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OSCILLATIONS, FLUCTUATIONS, AND THE HOPF BIFURCATION.(U)

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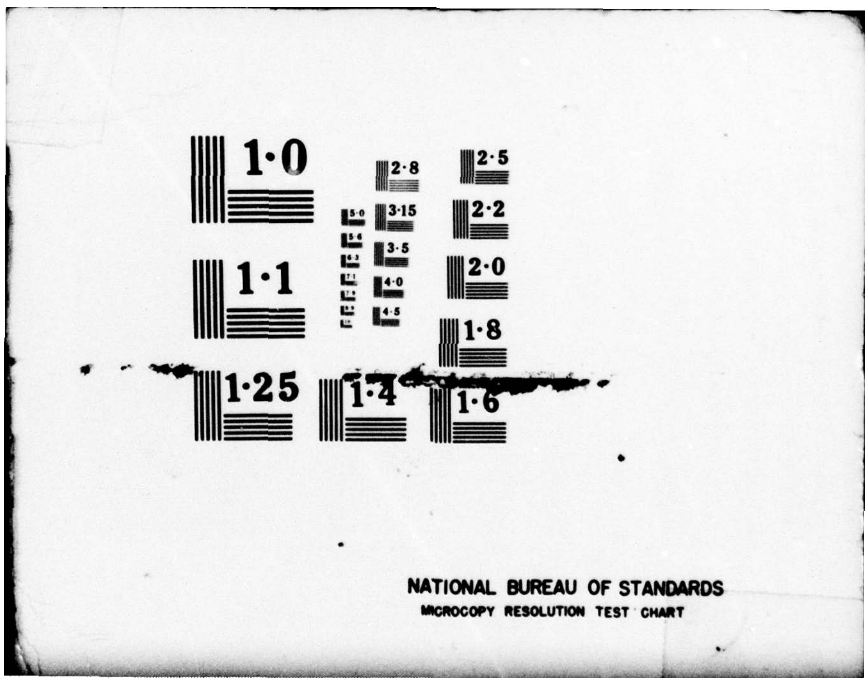
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Marc Mangel

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OSCILLATIONS, FLUCTUATIONS, AND THE HOPF BIFURCATION*

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June 1978

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ABSTRACT

We consider the effects of small random perturbations on deterministic systems of differential equations. The deterministic systems of interest have oscillatory dynamics and may undergo a bifurcation (the Hopf bifurcation). We formulate a first exit problem for experiments beginning near stable and unstable limit cycles. The unstable limit cycle is surrounded by an annulus. Of interest is the probability of first exit from the annulus through a specified boundary, conditioned on initial position. The diffusion approximation is used, so that the conditional probability satisfies a backward diffusion equation. Appropriate solutions on the backward equation are constructed by an asymptotic method. The behavior of the stochastic system in the vicinity of stable and unstable limit cycles is compared. When the deterministic system exhitits the Hopf bifurcation, the above analysis must be modified.

Uniform solutions of the backward equation are constructed. The solutions are analogous to Hadamard's solution of the point source problem for the wave equation. Numerical examples are used to compare the theory with Monte Carlo experiments.

INTRODUCTION

In the past few years, the anlaysis of oscillatory nonlinear dynamical systems has received considerable attention, due to a variety of biological, physical, and chemical applications. Mainly, the analysis has been based on systems of deterministic differential equations and involved classification of dynamical behavior of the systems. Most of the analyses ignored fluctuations that are always present in such systems (Ludwig,(1975), Van Kampen (1976), and White (1977) are exceptions). In this paper, we consider the effects of fluctuations on systems with oscillatory behavior. We consider an autonomous system

$$\dot{x} = b(x) \qquad x \varepsilon R^2 \tag{1.1}$$

that has a periodic solution(s). Three types of periodic solutions are of interest here: 1)a fixed, stable limit cycle, surrounding an unstable focus (figure 1A); 2)a fixed unstable limit cycle, surrounding a stable focus and enclosed by a fixed, stable limit cycle (figure 1B); 3)the Hopf bifurcation problem: the deterministic dynamics depend upon a parameter μ. As $\mu \downarrow 0$, a stable limt cycle coalesces with an unstable focus (at $\mu=0$). The limit cycle disappears and the focus becomes stable (figure 1C). A "dual" bifurcation, in which an unstable cycle and stable focus coalesce, is shown in figure 1D .

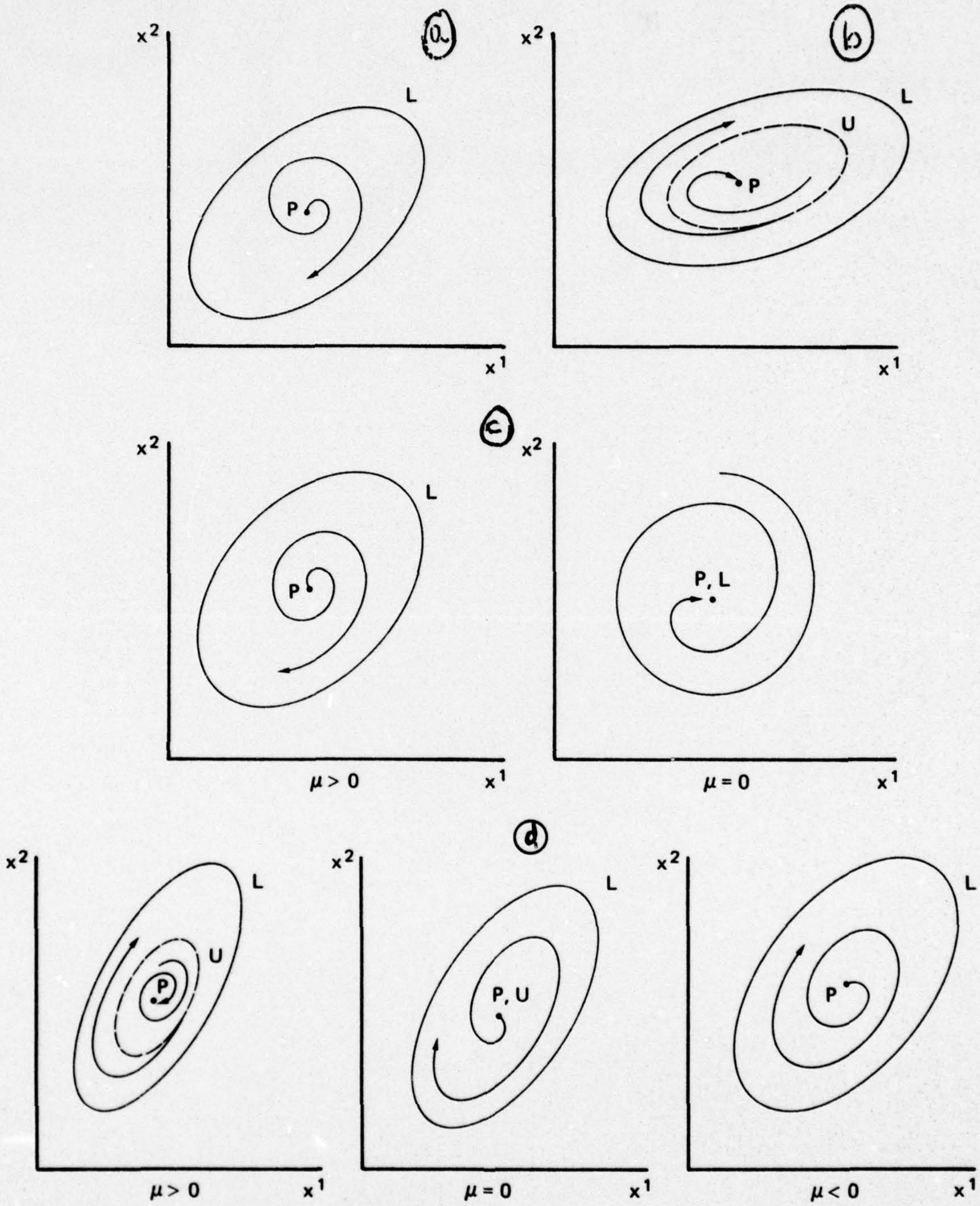

FIGURE 1: PHASE PORTRAITS OF THE DYNAMICAL SYSTEMS STUDIED IN THIS PAPER

The three types of oscillatory solutions arise, for example, in theoretical population dynamics (e.g., Bazekin, 1975) and chemical reaction dynamics (e.g., Cohen, 1972). The fixed unstable limit cycle occurs in the treatment of molecule ion-molecule collisions.

