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SHABON HAROLD JAVID

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by

Shabon Harold Javid

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THE TIME-OPTIMAL CONTROL OF SINGULARLY
PERTURBED SYSTEMS

BY

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B.S., University of Illinois, 1973
M.S., University of Illinois, 1976

THESIS

Submitted in partial fulfillment of the requirements
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Thesis Adviser: Professor Petar Kokotovic

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THE TIME-OPTIMAL CONTROL OF SINGULARLY
PERTURBED SYSTEMS

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University of Illinois at Urbana-Champaign, 1978

The problem of finding the time-optimal control of the systems

$$\dot{x} = A_{11}(t)x + A_{12}(t)z + B_1(t)u$$

$$\mu\dot{z} = A_{21}(t)x + A_{22}(t)z + B_2(t)u$$

and

$$\dot{x} = f(x,t) + F(x,t)z + B_1(x,t)u$$

$$\mu\dot{z} = g(x,t) + G(x,t)z + B_2(x,t)u$$

is treated where $x \in R^n$, $z \in R^m$, $\mu > 0$ is a small positive parameter and $u \in R^r$ is constrained. The time-optimal control of these systems is shown to possess a two time-scale property when the "fast" state z is stable. This property is that the optimal control is composed of a control in a slow time-scale followed by a control time in a fast time-scale. The "slow" control is primarily concerned with steering the "slow" state x. Based on the two time-scale property a near optimal control is presented which can be calculated on reduced order time-optimal control problems of order n and m. Some new stability bound results and some examples illustrating the near optimal control are presented.

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TABLE OF CONTENTS

1. INTRODUCTION

1.1 Problem Description

Control problems for singularly perturbed systems with unconstrained control have been given considerable attention and major results have been obtained [1]. For these problems the concept of time scale decomposition has been developed providing a separation of slow and fast dynamics and a reduction in problem order.

More recently, results [2-4] have been obtained for the time-optimal control of linear time-invariant singularly perturbed systems. For these problems, time scale decomposition again implies a separation of slow and fast dynamics and a reduction of problem order. Due to the control constraint invariably present in time-optimal problems, the control is characterized by a slow control, primarily dependent on slow dynamics, followed by a fast control primarily dependent on fast dynamics. This is referred to as the two time-scale property. The implication of this property is that the time-optimal control of systems with slow and fast states should first concentrate on steering the slow states near to their desired final state. Then the control should concentrate on rapidly steering the fast states to their desired final state while steering the slow states the remaining small distance to their final state.

The time-optimal problem treated here is that of finding the control which steers the slow and fast states of a singularly perturbed system from a fixed initial point to a fixed final point in minimum time. A major concern of this thesis is the extension of the classes of systems, for which the two time-scale property of the time-optimal control of singularly perturbed systems has been revealed, to include linear time-varying systems and a class of nonlinear systems. This is done by taking expansions in the singular perturbation parameter μ, of the necessary conditions provided by the minimum principle. Once the two time-scale property is revealed, two systems of reduced order are defined. These systems are referred to as the reduced order system, or slow subsystem, and the fast subsystem. It is computationally simpler to solve control problems for these systems due to the order reduction and the removal of the full order system's stiffness associated with the singular perturbation parameter μ. The purpose in revealing the two time-scale property is to make it possible to demonstrate that a near time-optimal control can be constructed, for singularly perturbed linear time-varying systems and a class of nonlinear systems, by concatenating a time-optimal control for the reduced order system and a time-optimal control for the fast subsystem.

One method often used in improving the near-optimal controls which are developed for singularly perturbed systems is asymptotic expansions in the singular perturbation parameter μ. In this thesis another method is presented for time-optimal problems. This method is called the intermediate point algorithm and was originally developed [5,6] for the time-optimal control of linear time-invariant systems. Originally the algorithm was designed for singularly perturbed systems which are block diagonalized. Since it is often difficult to block diagonalize nonlinear systems, the intermediate point algorithm is altered for the treatment of some nonlinear examples.

This thesis also contains some new stability results for linear time-varying singularly perturbed systems. Previously [7,8], it was shown via Liapunov functions that for μ small enough, the stability of two subsystems is sufficient for the stability of a full order system. The main thrust of the results presented here is the discovery of upper bounds for the singular perturbation parameter μ such that for μ smaller than these bounds the systems under consideration are uniformly asymptotically stable.

1.2 Research in Time-Optimal Control

According to Fel'dbaum one of the earliest theoretical papers on time-optimal control was published in 1949 [9]. In [10] Fel'dbaum presents the phase plane approach to time-optimal control. It appears to be in response to the difficulties encountered in finding necessary conditions for the optimality of constrained control problems, such as the time-optimal control problem, that the maximum principle was developed by Pontryagon, et al. [11]. In [12] Boltyanskii has produced a highly readable book in which a proof of the maximum principle is given in terms of the time-optimal problem. From this proof he generalizes to treat other problems. Thus one might say that time-optimal control can be looked at as being at the base of the maximum principle.

At the same time as the above work was going on, Neustadt, Lasalle and Hermes was treating time-optimal problems in the United States. In 1952 Bushaw [39] for some simple systems demonstrated that a bang-bang control was time-optimal. In 1959 Lasalle [13] presented the proof of the bang-bang principle which in essence states that for linear time-varying systems if there exists a time-optimal control, there exists a bang-bang time-optimal control. A bang-bang control is one which utilizes all the control available by only taking values on the edge of the control constraint. This and other results are presented in [14].

