

AD-A058 536

CENTER FOR NAVAL ANALYSES ARLINGTON VA
AGGREGATION BIFURCATION, AND EXTINCTION IN EXPLOITED ANIMAL POP--ETC(U)
MAR 78 M MANGEL

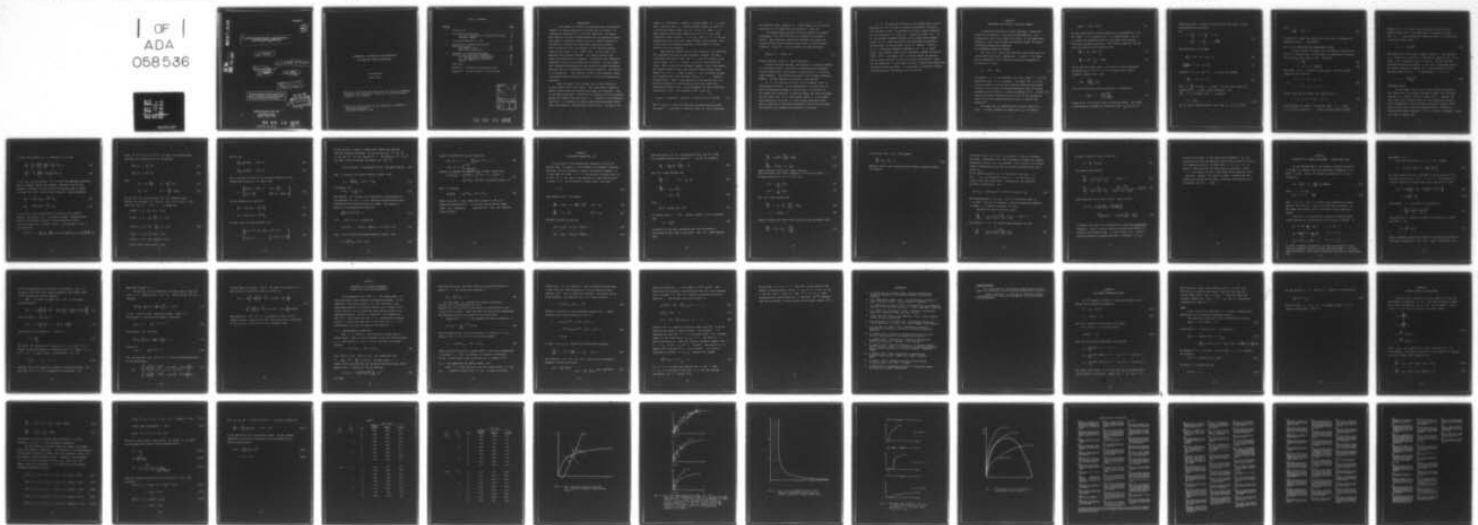
F/G 8/1

UNCLASSIFIED

CNA-PP-224

NL

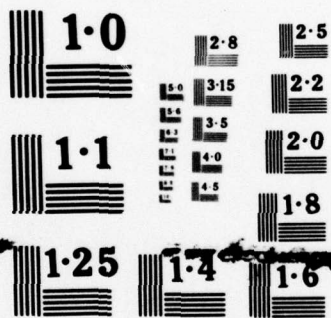
OF
ADA
058536



END
DATE
FILMED

11-78

DDC



NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

ADA 058536

55 000224.00

2
NW

6 AGGREGATION, BIFURCATION, AND EXTINCTION IN
EXPLOITED ANIMAL POPULATION,

10 Marc Mangel

14 CNA-PP-224

9 Professional Paper No. 224

11 Mar 78

12 53 p.

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

The ideas expressed in this paper are those of the
author. The paper does not necessarily represent
the views of the Center for Naval Analyses.

DDC
RECEIVED
SEP 13 1978
A

CENTER FOR NAVAL ANALYSES
1401 Wilson Boulevard
Arlington, Virginia 22209

78 09 12 026
077270

AGGREGATION, BIFURCATION, AND EXTINCTION
IN EXPLOITED ANIMAL POPULATIONS*

Marc Mangel**

March 1978

*Portions of this work were started at the Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, B.C., Canada

**Center for Naval Analyses of the University of Rochester
1401 Wilson Boulevard
Arlington, Virginia 22209

TABLE OF CONTENTS

<u>Section</u>		<u>Page</u>
1	Introduction	1
2	Derivations and Scalings	5
	The (n,q) Phase Plane & Bifurcation Picture. .	8
	Stochastic Models.	9
3	A Two-Timing Method for n, q	15
4	Extinction in a Random Environment:	
	Steady State Case.	21
	The Density $v(n,t)$	22
	Mean Time to Cross $n=\delta$	24
5	Extinction in a Random Environment	
	(Without the Steady State Assumption).	26
	$\alpha < 1$: The Ray Method (Ludwig, 1975).	26
	$\alpha \geq 1$: The Generalized Ray Method	
	(Mangel, 1977)	27
	References	31
	Appendix A: The General Aggregation Model	
	Appendix B: A Kinetic Model for Fluctuations	

ACCESSION for		
NTIS	White Section	<input checked="" type="checkbox"/>
DDC	Buff Section	<input type="checkbox"/>
UNANNOUNCED		<input type="checkbox"/>
JUSTIFICATION		
BY		
DISTRIBUTION/AVAILABILITY CODES		
Orig.	AVAIL. and/or SPECIAL	
A		

78 09 12 026

INTRODUCTION

In this paper, we consider the following type of harvesting problem. An animal population is divided into two stocks: an "underlying" population and a "surface" population. We assume that there is a natural exchange between the two population levels. The predator or harvester affects only the "surface" population and does not influence the "underlying" population directly. Such a situation occurs, for example, in the off-shore Eastern Tropical Tuna Fishery (e.g., IATTC, 1975). In this case, tuna associate with porpoise schools. The fishery harvests only those tuna associated with porpoise. Consequently, the underlying population of tuna is not sampled by the fishery. One may wonder what information measurements on the surface, harvested population provides about the unobservable underlying population. Furthermore, it is interesting and important to know if the standard, linear relationship between harvest and effort is valid in an aggregating population.

Clark and Mangel (1977) studied some of these questions, as they relate to the tuna fishery. They constructed a number of models of the fishery and analyzed the steady state behavior. Some of the models exhibited a bifurcation as harvesting effort increases. Namely, the steady state tuna level, N_{∞} , depends on two parameters. The first, α , characterizes natural association and growth rates, while the second, E , characterizes fishing effort. In some of the

models, no bifurcation occurred. In other models, if α is less than a critical value α_c , Clark and Mangel showed that there is a non-trivial steady state $N_\infty(\alpha, E)$ for all E . When $\alpha > \alpha_c$, there exists a non-trivial steady state only if $E < E_b$, some bifurcation value of effort. If $E > E_b$, then $N_\infty = 0$ is the only steady state. Clark and Mangel used a steady state analysis. Here we will study the dynamical equations in more detail, and thus extend the previous analysis.

When the only steady state is the trivial one, the population will approach extinction as time increases. However, for many situations, the time to reach $N(t) = 0$ is not as relevant as the time to reach $N(t) = \delta$, where δ is a given (low) population level. This is true for the following reasons. First, the predator will cease to seek the prey once the prey becomes scarce enough. Second, as $N(t) \rightarrow 0$ a model in which N is a continuous variable is no longer valid, since the small value of N demands a discrete model. Third, if the population level becomes very low, the entire dynamics of the species may shift into some other, stable configuration (e.g., a second predator may become important, (see Holling, 1973). We introduce a time $T_\delta(N_0, E)$:

$$T_\delta(N_0, E) = \min[t: N(0) = N_0, N(t) \leq \delta, N(S) > \delta, \text{ for } S < t, E(t) = E] \quad (1)$$

Then $T(N_0, E)$ is the first time that the population goes below the level δ , given that it starts at level N_0 and is harvested

with constant effort intensity E . In this paper, we will provide techniques for the approximate calculation of $T_\delta(N_0, E)$.

All predator-prey systems, of course, operate in a random environment. Clark and Mangel did not consider any stochastic effects. When stochastic effects are included, the proper description of the system involves probabilities of certain events. Instead of $T_\delta(N_0, E)$ we must consider the ensemble average

$$\bar{T}_\delta(N_0, E) = \langle T_\delta(N_0, E) \rangle \quad (2)$$

We will show how $\bar{T}_\delta(N_0, E)$ can be calculated.

In most biological systems, the modeling of stochastic effects is a difficult problem. We present a phenomenological approach, in which deterministic equations are reinterpreted in a birth and death framework. This technique provides a way to model stochastic effects. A second approach is outlined in the appendix.

In §2, we rederive model B of Clark and Mangel (1977) in a more general setting. We non-dimensionalize the equations, so that the appropriate parameters become clear. The validity of the steady state assumption of Clark and Mangel can then be studied. We also discuss the (N, Q) phase plane, where N is the underlying population level and Q is the surface population level. Next, we show how to incorporate stochastic effects into the model and formulate the probabilistic questions of interest.

In §3 , we construct solutions of the deterministic kinetic equations for N, Q by using a singular perturbation technique (Lin and Segel, 1973). In particular, we give a simple technique for the calculation of $T_\delta(N_0, E)$, accurate to order $1/E$. In §4 , we consider stochastic effects in the steady state approximation $\dot{Q} = 0$. Exact solutions of many problems are possible. In particular, we can calculate $\bar{T}_\delta(N_0, E)$ exactly. Finally, in §5 , we discuss stochastic problems in the (N, Q) phase plane. The relevant theory for the stochastic problem is given by Ludwig (1975) and Mangel (1977). We will sketch the application of those theories, but will not go into great detail. There are two appendices. In the first, we prove a theorem that generalizes the work of Clark and Mangel. In the second, we provide an alternative technique for the modeling of fluctuations.

SECTION 2

DERIVATIONS AND SCALING, STOCHASTIC MODELS

We generalize the model of Clark and Mangel, without the steady state assumption; and then non-dimensionalize. This procedure will allow us to study the validity of the steady state assumption. Next, we introduce the stochastic model and certain associated probabilistic questions.

THE AGGREGATION MODEL AND SCALING

Let $N(t)$, $Q(t)$ denote the underlying population level and surface population level, respectively; at time t . The fundamental assumption is that underlying population aggregates at the surface at a rate

$$R_a = \tilde{\alpha}N(1 - Q/\bar{Q}) \quad (3)$$

In equation (3), $\tilde{\alpha}$ is a parameter with units $(\text{time})^{-1}$; it is the association rate. \bar{Q} is a constant. One might view \bar{Q} as the carrying capacity of the surface level. The important point about the form of (3) is that the surface population level is limited, regardless of the value of N (Clark and Mangel, 1977). In fact, the qualitative results reported here will hold for any aggregation model in which the surface population level is limited (see appendix A).

We assume that the population has a natural growth rate $g_N(N)$. In many instances, we assume that g_N is logistic:

$$g_N(N) = rN(1 - N/\bar{N}). \quad (4)$$

We also assume that the surface population is harvested at a rate bE , where b is a constant and E is fishing effort. We let X_0 denote the average fraction of the surface level removed during one harvesting operation. The dynamical equations for surface and underlying stock evolution are:

$$\frac{dN}{dt} = g_N - \tilde{\alpha}N(1 - \frac{Q}{\bar{Q}}) \quad (5)$$

$$\frac{dQ}{dt} = \tilde{\alpha}N(1 - \frac{Q}{\bar{Q}}) - bEX_0Q. \quad (6)$$

We now show that these lead to the results of Clark and Mangel. We assume that $dQ/dt = 0$, i.e., the surface level rapidly achieves a steady state. Then

$$Q_{ss} = \frac{\tilde{\alpha}N\bar{Q}}{\bar{Q}bEX_0 + \tilde{\alpha}N}. \quad (7)$$

The rate at which the surface population is harvested is:

$$Y = bEX_0Q \cdot K = \frac{bEX_0 \tilde{\alpha}N\bar{Q}}{\tilde{\alpha}N + bEX_0\bar{Q}} \quad (8)$$

Equation (8) is the result (15B) of Clark and Mangel. The steady state assumption assumes that deviations from Q_{ss} die off on a

rapid time scale. In order to calculate this time scale, we non-dimensionalize the equations.