The stable limit cycle, with superposed fluctuations, was treated by Ludwig, White, and Van Kampen. We include it here for two reasons. First, our treatment is slightly different from the others. Second, it is interesting to compare the stochastic dynamics of stable and unstable limit cycles. The unstable cycle contained by a stable cycle arises in chemical dynamics (Tyson, 1977, Uppal et al, 1974). In the engineering literature, the unstable limit cycle is called a "hard" oscillation, and the stable limit cycle a "soft" oscillation. The Hopf bifurcation, and dual Hopf bifurcation, arise in many situations (Marsden and McCracken, 1976). Our interest is again motivated by chemical reaction dynamics (Cohen, 1972, Uppal et al, 1974). Fluctuation effects have not been considered in these systems.

When fluctuations are superposed upon the deterministic dynamics (1.1), a number of interesting questions arise. The type of question that should be posed depends upon the type of deterministic dynamics, as is to be expected. First, consider the stable limit cycle (figure 1A). Since the deterministic attraction is always towards L, the question of interest involves how fluctuations may derive the system away from L. Let $\tilde{x}(t)$ denote the random variable obtained by superposing fluctuations on (1.1).

In this case, $x(t)$ in (1.1) is the appropriate conditional average of $\tilde{x}(t)$. Let:

$$v(t, x)dx = Pr\{x \leq \tilde{x}(t) \leq x + dx\} \quad (1.2)$$

Thus $v(t,x)$ is the probability density for $\tilde{x}(t)$. It represents a natural function describing the stochastic dynamical system obtained from figure 1A. If we let $t \to \infty$, then $v(t,x) \to v(x)$, the equilibrium or stationary density of eventually finding the process between $\{x, x + dx\}$. We note that $v(t,x)$ is independent of $\tilde{x}(0)$. Namely, we are solely interested in the forward time density, given some initial distribution, $v_0(\tilde{x}(0))$.

On the other hand, the initial point is crucial when considering an unstable limit cycle U (figure 2). A phase point initially in the vicinity of U leaves any neighborhood of U with probability 1. Even if $x(0) \in U$, fluctuations will drive the point away from U. Ideally, we would like to calculate the probability that, given $\tilde{x}(0) = x$, the phase point reaches a neighborhood of the node P rather than a neighborhood of the stable limit cycle L. In general, this problem is too difficult to solve. We can, however, analyze the following problem. Let s measure distance normal to U. Consider two contours:

$$S_1: \{x: s(x) = s_1\}, \quad S_2 = \{x: s(x) = s_2\} \quad (1.3)$$

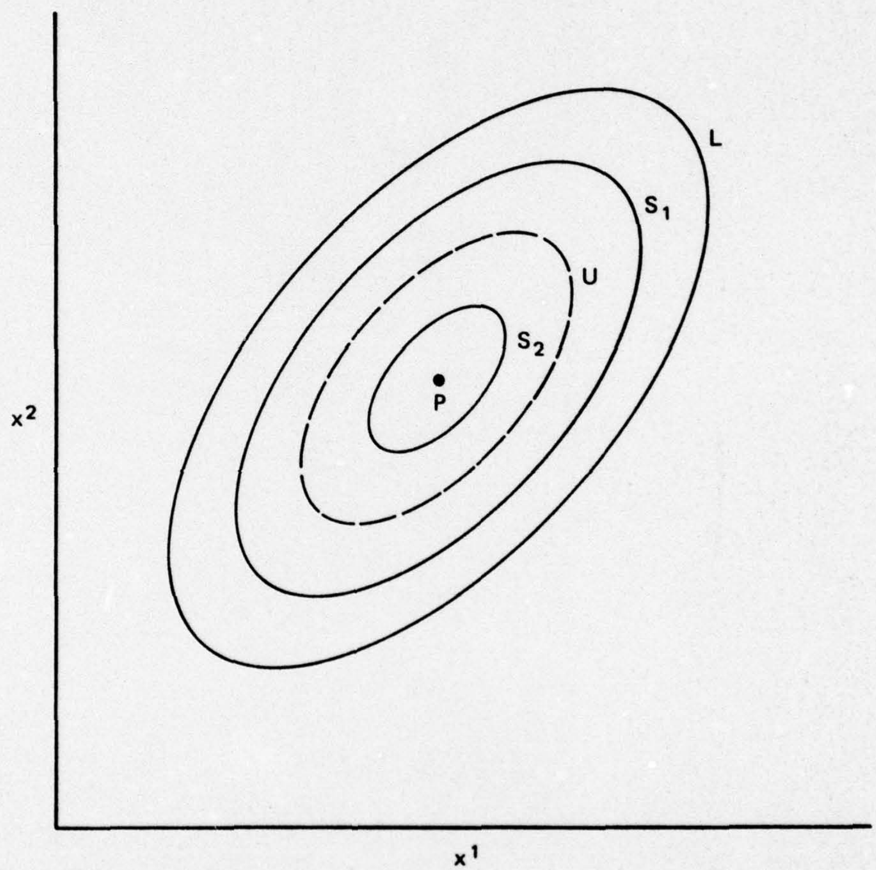

FIGURE 2: THE FIRST EXIT PROBLEM FOR THE UNSTABLE LIMIT CYCLE

(see figure 2), with $s_1 > 0$, $s_2 < 0$. Consider the probability

$$u(x,t) = \Pr\{\text{by time } t,\ \tilde{x}(\tau) \text{ has left the annulus } (S_1,S_2), \text{ through } S_2 \mid \tilde{x}(0) = x\} \tag{1.4}$$

This probability is dependent solely upon initial position. The stationary equivalent of (1.4) is $u(x)$, which is the probability that $\tilde{x}(t)$ first exits from (S_1,S_2) through S_2.

For the case of a system exhibiting the Hopf bifurcation, (figure 1-C), we are again interested in the density for $\tilde{x}(t)$. Now the density $v = v(x,t;\mu)$, where μ is the parameter characterizing the deterministic bifurcation. Now consider the dual Hopf system (figure 1-D). For small μ, a phase point will leave a neighborhood of P or U and approach L with probability 1. The singularity at P/U for $\mu=0$ will be evidenced by very slow deterministic repulsion from P. Let $\bar{L}$ be a neighborhood of L and

$$T(x) = E\{t: \tilde{x}(t) \in \bar{L},\ x(s) \notin \bar{L},\ s<t \mid \tilde{x}(0) = x,\ \tilde{x}(t) \text{ reaches } \bar{L}\} \tag{1.5}$$

Thus, $T(x)$ is the expected time to reach $\bar{L}$, given that $\tilde{x}(0) = x$.

In order to calculate the above quantities, we need to introduce a stochastic kinetic equation. In section 2, we first specify the deterministic dynamical equations corresponding to the systems pictured in figure 1.

Next, we modify these equations by the addition of a random function. This is the Langevin approach. We obtain a stochastic kinetic equation that is, usually, too difficult to treat directly. We treat the kinetic equation by the diffusion approximation of Papanicolaou and Kohler (1974). In this approximation, $v(x,t)$, $u(x,t)$ and $T(x)$ all satisfy partial differential equations. A small parameter, characterizing the intensity of the fluctuations, arises in derivation of the stochastic equations.

In section 3, we analyze canonical problems corresponding to stable and unstable limit cycles and the Hopf bifurcations. Various incomplete special functions arise in the analysis of these canonical problems. These functions are generalized in section 4, where we calculate $v(t,x)$ and $u(t,x)$ by the use of formal asymptotic methods, for stable and unstable (fixed) limit cycles. The stationary solutions $v(x)$, $u(x)$ have interesting interpretations in terms of "hindsight" and foresight." In section 5, we construct $v(t, x)$ and $u(t,x)$ for Hopf-type dynamical systems. We show that the solutions in section 4 breakdown and how the uniformly valid solutions can be obtained. In section 6, we present some numerical solutions indicating the phenomena discussed in sections 4-6.

SECTION 2

DETERMINISTIC AND STOCHASTIC KINETIC EQUATIONS

We first characterize the deterministic equations that lead to the phase portraits of interest. Then we formulate the stochastic kinetic equations and the diffusion approximation.

DETERMINISTIC KINETIC EQUATIONS

We assume that

$$x^i = b^i(x,\mu) \qquad x \in R^2 \qquad \eta \in R^1 \tag{2.1}$$

has a periodic solution Φ . Let P be a steady state of (2.1): $b^i(P,\mu) = 0$ for all i . Introduce new coordinates: s, which measures distance normal to Φ , and θ which measures distance along Φ . (If Φ were a circle, then $s = r$ and $\theta = \theta$).
From (2.1), we obtain equations for s and θ

$$\left.\begin{aligned} \dot{s} &= \tilde{b}^s(s,\theta,\mu) \\ \dot{\theta} &= \tilde{b}^\theta(s,\theta,\mu) \end{aligned}\right\} \tag{2.2}$$

The variable θ is periodic, with period Θ.
The limit cycle is stable if

$$\frac{\partial}{\partial s}(\tilde{b}^s(0,\theta,\mu)) < 0 \tag{2.3}$$

and is unstable if

$$\frac{\partial}{\partial s}(\tilde{b}^s(0,\theta,\mu)) > 0 . \tag{2.4}$$

If $\partial/\partial s\,(\tilde{b}^s(0,\theta,\mu)) = 0$, then the limit cycle is neutrally stable (or unstable). This phenomenon occurs at the Hopf bifurcation.