Neustadt and Eaton developed an iterative method for determining the initial values of the costates. Neustadt's work treats continuous systems and Eaton considered discretized systems [12]. Plant [15] later treated a number of iterative methods which may be applied to time-optimal control problems.

A number of papers have been published which consider specific problems or systems [16,17,18,19]. It is interesting to note that in [18] the time-optimal control for a triple integrator is calculated. This control law was implemented in the lunar module of an Apollo spacecraft and performed very well. In [20] the sensitivities in cost and position to small changes in the switching times of bang-bang controls are derived. Two papers which consider computational techniques for finding time-optimal controls are of interest [21,22]. The method treated in [22] is essentially a switching time sensitivity method. Beginning with a guess for the time-optimal control a successive approximation for changes in switching times is set up such that the error in reaching the final point is decreased. In [21], a hill climbing method is applied in nested iterations to find time-optimal controls.

1.3 Singular Perturbation Results

The singular perturbation approach to differential equations has been used for some time. A long list of such references is contained in [23]. More recently singularly perturbed optimal control problems have been treated and a survey of major results is contained in [1]. In this thesis two major areas of singular perturbation work have been touched on. The first area is covered in Chapter 2 and deals with the uniform asymptotic stability of singularly perturbed linear time-varying systems [7,8,24,25, 26,27].

The rest of the thesis which at times relies on the results contained in Chapter 2 deals with the development of decomposed time-optimal control for singularly perturbed systems. As mentioned earlier, the initial work for the time-optimal control of singularly perturbed system is contained in [2,3,4]. In [5,6] an iterative method which is based on the theory in [2,3,4] is developed. This method computes time-optimal controls for full order systems from the solutions to reduced order time-optimal control problems. It is called the intermediate point algorithm in this thesis and is described in Chapter 4.

A recent paper [28] treats the problem of finding the time-optimal control of singularly perturbed time-varying systems which are nonlinear in the slow state and linear in the fast state and control. In [29] the feedback time-optimal control of the reduced order system of a linear time-invariant singularly perturbed system is treated. There, it is shown that this control steers the slow state to within a compact subset of the origin (the final desired point) and the fast state to some point which is a finite distance from the origin. Two results for related problems are [30,31] where the time-optimal controls of systems with regular perturbations are treated.

1.4 Chapter Review

Chapter 2 might more properly be called Preliminaries. It contains the derivation of bounds for singularly perturbed systems. The stability of the fast subsystem takes on added importance in the study of time-optimal control. It is shown in Chapter 3 that the stability of the fast subsystem is a necessary condition for the two time-scale property and therefore these bounds guarantee this property. In the next section of

Chapter 2 an example of the application of these bounds is presented followed by the discussion of a stability assumption for a class of nonlinear singularly perturbed systems. This assumption is later used in the development of the two time-scale property in Chapter 5. Then the chapter is ended with a presentation of a block diagonalization transformation (used in Chapter 3) and some lemmas concerning the expansion of two common integrals via integration by parts.

Chapter 3 is first concerned with revealing the two time-scale property of time-optimal control for linear time-varying singularly perturbed systems. Then a near optimal control is defined based on this property.

In Chapter 4 the intermediate point algorithm is defined and discussed. Following this an example of its application is presented.

Chapter 5 begins by showing that a class of nonlinear systems possesses the two time-scale property of time-optimal control. Then a near optimal control for these systems is proposed and its near optimality proved.

Nonlinear examples are presented in Chapter 6. One of these examples is not within the classes of systems treated earlier in the thesis. However, based on the intuition gained during the earlier work, a near-optimal control is found for these systems and the intermediate point algorithm is applied in revised form to improve on the near optimal control.

Finally, Chapter 7 contains the conclusions and possible directions of future research.

1.5 Notation

Vectors and scalars are generally represented by lower case letters and matrices by upper case letters. The derivative with respect to time of a vector x or a matrix A is denoted by $\dot{x}$ or $\dot{A}$. The transpose of a matrix A is denoted by A'. The norm of a matrix A is written $|A|$. The norm used here is

$$|A| = (\Sigma\, \alpha_{ij}{}^2)^{1/2}$$

where the α_{ij} are the elements of the matrix A. For the norm of a scalar this reduces to the absolute value. A set U composed of objects u characterized by some property P is defined by

$$U = \{u : u \text{ has the property } P\} \; .$$

Throughout this thesis a vector f(x,t) is written as f in cases where no confusion will result from dropping the dependence on x and t.

2. STABILITY PROPERTIES

2.1 Introduction

In this chapter some bounds on μ are found which guarantee the uniform asymptotic stability of

$$\dot{x} = A_{11}(t)x + A_{12}(t)z + B_1(t)u \tag{2.1a}$$

$$\mu\dot{z} = A_{21}(t)x + A_{22}(t)z + B_2(t)u \tag{2.1b}$$

where $x \in R^n$, $z \in R^m$, $u \in R^r$ and μ is greater than zero. The following assumptions are made:

(i) The matrices $A_{ij}(t)$, $i,j = 1,2$, are bounded and have bounded first derivatives for all t.

(ii) The eigenvalues $\lambda_i(t)$ of $A_{22}(t)$ satisfy

$$\mathrm{Re}(\lambda_i(t)) < -\gamma < 0 \tag{2.2}$$

for all t where γ is a constant.

(iii) The reduced system

$$\begin{aligned}\dot{w} &= (A_{11}(t) - A_{12}(t)A_{22}^{-1}(t)A_{21}(t))W \\ &\triangleq A_0(t)w\end{aligned} \tag{2.3}$$

has $w = 0$ as the uniformly asymptotically stable equilibrium.