Let

$$\begin{aligned} n &= \frac{N}{\bar{N}} & q &= \frac{Q}{\bar{Q}} & \tau &= rt \\ \alpha &= \frac{\tilde{\alpha}}{r} & \beta &= \frac{\bar{N}}{\bar{Q}} & \bar{\lambda} &= \frac{bEX_0}{r} \end{aligned} \quad (9)$$

Then equations (5, 6) become

$$\frac{dn}{d\tau} = n(1 - n) - \alpha n(1 - q) \quad (10)$$

$$\left(\frac{1}{\alpha\beta}\right)\frac{dq}{d\tau} = n(1 - q) - \frac{\bar{\lambda}}{\beta} q \quad (11)$$

We define $\alpha\beta = 1/\varepsilon$ and $\bar{\lambda}/\beta = \lambda$ so that (11) becomes

$$\varepsilon \frac{dq}{d\tau} = n(1 - q) - \lambda q. \quad (12)$$

When $\alpha\beta = \frac{\tilde{\alpha}\bar{N}}{r\bar{Q}}$ is large, ε is small. For the tuna fishery, we believe that ε is small (Clark and Mangel, 1977).

When $\varepsilon = 0$, the steady state \bar{q} satisfies

$$\bar{q} = \frac{n}{\lambda + n} \quad (12a)$$

Let \hat{q} denote the deviation of $q(\tau)$ from \bar{q} : $q(\tau) = \bar{q} + \hat{q}(\tau)$.

Then,

$$\epsilon \frac{d\bar{q}}{d\tau} = -\bar{q}(n + \lambda) \quad (13)$$

Equation (13) allows the calculation of the rate of relaxation of deviations from \bar{q} , (see below).

THE (n, q) PHASE PLANE AND BIFURCATION PICTURE

We now consider the phase plane described by equations (10) and (12). Steady states (or equilibria) are determined by setting the left-hand sides equal to zero. We obtain:

$$q = \frac{n}{\lambda + n} \quad (14)$$

$$0 = n(1 - n) - \alpha n \left(\frac{\lambda}{\lambda + n} \right) \quad (15)$$

The origin $(0, 0)$ is always a steady state. The other steady states are the roots of:

$$0 = (1 - n)(\lambda + n) - \alpha\lambda$$

or

$$0 = -n^2 + n(1 - \lambda) - \alpha\lambda + \lambda \quad (16)$$

Hence, there will be another real steady state if

$$D = (1 - \lambda)^2 - 4\lambda(\alpha - 1) > 0 \quad (17)$$

We distinguish two cases. In the first case, $\alpha < 1$. Then $D > 0$ for all values of λ . Hence, we always observe another

steady state. It is easy to show that there is only one non-negative steady state. The phase plane is sketched in figure 1. When $\alpha > 1$, a more interesting phenomenon occurs. We obtain two non-trivial steady states if

$$\alpha < 1 + \frac{(\lambda - 1)^2}{4\lambda} \quad (18)$$

The phase plane in this case is shown in figure 2. The condition $D = 0$ allows us to construct a "bifurcation" diagram in the (α, λ) plane. Such a diagram is shown in figure 3. We note that this bifurcation diagram corresponds to the "fold" catastrophe and not the "cusp" catastrophe (compare, e.g., Jones and Walters (1976)). The bifurcation value of λ is determined by $D = 0$, i.e.,

$$\alpha = 1 + \frac{(\lambda_B - 1)^2}{4\lambda_B} \quad (18a)$$

STOCHASTIC MODELS

At the present time, there is much discussion concerning the "proper" way of introducing stochastic models into biological problems (e.g., Turelli (1977) or Ludwig (1975)). Ideally, one would start with a discrete model and take appropriate limits to obtain deterministic and/or stochastic equations (Ludwig, (1975)). Here, we shall take a different approach and will formally reinterpret the kinetic equations (10, 12) to obtain a stochastic model, as is done by Mangel (1977).

We start with equations (5, 6) rewritten in the form

$$\frac{dN}{dt} = \left[rN + \frac{\tilde{\alpha}NQ}{\bar{Q}} \right] - \left[\frac{rN^2}{\bar{N}} + \tilde{\alpha}N \right] \equiv b^N(N, Q) \quad (19)$$

$$\frac{dQ}{dt} = \tilde{\alpha}N - \left[\frac{\tilde{\alpha}NQ}{\bar{Q}} + bEX_O Q \right] \equiv b^Q(N, Q) \quad (20)$$

In (19, 20), we have rescaled N, Q so that they represent population level in numbers (rather than biomass). Equations (19, 20) are assumed to represent the evolution of the average value of random variables $\tilde{N}(t), \tilde{Q}(t)$ that satisfy stochastic kinetic equations

$$d\tilde{N} = b^N(\tilde{N}, \tilde{Q})dt + \sqrt{A^{Nj}} dW_j \quad (21)$$

$$d\tilde{Q} = b^Q(\tilde{N}, \tilde{Q})dt + \sqrt{A^{Qj}} dW_j \quad (22)$$

In (21, 22), we use the convention of summation over repeated indices. The process $W(t)$ is a Wiener process. Although we are using an Ito equation, we shall use the Stratonovich interpretation (Wong (1971), (Turelli (1977))). The functions $A^{ij}(N, Q)$ are defined by,

$$A^{ij}(N, Q) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} E \left\{ (x^i(t+\Delta t) - x^i(t)) (x^j(t+\Delta t) - x^j(t)) | x^k = x \right\} \quad (23)$$

where x^i, x^j, x^k are \tilde{N} or \tilde{Q} . In order to calculate these variances, we reinterpret (19, 20) as follows:

$$b^N(N, Q) = b_+^N - b_-^N \quad (24)$$

$$b^Q(N, Q) = b_+^Q - b_-^Q \quad (25)$$

where

$$b_+^N = rN + \frac{\tilde{\alpha}_{NQ}}{\bar{Q}} \quad b_-^N = \frac{rN^2}{\bar{N}} + \tilde{\alpha}_N \quad (26)$$

$$b_+^Q = \tilde{\alpha}_N \quad b_-^Q = \frac{\tilde{\alpha}_{NQ}}{\bar{Q}} + bEX_OQ \quad (27)$$

We view b_+^N, b_+^Q as "birth rates"; b_-^N, b_-^Q as "death rates".

Consider the increments $\Delta\tilde{N} = \tilde{N}(t+\Delta t) - \tilde{N}(t)$, $\Delta\tilde{Q} = \tilde{Q}(t+\Delta t) - \tilde{Q}(t)$, given that $\tilde{N}(t) = N$, $\tilde{Q}(t) = Q$, we assume that

$$\Pr \{ \Delta\tilde{N} = 1, \Delta\tilde{Q} = 0 \} = rN\Delta t + o(\Delta t)$$

$$\Pr \{ \Delta\tilde{N} = 1, \Delta\tilde{Q} = -1 \} = \frac{\tilde{\alpha}_{NQ}}{\bar{Q}} \Delta t + o(\Delta t)$$

$$\Pr \{ \Delta\tilde{N} = -1, \Delta\tilde{Q} = 0 \} = \frac{rN^2}{\bar{N}} \Delta t + o(\Delta t) \quad (27a)$$

$$\Pr \{ \Delta\tilde{N} = -1, \Delta\tilde{Q} = 1 \} = \tilde{\alpha}_N \Delta t + o(\Delta t)$$

$$\Pr \{ \Delta\tilde{N} = 0, \Delta\tilde{Q} = -1 \} = bEX_OQ \Delta t + o(\Delta t)$$

$$\Pr \{ \text{all other transitions} \} = o(\Delta t)$$

We note that

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} E\{\Delta \tilde{N}\} = b^N(N, Q) \quad (28)$$

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} E\{\Delta \tilde{Q}\} = b^Q(N, Q) \quad (29)$$

Hence, the mean evolution of the stochastic equation is the deterministic evolution. We easily find

$$A = \begin{bmatrix} b_+^N(N, Q) + b_-^N(N, Q) & -\tilde{\alpha}N(1 + \frac{NQ}{Q}) \\ -\tilde{\alpha}N(1 + \frac{NQ}{Q}) & b_+^Q(N, Q) + b_-^Q(N, Q) \end{bmatrix} \quad (30)$$

For the dimensionless equations,

$$d\tilde{n} = b^n(\tilde{n}, \tilde{q})dt + \sqrt{a^{nj}} d\tilde{w}_j \quad (31)$$

$$d\tilde{q} = b^q(\tilde{n}, \tilde{q})dt + \sqrt{a^{qj}} d\tilde{w}_j \quad (32)$$

we find, using the same procedure, that

$$a = \begin{bmatrix} \frac{1}{N} \{n + n^2 + \alpha n + \alpha n q\} & -\alpha n(1 + q) \\ -\alpha n(1 + q) & \frac{1}{Q} \cdot \frac{1}{\epsilon} \{n + q + \lambda q\} \end{bmatrix} \quad (33)$$

We now consider a number of probabilistic quantities connected with the stochastic equations. We use the notation, b^i for b^N or b^Q and a^{ij} for the elements of a . We denote by $x^1 = \tilde{N}$ or \tilde{n} and $x^2 = \tilde{Q}$ or \tilde{q} . First consider the density for (x^1, x^2) :

$$v(x^1, x^2, t) dx^1 dx^2 = \Pr\{\tilde{x}^1(t) \in (x^1, x^1 + dx^1), \tilde{x}^2(t) \in (x^2, x^2 + dx^2)\} \quad (34)$$

Then v satisfies the forward equation (Ludwig, 1975):

$$v_t = \left(\frac{a^{ij}}{2} v\right)_{ij} - ((b^i + c^i) v)_i \quad (35)$$

In equation (35),

$$c^i(x) = \frac{1}{4} \frac{\partial}{\partial x^j} a^{ij} \quad (36)$$

The function $c(x)$ arises in the Stratonovich interpretation of the equations (31, 32). In (35), subscripts indicate partial differentiation and repeated indices are summed. The density is subject to the normalization condition,

$$\iint v(x^1, x^2) dx^1 dx^2 = 1. \quad (37)$$

Let $p(x^1, x^2, t)$ be defined by

$$p(x^1, x^2, t) = \Pr\{\tilde{x}^1(t) \leq \delta \mid \tilde{x}^1(0) = x^1, \tilde{x}^2(0) = x^2\} \quad (38)$$

Then p will satisfy the backward equation (Ludwig, 1975)

$$p_t = \frac{1}{2} a^{ij} p_{ij} + (b^i + c^i) p_i \quad (39)$$

subject to boundary and initial conditions

$$p(\delta, x^2, t) = 1 \quad \lim_{x^1 \rightarrow \infty} p(x^1, x^2, t) = 0 \quad (40)$$

$$p(x^1, x^2, 0) = \begin{cases} 0 & x^1 > \delta \\ 1 & \text{otherwise} \end{cases}$$

Finally, we consider the expected time to cross a given line

$$T(\delta; x_0^1, x_0^2) = E\{t: \tilde{x}^1(t) \leq \delta, \tilde{x}^1(s) > \delta, s < t | \tilde{x}^1(0) = x_0^1, \tilde{x}^2(0) = x_0^2, \tilde{x}^1(t) \text{ eventually crosses } \delta\} \quad (41)$$

Then T satisfies

$$-p(x_0^1, x_0^2) = \frac{1}{2} a^{ij} T_{ij} + (b^i + c^i) T_i \quad (42)$$

where $p(x_0^1, x_0^2)$ is the steady state solution of (38) (e.g., Gihman and Skorohod(1972)). In the case that we use the steady state ($\epsilon=0$) assumption equations (36), (39), (42) simplify, as will be seen.

SECTION 3

A TWO-TIMING METHOD FOR n, \hat{q}

We now return to the deterministic equations (10),(12) and analyze them. In section 4, 5 we consider the stochastic equations. Equations (10),(12) represent a singular perturbation problem. It is clear that for times $\tau \gg \epsilon$, q has achieved its steady state \bar{q} , so that $\hat{q} = 0$. Our goal in this section is to calculate the rate at which $\hat{q} \rightarrow 0$. We introduce an "initial layer" time scale

$$S = \frac{\tau}{\epsilon} \quad (43)$$

Then equations (10), (12) become

$$\frac{dn}{dS} = \epsilon \left\{ n(1-n) - \frac{\alpha \lambda n}{\lambda + n} + \alpha n \hat{q} \right\} \quad n(0) = N_0 \quad (44)$$

$$\frac{d\hat{q}}{dS} = -(n + \lambda) \hat{q} \quad \hat{q}(0) = Q_0 \quad (45)$$

We seek a solution of the form

$$n(S) = n_0(S) + \epsilon n_1(S) + \epsilon^2 n_2(S) + \dots \quad (46)$$

$$\hat{q}(S) = \hat{q}_0(S) + \epsilon \hat{q}_1(S) + \epsilon^2 \hat{q}_2(S) + \dots \quad (47)$$

After the series (46, 47), are substituted into (44, 45), terms are collected according to powers of ϵ . We use the expansion

$$\frac{1}{\lambda+n} = \frac{1}{\lambda+n_0} \left(1 - \frac{\epsilon n_1}{\lambda+n_0} + \dots \right) \quad (48)$$

The $O(1)$ terms indicate that

$$\frac{dn_0}{ds} = 0 \quad \text{i.e., } n_0 = N_0 \quad (49)$$

$$\begin{aligned} \frac{d\hat{q}_0}{ds} &= -(n_0 + \lambda) \hat{q}_0 \\ &= -(N_0 + \lambda) \hat{q}_0 \end{aligned} \quad (50)$$

Thus,

$$\hat{q}_0(s) = Q_0 \exp[-(N_0 + \lambda)s] \quad (51)$$

To leading order in ϵ , $\hat{q}(s)$ relaxes towards 0 with a relaxation time

$$\tau_R = \frac{1}{N_0 + \lambda} \quad (52)$$

Corrections to the above relaxation time can be obtained by considering further terms in the series. The $O(\epsilon)$ terms indicate that,

$$\frac{dn_1}{dS} = N_o - N_o^2 - \frac{\alpha \lambda N_o}{\lambda + N_o} + \alpha N_o \hat{q}_o \quad (53)$$

$$\frac{dq_1}{dS} = -\hat{q}_1 (N_o + \lambda) - \hat{q}_o n_1. \quad (54)$$

These equations do not have a simple solution.