To consider the Hopf bifurcation, we return to (2.1). Let $B = (b^i_{,j}(P,\mu))$, and let $\lambda(\mu)$, $\lambda^*(\mu)$ denote the eigenvalues of B. The Hopf bifurcation is characterized by the following

1) When $\mu < 0$, $\lambda(\mu) \neq \lambda^*(\mu)$ are located in the left-half plane.
2) When $\mu = 0$, the eigenvalues are located on the imaginary axis.
3) When $\mu > 0$, the eigenvalues are located in the right-half plane. Also, the condition

$$\frac{d}{d\mu} \mathrm{Re}\lambda(\mu)\Big|_{\mu=0} \equiv \gamma_1 \neq 0 \tag{2.5}$$

holds. There are analogous (dual) conditions for the dual Hopf bifurcation. Let

$$z = re^{i\theta} = x^1 + ix^2 . \tag{2.6}$$

Fenichel (1975) (see also Arnold, 1972) has shown that (2.1) can be put into the form (for small μ)

$$\begin{aligned} \dot{r} &= \pm(b_1 r^3 - \eta\gamma_1 r) \equiv b^r(r,\phi,\eta) \\ \dot{\phi} &= \lambda_2 + b_2 r^2 + \eta\gamma_2 r \equiv b^\phi(r,\phi,\eta) \end{aligned} \tag{2.7}$$

where $r = r(s,\theta)$, $\phi = \phi(s,\theta)$ are regular functions, γ_1 is defined in (2.5), $\lambda_2 > 0$ and $b_1, b_2 \neq 0$. The (±) sign in (2.7) is included so that both the Hopf bifurcation and dual Hopf bifurcation can be treated. The function $\eta = \eta(\mu)$ is regular and

$$\eta(0) = 0 \tag{2.8}$$

At the bifurcation, $\eta = 0$, we note that

$$\begin{aligned} b^r(0,\phi,0) = b^r_{,r}(0,\phi,0) = b^r_{,rr}(0,\phi,0) = 0 \\ b^r_{,rrr}(0,\phi,0) \neq 0 \ . \end{aligned} \tag{2.9}$$

These conditions will be used later (sections 3 and 4).

STOCHASTIC KINETIC EQUATION AND DIFFUSION APPROXIMATION

Equation (2.1) is approximate in that it completely ignores fluctuations. On the other hand, deterministic equations (e.g., the "law of mass action" in chemistry) often yield correct predictions. Such deterministic equations are successful for two reasons. First, the fluctuations are of small intensity. Second, the fluctuations occur on a time scale rapid compared to the macroscopic equation. Accordingly, we replace (2.1) by a Langevin-like equation for the random variable $\tilde{x}_\alpha(t)$:

$$\frac{d\tilde{x}^i_\alpha}{dt} = b^i(\tilde{x}_\alpha,\mu) + \sqrt{\frac{\varepsilon}{\alpha}}\, f^i_j(\tilde{x}_\alpha) Y^j(t/\alpha^2) \ . \tag{2.10}$$

In (2.10), $Y^j(s)$ is a stationary, zero mean process, satisfying the mixing condition of Papanicolaou and Kohler (1974). The parameter ε, $0 < \varepsilon \ll 1$, characterizes the intensity of the fluctuations. In chemical systems

$$\varepsilon = V_e/V \tag{2.11}$$

where V is the volume of the reacting system and V_e is the elementary volume, i.e., the volume of a sub-unit of the reacting system (Kubo et al, 1973, Van Kampen, 1976). The parameter α, $0 \le \alpha \ll 1$ characterizes the time scale of the fluctuations. As $\alpha \to 0$, $\tilde{x}_\alpha(t)$ converges to a diffusion (Papanicolaou and Kohler, 1974). The density $v(t,x)$ defined in section 1, satisfies (Papanicolaou and Kohler, (1974)):

$$v_t = \frac{\varepsilon}{2}(a^{ij}v)_{ij} - ((b^i + \varepsilon c^i)v)_i . \tag{2.12}$$

The probability $u(t,x)$ satisfies

$$u_t = \frac{\varepsilon}{2}a^{ij}u_{ij} + (b^i + \varepsilon c^i)u_i . \tag{2.13}$$

The expected time, $T(x)$, satisfies

$$-1 = \frac{\varepsilon}{2}a^{ij}T_{ij} + (b^i + \varepsilon c^i)T_i \tag{2.14}$$

In (2.12 - 2.14), subscripts indicate partial derivatives and repeated indices are summed from 1 to n. Also,

$$\begin{aligned} a^{ij}(x) &= f^i_k f^j_l(\gamma^{kl} + \gamma^{lk}) \\ c^i(x) &= \gamma^{kl} f^j_k \frac{\partial}{\partial x^j} f^i_l \end{aligned} \tag{2.15}$$

where

$$\gamma^{kl} = \int_0^\infty E[Y^k(s)Y^l(0)]\, ds$$

In later sections, we will specify the appropriate boundary conditions for (2.12 - 2.13). Equation (2.14) is treated elsewhere (Mangel, 1977).

SECTION 3

CANONICAL PROBELMS AND SPECIAL FUNCTIONS

We now consider $x \in R^1$, $c^i \equiv 0$. The stationary solutions versions of (2.12, 2.13) and appropriate boundary conditions are

$$0 = \frac{\varepsilon}{2}(av)_{xx} - (bv)_x \; ; \int_{-\infty}^{\infty} v(x)dx = 1; v \to 0 \text{ as } |x| \to \infty \tag{3.1}$$

$$0 = \frac{\varepsilon}{2}au_{xx} + bu_x \qquad u(x_1) = 0 \qquad u(x_2) = 1 \qquad x_1 < x_2 \tag{3.2}$$

We assume that the deterministic system

$$\dot{x} = b(x, \mu) \tag{3.3}$$

has a steady state at $x = x_0$. The deterministic equation (3.3) corresponds to the following degenerate "planar" dynamical system. Let (r, θ) denote the usual polar coordinates and consider

$$\begin{aligned} \dot{r} &= b(r, \theta; \mu) \\ \dot{\theta} &= 0 \end{aligned} \tag{3.3a}$$

If we further require that there are no fluctuations in θ, then the system in (3.3a) reduces to the one-dimensional equation (3.3). We assume that for $\mu > 0$, $|b'(x_0, \mu)| > 0$ and that at $\mu = 0$, the equation exhibits a Hopf type bifurcation. Equation (3.1) corresponds to the stable limit cycle; and equation (3.2) corresponds to the unstable limit cycle. The solutions are:

$$v(x) = k \exp\left[\int^{x} \frac{2b}{\varepsilon a}\, dz\right] ; \quad b'(x_0) < 0 \tag{3.4}$$

$$u(x) = k' \int_{x_1}^{x} \exp\left[-\int^{y} \frac{2b}{\varepsilon a}\, dz\right] dy; \quad b'(x_0) > 0 \tag{3.5}$$

The constants k, k' are determined by the integrability conditions and boundary conditions. For small ε, we use Laplace's method to analyze (3.4, 3.5). We obtain

$$v(x) \sim k \exp\left[\frac{-|b'(x_0)|(x - x_0)^2}{\varepsilon a(x_0)}\right] + 0(\sqrt{\varepsilon}) \tag{3.6}$$

$$u(x) = k' \int_{x_1}^{x} \exp\left[\frac{-b'(x_0)(y - x_0)^2}{\varepsilon a(x_0)}\right] dy + 0(\sqrt{\varepsilon}) \tag{3.7}$$

Thus, in the case of a stable limit cycle, we obtain a locally Gaussian density. The integral appearing in (3.7) is the error integral

$$E(z) = \int_{z_0}^{z} e^{-s^2/2} ds . \tag{3.8}$$

The correction term in (3.7) involves $E'(z) = e^{-z^2/2}$. The error integral satisfies

$$\frac{d^2E}{dz^2} = -z \frac{dE}{dz} \qquad -\infty \le z_0 \le z \le z_1 \le \infty \tag{3.9}$$

The appearance of Gaussian forms in (3.6, 3.7) is due to the linearization process involved in Laplace's method. (Namely the assumption that $|b'(x_0)|>0$.)