In Lemmas 2.2 and 2.3 under assumptions (i) and (ii), a bound μ_0 is found such that for $\mu \in (0,\mu_0)$ the equilibrium $y = 0$ of the fast subsystem

$$\mu\dot{y} = A_{22}(t)y \tag{2.4}$$

is uniformly asymptotically stable. Then in Lemma 2.4, under assumptions (i), (ii) and (iii), a bound μ_1 is found such that for $\mu \in (0,\mu_1)$ the equilibrium, $u = 0$, $x = 0$ and $z = 0$ of the full order system (2.1) is uniformly asymptotically stable. Under (i), (ii) and (iii) it has been known for some time [7,8] that (2.1) is uniformly asymptotically stable for μ small enough. In [24] it is shown that under (i) and (ii), (2.4) is uniformly asymptotically stable. The new results here are the bounds μ_0 and μ_1.

After these stability lemmas have been stated and proved, a stability assumption concerning the nonlinear system

$$\dot{x} = f(x,t) + F(x,t)z + B_1(x,t)u \tag{2.5a}$$

$$\mu\dot{z} = g(x,t) + G(x,t)z + B_2(x,t)\,u \tag{2.5b}$$

is presented and discussed in preparation for Chapter 5 where the time-optimal control of (2.5) is treated.

Finally a diagonalization transformation and two lemmas are presented. In these lemmas expansions in μ are presented for two integrals which appear often in the development in this thesis.

The stability lemmas for systems (2.1) and (2.4) are presented in Section 2.2 and an example illustrating the use of these lemmas is presented in Section 2.3. Section 2.4 contains the discussion of a stability assumption for system (2.5). Section 2.5 contains the diagonalization transformation and in Section 2.6 expansions in μ are derived for two integrals.

2.2 The Stability of Linear Time-Varying Systems

In the presentation of this section we will need the well-known lemma of Gronwall.

Lemma 2.1 (Gronwall's Lemma [32]). Let $\lambda(t)$ be a real continuous function and $\gamma(t)$ a non-negative continuous function on the interval $[t_0,t_1]$. If a continuous function $y(t)$ has the property that

$$y(t) \leq \lambda(t) + \int_{t_0}^{t} \gamma(s)y(s)ds$$

for $t_0 \leq t \leq t_1$, then on the same interval

$$y(t) \leq \lambda(t) + \int_{t_0}^{t} \lambda(s)\gamma(s)\exp\left(\int_{s}^{t} \gamma(t)dt\right)ds \ .$$

The proof of this lemma is contained in [32].

In the next lemma a bound $\hat{\mu}$ on μ is found to guarantee that the equilibrium $y = 0$ of system (2.4) is uniformly asymptotically stable under assumptions (i) and (ii). Assumption (i) implies

$$|A_{22}(t) - A_{22}(t_0)| \leq \beta(t-t_0) \tag{2.6}$$

where β is a constant equal to the maximum of $|\dot{A}_{22}(t)|$ for all t by the mean value theorem. Also for $t \geq t_0$ there exists a K such that

$$e^{A_{22}(t_0)\left(\frac{t-t_0}{\mu}\right)} \leq Ke^{-\gamma\left(\frac{t-t_0}{\mu}\right)} \tag{2.7}$$

when (ii) is satisfied [38].

Let $\Phi_{22}(t,t_0)$ be the state transition matrix of (2.4) and define $\varphi(t,t_0)$

$$\varphi(t,t_0) \triangleq \Phi_{22}(t,t_0) - e^{A_{22}(t_0)\left(\frac{t-t_0}{\mu}\right)} . \tag{2.8}$$

Lemma 2.2. Let (i) and (ii) be satisfied and for any $\alpha \in (0,\gamma)$, let $\hat{\mu} = \alpha^2/\beta K$. Then for $\mu \in (0,\hat{\mu})$, $\varphi(t,t_0)$ satisfies

$$\varphi(t_0,t_0) = 0 \tag{2.9}$$

and

$$|\varphi(t,t_0)| \leq 2 \frac{\mu K^2 \beta}{e^2(\alpha^2-\mu K\beta)} e^{-\sigma\left(\frac{t-t_0}{\mu}\right)} \tag{2.10}$$

where $\sigma = \gamma - \alpha > 0$.

Proof of Lemma 2.2. The definition of $\varphi(t,t_0)$ implies (2.9) and (2.11),

$$\dot{\varphi}(t,t_0) = \frac{A_{22}(t)}{\mu} \varphi(t,t_0) + \frac{A_{22}(t) - A_{22}(t_0)}{\mu} e^{A_{22}(t_0)\left(\frac{t-t_0}{\mu}\right)}. \tag{2.11}$$

Applying the variation of constants formula to (2.11) we obtain

$$\begin{aligned} \varphi(t,t_0) = {} & \frac{1}{\mu}\int_{t_0}^{t} e^{A_{22}(t_0)\left(\frac{t-\tau}{\mu}\right)}(A_{22}(\tau) - A_{22}(t_0))e^{A_{22}(t_0)\left(\frac{\tau-t_0}{\mu}\right)} d\tau \\ & + \frac{1}{\mu}\int_{t_0}^{t} \varphi(t,\tau)(A_{22}(\tau) - A_{22}(t_0))e^{A_{22}(t_0)\left(\frac{\tau-t_0}{\mu}\right)} d\tau . \end{aligned} \tag{2.12}$$