Next we consider the "outer" equations, in the time variable τ .

We seek solutions of (10, 12) of the form

$$n(\tau) = \sum_{j=0} \epsilon^j \bar{n}_j(\tau) \quad (55)$$

$$q(\tau) = \sum_{j=0} \epsilon^j \bar{q}_j(\tau) \quad (56)$$

The $O(1)$ terms indicate that

$$\frac{d\bar{n}_o}{d\tau} = \bar{n}_o - \bar{n}_o^2 - \frac{\alpha \lambda \bar{n}_o}{\lambda + \bar{n}_o} + \alpha \bar{n}_o \bar{q}_o \quad (57)$$

$$0 = -\bar{q}_o (\bar{n}_o + \lambda) \quad (58)$$

Hence, we obtain the steady state result of Clark and Mangel (1977)

$$\frac{d\bar{n}_o}{d\tau} = \bar{n}_o - \bar{n}_o^2 - \frac{\alpha \lambda \bar{n}_o}{\lambda + \bar{n}_o} \quad (59)$$

In the limit that $\lambda \ll \bar{n}_0$, (59) becomes

$$\frac{d\bar{n}_0}{d\tau} = \bar{n}_0 - \bar{n}_0^2 - \alpha\lambda. \quad (59a)$$

Equation (59a) is the "constant-yield" equation studied by Brauer and Soudack.

No simple solution for $n_o(T)$ is available in terms of elementary functions. Consequently, the usual "matching" procedures of singular perturbation theory (e.g., Lin and Segel (1977)) are not very helpful. (Simple considerations show that the matching procedure must yield $\bar{n}_o(0) = N_o$.)

We now consider equation (59) for the special case that $\alpha > 1$ and $\lambda > \lambda_B$, the bifurcation value of λ . In this case, the origin is the only steady state. Consequently, the population is driven to extinction. Let

$$T(\delta, N_o) = \min\{t: \bar{n}_o(t) \leq \delta, \bar{n}_o(s) > \delta \text{ } s < t, \bar{n}_o(0) = N_o\} \quad (60)$$

The interpretation of $T(\delta, N_o)$ is as an "extinction time", if δ is small. We will now construct a second perturbation expansion, for large λ , in order to calculate $T(\delta, N_o)$.

Equation (59) can be inverted to give:

$$\frac{dT}{d\bar{n}_o} = \frac{\lambda + \bar{n}_o}{\bar{n}_o(1 - \bar{n}_o)(\lambda + \bar{n}_o) - \alpha\lambda\bar{n}_o} \quad T(N_o) = 0 \quad (61)$$

We introduce $\Gamma = 1/\lambda$ as a second small parameter, so that

$$\frac{dT}{d\bar{n}_o} = \frac{1 + \Gamma\bar{n}_o}{\bar{n}_o(1 - \bar{n}_o)(1 + \Gamma\bar{n}_o) - \alpha\bar{n}_o} \quad (62)$$

We seek a solution of (62) in the form

$$T = \sum_j \Gamma^j T_j(\bar{n}_0). \quad (63)$$

The leading terms satisfy

$$\frac{dT_0}{d\bar{n}_0} = \frac{1}{\bar{n}_0(1 - \bar{n}_0) - \alpha\bar{n}_0} \quad T_0(N_0) = 0 \quad (64)$$

$$\frac{dT_1}{d\bar{n}_0} = \frac{\bar{n}_0}{\bar{n}_0(1 - \bar{n}_0) - \alpha\bar{n}_0} - \frac{\bar{n}_0^2(1 - \bar{n}_0)}{[\bar{n}_0(1 - \bar{n}_0) - \alpha\bar{n}_0]^2} \quad T_1(N_0) = 0 \quad (65)$$

These equations can be easily solved. Thus, we find:

$$T(\delta, N_0) = \frac{1}{1-\alpha} \left[\ln\left(\frac{\delta}{N_0}\right) - \ln\left(\frac{1-\alpha-\delta}{1-\alpha-N_0}\right) \right] + \frac{\alpha^2}{\lambda} \left\{ \frac{1}{(\alpha-1) + \delta} - \frac{1}{(\alpha-1) + N_0} \right\} + o\left(\frac{1}{\lambda^2}\right) \quad (66)$$

In summary, the result (66) was obtained by using two perturbation procedures. First, we used a singular perturbation procedure to calculate the relaxation time τ_R and to obtain (59). Second, a regular perturbation procedure was used to calculate $T(\delta, N_0)$.

To check the accuracy of the steady state assumption, $T(\delta, N_0)$ calculated from (66) was compared with the exact solution of (10,13), by a Runge-Kutta method. We chose $\tilde{q}(0) = 1$, $\epsilon = 1$, $\delta = .1$. In table 1, some results of the calculation are presented. Since $\epsilon = 1$, in principle, the use of the steady state assumption is not warranted. However, the results in table 1 indicate that the steady state assumption provides a relatively good description of the system, even for ϵ order 1.

SECTION 4

EXTINCTION IN A RANDOM ENVIRONMENT: STEADY STATE CASE

We now consider some of the stochastic problems presented in section 2.3, assuming that the steady state assumption ($\epsilon = 0$) about \hat{q} holds. Hence, the stochastic equations (31, 32) are replaced by,

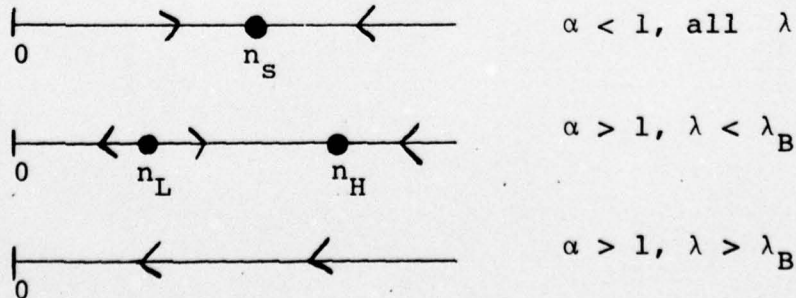
$$d\tilde{n} = \left[\tilde{n}(1 - \tilde{n}) - \frac{\alpha\lambda\tilde{n}}{\lambda + \tilde{n}} \right] dt + \sqrt{\sigma a(\tilde{n})} dW \quad (67)$$

where $\sigma = 1/\bar{N}$ and

$$a(n) = n + n^2 + \frac{\alpha\lambda n}{\lambda + n} \quad (68)$$

When $\sigma = 0$ (i.e., $1/\bar{N} = 0$) we obtain the deterministic kinetic equation (59) from (67). This is in accord with intuition: as the sample size becomes larger, fluctuation effects should disappear.

The flow of the deterministic equation is sketched below (these pictures are transcriptions of the results in section 2):



In order to simplify calculations, we shall approximate $\alpha(n)$ by n . Since the stochastic effects are most interesting when n is small, this approximation (which greatly simplifies calculations) seems reasonable.

THE DENSITY $v(n, t)$

The forward equation for $v(n, t)$ (35) becomes,

$$v_t = \frac{\sigma}{2} (nv)_{nn} - \left[\{n(1-n) - \frac{\alpha\lambda n}{\lambda+n} + \frac{\sigma}{4}\}v \right]_n \quad (68)$$

The first question we wish to consider is the calculation of the stationary distribution, $v(n)$. If $v_t = 0$, (68) becomes an ordinary differential equation that can be integrated to yield,

$$v(n) = c \exp \left[\frac{2}{\sigma} \left\{ n - \frac{n^2}{2} - \alpha\lambda \ln \left(\frac{1+n}{\lambda} \right) - \frac{\sigma^2}{2} \ln n \right\} \right] \quad (69)$$

$$= c \exp \left[\frac{-2}{\sigma} \phi(n) \right] \quad (69a)$$

The constant c is determined by normalization:

$$c = \left[\int_0^\infty \exp \left[\frac{-2}{\sigma} \phi(n) \right] dn \right]^{-1} \quad (70)$$

The probability of eventually finding the population at a level less than δ is,

$$P_\delta = \int_0^\delta v(n) dn. \quad (71)$$

Natural questions concern the location of concentration points of $v(n)$ and the behavior of $v(n)$ for λ large. The density $v(n)$

will be concentrated near the minima of the function $\phi(n)$. It is easy to show that these minima correspond to the steady states of the deterministic kinetic equation.

When λ is large, we expand $\ln(1 + n/\lambda)$ in its Taylor series and obtain,

$$v(n) = c \exp \left[\frac{2}{\sigma} \left\{ n - \frac{n^2}{2} - \alpha \lambda \left(\frac{n}{\lambda} - \frac{n^2}{2\lambda^2} \right) + O\left(\frac{1}{\lambda^3}\right) - \frac{\sigma^2}{2} \ln n \right\} \right] \quad (72)$$

Thus, for small σ and large λ :

$$v(n) \sim c \exp \left[\frac{2}{\sigma} \left\{ n(1 - \alpha) - \frac{n^2}{2} \left(1 - \frac{\alpha}{\lambda} \right) \right\} \right] \quad (73)$$

In this case, the extremum of ϕ occurs at

$$n^* = \frac{1-\alpha}{1-\frac{\alpha}{\lambda}} \quad (74)$$

The result (76) indicates the following. If $\alpha < 1$, then $n^* > 0$ always, so that the density is concentrated near n^* . When $\alpha > 1$, $n^* > 0$ if λ is small enough. In particular, $n^* > 0$ if

$$1 - \frac{\alpha}{\lambda} < 0 \quad \text{i.e., } \lambda < \alpha \quad (76)$$

Equation (76) can be viewed as a stochastic bifurcation result. We note that, for $\alpha > 1$, n^* corresponds to a minimum of $v(n)$.

MEAN TIME TO CROSS $n = \delta$

We now turn to the calculation of the mean time to cross the level $n = \delta$, conditioned on $n(0) = N_0$. This function $\bar{T}(\delta, N_0)$ satisfies

$$\frac{\sigma n}{2} \bar{T}_{nn} + \left[n(1-n) - \frac{\alpha \lambda n}{\lambda + n} + \frac{\sigma}{4} \right] \bar{T}_n = -p(n) \quad (77)$$

In (77), $p(n) = \Pr \{ \tilde{n}(t) \text{ eventually crosses } \delta | \tilde{n}(0) = n \}$.

The function \bar{T} satisfies the boundary conditions:

$$\bar{T}(\delta, \delta) = 0 \quad \lim_{n \rightarrow \infty} \bar{T}_n(\delta, n) = 0 \quad (78)$$

The function $p(n)$ satisfies:

$$\frac{\sigma n}{2} p_{nn} + \left[n(1-n) - \frac{\alpha \lambda n}{\lambda + n} + \frac{\sigma}{4} \right] p_n = 0 \quad (79)$$

subject to

$$p(\delta) = 1 \quad \lim_{n \rightarrow \infty} p(n) = 0 \quad (80)$$

Both the functions $p(n)$ and $\bar{T}(\delta, n)$ can be calculated explicitly by two integrations:

$$p(n) = \frac{\int_n^\infty \exp \left[-\frac{2}{\sigma} \left\{ n' - \frac{(n')^2}{2} - \alpha \lambda \ln \left(1 + \frac{n'}{\lambda} \right) - \frac{\sigma}{2} \ln n' \right\} \right] dn'}{\int_\delta^\infty \exp \left[-\frac{2}{\sigma} \left\{ n' - \frac{(n')^2}{2} - \alpha \lambda \ln \left(1 + \frac{n'}{\lambda} \right) - \frac{\sigma}{2} \ln n' \right\} \right] dn'} \quad (81)$$

In many cases of interest, $p(n) \approx 1$ for almost all values of n .