There are, however, instances where the linearized forms (3.6, 3.7) break down, as in the Hopf bifurcation.

When $\mu = 0$, i.e., at the bifurcation value, a one-dimensional Hopf bifurcation occurs:

$$\begin{aligned} &b'(x_0, 0) = b''(x_0, 0) = 0 \\ &b'''(x_0, 0) \neq 0 \end{aligned} \tag{3.10}$$

Hence, when applying Laplace's method, for μ near 0 , we must use four terms in the Taylor expansion of $\int^x b/\varepsilon a \, ds$.

Instead of the error integral, we find (Mangel, 1977)

$$\begin{aligned} u(x) = k' \int_{x_1}^{x} \exp\Bigg[-\Bigg(&\frac{b'''(x_0,\mu)(y-x_0)^4}{6\varepsilon a(x_0)} + \frac{b''(x_0,\mu)(y-x_0)^3}{3\varepsilon a(x_0)} \\ &+ \frac{b'(x_0\mu)(y-x_0)^2}{\varepsilon a(x_0)}\Bigg)\Bigg] dy + O(\varepsilon^{3/4}) \end{aligned} \tag{3.11}$$

By a change of variables, we obtain

$$u(x) \sim c \int_{\tilde{x}_1(x_1)}^{\tilde{x}(x)} \exp\left[-\frac{y^4}{4} + \frac{\eta(\mu)y^2}{2}\right] dy \tag{3.12}$$

where $\tilde{x}_1(x_1)$, $\tilde{x}(x)$ and $\eta(\mu)$ are regular functions of their arguments and $\eta(0) = 0$. The result (3.12) can be obtained by applying Levinson's result (Levinson, 1962) directly to (3.5). Similarly, we find

$$v(x) \sim c' \exp\left(\frac{-y^4}{4} + \frac{\eta(y)}{2} y^2 \right) \tag{3.13}$$

Thus, we are led to a new special function, the incomplete Hopf integral,

$$H_{\pm}(z, \beta) \equiv \int_{z_0}^{z} \exp\left[\pm\left(\frac{s^4}{4} - \frac{\beta s^2}{2} \right) \right] ds \qquad z_0 \leq z \leq z_1 \tag{3.14}$$

These integrals satisfy

$$\frac{d^2 H_{\pm}}{dz^2} = \pm(z^3 - \beta z) \frac{dH_{\pm}}{dz} \tag{3.15}$$

It can be shown that $H_{\pm}(z, \beta)$ are related to the modified Bessel functions K_n, I_n (Abramowitz and Stegun, 1965). It can also be shown that, for β large, $H_{\pm}(z) \sim E(\tilde{z}(z))$, where $\tilde{z}(z)$ is a regular function of z.

SECTION 4

FIXED LIMIT CYCLES: STATIONARY ASYMPTOTIC SOLUTIONS. "HINDSIGHT AND FORESIGHT"

In this section, we construct formal asymptotic solutions of the stationary versions of (2.12, 2.13) for fixed limit cycles. Namely, μ is bounded away from any bifurcation values. In the next section, we allow μ to vary and consider the bifurcation case.

UNSTABLE LIMIT CYCLE

We seek a solution of the stationary version of (2.13) in the form

$$u(x) = \sum \varepsilon^n g^n(x) E(\psi(x)/\sqrt{\varepsilon}) + \varepsilon^{n+\frac{1}{2}} h^n(x) E'(\psi/\sqrt{\varepsilon}) \tag{4.1}$$

In (4.1), $g^n(x)$, $h^n(x)$, and $\psi(x)$ are to be determined. When derivatives are evaluated, (3.9) is used to replace $E''(\psi/\sqrt{\varepsilon})$ by $-E'(\psi/\sqrt{\varepsilon}) \cdot \psi/\sqrt{\varepsilon}$. After substitution into (4.2), terms are collected according to powers of ε. We obtain:

$$0 = \sum_{n=0} \varepsilon^{n-\frac{1}{2}}(g^n - \psi h^n)(b^i\psi_i - \frac{a^{ij}}{2}\psi_i\psi_j\psi)E'(\psi/\sqrt{\varepsilon})$$

$$+ \varepsilon^n(b^i g_i^n + \frac{a^{ij}}{2} g_{ij}^{n-1} + c^i g_i^{n-1})E(\psi/\sqrt{\varepsilon})$$

$$+ E'(\psi/\sqrt{\varepsilon})\varepsilon^{n+\frac{1}{2}}\Big\{b^i h_i^n + a^{ij} g_i^n \psi_j + \frac{a^{ij}}{2} g^n \psi_{ij} + g^n c^i \psi_i \tag{4.2}$$

$$- c^i h^n \psi_i \psi + c^i h_i^{n-1} - \psi a^{ij} h_i^n \psi_j + \frac{a^{ij}}{2} h_{ij}^{n-1} - \frac{a^{ij}}{2} h^n((\psi\psi_i),_j)\Big\}.$$

The leading term, $n=0$, vanishes if

$$b^i\psi_i - \frac{a^{ij}}{2}\psi_i\psi_j\psi = 0 \tag{4.3}$$

$$b^i g_i^0 = 0 \tag{4.4}$$

$$b^i h_i^0 + \frac{a^{ij}}{2} g^0\psi_{ij} + a^{ij} g_i^0 \psi_j - h_i^0 a^{ij}\psi_j\psi + g^0 c^i \psi_i - c^i h^0 \psi\psi_i$$
$$- \frac{a^{ij}}{2} h^0((\psi\psi_i)_j) = 0 \tag{4.5}$$

The transformation $\phi = -\frac{1}{2}\psi^2$ converts (4.3) to the Hamilton-Jacobi or eikonal equation

$$b^i\phi_i + \frac{a^{ij}}{2}\phi_i\phi_j = 0 \tag{4.6}$$

The interpretation of the eikonal equation (4.6) in the stochastic context is discussed by Ventcel and Freidlin (1970), Ludwig(1975), Mangel and Ludwig (1977), and Mangel (1977).

An argument using Hamliton-Jacobi theory (Mangel, 1977), shows that $\phi = \psi = 0$ on the limit cycle U. Since ϕ is constant on U, $\phi_\theta = 0$ there. We differentiate (4.6) with respect to x^k:

$$b^i_{jk}\phi_i + b^i\phi_{ik} + \frac{a^{ij}_{jk}}{2}\phi_i\phi_j + \frac{a^{ij}}{2}(\phi_{ik}\phi_j + \phi_i\phi_{kj}) = 0 \tag{4.7}$$

Since $\phi = -\frac{1}{2}\psi^2 = 0$ and $\psi = 0$ on U, $\phi_i = -\psi\psi_i = 0$ on U. Thus, (4.7) becomes

$$b^i\phi_{ik} = 0 = \frac{d}{d\theta}\phi \tag{4.8}$$

We differentiate (4.7) with respect to x^1, use the fact that $\phi_\theta = 0$ on U and obtain:

$$\frac{d}{d\theta}(\phi_{ss}) + 2b^s_{,s}\phi_{ss} = -a^{ss}(\phi_{ss})^2 . \tag{4.9}$$

In obtaining (4.9), we have switched to (s,θ) coordinates on U. (equation (2.2)). If we set $W = \phi^{-1}_{ss}$, equation (4.9) becomes a linear equation for W:

$$\frac{dW}{d\theta} - 2b^s_{,s}W = a^{ss} \tag{4.10}$$

The interpretations of ϕ, W are interesting. The leading order, $u(x)$ is constant if $\phi(x)$ is constant. Since $\phi(x(s)) = \phi_{ss}(\delta s)^2/2 = (\delta s)^2/2W$, level curves are obtained a distances proportional to $1/\sqrt{W}$. Hence W is a "local variance".

We introduce the integrating factor

$$\Gamma(\theta) = \exp\left[\int_0^{\theta} 2b^s_{,s}\, d\theta'\right] \tag{4.11}$$

and seek a periodic solution (of period Θ) of (4.9). We obtain:

$$W(\theta) = \left[\frac{\Gamma(\theta)}{1 - (1/\Gamma(\Theta))}\right]\left[\int_{\theta}^{\theta+\Theta} \frac{a^{ss}(r,\theta')d\theta'}{\Gamma(\theta')}\right] \tag{4.12}$$

Since $W(\theta)$ is a measure of the distance to a contour of $u(x)$, (4.12) has an interesting interpretation. The first factor is a purely deterministic factor that causes the contours to spread apart. The second factor is due to stochastic effects and causes the contours to close together.