Let $\gamma = \alpha + \sigma$, multiply (2.12) through by $e^{\sigma((t-t_0)/\mu)}$ and let

$$\eta(t,t_0) = e^{\sigma((t-t_0)/\mu)}\varphi(t,t_0)$$

to yield

$$\eta(t,t_0) = \frac{1}{\mu}\int_{t_0}^{t} e^{\sigma\left(\frac{t-t_0}{\mu}\right)} e^{A_{22}(t_0)\left(\frac{t-\tau}{\mu}\right)}$$

$$\times (A_{22}(\tau) - A_{22}(t_0))e^{A_{22}(t_0)\left(\frac{\tau-t_0}{\mu}\right)}d\tau$$

$$+ \frac{1}{\mu}\int_{t_0}^{t} \eta(t,\tau)e^{\sigma\left(\frac{\tau-t_0}{\mu}\right)}(A_{22}(\tau) - A_{22}(t_0))e^{A_{22}(t_0)\left(\frac{\tau-t_0}{\mu}\right)}d\tau .$$

We next construct the successive approximation

$$\eta^{(k+1)}(t,t_0) = \frac{1}{\mu}\int_{t_0}^{t} e^{\sigma\left(\frac{t-t_0}{\mu}\right)} e^{A_{22}(t_0)\left(\frac{t-\tau}{\mu}\right)}$$

$$\times (A_{22}(\tau) - A_{22}(t_0))e^{A_{22}(t_0)\left(\frac{\tau-t_0}{\mu}\right)}d\tau \tag{2.13}$$

$$+ \frac{1}{\mu}\int_{t_0}^{t} \eta^{(k)}(t,\tau)e^{\sigma\left(\frac{\tau-t_0}{\mu}\right)}$$

$$\times (A_{22}(\tau) - A_{22}(t_0))e^{A_{22}(t_0)\left(\frac{\tau-t_0}{\mu}\right)}d\tau$$

with initial guess $\eta^{(0)}(t,t_0) = 0$. Substituting (2.6) and (2.7) into (2.13) and integrating for $\eta^{(1)}(t,t_0)$ we obtain

$$|\eta^{(1)}(t,t_0)| \le \frac{\mu K^2\beta}{2} e^{-\alpha\left(\frac{t-t_0}{\mu}\right)}\left(\frac{t-t_0}{\mu}\right)^2 \le 2\,\frac{\mu K^2\beta}{\alpha^2 e^2}$$

for all t, t_0, $t \geq t_0$. Taking the difference between two successive terms for η we obtain

$$\eta^{(k+1)}(t,t_0) - \eta^{(k)}(t,t_0)$$

$$= \frac{1}{\mu}\int_{t_0}^{t} (\eta^{(k)}(t,\tau) - \eta^{(k-1)}(t,\tau))$$

$$\times e^{\sigma\left(\frac{\tau-t_0}{\mu}\right)}(A_{22}(\tau) - A_{22}(t_0))e^{A_{22}(t_0)\left(\frac{\tau-t_0}{\mu}\right)}d\tau .$$

Substituting in (2.6) and (2.7) yields

$$|\eta^{(k+1)}(t,t_0) - \eta^{(k)}(t,t_0)|$$
$$\leq \int_{t_0}^{t} |\eta^{(k)}(t,\tau) - \eta^{(k-1)}(t,\tau)|\beta K\left(\frac{\tau-t_0}{\mu}\right) e^{-\alpha\left(\frac{\tau-t_0}{\mu}\right)}d\tau . \tag{2.14}$$

Suppose for $k \leq p$

$$|\eta^{(k)}(t,\tau) - \eta^{(k-1)}(t,\tau)| \leq c^{(k)}$$

where $c^{(k)}$ are constants. Then by (2.14)

$$|\eta^{(k+1)}(t,t_0) - \eta^{(k)}(t,t_0)| \leq \frac{\mu\beta K}{\alpha^2} c^{(k)}$$

for all t, t_0 and $t \geq t_0$. Since for $k = 1$

$$|\eta^{(k)}(t,\tau) - \eta^{(k-1)}(t,\tau)| \leq 2\,\frac{\mu K^2\beta}{\alpha^2 e^2} ,$$

we have by induction

$$|\eta^{(k+1)}(t,t_0) - \eta^{(k)}(t,t_0)| \leq 2 \left(\frac{\mu\beta K}{\alpha^2}\right)^k \left(\frac{\mu K^2\beta}{\alpha^2 e^2}\right) .$$

Define $\rho = \frac{\mu\beta K}{\alpha^2}$. Since

$$|\eta^{(k)}(t,t_0) - \eta^{(0)}(t,t_0)| \leq \sum_{j=1}^{k} |\eta^{(j)}(t,t_0) - \eta^{(j-1)}(t,t_0)|$$

$$\leq 2(\rho^{k-1} + \dots + \rho + 1) \frac{\mu K^2\beta}{\alpha^2 e^2}$$

$$= 2 \left(\frac{1-\rho^k}{1-\rho}\right) \frac{\mu K^2\beta}{\alpha^2 e^2}$$

then for $\rho < 1$ or $\mu < \alpha^2/\beta K$

$$\lim_{k\to\infty} |\eta^{(k)}(t,t_0)| = \lim_{k\to\infty} |\eta^{(k)}(t,t_0) - \eta^{(0)}(t,t_0)|$$

$$\leq \left(\frac{2}{1-\rho}\right) \frac{\mu K^2\beta}{\alpha^2 e^2} = 2 \frac{\mu K^2\beta}{e^2(\alpha^2-\mu\beta K)} .$$

