(T1-(CONJG(LAMBDA(L))))
      S2=S2-(CONJG(T))/(T2-(CONJG(LAMBDA(L))))
   20 CONTINUE
      Y1(I)=S1
      Y2(I)=S2
   30 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE RECVR (U,V,LAMBDA,M,EM,KA,C,T1,T2,T3)
C
C          THIS SUBROUTINE COMPUTES THE MASS, DAMPING, AND
C     STIFFNESS MATRICES, M, C, AND K, GIVEN THE FINAL VALUES
C     OF ALL LAMBDA'S, V'S, AND U'S.
C
      COMMON T,L,VV,M2,N
      COMPLEX U(M,M),V(M,M),LAMBDA(1),T,L,VV
      DIMENSION EM(1),KA(1),C(1),T1(1),T2(1),T3(1)
      M2=M*M
      DO 10 I=1,M2
   10 T1(I)=T2(I)=T3(I)=0.
C
C          COMPUTE MATRICES: H(0), -I*H'(0), .5*H''(0)
C
      DO 20 K=1,M
      L=LAMBDA(K)
      N=0
      DO 20 J=1,M
      VV=V(J,K)
      DO 20 I=1,M
      N=N+1
      T=-U(I,K)*VV/L
      T1(N)=T1(N)+REAL(T)
      T=T/L
      T2(N)=T2(N)+REAL(T)
      T=T/L
      T3(N)=T3(N)-REAL(T)
   20 CONTINUE
      DO 30 I=1,M2
      T1(I)=2.*T1(I)
      T2(I)=2.*T2(I)
      T3(I)=2.*T3(I)
   30 CONTINUE
C
C          COMPUTE REAL MATRICES M, C, K
C
      CALL INV (T1,KA,M,EM)
      CALL MTMT (M,M,M,T2,KA,T1)
      CALL MTMT (M,M,M,KA,T1,C)
      CALL MTMT (M,M,M,C,T2,T1)
      DO 40 I=1,M2
   40 C(I)=-C(I)
      CALL MTMT (M,M,M,KA,T3,T2)
      DO 50 I=1,M2
   50 T3(I)=T1(I)+T2(I)
      CALL MTMT (M,M,M,T3,KA,EM)
      RETURN
      END
```

REFERENCES

1. Klosterman, A. L., McClelland, W. A., "Combining Experimental and Analytical Techniques for Dynamic System Analysis," Tokyo Seminar on Finite Element Analysis, Nov. 5-10, 1973.

2. Richardson M., Potter, R., "Identification of the Modal Properties of an Elastic Structure from Measured Transfer Function Data," Proc. 20th International Instrumentation Symposium, May 21-23, 1974, Albuquerque, New Mexico.

3. Richardson, M., Potter, R., "Mass, Stiffness and Damping Matrices from Measured Modal Parameters," I. S. A. Conference and Exhibit, New York, NY, October 1974.

4. Richardson, M., "Modal Analysis Using Digital Test Systems," Seminar on Understanding Digital Control and Analysis in Vibration Test Systems. A publication of the Shock and Vibration Information Center, Naval Research Laboratory, May 1975, Washington, D.C.

5. Klosterman, A., Zimmerman, R., "Modal Survey Activity Via Frequency Response Functions," Society of Automotive Engineers, Paper 751068, National Aerospace Engineering and Manufacturing Meeting, Culver City, CA, Nov 17-20, 1975.

6. "Recent Developments in Dynamic Structural Testing," ASIAC Newsletter vol 2, Nos. 5&6, June 1977, AFFDL/FBR, Wright-Patterson AFB, Ohio, 45433.

7. Lanczos, C., "Applied Analysis," Englewood Cliffs, N.J., Prentice Hall, Inc., 1956.

8. Ibrahim, S. R., Mikulcik, E. C., "A Time Domain Modal Vibration Test Technique," Shock and Vibration Bulletin, Bulletin 43, Part 4, 1973.

9. Ibrahim, S. R., Mikulcik, E. C., "The Experimental Determination of Vibration Parameters from Time Responses," The Shock and Vibration Bulletin, Bulletin 46, 1976.

10. Ibrahim, S. R., Mikulcik, E. C., "A Method for the Direct Indentification of Vibration Parameters from the Free Response," The Shock and Vibration Bulletin, Bulletin 47, Part 4, September, 1977.

11. Audley, D. R., and Lee, D. A., "Ill-Posed and Well-Posed Problems in System Identification," IEEE Trans. Automat. Contro., vol. AC-19, pp. 738-747, No. 6, Dec. 1974.

12. Frazer, R. A., Duncan, W. J., Collar, A. R., "Elementary Matrices," The MacMillan Co., New York, NY, 1946.

13. Kaplan, W., "Operational Methods for Linear Systems," Addison-Wesley Publishing Co., Reading, Mass., 1962.

14. Coddington, E. A., Levinson, N., "Theory of Ordinary Differential Equations," McGraw-Hill Book Co., New York, 1955.

15. Béliveau, Jean-Guy, "Eigenrelations in Structural Dynamics," AIAA Journal vol. 15, No. 7, July 1977.

16. Lanczos, C., "Discourse on Fourier Series," Hafner Publishing Co., New York, NY, 1966.

17. Papoulis, A., "The Fourier Integral and its Application," McGraw-Hill Book Co. Inc., New York, NY, 1962.

18. Cooley, J. W., Tukey, J. W., "An Algorithm for the Machine Calculation of Complex Fourier Series," Mathematic of Computation, vol. 19, April 1965.

*U.S.Government Printing Office: 1978 — 757-080/25