Now consider (4.4). Since $b^i = dx^i/dt$, equation (4.5) indicates that g^0 is constant on trajectories. Following Mangel (1977) we set g^0 to be the same constant on all trajectories. This constant is determined so that the leading part of (4.1) satisfies the boundary conditions. We set $u(x) = 0$ if $x \in S_2$ and $u(x) = 1$ if $x \in S_1$. Suppose that S_1, S_2 are level curves of ψ, with $\psi = \psi_1$ on S_1 and $\psi = \psi_2$ on S_2. In (3.9), we set

$$z_0 = \frac{\psi_2}{\sqrt{\varepsilon}} \qquad z_1 = \frac{\psi_1}{\sqrt{\varepsilon}} \tag{4.13}$$

and

$$g^0 = 1/E(\psi_1/\sqrt{\varepsilon}) \ . \tag{4.14}$$

Then, to leading order $u=0$ on S_2 and $u=1$ on S_1. If S_1, S_2 are not level curves of ψ, we proceed as follows. Let ψ_1^m be the maximum value of ψ on S_1 and ψ_2^u be the minimum value of ψ on S_2, then set

$$z_0 = \frac{\psi_2^u}{\sqrt{\varepsilon}} \qquad z_1 = \frac{\psi_2^m}{\sqrt{\varepsilon}} \qquad g^0 = 1/E(\psi_1^m/\sqrt{\varepsilon}) \ . \tag{4.15}$$

It can be shown that, if ψ is bounded away from zero on S_1 and S_2 then $u(x)$ is exponentially small on S_1 and $1-u(x)$ is exponentially small on S_2 (Mangel, 1977).

Next, consider equation (4.5). On U, where $\psi=0$ we obtain

$$\frac{dh^0}{d\theta} - \frac{h^0}{2} a^{ij}\psi_i\psi_j = -\left[\frac{a^{ij}}{2}\psi_{ij} + c^i\psi_i\right] g^0 \tag{4.16}$$

The periodic solution of (4.16) is

$$h^0(\theta) = \frac{\displaystyle\int_\theta^{\Theta+\theta} g^0\left(\frac{a^{ij}}{2}\psi_{ij} + c^i\psi_i\right)\exp\left\{-\int^{\theta'} \frac{a^{ij}}{2}\psi_i\psi_j ds\right\} d\theta'}{\exp\left\{-\displaystyle\int_\theta^{\Theta+\theta} \frac{a^{ij}}{2}\psi_i\psi_j ds\right\}\left\{\exp\left\{-\displaystyle\int_\theta^{\theta+\Theta} \frac{a^{ij}}{2}\psi_i\psi_j ds\right\} - 1\right\}} \tag{4.17}$$

Once h^0 is known on U, it can be determined off U by the method of characteristics (see Mangel,1977).

The leading part of the expansion (4.1) is

$$u(x) \sim g^0 E(\psi/\sqrt{\varepsilon}) + 0(\sqrt{\varepsilon}) . \tag{4.18}$$

Hence, once g^0 and ψ are known, we can construct contours of $u(x)$.

STABLE LIMIT CYCLE

We now consider a stable limit cycle, so that we seek a solution of the stationary version of (2.12):

$$\frac{\varepsilon}{2} (a^{ij}v)_{ij} - ((b^i + \varepsilon c^i)v)_i = 0 . \tag{4.19}$$

Our treatment is slightly different from that of Ludwig (1975). We seek a Gaussian solution of the form

$$v(x) = e^{-\psi(x)^2/\varepsilon} (z_0 + \varepsilon z_1 + \dots) . \tag{4.20}$$

After evaluation of derivatives and substitution into (4.19), terms are collected according to powers of ε (see Ludwig, 1975). The leading term will vanish if ψ satisfies

$$b^i\psi_i + \frac{a^{ij}}{2} \psi_i\psi_j\psi = 0 . \tag{4.23}$$

The change in sign in going from (4.3) to (4.23) is important (see section 6). If $\phi = \frac{1}{2}\psi^2$, we obtain the eikonal equation (4.6), so that the analysis of (4.23) is identical to the analysis in the previous section. We find

$$v(x) \sim z_0 \exp\left[\frac{-\phi_{ss}(\delta s)^2}{2\varepsilon}\right] = z_0 \exp\left[\frac{-(\delta s)^2}{2\varepsilon W}\right] . \tag{4.24}$$

Hence, $\varepsilon W/2$ has the interpretation of a local variance about the stable limit cycle. Such an interpretation has been given by Ludwig (1975).

The function z_0 can be determined by integration along the characteristics of (4.23) (Ludwig, 1975, Mangel, 1977). Then to leading order

$$v(x) \sim \frac{z_o e^{-\psi^2(x)/\varepsilon}}{\int z_o e^{-\psi^2(x)/\varepsilon} dx} + 0(\varepsilon) \tag{4.25}$$

Ludwig (1975) shows how to determine z^0 by the method of characteristics.

SECTION 5

HOPF BIFURCATION

The analysis of the preceeding section breaks down at the Hopf bifurcation, because the linear dynamics vanish at the bifurcation point. The analysis of section 3 suggests a possible form of the correct asymptotic solution. The Hopf problem is closely related to a point source problem for the wave equation. Zauderer (1970) used Hadamard's method (Hadamard, 1951) for such problems. Our construction is considerably simpler than Zauderer's, and can be shown to be equivalent to his.

UNSTABLE LIMIT CYCLE, STABLE FOCUS

We seek an asymptotic solution of the stationary backward equation of the form

$$u(x) = \sum \varepsilon^n g^n(x)\, H(\psi/\varepsilon^{\frac{1}{4}}, \beta/\varepsilon^{\frac{1}{2}}) + \varepsilon^{n+\frac{1}{2}} h^n(x) H'(\psi/\varepsilon^{\frac{1}{4}}, \beta/\varepsilon^{\frac{1}{2}}), \quad (5.1)$$

where $H(z,\beta) = H_-(z,\beta)$, defined in (3.14). When derivatives are evaluated, (3.15) is used to replace H'' by $H'(\psi^3 - \beta\psi)/\varepsilon^{3/4}$.
We assume that β has an asymptotic expansion

$$\beta = \sum \varepsilon^k \beta_k \quad . \quad (5.2)$$

After terms are collected according to powers of ε, we obtain:

$$0 = \sum \varepsilon^{n-\frac{1}{4}} H'(\psi/\varepsilon^{\frac{1}{4}}, \beta/\varepsilon^{\frac{1}{2}}) \left[(b^i \psi_i - \frac{a^{ij}}{2} \psi_i \psi_j (\psi^3 - \beta_o \psi) \right] \left[g^n - h^n (\psi^3 - \beta_o \psi) \right]$$

$$+ \varepsilon^n H(\quad) \left[b^i g_i^n + \frac{a^{ij}}{2} g_{ij}^{n-1} + c^i g_i^{n-1} \right]$$

$$+ \varepsilon^{h+3/4} H'(\quad) \Big\{ b^i h_i^n + c^i g^n \psi_i + c^i \psi_i h^h (\psi^3 - \beta_o \psi)$$

$$+ \frac{a^{ij}}{2} \Big(2 g_i^n \psi_j + g^n \psi_{ij} + h_{ij}^{n-1} - 2 h_j^n \psi_i (\psi^3 - \beta_o \psi)$$

$$- h^n \psi_j (\psi^3 - \beta_o \psi) - h^n \psi_i (\psi^3 - \beta_o \psi)_j \Big)$$

$$+ -\sum_{k=1}^{n+1} (\psi \beta_k) \Big\{ \frac{a^{ij}}{2} \psi_i \psi_j \Big(g^{n+1-k} + h^{n+1-k} (\psi^3 - \beta_o \psi) \qquad (5.3)$$

$$+ h^{n+1-k} (b^i \psi_i + \frac{a^{ij}}{2} \psi_i \psi_j (\psi^3 - \beta_o \psi) \Big) \Big\}$$

$$+ \sum_{k=2}^{n+1} \frac{a^{ij}}{2} \psi_i \psi_j h^{n-k+1} \Big(\sum_{j=1}^{k-1} \psi \beta_j (\psi \beta_{k-j}) \Big)$$

$$- \sum_{k=1}^{n} \psi \beta_k \Big(c^i \psi_i h^{n-k} - \frac{a^{ij}}{2} (2 h_i^{n-k} \psi_j + h^{h-k} \psi_{ij}) \Big)$$

$$- \sum_{k=1}^{n} \beta_k \frac{a^{ij}}{2} \psi_i \psi_j h^{n-k} \Big\} .$$

In (5.3), if a superscript is less than zero, that term is set equal to zero. The leading term, $n=0$, is composed of three parts and vanishes if

$$b^i\psi_i - \frac{a^{ij}}{2}\psi_i\psi_j(\psi^3 - \beta\psi) = 0 \tag{5.4}$$

$$b^i g_i^o = 0 \tag{5.5}$$

$$b^i h_i^0 + \frac{a^{ij}}{2} g^0\psi_{ij} + (\psi^3 - \beta_0\psi)a^{ij}h_i^0\psi_j + \frac{a^{ij}}{2} h^0\psi_{ij}(\psi^3 - \beta_0\psi)$$

$$+ h^0 \frac{a^{ij}}{2}\psi_i\psi_j(3\psi^2 - \beta_0) - (\psi\beta_i)f^0(\psi,1) + g^0c^i\psi_i + c^i\psi_i h^0(\psi^3-\beta_0\psi) = 0. \tag{5.6}$$