Thus for $\mu < \alpha^2/\beta K$ the successive approximation (2.13) converges to a solution which satisfies

$$|\eta(t,t_0)| \leq 2 \frac{\mu K^2\beta}{e^2(\alpha^2-\mu\beta K)} .$$

Now $e^{\sigma\left(\frac{t-t_0}{\mu}\right)}\varphi(t,t_0) = \eta(t,t_0)$ and therefore

$$|\varphi(t,t_0)| \leq \frac{\mu 2K^2\beta}{e^2(\alpha^2-\mu\beta K)} e^{-\sigma\left(\frac{t-t_0}{\mu}\right)} . \tag{2.15}$$

This completes the proof of Lemma 2.2.

This lemma implies that for $\mu \in (0,\hat{\mu})$ the fast subsystem (2.4) is uniformly asymptotically stable for any $\alpha \in (0,\gamma)$ where $-\gamma$ is the constant upper bound on the real parts of the eigenvalues of $A_{22}(t)$. A consequence of the proof of this lemma is equation (2.10) which provides a bound on the error incurred in approximating the state transition matrix $\Phi_{22}(t,t_0)$ of (2.4) by the transition matrix

$$e^{A_{22}(t_0)\frac{t-t_0}{\mu}}$$

of the time-invariant system

$$\mu\dot{\tilde{y}} = A_{22}(t_0)\tilde{y} \; . \tag{2.16}$$

As can be seen this error is $O(\mu)$ and exponentially decaying with an $O(\mu)$ time constant.

Lemma 2.3. Let assumptions (i) and (ii) be satisfied and set $\mu_0 = \gamma^2/\beta K$. Then for $\mu \in (0,\mu_0)$ system (2.4) is uniformly asymptotically stable.

Proof of Lemma 2.3. *Define* $\mu_0 = \gamma^2/\beta K$ *and* $\alpha = \frac{1}{2}\sqrt{\mu\beta K} + \gamma/2$ where $\mu \in (0,\mu_0)$. Then $\sigma = \gamma - \alpha > 0$ and for $\mu \in (0,\mu_0)$ equation (2.15) implies that (2.4) is uniformly asymptotically stable since the definition of α implies that $\alpha^2 - \mu K\beta$ is never equal to zero. The proof is finished.

In Lemma 2.4 it is assumed that (iii) is satisfied and the fast subsystem (2.4) is uniformly asymptotically stable. Under these assumptions the bound μ_1 is found such that for $\mu \in (0,\mu_1)$ the equilibrium $u = 0$, $x = 0$ and $z = 0$ of (2.1) is uniformly asymptotically stable. Then in

Lemma 2.5 the bounds of Lemmas 2.3 and 2.4 are combined to find the bound μ^* such that under (i), (ii) and (iii) and for $\mu \in (0,\mu^*)$ the above equilibrium is uniformly asymptotically stable.

In order to simplify the statement and proof of the lemma, the transformed system

$$\begin{aligned} \dot{x} &= A_0(t)x + A_{12}(t)\eta \\ \mu\dot{\eta} &= \mu(\dot{L}(t) + L(t)A_0(t))x + A_{22}(t)\eta + \mu L(t)A_{12}(t)\eta \end{aligned} \tag{2.17}$$

is treated which is the result of applying the transformation

$$\eta = z + A_{22}(t)^{-1}A_{21}(t)x \triangleq z + L(t)x \tag{2.18}$$

to (2.1) with u set equal to zero. Here A_0 is as defined in (2.3). Clearly if (2.17) is uniformly asymptotically then the equilibrium $u = 0$, $x = 0$ and $z = 0$ of (2.1) is uniformly asymptotically stable.

<u>Lemma 2.4</u>. Let (2.3) and (2.4) be uniformly asymptotically stable systems so that their state transition matrices satisfy (2.19) and (2.20) respectively

$$|\Phi_0(t,t_0)| \le K_1 e^{-\sigma_1(t-t_0)} \tag{2.19}$$

$$\forall\ t_0,\ t \ge t_0$$

$$|\Phi_{22}(t,t_0)| \le K_2 e^{-\sigma_2\left(\frac{t-t_0}{\mu}\right)} \ . \tag{2.20}$$

If constants M_1, M_2 and M_3 exist such that for all t

$$|A_{12}(t)| \le M_1, \quad |L(t)A_{12}(t)| \le M_2,$$

$$|\dot{L}(t) + L(t)A_0(t)| \le M_3 \ , \tag{2.21}$$

then for all $\mu \in (0,\mu_1)$, where

$$\mu_1 = \frac{\sigma_1\sigma_2}{\sigma_1 K_2 M_2 + K_1 M_1 K_2 M_3}$$

the equilibrium $u = 0$, $x = 0$ and $z = 0$ of (2.1) is uniformly asymptotically stable.