The solution of the problem posed by (77,78) is,

$$T(n) = \frac{2}{\sigma} \int_0^n \exp \left[\frac{-2}{\sigma} \left\{ s - \frac{s^2}{2} - \alpha \lambda \ln \left(1 + \frac{s}{\lambda} \right) - \frac{\sigma}{2} \ln s \right\} \right] \times \quad (82)$$

$$\int_s^\infty \exp \left[\frac{2}{\sigma} \left\{ y - \frac{y^2}{2} - \alpha \lambda \ln \left(1 + \frac{y}{\lambda} \right) - \frac{\sigma}{\lambda^2} \ln y \right\} \right] p(y) dy ds$$

The behavior of $p(n)$, $T(\delta, n)$ is sketched in figure 4, for a number of cases. Equations (81, 82) allow the explicit calculation of the mean time to extinction in the stochastic case.

SECTION 5

EXTINCTION IN A RANDOM ENVIRONMENT, WITHOUT THE STEADY STATE ASSUMPTION

In two dimensions the $[(n, \hat{q})$ or (n, q) phase space] no exact solutions of the stochastic problems are possible. If the intensity of the noise is small (i.e., $1/\bar{N}$, $1/\bar{Q}$ large), then approximate theories are available (Ludwig (1975), Mangel (1977)). In this section, we will sketch how those theories apply to the aggregation problem. We wish to calculate: 1) the probability that $\tilde{n}(t)$ ever crosses the line $n = \delta$, conditioned on initial value of n , and 2) the mean time to cross the line $n = \delta$. We differentiate two cases, according to the values of α .

$\alpha < 1$; THE RAY METHOD (LUDWIG, 1975)

When $\alpha < 1$ (figure 1), the non-trivial steady state is always stable. Hence, we are interested in the exit from the domain $n > \delta$. The density of (n, q) , $v(\tau, n, q)$, satisfies the forward equation

$$v_{\tau} = \frac{\sigma}{2} (a^{ij} v)_{ij} - ((b^i + \sigma c^i) v)_i \quad (83)$$

with $a(n, q)$, $c(n, q)$ given by (33), (36) respectively and

$$b^1 = \frac{dn}{d\tau}, \quad b^2 = \frac{dq}{d\tau} \quad \text{in (10,12). We assume that } \epsilon = 1.$$

Ludwig (1975) has shown that the ray method (Cohen and Lewis, 1967) applies here. A solution of (83) of the form,

$$v(\tau, n, q) = \exp\left[\frac{-\phi(\tau, n, q)}{\sigma}\right] \sum_{j=0}^{\infty} z_j \sigma^j \quad (84)$$

is sought.

After derivatives are evaluated, terms are collected according to powers of σ . The leading term vanishes if

$$b^i \phi_i + \frac{a^{ij}}{2} \phi_i \phi_j = 0 \quad (85)$$

To first order, Z_0 satisfies an ordinary differential equation on the characteristics of (85).

Equation (85) can be solved by the method of characteristics, once initial data are given. Ludwig has shown how this may be accomplished. The characteristics of the system (85) are called rays.

Suppose that $v \sim 0$ on the boundary and that we assume for v ,

$$v(\tau, n, q) = \sum_{k=0} e^{-\lambda_k \tau} \bar{v}(n, q) \quad (86)$$

Then the expected time to hit the boundary is asymptotic to $1/\lambda_0$ (Ludwig, 1975). We are led to the eigenvalue problem

$$-\lambda_0 = \frac{\sigma}{2} (a^{ij} v)_{ij} - ((b^i + \sigma c^i) v)_i \quad (87)$$

Ludwig has shown how Miller's method (Miller, 1962) may be generalized to calculate λ_0 . Thus, in principle the problem is completely solved. Further details and calculations are in (Ludwig, 1975).

$\alpha \geq 1$: THE GENERALIZED RAY METHOD (MANGEL, 1977)

When $\alpha \geq 1$, there may be two non-zero steady states ($\lambda < \lambda_B$), one degenerate steady state ($\lambda = \lambda_B$), or only the trivial

steady state ($\lambda > \lambda_B$) (figure 2). The ray method will break down in this case, but a generalization of the ray method does work (Mangel, 1977). Here we sketch the technique. The details can be found elsewhere. We consider $\bar{T}(\delta; n, q)$, which satisfies

$$-1 = \frac{\sigma}{2} a^{ij} T_{ij} + b^i T_i + c^i T_i \quad (88)$$

Asymptotic analysis of a one dimensional problem (for σ small) indicates that a formal solution of (88) is

$$\begin{aligned} \bar{T} = & g(x) B(\psi/\sigma^{1/3}, \alpha/\sigma^{2/3}, 1/\sigma^{1/3}, \eta) \\ & + \epsilon^{2/3} h(x) B'(\psi/\sigma^{1/3}, \alpha/\sigma^{2/3}, 1/\sigma^{1/3}, \eta) \\ & + \epsilon k(x) \end{aligned} \quad (89)$$

In (89), $B(Z, \alpha, \xi_1, \xi_2)$ satisfies the differential equation

$$\frac{d^2 B}{dZ^2} = -(Z^2 - \alpha) \frac{dB}{dZ} - \xi_1 + \xi_2 Z \quad Z_0 \leq Z \quad (90)$$

The functions, $g(x)$, $h(x)$, and $k(x)$ should also be expanded in asymptotic series of the form,

$$\begin{aligned} g(x) &= \sum \epsilon^n g^n(x) \\ k(x) &= \sum \epsilon^n k^n(x) \quad h(x) = \sum \epsilon^n h^n(x) \end{aligned} \quad (91)$$

where the superscript n is an index on g^n, h^n , and k^n . After derivatives are evaluated, and substituted into (88) (using equation (90) to eliminate B'' and B''') terms are collected according to powers of ϵ . The leading coefficients vanish if

$$O(\gamma^{-2/3} B): b^i \psi^i - \frac{a^{ij}}{2} \psi_i \psi_j (\psi^2 - \alpha) = 0 \quad (92)$$

$$O(\gamma^0 B): b^i g_i = 0 \quad (93)$$

$$O(\gamma^0): b^i k_i + \frac{a^{ij}}{2} g \psi_i \psi_j (-1 + \psi \eta) = -1 \quad (94)$$

Equation (92) is a generalized Hamilton-Jacobi equation. It can be solved by the method of characteristics. The parameter α is determined by requiring $\psi^2 = \alpha$ at the points where $b(x)$ vanishes. Namely, at the steady states P_0, P_1 , $\psi^2 = \alpha$. The value of α must be determined by a numerical iterative procedure (Mangel, 1977). At $\lambda = \lambda_B$, so that P_0 and P_1 coalesce, the value of $\alpha = 0$. Equation (93) indicates that g is a constant. Its value is determined as follows. At P_0, P_1 , equation (94) becomes

$$g \frac{a^{ij}}{2} \psi_i \psi_j (-1 + \psi \eta) \Big|_{P_k} = -1 \quad (95)$$

For $k = 0, 1$, we obtain two equations for g and η . When $\lambda = \lambda_B$, it is possible to show that $\eta = 0$ and (95) provides one equation for g . (Mangel, 1977).

We give data $k = 0$ on $n = \delta$. Then (94) can be solved by the method of characteristics. We let $\psi = \psi_0$ on $n = \delta$, set $\psi_0 = z_0$ in (90) and $B(z_0) = B'(z_0) = 0$. Then $\bar{T} = 0$ on $n = \delta$. This construction will yield values for T which are $O(\sigma^{2/3})$ different from the true values. Further details can be found in Mangel (1977).

REFERENCES

1. F. Brauer and A.C. Soudack (1978), "Stability Regions and Transition Phenomena for Harvested Predator Prey Systems," Preprint.
2. C.W. Clark and M. Mangel (1977), "On the Schooling Strategy of Tuna: Effects on Purse - Seine Fisheries," Preprint.
3. J.K. Cohen and R.M. Lewis (1967), "A Ray Method for the Asymptotic Solution of the Diffusion Equation," J. Inst. Math Appl. 3:266-290.
4. I.I. Gihman and A.V. Skorohod (1972), "Stochastic Differential Equations," Springer Verlag, New York, 354 pp.
5. "Inter American Tropical Tuna Commission (1975)," Annual Report (1974), La Jolla, California.
6. D.D. Jones and C.J. Walters (1976), "Catastrophe Theory and Fishery Regulation," J. Fish. Res. Board, Canada, 33(12):2829-33.
7. C.C. Lin and L.A. Segel (1973), "Mathematics Applied to Deterministic Problems in the Natural Sciences," MacMillan, New York.
8. D. Ludwig (1974), "Stochastic Population Theories," Lecture Notes on Biomathematics 3, Springer-Verlog, New York.
9. D. Ludwig (1975), "Persistence of Dynamical Systems Under Random Perturbations," Siam Rev. 17(4): 605-640.
10. M. Mangel (1977), "Small Fluctuations at the Unstable Steady State," Technical Report 77-6, Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, Canada.
11. M. Mangel (1978) "Small Fluctuations in Systems with Multiple Steady States," Submitted to SIAM J. Applied Math.
12. M. Turelli (1977), "Random Environments and Stochastic Calculus," Theor. Pop. Biol., in press.
13. E. Wong (1971), "Stochastic Processes in Information Theory and Dynamical Systems," McGraw-Hill.

ACKNOWLEDGEMENTS

The initial phase of this work was supported by contract number 03-6-208-35341, from the National Marine Fisheries Service.

I thank Professors C.W. Clark and D. Ludwig for helpful discussions and C.W. Clark for his comments on a previous draft of the manuscript.

APPENDIX A
THE GENERAL AGGREGATION MODEL

In this appendix, we prove a theorem that applies to the general set of aggregation equations

$$\dot{N} = g_N - \alpha f(N, Q) \quad (A-1)$$

$$\dot{Q} = \alpha f(N, Q) - \lambda Q \quad (A-2)$$

with N, Q defined as in the body of the paper.

Let $Q^*(N, \alpha, \lambda)$ be the solution of

$$\alpha f(N, Q) = \lambda Q \quad (A-3)$$

such that the following hypotheses are satisfied:

$$a.) \quad \lim_{N \rightarrow \infty} Q^*(N, \alpha, \lambda) = c_0 < \infty$$

$$b.) \quad \lim_{N \rightarrow \infty} \alpha f(N, Q^*(N, \alpha, \lambda)) = c_1(\alpha, \lambda) < \infty \quad \text{for all } \alpha, \lambda$$

$$c.) \quad \lim_{\lambda \rightarrow \infty} \alpha f(N, Q^*(N, \alpha, \lambda)) = c_2(\alpha, N) < \infty \quad \text{for all } \alpha, N$$

Then there exist values α^*, λ^* such that the tuna steady state level exhibits a bifurcation. Namely, for $\alpha > \alpha^*$ and $\lambda < \lambda^*$

there are three steady state solutions, $N(\alpha, \lambda)$ of (A-1, A-2). The origin is a stable steady state. There are two other non-trivial steady states $N_S(\alpha, \lambda)$, $N_U(\alpha, \lambda)$ which are stable and unstable, respectively. When $\lambda = \lambda^*$, N_S and N_U coalesce and annihilate each other for $\lambda > \lambda^*$.

PROOF

First, we note that conditions a, b insure a finite surface population, for all underlying population levels.

The steady state underlying level is determined by solving

$$g_N(N) = \alpha f(N, Q^*(N, \alpha, \lambda)) \quad (A-4)$$

We determine α^* by solving for α the equation:

$$g'_N(0) = \alpha^* \frac{\partial}{\partial N} f(N, Q^*(N, \alpha, \lambda)) \Big|_{N=0} \quad (A-5)$$

$$= \alpha^* \left\{ f_N(0, Q^*(0, \alpha^*, \lambda)) + f_Q(0, Q^*(0, \alpha^*, \lambda)) \frac{\partial Q^*}{\partial N}(0, \alpha^*, \lambda) \right\} \quad (A-6)$$

We determine $\lambda^*(\alpha)$ as follows. For any fixed α , $\alpha > \alpha^*$ consider the function

$$Y(N, \lambda) = \alpha f(N, Q^*(N, \lambda, \alpha)) - g_N(N) \quad (A-7)$$

We choose λ^* by requiring that

$$Y(N, \lambda^*) = 0 \quad (A-8)$$

has two solutions: $N = 0$ and $N = N^*$, where N^* is determined by

$$\left. \frac{\partial}{\partial N} Y(N, \lambda^*) \right|_{N^*} = 0 . \quad (A-9)$$

Then for fixed α , $\alpha > \alpha^*$, as λ increases through λ^* , we observe a bifurcation (figure 5).