In (5.6), we have introduced

$$f^n(\psi,k) \equiv \sum_{k=1}^{n+1} \frac{a^{ij}}{2}\psi_i\psi_j\left(g^{n+1-k} + h^{n+1-k}(\psi^3 - \beta_0\psi)\right)$$

$$+ h^{n+1-k}\left(b^i\psi_i + \frac{a^{ij}}{2}\psi_i\psi_j(\psi^3 - \beta_0\psi)\right) \tag{5.7}$$

First consider (5.4). Since b^i vanishes at the stable focus P, we set $\psi^3 - \beta_0\psi = 0$ at the focus. This insures that ψ will have non-vanishing first derivatives. On the limit cycle U, we also set $\psi^3 - \beta_0\psi = 0$. Since $u(U) > u(P)$ we require that

$$\psi(P) = 0 \qquad \psi(U) = \sqrt{\beta_0}\ . \tag{5.8}$$

When the limit cycle and focus coalesce, we obtain $0 = \sqrt{\beta_o}$, i.e., $\beta_0 = 0$. The sigularities of $F(\psi) = \psi^3 - \beta_0\psi$ now match the

the singularities of the deterministic system. This is the reason that our method can be used to produce a uniform solution.

The value of β_0 is still undetermined. It can be calculated by the following iterative procedure. Since (5.4) is a first order partial differential equation, the method of characteristics can be used to solve it, starting just off U, where $\psi = \sqrt{\beta_0^{(0)}}$ and $\beta_0^{(0)}$ is the initial estimate for β_0. We follow characteristics (called rays) that approach P. If ψ does not approach 0, then $\beta_0^{(0)}$ must be replaced by a better estimate $\beta_0^{(1)}$. The method of false position can be used to calculate iterates of β_0. In this fashion, β_0 can be determined to any order of accuracy. An alternative procedure would follow rays from P to U. The choice of method must be made on the basis of numerical practicality.

Although (5.4) can be solved by the method of characteristics, our main interest is in experiments beginning near U. Consequently, we determine ψ in a vicinity of the limit cycle by a Taylor expansion. We assume that $\beta > 0$. Equation (5.4) is differentiated with respect to x^k and then changed to (s,θ) coordinates. We obtain:

$$\frac{d\psi_s}{d\theta} + b^s_{,s}(0,\theta,\mu)\psi_s - a\beta_0\psi_s^3 = 0 . \tag{5.9}$$

In deriving (5.9), we have used the fact that $\psi = \sqrt{\beta_0}$ on U (so that $3\psi^2 - \beta_0 = 2\beta_0$ on U).

Equation (5.9) is a version of Abel's equation (Davis, 1962). We introduce a new variable z, defined by

$$\psi = 1/Bz \qquad \text{where } B' = b^s_{,s} B \ . \tag{5.10}$$

The periodic solution of (5.9) is then

$$\psi_s(0, \theta, \mu) = \left\{ -2\beta \int_\theta^{\theta+\Theta} \frac{a(s)}{B(s)^2}\, ds - \frac{\beta \int_0^\Theta \frac{a(s)}{B(s)}\, ds}{\Gamma(\Theta) - 1} \right\}^{-1} \tag{5.11}$$

where

$$B(s) = \exp\left[\int_s^{\Theta+s} b^s_{,s}(0, \theta, \mu)\, d\theta \right] \tag{5.12}$$

$$\Gamma(s) = 1/B(s) \ . \tag{5.13}$$

Equation (5.5) indicates the g^0 is a constant. The value of g^0 can be determined exactly as in section 4. Equation (5.6) is analogous to (4.5). It is slightly more complicated since it contains the unknown paramter β_1. This parameter can be determined in the same fashion as β_0 was determined.

It can be shown that all of these constrictions are regular at the bifurcation point $\mu = 0$. The proof is analygous to the proof given in Mangel (1977) for other stochastic dynamical systems.

STABLE LIMIT CYCLE, UNSTABLE FOCUS

In this case, we are interested in uniform solutions of the forward equation (4.9). It is clear that the Gaussin ansatz in section 4.2 breaks down for μ small. We seek a solution of the form

$$v(x) \sim \exp\left[-\frac{1}{\varepsilon}\left(\frac{\psi(x)^4}{4} - \beta\frac{\psi(x)^2}{2}\right)\right] (z^0(x) + \varepsilon z^1(x) + \ldots) \quad (5.14)$$

Following the procedure of section 4.2, we are led to

$$b^i\psi_i + \frac{a^{ij}}{2}\psi_i\psi_j(\psi^3 - \beta_0\psi) = 0 \quad (5.15)$$

The change is sine in going from (5.4) to (5.19) is important. Equation (5.19) can be treated by the method of characteristics or by a Taylor expansion. The function $z^0(x)$ can be determined by integration along the characteristics of 5.15 (Ludwig, 1975).

Thus, the stationary distributions for the Hopf bifurcation problem have been determined. These distributions are regular functions of μ, the deterministic bifurcation parameter.

SECTION 6

FIXED LIMIT CYCLES: SOME EXAMPLES

In this section, we present a number of numerical examples that illustrate the behavior of $u(x)$ and $v(x)$, as determined in section 4. For convenience, we use systems already in polar coordinates.

<u>EXAMPLE 6.1</u>

$$\dot{r} = r(r-1)(2-r)(1.1 + \cos\theta) \tag{6.1}$$

$$\dot{\theta} = 1 \tag{6.2}$$

with covariance

$$\varepsilon a^{rr} = .1[r^2 + (2-r)^2](1.5 + \cos\theta)^2 \tag{6.3}$$

The circle $r = 1$ is an unstable limit cycle. Let

$$u(x) = \Pr\{\text{process hits } r = 1.98 \text{ before } r = .02 | \tilde{x}(0) = x\}$$

In figure 3, we show the $u = .8, .9$ contours $\delta r(\theta)$ where θ measures distance along the cycle and δr is the distance from $r = 1$ to the contour. The noise and deteminiistic dynamics are in phase. Both contours are of the form $\delta r(\theta) = k(k' + \cos\theta)$ where k, k' are constants and $k' > 1$. In table 1, we compare the theory with Monte Carlo experiments.