Proof of Lemma 2.4. Applying the variation-of-constants formula to (2.17) yields

$$x(t) = \Phi_0(t,t_0)x_0 + \int_{t_0}^{t} \Phi_0(t,\tau)A_{12}(\tau)\eta(\tau)d\tau$$

$$\eta(t) = \Phi_{22}(t,t_0)\eta_0 + \int_{t_0}^{t} \Phi_{22}(t,\tau)L(\tau)A_{12}(\tau)\eta(\tau)d\tau$$

$$+ \int_{t_0}^{t} \Phi_{22}(t,\tau)(\dot{L}(\tau) + L(\tau)A_0(\tau))x(\tau)d\tau$$

where

$$\eta_0 = z_0 + A_{22}^{-1}(t_0)A_{21}(t_0)x_0 \ .$$

The bounds of equations (2.19), (2.20) and (2.21) imply

$$|x(t)| \le K_1 e^{-\sigma_1(t-t_0)}|x_0| + \int_{t_0}^{t} K_1 e^{-\sigma_1(t-\tau)} M_1 |\eta(\tau)| d\tau \qquad (2.22)$$

$$|\eta(t)| \le K_2 e^{-\sigma_2\left(\frac{t-t_0}{\mu}\right)} |\eta_0| + \int_{t_0}^{t} K_2 e^{-\sigma_2\left(\frac{t-\tau}{\mu}\right)} M_2 |\eta(\tau)| d\tau$$
$$+ \int_{t_0}^{t} K_2 e^{-\sigma_2\left(\frac{t-\tau}{\mu}\right)} M_3 |x(\tau)| d\tau \ . \tag{2.23}$$

In this proof we apply Gronwall's Lemma to (2.23) and then to (2.22) to derive an upper bound μ_1 such that for $\mu \in (0,\mu_1)$, the absolute values $|x(t)|$ and $|\eta(t)|$ are bounded by a decreasing exponential. Letting $w(t) = e^{\sigma_2 t/\mu} |\eta(t)|$ in equation (2.23) yields

$$w(t) \le K_2 e^{\sigma_2 t_0/\mu} |\eta_0| + \int_{t_0}^{t} K_2 e^{\sigma_2 \tau/\mu} M_3 |x(\tau)| d\tau$$
$$+ \int_{t_0}^{t} K_2 M_2 w(\tau) d\tau \ .$$

Applying Gronwall's Lemma and integrating, we obtain

$$w(t) \le K_2 e^{\sigma_2 t_0/\mu} |\eta_0| e^{K_2 M_2 (t-t_0)}$$
$$+ \int_{t_0}^{t} e^{K_2 M_2 (t-\tau)} K_2 e^{\sigma_2 \tau/\mu} M_3 |x(\tau)| d\tau$$

which yields

$$|\eta(t)| \le K_2 e^{-\sigma_3 (t-t_0)} |\eta_0| + \int_{t_0}^{t} K_2 M_3 e^{-\sigma_3 (t-\tau)} |x(\tau)| d\tau \tag{2.24}$$

where $\sigma_3 = \sigma_2/\mu - K_2 M_2$. In the following we will need $\sigma_3 > 0$.

Substituting (2.24) into (2.22) yields

$$|x(t)| \leq K_1 e^{-\sigma_1(t-t_0)} |x_0| + \int_{t_0}^{t} K_1 e^{-\sigma_1(t-\tau)} M_1 K_2 |\eta_0| e^{-\sigma_3(\tau-t_0)} d\tau$$

$$+ \int_{t_0}^{t} K_1 e^{-\sigma_1(t-\tau)} M_1 (\int_{t_0}^{t} K_2 M_3 e^{-\sigma_3(t-s)} |x(s)| ds) d\tau$$

which implies

$$|x(t)| \leq \left[K_1 |x_0| - \frac{K_1 M_1 K_2 |\eta_0|}{\sigma_3 - \sigma_1}\right] e^{-\sigma_1(t-t_0)} + \frac{K_1 M_1 K_2 |\eta_0|}{\sigma_3 - \sigma_1} e^{-\sigma_3(t-t_0)}$$

$$+ \frac{K_1 M_1 K_2 M_3}{\sigma_1} \int_{t_0}^{t} e^{-\sigma_3(t-\tau)} |x(\tau)| d\tau \ .$$

Let $y(t) = e^{\sigma_3 t} |x(t)|$, apply Gronwall's Lemma and integrate to obtain

$$|x(t)| \leq \frac{K_1 M_1 K_2 |\eta_0|}{\sigma_3 - \sigma_1} \left[1 - \frac{\sigma_1}{K_1 M_1 K_2 M_3}\right] e^{-\sigma_3(t-t_0)}$$

$$+ \left[K_1 |x_0| - \frac{K_1 M_1 K_2 |\eta_0|}{\sigma_3 - \sigma_1}\right] (1 + \frac{1}{\sigma}) e^{-\sigma_1(t-t_0)}$$

$$+ \left[\frac{\sigma_1 |\eta_0|}{M_3(\sigma_3 - \sigma_1)} - \frac{1}{\sigma} \left(K_1 |\eta_0| - \frac{K_1 M_1 K_2 |\eta_0|}{\sigma_3 - \sigma_1}\right)\right] e^{\left(\frac{K_1 M_1 K_2 M_2}{\sigma_1} - \sigma_3\right)(t-t_0)}$$

where

$$\sigma = \sigma_3 - \sigma_1 - \frac{K_1 M_1 K_2 M_3}{\sigma_1} \ .$$

Thus for (2.17) to be uniformly asymptotically stable we need inequalities (2.25) and (2.26) to be satisfied,

$$\sigma_3 = \frac{\sigma_2}{\mu} - K_2M_2 > 0 \tag{2.25}$$

$$\sigma_3 - \frac{K_1M_1K_2M_3}{\sigma_1} > 0 \ . \tag{2.26}$$

Let

$$\mu_1 = \frac{\sigma_1\sigma_2}{\sigma_1K_2M_2 + K_1M_1K_2M_3} \ .$$

If $\mu \in (0,\mu_1)$, inequalities (2.25) and (2.26) are satisfied and therefore (2.17) *is uniformly* asymptotically stable which implies that the equilibrium $u = 0$, $x = 0$ and $z = 0$ as the full order system (2.1) is uniformly asymptotically stable. Lemma 2.4 is proved.