APPENDIX B

A KINETIC MODEL FOR FLUCTUATIONS

Clark and Mangel considered a kinetic model that yields the same behavior as equations (11, 12). This model provides a natural way to treat fluctuations. We let $T(t)$ be the number of "core" schools in the ocean and consider the kinetic scheme.



where C_1, C_2 are, respectively, surface complexes with 1 and 2 core schools. The kinetic equations for the scheme (B-1), derived by a "law of mass action" approach, are:

$$\frac{dT}{dt} = g_T - \alpha_1 TK - \alpha_2 TC_1 + \beta_1 C_1 + \beta_2 C_2
 \tag{B-2}$$

$$\frac{dK}{dt} = g_K - \alpha_1 TK + \beta_1 C_1 + bEX_o (C_1 + C_2)
 \tag{B-3}$$

$$\frac{dC_1}{dt} = \alpha_1 K T - \beta_1 C_1 + \beta_2 C_2 - \alpha_2 C_1 T - bEX_O C_1 \quad (B-4)$$

$$\frac{dC_2}{dt} = \alpha_2 C_1 T - \beta_2 C_2 - bEX_O C_1 \quad (B-5)$$

The analysis of (B.2-B.5) yields results analogous to (10-12); if we set $\beta_1 = \beta_2 = 0$ (Clark and Mangel, Appendix B).

For the purpose of stochastic modeling, however, the scheme (B-1) has a number of advantages. In particular, no re-interpretation of the kinetic equation is needed, since the elementary events (B-1) are clearly a birth and death process. Consider a time increment $t \rightarrow t + \Delta t$, with the values of $T(t)$, $C_1(t)$, $C_2(t)$ and $K(t)$ known. Then in the increment Δt , we assume that the following transition probabilities exist:

$$\Pr\{\Delta T = 1, \Delta K = 1, \Delta C_1 = -1, \Delta C_2 = 0\} = \beta_1 C_1 \Delta t + o(\Delta t) \quad (B-6)$$

$$\Pr\{\Delta T = -1, \Delta K = -1, \Delta C_1 = 1, \Delta C_2 = 0\} = \alpha_1 T K \Delta t + o(\Delta t) \quad (B-7)$$

$$\Pr\{\Delta T = -1, \Delta K = 0, \Delta C_1 = -1, \Delta C_2 = 1\} = \alpha_2 T C_1 \Delta t + o(\Delta t) \quad (B-8)$$

$$\Pr\{\Delta T = 1, \Delta K = 0, \Delta C_1 = 1, \Delta C_2 = -1\} = \beta_2 C_2 \Delta t + o(\Delta t) \quad (B-9)$$

$$\Pr\{\Delta T = 0, \Delta K = 0, \Delta C_1 = -1, \Delta C_2 = 0\} = bEX_O C_1 \Delta t + o(\Delta t) \quad (B-10)$$

$$\Pr\{\Delta T = 0, \Delta K = 0, \Delta C_1 = 0, \Delta C_2 = -1\} = bEX_0 C_2 \Delta t + o(\Delta t) \quad (B-11)$$

$$\Pr\{\text{all other transitions}\} = o(\Delta t), \quad (B-12)$$

where $\Delta T = T(t + \Delta t) - T(t)$, etc.

If we use a quasi-steady state approach, we replace C_1 , C_2 , and K by the steady state values (Clark and Mangel(1977)):

$$K = K_{eq} \quad (B-13)$$

$$C_2 = \frac{\alpha_2 C_1 T}{\beta_2 + bEX_0} \quad (B-14)$$

$$C_1 = \frac{\alpha_1 TK}{\beta_1 + \alpha_2 T + bEX_0 - \left\{ \frac{\alpha_2 \beta_2 T}{\beta_2 + bEX_0} \right\}} \quad (B-15)$$

Then the transition probabilities are functions of $T(t)$ only.

We obtain

$$\Pr\{\Delta T = 1\} = \Delta t\{g_T + \beta_1 C_1 + \beta_2 C_2\} + o(\Delta t) \quad (B-16)$$

$$\equiv \lambda(T)\Delta t + o(\Delta t) \quad (B-17)$$

$$\Pr\{\Delta T = -1\} = \Delta t\{\alpha_1 KT + \alpha_2 C_1 T\} \quad (B-18)$$

$$\equiv \mu(T)\Delta t + o(\Delta t)$$

with C_1 , C_2 , and K given by (B.13-15). It is easy to show that

$$\frac{dT}{dt} = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} E\{\Delta T\} = \lambda(T) - \mu(T) \quad (B-19)$$

is the same as the law of mass action result. We can, however, immediately calculate the incremental variance needed for the diffusion approximation:

$$a(T) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} E\{(\Delta T)^2\} \quad (B-20)$$

$$= \lambda(T) + \mu(T) \quad (B-21)$$

TABLE 1

α	N_o	$1/\lambda$	$T(\delta, N_o)$		
			Steady State	Exact	% Diff
10.05	1.0	.1	.234	.225	4.9
		.2	.244	.253	3.5
		.4	.266	.271	1.8
	.6	.1	.186	.199	6.5
		.2	.192	.204	5.8
		.4	.203	.213	4.7
	.2	.1	.074	.078	5.1
		.2	.076	.079	3.8
		.4	.078	.081	3.7
7.05	1.0	.1	.343	.373	8.0
		.2	.358	.386	5.4
		.4	.387	.410	5.6
	.6	.1	.274	.292	6.2
		.2	.282	.300	6.0
		.4	.300	.316	8.1
	.2	.1	.110	.114	3.8
		.2	.112	.116	3.4
		.4	.116	.119	2.5

α	N_O	$1/\lambda$	$T(\delta, N_O)$			
			Steady State	Exact	% Diff	
4.05	1.0	.1	.644	.701	8.1	
		.2	.673	.726	7.3	
		.4	.730	.773	5.6	
	.6	.1	.522	.577	6.3	
		.2	.540	.573	5.8	
		.4	.575	.605	5.0	
	.2	.1	.213	.221	3.6	
		.2	.217	.225	3.6	
		.4	.225	.233	3.4	
	1.05	1.0	.1	6.53	7.74	15.6
			.2	7.13	8.32	14.3
			.4	8.33	9.47	12.0
.6		.1	5.97	7.05	15.3	
		.2	6.51	7.59	14.2	
		.4	7.51	8.66	13.3	
.2		.1	3.37	3.94	14.4	
		.2	3.65	4.25	14.1	
		.4	4.21	4.91	14.2	

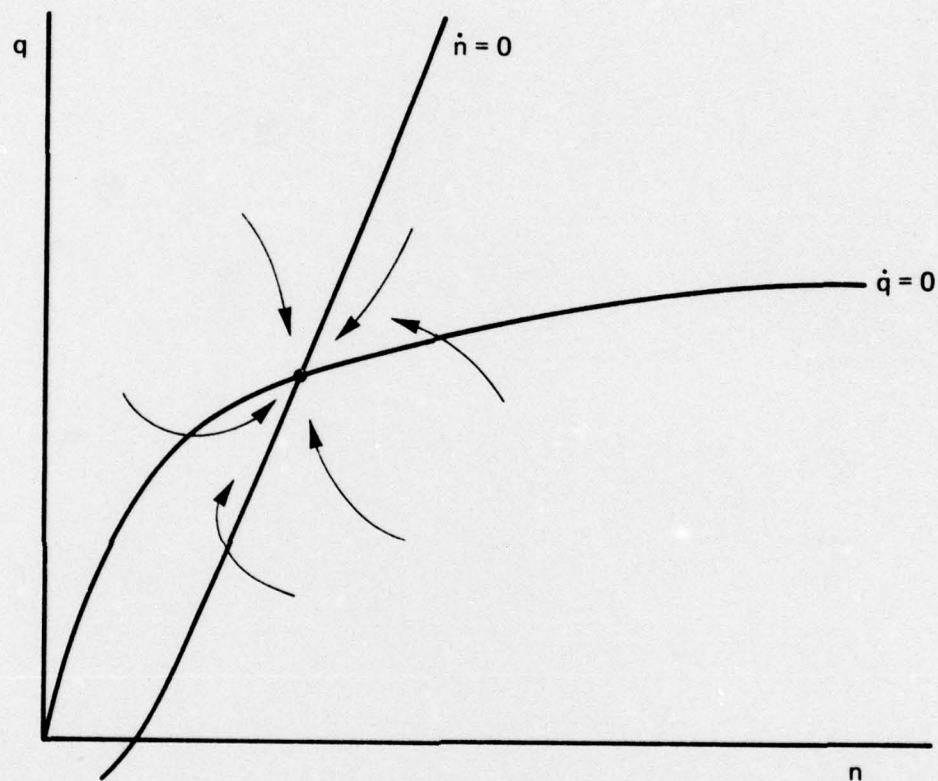
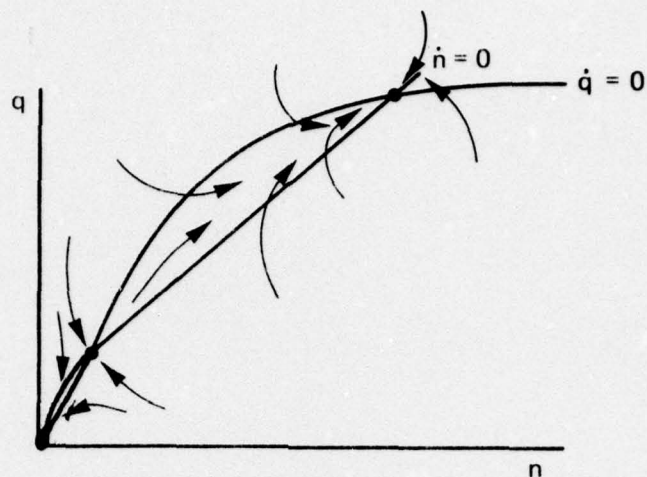
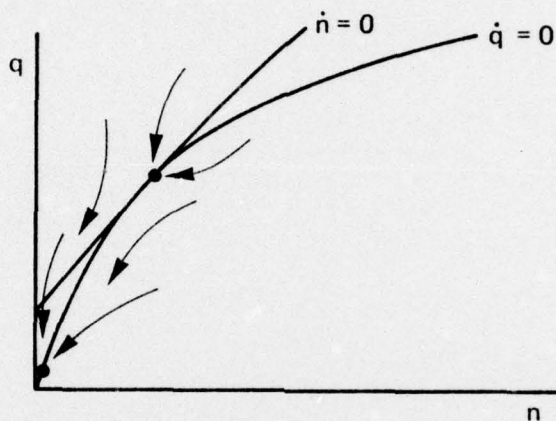


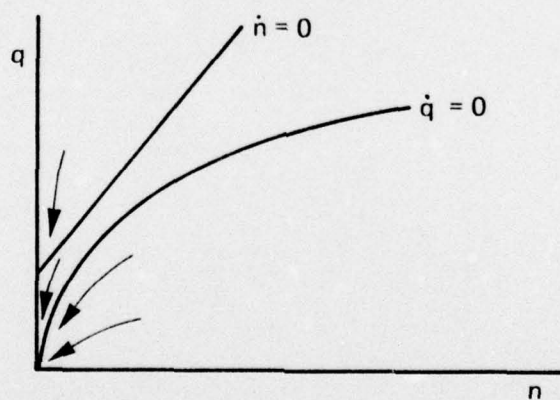
FIG. 1: The (n, q) phase plane for the case $\alpha < 1$. There is a unique, stable steady state.



(a) $D > 0$



(b) $D = 0$



(c) $D < 0$

FIG. 2: The (n, q) phase plane for the case $\alpha > 1$. In this case, the deterministic system is of the marginal type (Mangel, 1977). a) For $D > 0$ ($\lambda < \lambda_B$) P_0 is unstable. (b) When $D = 0$ ($\lambda = \lambda_B$) P_0 and P_1 coalesce to form a state of marginal stability. c) For $D < 0$ ($\lambda > \lambda_B$) P_0 and P_1 have annihilated each other. The only steady state is the origin; it is stable. The $\dot{n} = 0$ is really more parabolic than shown.