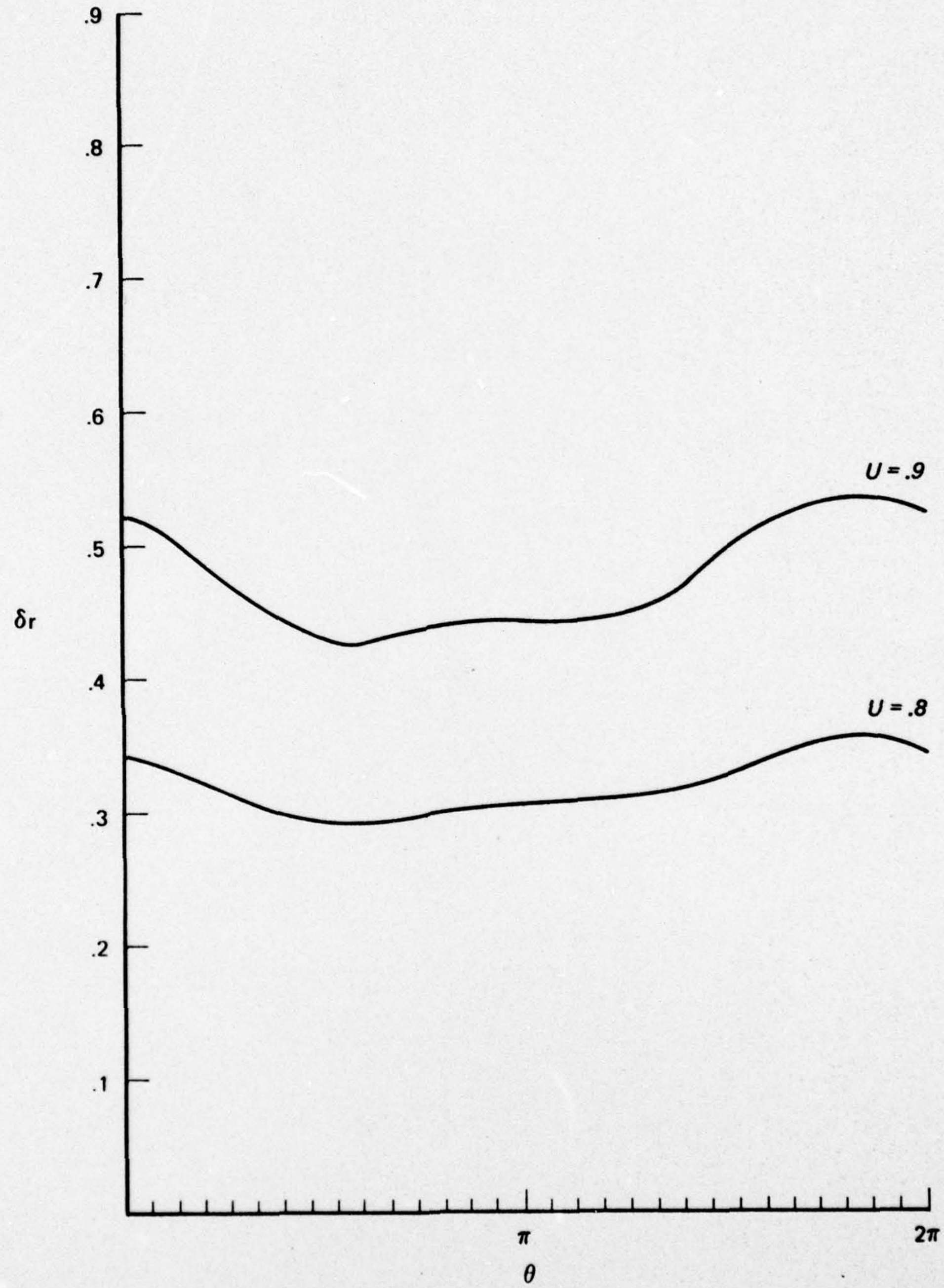

FIGURE 3: EQUAL PROBABILITY CONTOURS FOR EXAMPLE 6.1

TABLE 1

COMPARISON OF THEORETICAL AND MONTE CARLO RESULTS
FOR EXAMPLE 6.1

Initial point	u(Theory)	u(Monte Carlo)*
(1.10, 0)	.60	.57
(1.21, .21)	.70	.68
(1.50, .42)	.90	.92
(1.30, 3.97)	.80	.79
(1.53, 5.86)	.90	.92
(1.08, 1.88)	.60	.58
(1.19, 2.30)	.70	.69

*2,500 simulations were performed

EXAMPLE 6.2

We now consider the system

$$\dot{r} = r(r-1)(r-2)(1.1 + \cos\theta) \tag{6.5}$$

$$\dot{\theta} = 1 \tag{6.6}$$

with εa^{rr} given by (6.3). The deterministic system has a stable limit cycle at $r = 1$. Let

$$v(x)dx = \Pr\{\text{process is eventually found between } (x, x + dx)\} \tag{6.7}$$

In figure 4, we plot the .91, .99 contours of $v(x)$ as a function of $\delta r(\theta)$, where δr is the distance from the cycle to the contour.

EXAMPLE 6.3

We now take

$$\dot{r} = r(r-1)(2-r)(1.1 + \sin\theta) \tag{6.8}$$

with εa^{rr} given by (6.3). In this case, the noise is out of phase with the deterministic cycling. In figure 5, we plot the $u = .8, .9$ contours and in table 2, compare Monte Carlo and theoretical results.