The corollary follows directly from Lemmas 2.3 and 2.4.

Corollary 2.5. Let $\mu^* = \min(\mu_0,\mu_1)$. Assumptions (i), (ii) and (iii) guarantee that for $\mu \in (0,\mu^*)$, the *equilibrium* $u = 0$, $x = 0$ and $z = 0$ of (2.1) is uniformly asymptotically stable.

2.3 Example

The example illustrates the calculation of the stability bounds for μ derived in the last section for the system

$$\begin{bmatrix} \dot{x} \\ \mu\dot{z}_1 \\ \mu\dot{z}_2 \end{bmatrix} = \begin{bmatrix} -4 + \cos t & 1 & 0 \\ 1 & -1 + 1.1 \cos^2 t & 1 - 1.1 \sin t \cos t \\ 0 & -1 = 1.1 \sin t \cos t & -1 + 1.1 \sin^2 t \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} . \tag{2.27}$$

The reduced system is

$$\dot{\bar{x}} = (-3.474 + \cos t - 1.222 \sin^2 t)\bar{x} \tag{2.28}$$

and the fast subsystem is

$$\mu\dot{\eta} = \begin{bmatrix} -1 + 1.1 \cos^2 t & 1 - 1.1 \sin t \cos t \\ -1 - 1.1 \sin t \cos t & -1 + 1.1 \sin^2 t \end{bmatrix} \eta . \tag{2.29}$$

When $\mu = 1$ *an* unstable fundamental solution of (2.29) is

$$\Phi_2(t,0) = \begin{bmatrix} e^{.1t} \cos t & e^{-t} \sin t \\ -e^{.1t} \sin t & e^{-t} \cos t \end{bmatrix}$$

even though the eigenvalues of $A_{22}(t)$ have real parts = -0.45 for all t [33, p. 147]. Since system (2.27) satisfies (i) and (ii) and system (2.28) is uniformly asymptotically stable, we know that for μ sufficiently small, both systems (2.29) and (2.27) are uniformly asymptotically stable.

Fixing the coefficients of the fast subsystem at any $t = t_0$, we obtain the linear time-invariant system

$$\mu\dot{\eta} = \begin{bmatrix} -1 + 1.1 \cos^2 t_0 & 1 + 1.1 \sin t_0 \cos t_0 \\ -1 - 1.1 \sin t_0 \cos t_0 & -1 + 1.1 \sin^2 t_0 \end{bmatrix} \eta . \tag{2.30}$$

The state transition matrix for (2.30) is

$$e^{A_{22}(t_0)\tau}$$

$$= e^{-.45\tau}\begin{bmatrix} \alpha_{11}(t_0)\cos(.835\tau-\delta_{11}(t_0)) & \alpha_{12}(t_0)\sin(.835\tau) \\ \alpha_{21}(t_0)\sin(.835\tau) & \alpha_{22}(t_0)\cos(.835\tau-\delta_{22}(t_0)) \end{bmatrix}$$

where $\tau = \dfrac{t-t_0}{\mu}$,

$$\alpha_{11}(t_0) = (1.377 - 1.617 \sin^2 t_0 + 1.734 \sin^4 t_0)^{1/2} ,$$

$$\alpha_{12}(t_0) = (1.198 - 1.317 \sin t_0 \cos t_0) ,$$

$$\alpha_{21}(t_0) = (-1.198 - 1.317 \sin t_0 \cos t_0) ,$$

$$\alpha_{22}(t_0) = (1.377 - 1.617 \cos^2 t_0 + 1.734 \cos^4 t_0)^{1/2} ,$$

$$\delta_{11}(t_0) = \tan^{-1}(.614 - 1.317 \sin^2 t_0) ,$$

$$\delta_{22}(t_0) = \tan^{-1}(.614 - 1.317 \cos^2 t_0) .$$

Using as a norm $(\Sigma\alpha_{ij}^{\ 2}(t_0))^{1/2}$ we find that $K = 7.358$ and $\gamma = 0.45$. Correspondingly we find the max $|\dot{A}_{22}(t)| = 1.555 = \beta$. The values of β, K and γ and Lemma 2.3 imply that $\mu_0 = .0177$ and that for $0 < \mu < \mu_0$, system (2.29) is uniformly asymptotically stable.

Since μ_0 is found by taking various matrix norms it is conservative. In a computer simulation the fast subsystem (2.29) was numerically integrated for the initial condition $z_1(0) = z_2(0) = 1$ for various values of μ. It was found that for $\mu < .65$, the solution was exponentially decaying. In Figures 2.1, 2.2 and 2.3 the solution for z_1 is shown for μ equal 0.1, 0.4 and 0.7 respectively. From Figures 2.1 and 2.2 it can be seen as expected, that for smaller μ the exponential decay is more rapid.