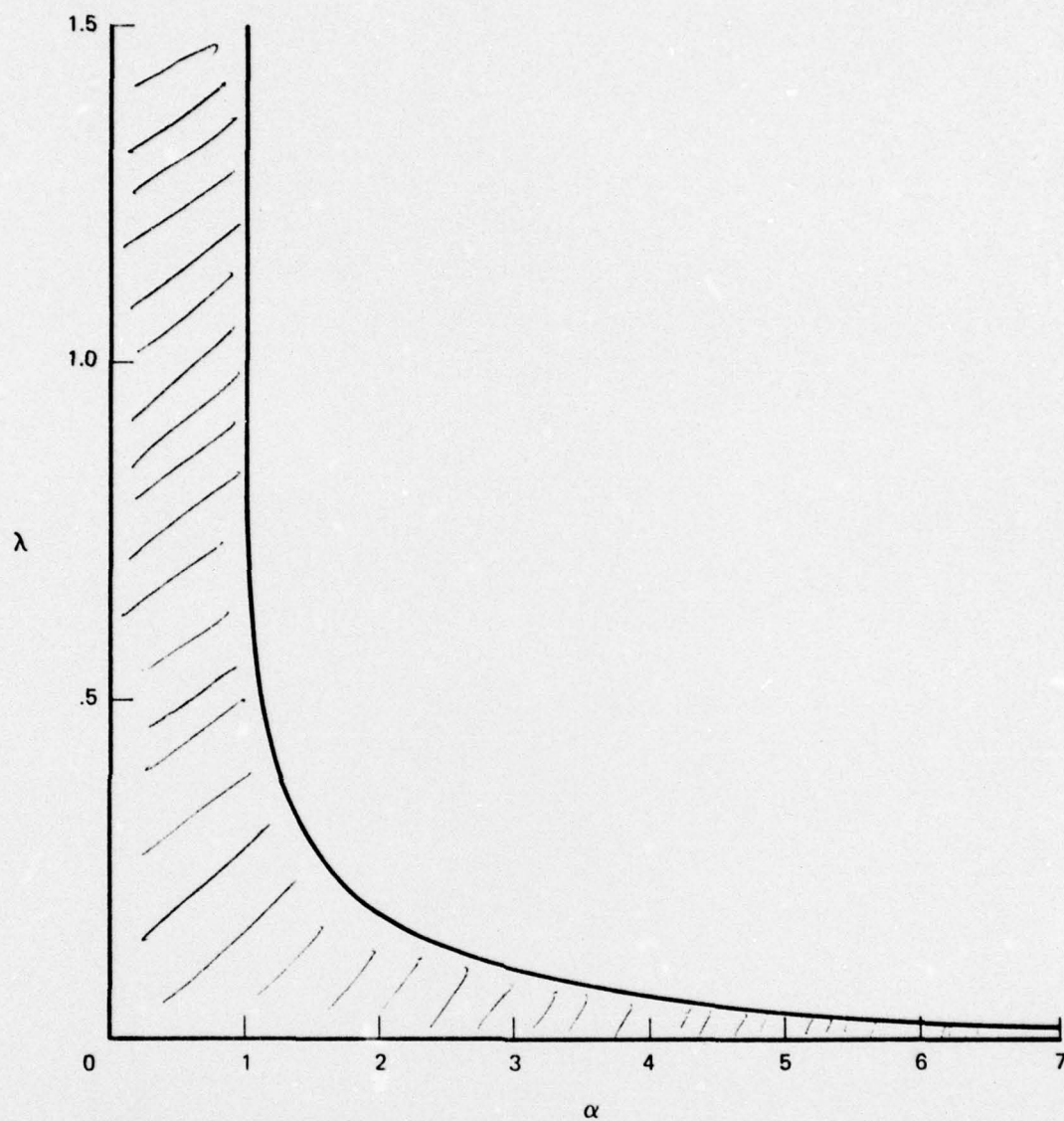


FIG. 3: The (α, λ) bifurcation picture. For points in the hatched region, there is always a nontrivial, stable steady state.

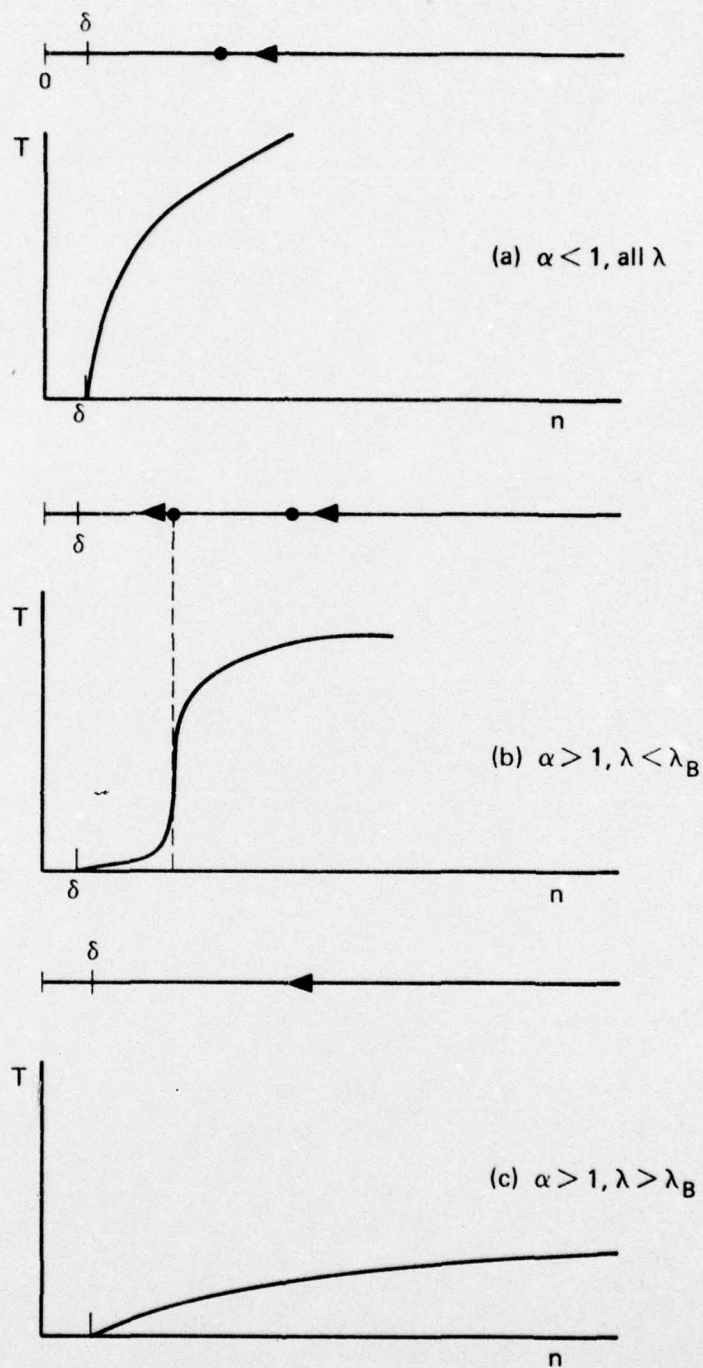


FIG. 4: The mean time to cross $n = \delta$ as a function of initial position, for the values of α ($\alpha < 1, \alpha > 1$) and λ ($\lambda > \lambda_B$ or $\lambda < \lambda_B$).

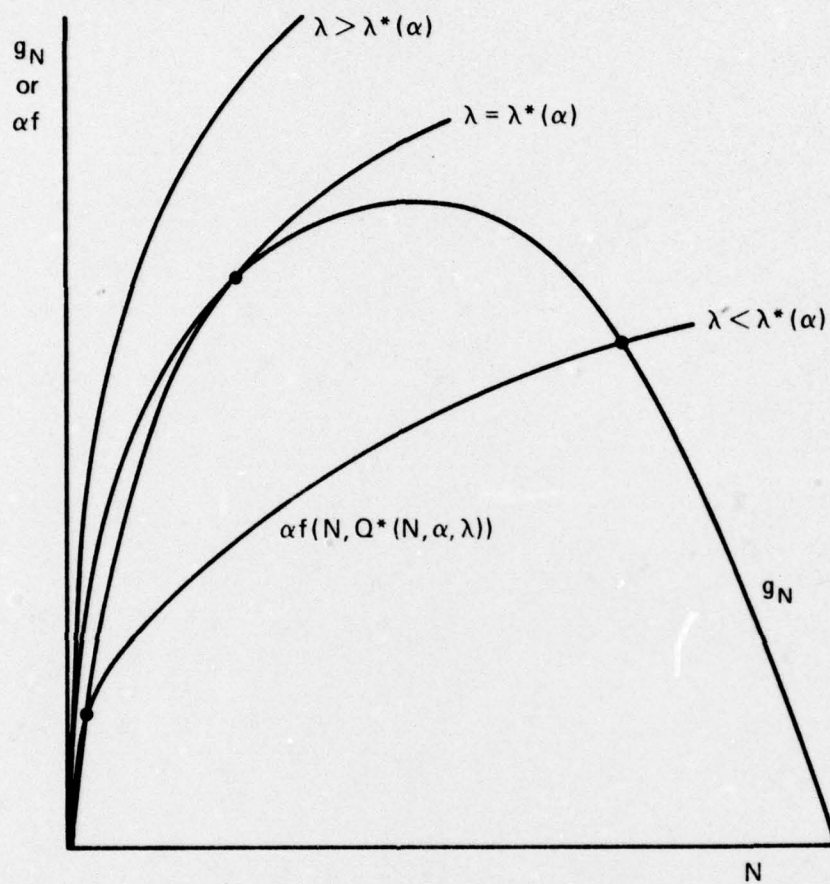


FIG. 5: Illustration of the calculation of bifurcation values α^* and λ^* .