If we take

$$\dot{r} = r(r-1)(r-2)(1.1 + \sin\theta) \tag{6.9}$$

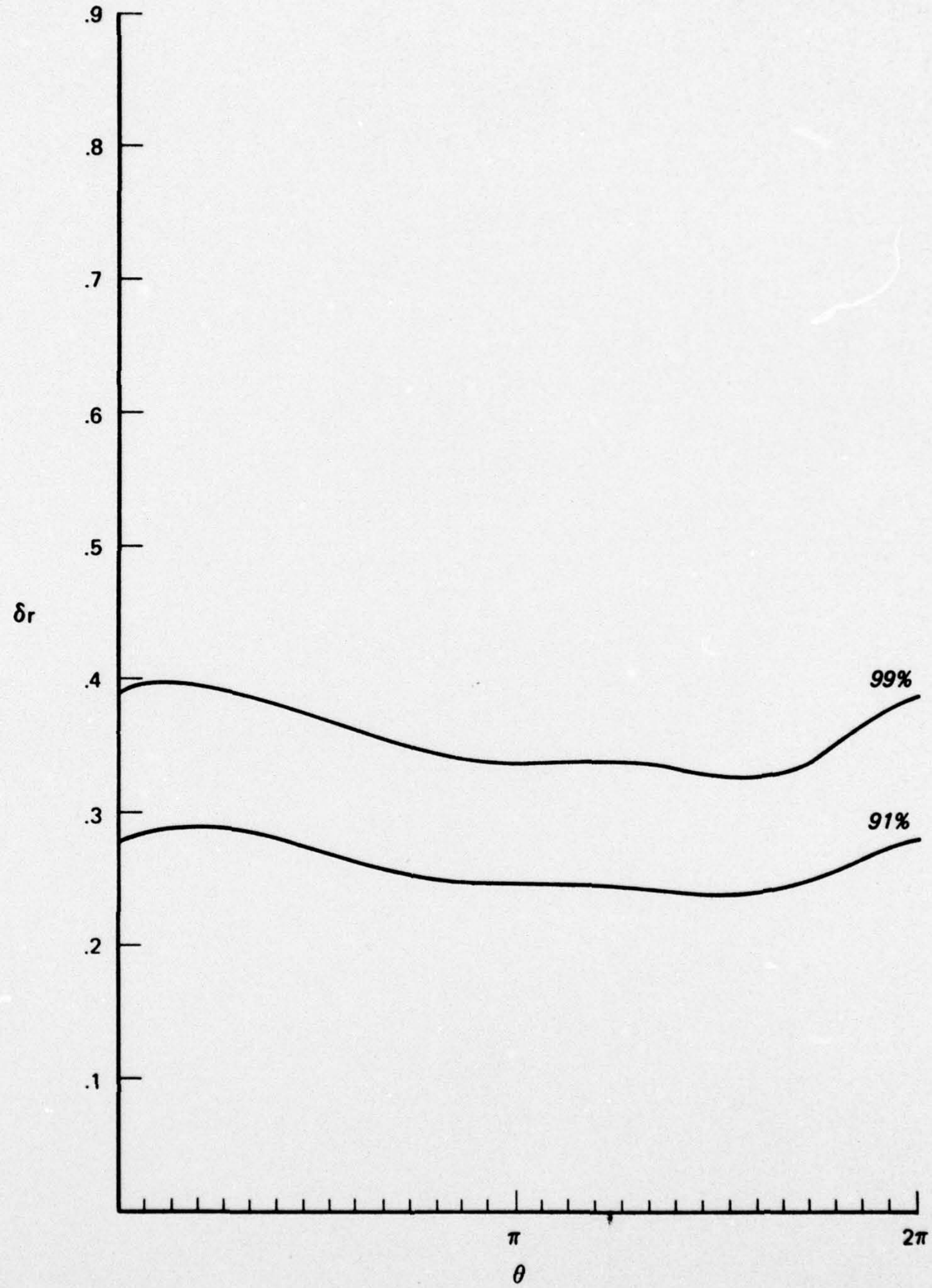

FIGURE 4: EQUAL PROBABILITY CONTOURS FOR EXAMPLE 6.2

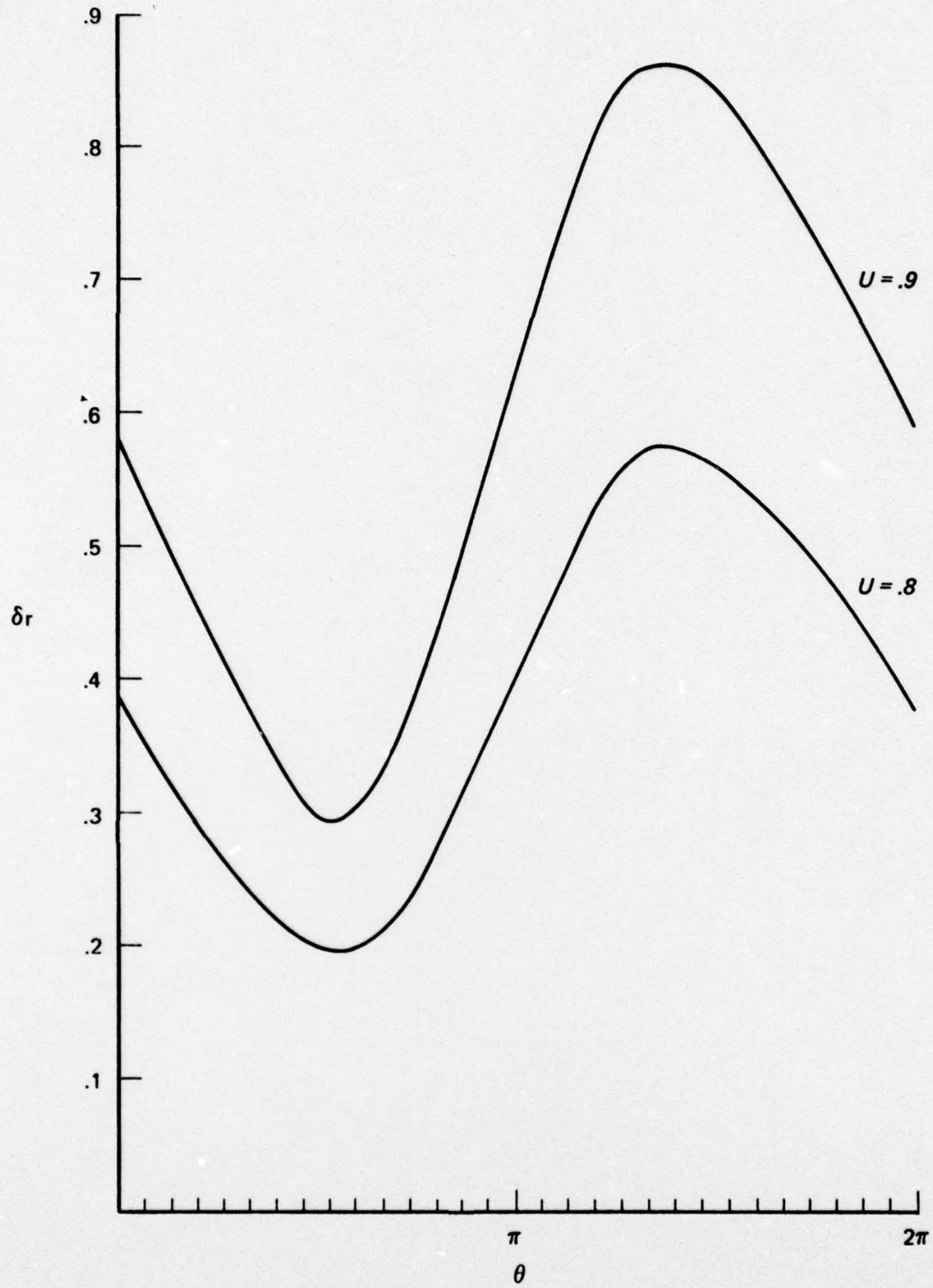

FIGURE 5: EQUAL PROBABILITY CONTOURS FOR EXAMPLE 6.3

TABLE 2

COMPARISON OF THEORETICAL AND MONTE CARLO RESULTS
FOR EXAMPLE 6.3

Initial Point	u(Theory)	u(Monte Carlo)*
(1.14, 0)	.60	.62
(1.22, .21)	.70	.73
(1.49, .42)	.90	.90
(1.55, 3.97)	.80	.82
(1.66, 5.86)	.90	.87
(1.12, 1.88)	.70	.67
(1.06, 1.47)	.60	.64

*2,500 simulations were performed

then $r = 1$ is a stable limit cycle. In figure 6, we plot the $v = .91, .99$ contours. We note the shift in phase of u,v. For fixed r, εa^{rr} reaches its maximum at $0, 2\Pi$. The behavior of the solution of the backward equation, $u(x)$, "anticipates" the noise, in that $\delta r(\theta)$ reaches its maximum before $\varepsilon a^{rr}(\theta)$ reaches its maximum. The density $v(x)$ solution of the forward equation, exhibits hindsight in the $\delta r(\theta)$ peaks after the maximum value of the noise.

Comparison of figures 3 and 5 is also interesting, in light of the interpretation given to W in section 4. Namely, the deterministic and stochastic terms in (4.12) "compete" with each other, the former increasing W and the latter decreasing W. For the situation exhibited in figure 3, the deterministic and stochastic terms are "in phase" and the contours are relatively constant. On the other had, for the situation exhibited in figure 5, the deterministic and stochastic terms are out of phase. The contours exhibit sinusoidal oscillations.

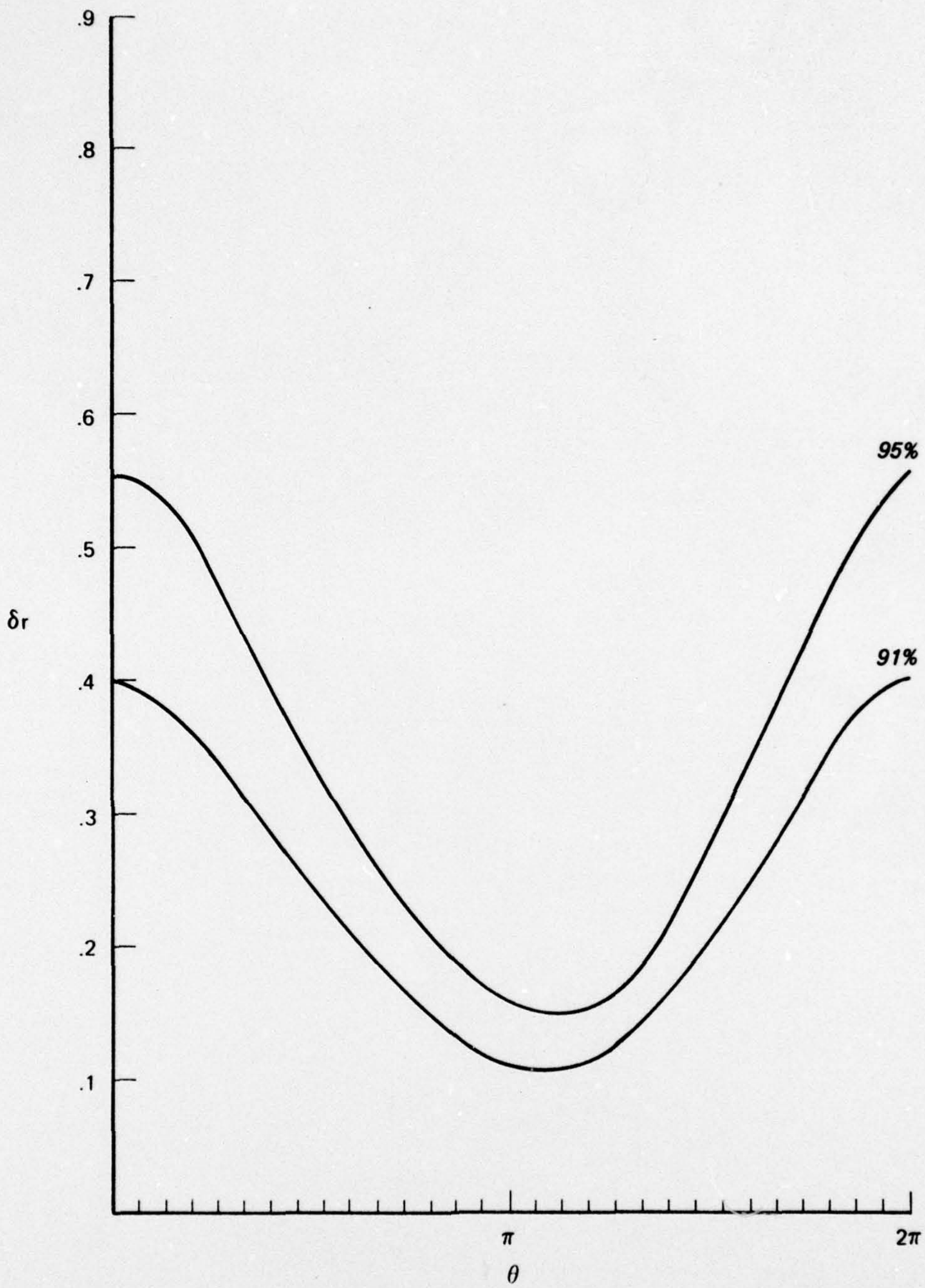

FIGURE 6: EQUAL PROBABILITY CONTOURS FOR EXAMPLE 6.4

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