CNA Professional Papers — 1973 to Present*

- PP 103
Friedheim, Robert L., "Political Aspects of Ocean Ecology" 48 pp., Feb 1973, published in *Who Protects the Oceans*, John Lawrence Hargrove (ed.) (St. Paul: West Publ'g Co., 1974), published by the American Society of International Law AD 757 936
- PP 104
Schick, Jack M., "A Review of James Cable, Gunboat Diplomacy Political Applications of Limited Naval Forces," 5 pp., Feb 1973, (Reviewed in the *American Political Science Review*, Vol. LXVI, Dec 1972)
- PP 105
Corn, Robert J. and Phillips, Gary R., "On Optimal Correction of Gunfire Errors," 22 pp., Mar 1973, AD 761 674
- PP 106
Stoloff, Peter H., "User's Guide for Generalized Factor Analysis Program (FACTAN)," 35 pp., Feb 1973, (Includes an addendum published Aug 1974) AD 758 824
- PP 107
Stoloff, Peter H., "Relating Factor Analytically Derived Measures to Exogenous Variables," 17 pp., Mar 1973, AD 758 820
- PP 108
McConnell, James M. and Kelly, Anne M., "Superpower Naval Diplomacy in the Indo-Pakistani Crisis," 14 pp., 5 Feb 1973, (Published, with revisions, in *Survival*, Nov/Dec 1973) AD 761 675
- PP 109
Berghoefer, Fred G., "Salaries—A Framework for the Study of Trend," 8 pp., Dec 1973, (Published in *Review of Income and Wealth*, Series 18, No. 4, Dec 1972)
- PP 110
Augusta, Joseph, "A Critique of Cost Analysis," 9 pp., Jul 1973, AD 766 378
- PP 111
Herrick, Robert W., "The USSR's 'Blue Belt of Defense' Concepts: A Unified Military Plan for Defense Against Seaborne Nuclear Attack by Strike Carriers and Polaris/Poseidon SSBNs," 18 pp., May 1973, AD 766 375
- PP 112
Ginsberg, Lawrence H., "ELF Atmosphere Noise Level Statistics for Project SANGUINE," 29 pp., Apr 1974, AD 786 969
- PP 113
Ginsberg, Lawrence H., "Propagation Anomalies During Project SANGUINE Experiments," 5 pp., Apr 1974, AD 786 968
- PP 114
Maloney, Arthur P., "Job Satisfaction and Job Turnover," 41 pp., Jul 1973, AD 768 410
- PP 115
Silverman, Lester P., "The Determinants of Emergency and Elective Admissions to Hospitals," 145 pp., 18 Jul 1973, AD 766 377
- PP 116
Rehm, Allan S., "An Assessment of Military Operations Research in the USSR," 19 pp., Sep 1973, (Reprinted from *Proceedings, 30th Military Operations Research Symposium (U)*, Secret Dec 1972) AD 770 116
- PP 117
McWhite, Peter B. and Rattliff, H. Donald,* "Defending a Logistics System Under Mining Attack,"** 24 pp., Aug 1976 (to be submitted for publication in *Naval Research Logistics Quarterly*), presented at 44th National Meeting, Operations Research Society of America, November 1973, AD A030 454
*University of Florida.
**Research supported in part under Office of Naval Research Contract N00014-68-0273-0017
- PP 118
Barfoot, C. Bernard, "Markov Duels," 18 pp., Apr 1973, (Reprinted from *Operations Research*, Vol. 22, No. 2, Mar-Apr 1974)
- PP 119
Stoloff, Peter and Lockman, Robert F., "Development of Navy Human Relations Questionnaire," 2 pp., May 1974, (Published in *American Psychological Association Proceedings*, 81st Annual Convention, 1973) AD 779 240
- PP 120
Smith, Michael W. and Schrimper, Ronald A.,* "Economic Analysis of the Intracity Dispersion of Criminal Activity," 30 pp., Jun 1974, (Presented at the *Econometric Society Meetings*, 30 Dec 1973) AD 780 538
*Economics, North Carolina State University.
- PP 121
Devine, Eugene J., "Procurement and Retention of Navy Physicians," 21 pp., Jun 1974, (Presented at the 49th Annual Conference, Western Economic Association, Las Vegas, Nev., 10 Jun 1974) AD 780 539
- PP 122
Kelly, Anne M., "The Soviet Naval Presence During the Iraq-Kuwaiti Border Dispute: March-April 1973," 34 pp., Jun 1974, (Published in *Soviet Naval Policy*, ed. Michael McGwire; New York: Praeger) AD 780 592
- PP 123
Petersen, Charles C., "The Soviet Port-Clearing Operation in Bangladesh, March 1972-December 1973," 35 pp., Jun 1974, (Published in Michael McGwire, et al. (eds) *Soviet Naval Policy: Objectives and Constraints*, (New York: Praeger Publishers, 1974) AD 780 540
- PP 124
Friedheim, Robert L. and Jehn, Mary E., "Anticipating Soviet Behavior at the Third U.N. Law of the Sea Conference: USSR Positions and Dilemmas," 37 pp., 10 Apr 1974, (Published in *Soviet Naval Policy*, ed. Michael McGwire; New York: Praeger) AD 783 701
- PP 125
Weinland, Robert G., "Soviet Naval Operations—Ten Years of Change," 17 pp., Aug 1974, (Published in *Soviet Naval Policy*, ed. Michael McGwire; New York: Praeger) AD 783 962
- PP 126 — Classified.
- PP 127
Dragnich, George S., "The Soviet Union's Quest for Access to Naval Facilities in Egypt Prior to the June War of 1967," 64 pp., Jul 1974, AD 786 318
- PP 128
Stoloff, Peter and Lockman, Robert F., "Evaluation of Naval Officer Performance," 11 pp., (Presented at the 82nd Annual Convention of the American Psychological Association, 1974) Aug 1974, AD 784 012
- PP 129
Holen, Arlene and Horowitz, Stanley, "Partial Unemployment Insurance Benefits and the Extent of Partial Unemployment," 4 pp., Aug 1974, (Published in the *Journal of Human Resources*, Vol. IX, No. 3, Summer 1974) AD 784 010
- PP 130
Dismukes, Bradford, "Roles and Missions of Soviet Naval General Purpose Forces in Wartime: PROSSBN Operation," 20 pp., Aug 1974, AD 786 320
- PP 131
Weinland, Robert G., "Analysis of Gorshkov's *Navies in War and Peace*," 45 pp., Aug 1974, (Published in *Soviet Naval Policy*, ed. Michael McGwire; New York: Praeger) AD 786 319
- PP 132
Kleinman, Samuel D., "Racial Differences in Hours Worked in the Market: A Preliminary Report," 77 pp., Feb 1975, (Paper read on 26 Oct 1974 at Eastern Economic Association Convention in Albany, N.Y.) AD A 005 517
- PP 133
Squires, Michael L., "A Stochastic Model of Regime Change in Latin America," 42 pp., Feb 1975, AD A 007 912
- PP 134
Root, R. M. and Cuniff, P. F.,* "A Study of the Shock Spectrum of a Two-Degree-of-Freedom Non-linear Vibratory System," 39 pp., Dec 1975, (Published in the condensed version of *The Journal of the Acoustic Society*, Vol 60, No. 6, Dec 1976, pp. 1314
*Department of Mechanical Engineering, University of Maryland.
- PP 135
Goudreau, Kenneth A.; Kuzmack, Richard A.; Wiedemann, Karen, "Analysis of Closure Alternatives for Naval Stations and Naval Air Stations," 47 pp., 3 Jun 1975 (Reprinted from "Hearing before the Subcommittee on Military Construction of the Committee on Armed Service," U.S. Senate, 93rd Congress, 1st Session, Part 2, 22 Jun 1973)
- PP 136
Staffings, William, "Cybernetics and Behavior Therapy," 13 pp., Jun 1975
- PP 137
Petersen, Charles C., "The Soviet Union and the Reopening of the Suez Canal: Mineclearing Operations in the Gulf of Suez," 30 pp., Aug 1975, AD A 015 376

*CNA Professional Papers with an AD number may be obtained from the National Technical Information Service, U.S. Department of Commerce, Springfield, Virginia 22151. Other papers are available from the author at the Center for Naval Analyses, 1401 Wilson Boulevard, Arlington, Virginia 22209.

- PP 138
Stallings, William, "BRIDGE: An Interactive Dialogue-Generation Facility," 5 pp., Aug 1975 (Reprinted from IEEE Transactions on Systems, Man, and Cybernetics, Vol. 5, No. 3, May 1975)
- PP 139
Morgan, William F., Jr., "Beyond Folklore and Fables in Forestry to Positive Economics," 14 pp., (Presented at Southern Economic Association Meetings November, 1974) Aug 1975, AD A 015 293
- PP 140
Mahoney, Robert and Druckman, Daniel, "Simulation, Experimentation, and Context," 36 pp., 1 Sep 1975, (Published in Simulation & Games, Vol. 6, No. 3, Sep 1975)
*Mathematica, Inc.
- PP 141
Mizrahi, Maurice M., "Generalized Hermite Polynomials," 5 pp., Feb 1976 (Reprinted from the Journal of Computational and Applied Mathematics, Vol. 1, No. 4 (1975), 273-277).
*Research supported by the National Science Foundation
- PP 142
Lockman, Robert F., Jehn, Christopher, and Shughart, William F. II, "Models for Estimating Premature Losses and Recruiting District Performance," 36 pp., Dec 1975 (Presented at the RAND Conference on Defense Manpower, Feb 1976; to be published in the conference proceedings) AD A 020 443
- PP 143
Horowitz, Stanley and Sherman, Allan (LCdr., USN), "Maintenance Personnel Effectiveness in the Navy," 33 pp., Jan 1976 (Presented at the RAND Conference on Defense Manpower, Feb 1976; to be published in the conference proceedings) AD A021 581
- PP 144
Durch, William J., "The Navy of the Republic of China - History, Problems, and Prospects," 66 pp., Aug 1976 (To be published in "A Guide to Asiatic Fleets," ed. by Barry M. Blechman; Naval Institute Press) AD A030 460
- PP 145
Kelly, Anne M., "Port Visits and the "Internationalist Mission" of the Soviet Navy," 36 pp., Apr 1976 AD A023 436
- PP 146
Palmour, Vernon E., "Alternatives for Increasing Access to Scientific Journals," 6 pp., Apr 1975 (Presented at the 1975 IEEE Conference on Scientific Journals, Cherry Hill, N.C., Apr 28-30; published in IEEE Transactions on Professional Communication, Vol. PC-18, No. 3, Sep 1975) AD A021 798
- PP 147
Kessler, J. Christian, "Legal Issues in Protecting Offshore Structures," 33 pp., Jun 1975 (Prepared under task order N00014-68-A-0091-0023 for ONR) AD A028 389
- PP 148
McConnell, James M., "Military-Political Tasks of the Soviet Navy in War and Peace," 62 pp., Dec 1975 (Published in Soviet Oceans Development Study of Senate Commerce Committee October 1976) AD A022 590
- PP 149
Squires, Michael L., "Counterforce Effectiveness: A Comparison of the Tsipis "K" Measure and a Computer Simulation," 24 pp., Mar 1976 (Presented at the International Study Association Meetings, 27 Feb 1976) AD A022 591
- PP 150
Kelly, Anne M. and Petersen, Charles, "Recent Changes in Soviet Naval Policy: Prospects for Arms Limitations in the Mediterranean and Indian Ocean," 28 pp., Apr 1976, AD A 023 723
- PP 151
Horowitz, Stanley A., "The Economic Consequences of Political Philosophy," 8 pp., Apr 1976 (Reprinted from Economic Inquiry, Vol. XIV, No. 1, Mar 1976)
- PP 152
Mizrahi, Maurice M., "On Path Integral Solutions of the Schrodinger Equation, Without Limiting Procedure," 10 pp., Apr 1976 (Reprinted from Journal of Mathematical Physics, Vol. 17, No. 4 (Apr 1976), 566-575).
*Research supported by the National Science Foundation
- PP 153
Mizrahi, Maurice M., "WKB Expansions by Path Integrals, With Applications to the Anharmonic Oscillator," 137 pp., May 1976, AD A025 440
*Research supported by the National Science Foundation
- PP 154
Mizrahi, Maurice M., "On the Semi-Classical Expansion in Quantum Mechanics for Arbitrary Hamiltonians," 19 pp., May 1976 (Published in Journal of Mathematical Physics, Vol. 18, No. 4, p. 786, Apr 1977), AD A025 441
- PP 155
Squires, Michael L., "Soviet Foreign Policy and Third World Nations," 26 pp., Jun 1976 (Prepared for presentation at the Midwest Political Science Association meetings, Apr 30, 1976) AD A028 388
- PP 156
Stallings, William, "Approaches to Chinese Character Recognition," 12 pp., Jun 1976 (Reprinted from Pattern Recognition (Pergamon Press), Vol. 8, pp. 87-98, 1976) AD A028 692
- PP 157
Morgan, William F., "Unemployment and the Pentagon Budget: Is There Anything in the Empty Pork Barrel?" 20 pp., Aug 1976 AD A030 455
- PP 158
Haskell, LCdr. Richard D. (USN), "Experimental Validation of Probability Predictions," 25 pp., Aug 1976 (Presented at the Military Operations Research Society Meeting, Fall 1976) AD A030 458
- PP 159
McConnell, James M., "The Gorshkov Articles, The New Gorshkov Book and Their Relation to Policy," 93 pp., Jul 1976 (Published in Soviet Naval Influence: Domestic and Foreign Dimensions, ed. by M. McGwire and J. McDonnell; New York; Praeger, 1977) AD A029 227
- PP 160
Wilson, Desmond P., Jr., "The U.S. Sixth Fleet and the Conventional Defense of Europe," 50 pp., Sep 1976 (Submitted for publication in Adelphi Papers, I.I.S.S., London) AD A030 457
- PP 161
Melich, Michael E. and Peet, Vice Adm. Ray (USN, Retired), "Fleet Commanders: Afloat or Ashore?" 9 pp., Aug 1976 (Reprinted from U.S. Naval Institute Proceedings, Jun 1976) AD A030 456
- PP 162
Friedheim, Robert L., "Parliamentary Diplomacy," 106 pp. Sep 1976 AD A033 306
- PP 163
Lockman, Robert F., "A Model for Predicting Recruit Losses," 9 pp., Sep 1976 (Presented at the 84th annual convention of the American Psychological Association, Washington, D.C., 4 Sep 1976) AD A030 459
- PP 164
Mahoney, Robert B., Jr., "An Assessment of Public and Elite Perceptions in France, The United Kingdom, and the Federal Republic of Germany," 31 pp., Feb 1977 (Presented at Conference "Perception of the U.S. - Soviet Balance and the Political Uses of Military Power" sponsored by Director, Advanced Research Projects Agency, April 1976) AD 036 599
- PP 165
Jondrow, James M., "Effects of Trade Restrictions on Imports of Steel," 67 pp., November 1976, (Delivered at ILAB Conference in Dec 1976)
- PP 166
Feldman, Paul, "Impediments to the Implementation of Desirable Changes in the Regulation of Urban Public Transportation," 12 pp., Oct 1976, AD A033 322
- PP 166 - Revised
Feldman, Paul, "Why It's Difficult to Change Regulation," Oct 1976
- PP 167
Kleinman, Samuel, "ROTC Service Commitments: a Comment," 4 pp., Nov 1976, (To be published in Public Choice, Vol. XXIV, Fall 1976) AD A033 305
- PP 168
Lockman, Robert F., "Revalidation of CNA Support Personnel Selection Measures," 36 pp., Nov 1976
- PP 169
Jacobson, Louis S., "Earnings Losses of Workers Displaced from Manufacturing Industries," 38 pp., Nov 1976, (Delivered at ILAB Conference in Dec 1976), AD A039 809
- PP 170
Brechling, Frank P., "A Time Series Analysis of Labor Turnover," Nov 1976, (Delivered at ILAB Conference in Dec 1976)
- PP 171
Ralston, James M., "A Diffusion Model for GeP Red LED Degradation," 10 pp., Nov 1976, (Published in Journal of Applied Physics, Vol. 47, pp. 4518-4527, Oct 1976)

- PP 172
Classen, Kathleen P., "Unemployment Insurance and the Length of Unemployment," Dec 1976, (Presented at the University of Rochester Labor Workshop on 16 Nov 1976)
- PP 173
Kleinman, Samuel D., "A Note on Racial Differences in the Added-Worker/Discouraged-Worker Controversy," 2 pp., Dec 1976, (Published in the American Economist, Vol. XX, No. 1, Spring 1976)
- PP 174
Mahoney, Robert B., Jr., "A Comparison of the Brookings and International Incidents Projects," 12 pp. Feb 1977 AD 037 206
- PP 175
Levine, Daniel; Stolloff, Peter and Spruill, Nancy, "Public Drug Treatment and Addict Crime," June 1976, (Published in Journal of Legal Studies, Vol. 5, No. 2)
- PP 176
Felix, Wendi, "Correlates of Retention and Promotion for USNA Graduates," 38 pp., Mar 1977, AD A039 040
- PP 177
Lockman, Robert F. and Warner, John T., "Predicting Attrition: A Test of Alternative Approaches," 33 pp. Mar 1977. (Presented at the OSD/ONR Conference on Enlisted Attrition Xerox International Training Center, Leesburg, Virginia, 4-7 April 1977), AD A039 047
- PP 178
Kleinman, Samuel D., "An Evaluation of Navy Unrestricted Line Officer Accession Programs," 23 pp. April 1977, (To be presented at the NATO Conference on Manpower Planning and Organization Design, Stresa, Italy, 20 June 1977), AD A039 048
- PP 179
Stolloff, Peter H. and Belut, Stephen J., "Vacates: A Model for Personnel Inventory Planning Under Changing Management Policy," 14 pp. April 1977, (Presented at the NATO Conference on Manpower Planning and Organization Design, Stresa, Italy, 20 June 1977), AD A039 049
- PP 180
Horowitz, Stanley A. and Sherman, Allan, "The Characteristics of Naval Personnel and Personnel Performance," 16 pp. April 1977, (Presented at the NATO Conference on Manpower Planning and Organization Design, Stresa, Italy, 20 June 1977), AD A039 050
- PP 181
Belut, Stephen J. and Stolloff, Peter, "An Inventory Planning Model for Navy Enlisted Personnel," 35 pp., May 1977, (Prepared for presentation at the Joint National Meeting of the Operations Research Society of America and The Institute for Management Science, 9 May 1977, San Francisco, California), AD A042 221
- PP 182
Murray, Russell, 2nd, "The Quest for the Perfect Study or My First 1138 Days at CNA," 57 pp., April 1977
- PP 183
Kassing, David, "Changes in Soviet Naval Forces," 33 pp., November, 1976, (Published as part of Chapter 3, "General Purpose Forces: Navy and Marine Corps," in Arms, Men, and Military Budgets, Francis P. Hoerber and William Schneider, Jr. (eds.), (Crane, Russak & Company, Inc.: New York), 1977), AD A040 106
- PP 184
Lockman, Robert F., "An Overview of the OSD/ONR Conference on First Term Enlisted Attrition," 22 pp., June 1977, (Presented to the 39th MORS Working Group on Manpower and Personnel Planning, Annapolis, Md., 28-30 June 1977), AD A043 618
- PP 185
Kassing, David, "New Technology and Naval Forces in the South Atlantic," 22 pp. (This paper was the basis for a presentation made at the Institute for Foreign Policy Analyses, Cambridge, Mass., 28 April 1977), AD A043 619
- PP 186
Mizrahi, Maurice M., "Phase Space Integrals, Without Limiting Procedure," 31 pp., May 1977, (Invited paper presented at the 1977 NATO Institute on Path Integrals and Their Application in Quantum Statistical, and Solid State Physics, Antwerp, Belgium, July 17-30, 1977) (Published in Journal of Mathematical Physics 19(1), p. 298, Jan 1978), AD A040 107
- PP 187
Coile, Russell C., "Nomography for Operations Research," 35 pp., April 1977, (Presented at the Joint National Meeting of the Operations Research Society of America and The Institute for Management Services, San Francisco, California, 9 May 1977), AD A043 620
- PP 188
Durch, William J., "Information Processing and Outcome Forecasting for Multilateral Negotiations: Testing One Approach," 53 pp., May 1977 (Prepared for presentation to the 18th Annual Convention of the International Studies Association, Chase Park Plaza Hotel, St. Louis, Missouri, March 16-20, 1977), AD A042 222
- PP 189
Coile, Russell C., "Error Detection in Computerized Information Retrieval Data Bases," July, 1977, 13 pp. Presented at the Sixth Cranfield International Conference on Mechanized Information Storage and Retrieval Systems, Cranfield Institute of Technology, Cranfield, Bedford, England, 26-29 July 1977, AD A043 580
- PP 190
Mahoney, Robert B., Jr., "European Perceptions and East-West Competition," 96 pp., July 1977 (Prepared for presentation at the annual meeting of the International Studies Association, St. Louis, Mo., March, 1977), AD A043 661
- PP 191
Sawyer, Ronald, "The Independent Field Assignment: One Man's View," August 1977, 25 pp.
- PP 192
Holen, Arlene, "Effects of Unemployment Insurance Entitlement on Duration and Job Search Outcomes," August 1977, 6 pp., (Reprinted from Industrial and Labor Relations Review, Vol., 30, No. 4, Jul 1977)
- PP 193
Horowitz, Stanley A., "A Model of Unemployment Insurance and the Work Test," August 1977, 7 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)
- PP 194
Classen, Kathleen P., "The Effects of Unemployment Insurance on the Duration of Unemployment and Subsequent Earnings," August 1977, 7 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)
- PP 195
Brechling, Frank, "Unemployment Insurance Taxes and Labor Turnover: Summary of Theoretical Findings," 12 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)
- PP 196
Ralston, J. M. and Lorimer, O. G., "Degradation of Bulk Electroluminescent Efficiency in Zn, O-Doped GaP LED's," July 1977, 3 pp. (Reprinted from IEEE Transactions on Electron Devices, Vol. ED-24, No. 7, July 1977)
- PP 197
Wells, Anthony R., "The Centre for Naval Analyses," 14 pp., Dec 1977, AD A049 107
- PP 198
Classen, Kathleen P., "The Distributional Effects of Unemployment Insurance," 25 pp., Sept. 1977 (Presented at a Hoover Institution Conference on Income Distribution, Oct 7-8, 1977)
- PP 199
Durch, William J., "Revolution From A F.A.R. - The Cuban Armed Forces in Africa and the Middle East," Sep 1977, 16 pp., AD A046 268
- PP 200
Powers, Bruce F., "The United States Navy," 40 pp. Dec 1977. (To be published as a chapter in The U.S. War Machine by Salamander Books in England during 1978), AD A049 108
- PP 201
Durch, William J., "The Cuban Military in Africa and The Middle East: From Algeria to Angola," Sep 1977, 67 pp., AD A045 675
- PP 202
Feldman, Paul, "Why Regulation Doesn't Work," (Reprinted from *Technological Change and Welfare in the Regulated Industries and Review of Social Economy*, Vol. XXIX, March, 1971, No. 1.) Sep 1977, 8 pp.
- PP 203
Feldman, Paul, "Efficiency, Distribution, and the Role of Government in a Market Economy," (Reprinted from *The Journal of Political Economy*, Vol. 79, No. 3, May/June 1971.) Sep 1977, 19 pp., AD A045 675

- PP 204
Wells, Anthony R., "The 1967 June War: Soviet Naval Diplomacy and The Sixth Fleet - A Re-appraisal," Oct 1977, 36 pp., AD A047 236
- PP 205
Coile, Russell C., "A Bibliometric Examination of the Square Root Theory of Scientific Publication Productivity," (Presented at the annual meeting of the American Society for Information Science, Chicago, Illinois, 29 September 1977.) Oct 1977, 6 pp., AD A047 237
- PP 206
McConnell, James M., "Strategy and Missions of the Soviet Navy in the Year 2000," 48 pp., Nov 1977, (Presented at a Conference on Problems of Sea Power as we Approach the 21st Century, sponsored by the American Enterprise Institute for Public Policy Research, 6 October 1977, and subsequently published in a collection of papers by the Institute), AD A047 244
- PP 207
Goldberg, Lawrence, "Cost-Effectiveness of Potential Federal Policies Affecting Research & Development Expenditures in the Auto, Steel and Food Industries," 36 pp., Oct 1977, (Presented at Southern Economic Association Meetings beginning 2 November 1977)
- PP 208
Roberts, Stephen S., "The Decline of the Overseas Station Fleets: The United States Asiatic Fleet and the Shanghai Crisis, 1932," 18 pp., Nov 1977, (Reprinted from The American Neptune, Vol. XXXVII, No. 3, July 1977), AD A047 245
- PP 209 - Classified.
- PP 210
Kassing, David, "Protecting The Fleet," 40 pp., Dec 1977 (Prepared for the American Enterprise Institute Conference on Problems of Sea Power as We Approach the 21st Century, October 6-7, 1977), AD A049 109
- PP 211
Mizrahi, Maurice M., "On Approximating the Circular Coverage Function," 14 pp., Feb 1978
- PP 212
Mangel, Marc, "On Singular Characteristic Initial Value Problems with Unique Solutions," 20 pp., Jun 1978 (To be submitted for publication in Journal of Mathematical Analysis and Its Applications)
- PP 213
Mangel, Marc, "Fluctuations in Systems with Multiple Steady States. Application to Lanchester Equations," 12 pp., Feb 78, (Presented at the First Annual Workshop on the Information Linkage Between Applied Mathematics and Industry, Naval PG School, Feb 23-25, 1978)
- PP 214
Weinland, Robert G., "A Somewhat Different View of The Optimal Naval Posture," 37 pp., Jun 1978 (Presented at the 1976 Convention of the American Political Science Association (APSA/IUS Panel on "Changing Strategic Requirements and Military Posture"), Chicago, Ill., September 2, 1976)
- PP 215
Coile, Russell C., "Comments on: *Principles of Information Retrieval* by Manfred Kochen, 10 pp., Mar 78, (Published as a Letter to the Editor, Journal of Documentation, Vol. 31, No. 4, pages 298-301, December 1975)
- PP 216
Coile, Russell C., "Lotka's Frequency Distribution of Scientific Productivity," 18 pp., Feb 1978, (Published in the Journal of the American Society for Information Science, Vol. 28, No. 6, pp. 366-370, November 1977)
- PP 217
Coile, Russell C., "Bibliometric Studies of Scientific Productivity," 17 pp., Mar 78, (Presented at the Annual meeting of the American Society for Information Science held in San Francisco, California, October 1976.)
- PP 218 - Classified.
- PP 219
Huntzinger, R. LaVar, "Market Analysis with Rational Expectations: Theory and Estimation," 60 pp., Apr 78 (To be submitted for publication in Journal of Econometrics)
- PP 220
Maurer, Donald E., "Diagonalization by Group Matrices," 26 pp., Apr 78
- PP 221
Weinland, Robert G., "Superpower Naval Diplomacy in the October 1973 Arab-Israeli War," 76 pp., Jun 1978
- PP 222
Mizrahi, Maurice M., "Correspondence Rules and Path Integrals," 30 pp., Jun 1978 (Invited paper presented at the CNRS meeting on "Mathematical Problems in Feynman's Path Integrals," Marseille, France, May 22-26, 1978)
- PP 223
Mangel, Marc, "Stochastic Mechanics of Molecule-Ion Molecule Reactions," 21 pp., Jun 1978 (To be submitted for publication in Journal of Mathematical Physics)
- PP 224
Mangel, Marc, "Aggregation, Bifurcation, and Extinction in Exploited Animal Populations," 48 pp., Mar 1978 (To be submitted for publication in American Naturalist)
*Portions of this work were started at the Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, B.C., Canada
- PP 225
Mangel, Marc, "Oscillations, Fluctuations, and the Hopf Bifurcation," 43 pp., Jun 1978
*Portions of this work were completed at the Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, Canada.
- PP 226
Ralston, J. M. and J. W. Mann*, "Temperature and Current Dependence of Degradation in Red-Emitting GaP LEDs," 34 pp., Jun 1978
- PP 227
Mangel, Marc, "Uniform Treatment of Fluctuations at Critical Points," 50 pp., May 1978 (To be submitted for publication in Journal of Statistical Physics)
- PP 228
Mangel, Marc, "Relaxation at Critical Points: Deterministic and Stochastic Theory," 54 pp., Jun 1978 (To be submitted for publication in Journal of Mathematical Physics)
- PP 229
Mangel, Marc, "Diffusion Theory of Reaction Rates, I: Formulation and Einstein-Smoluchowski Approximation," 50 pp., Jan 1978
- PP 230
Mangel, Marc, "Diffusion Theory of Reaction Rates, II Ornstein-Uhlenbeck Approximation," 34 pp., Feb 1978