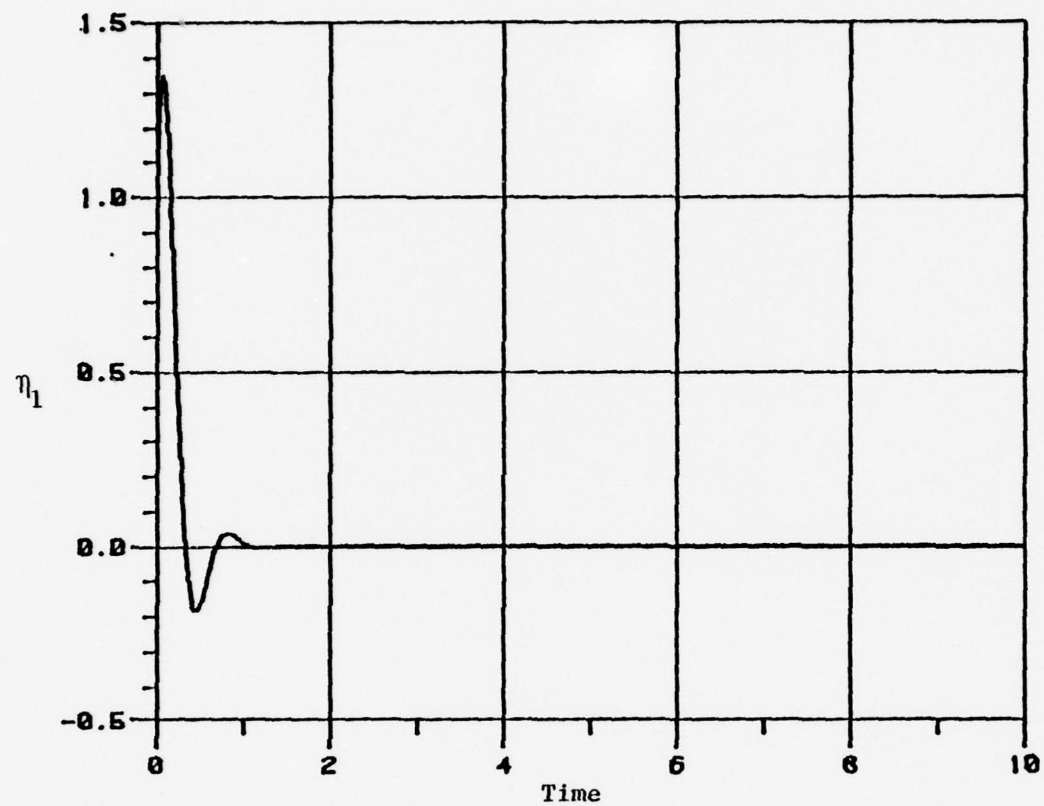

Figure 2.1. Solution for η_1 of equation (2.29) with μ = .1 and initial state (1,1).

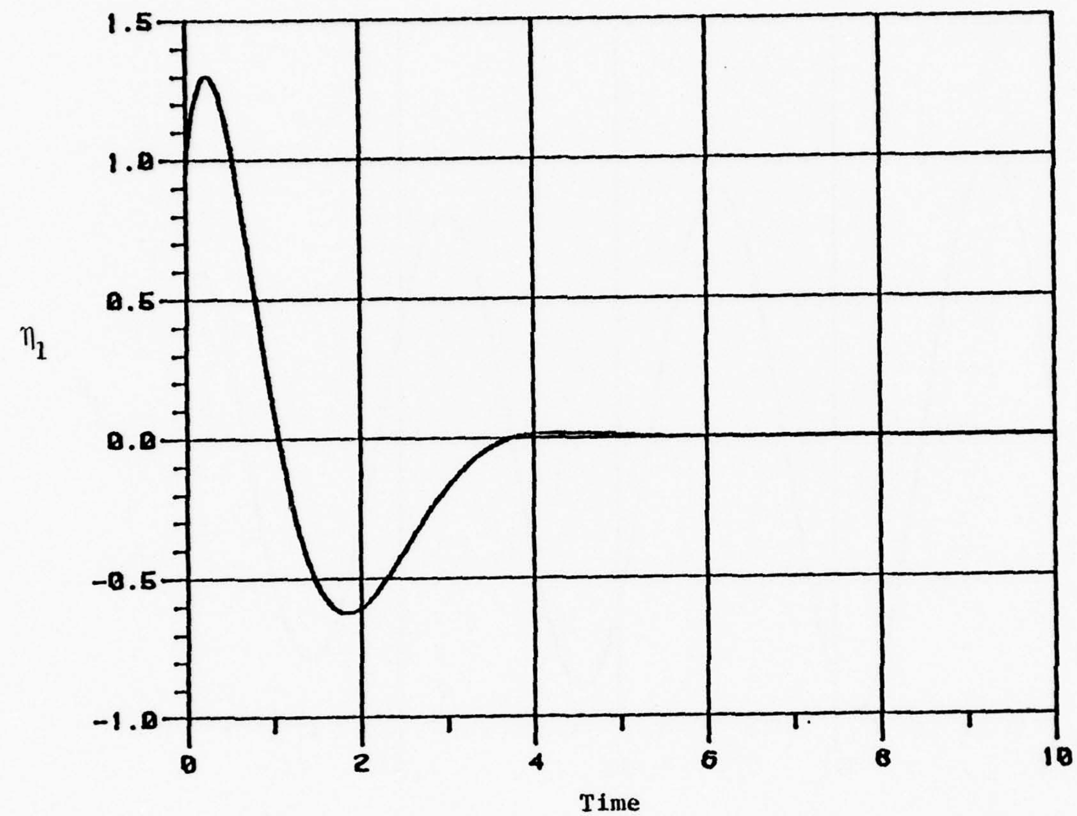

Figure 2.2. Solution for η_1 of equation (2.29) with μ = .4 and initial state (1,1).

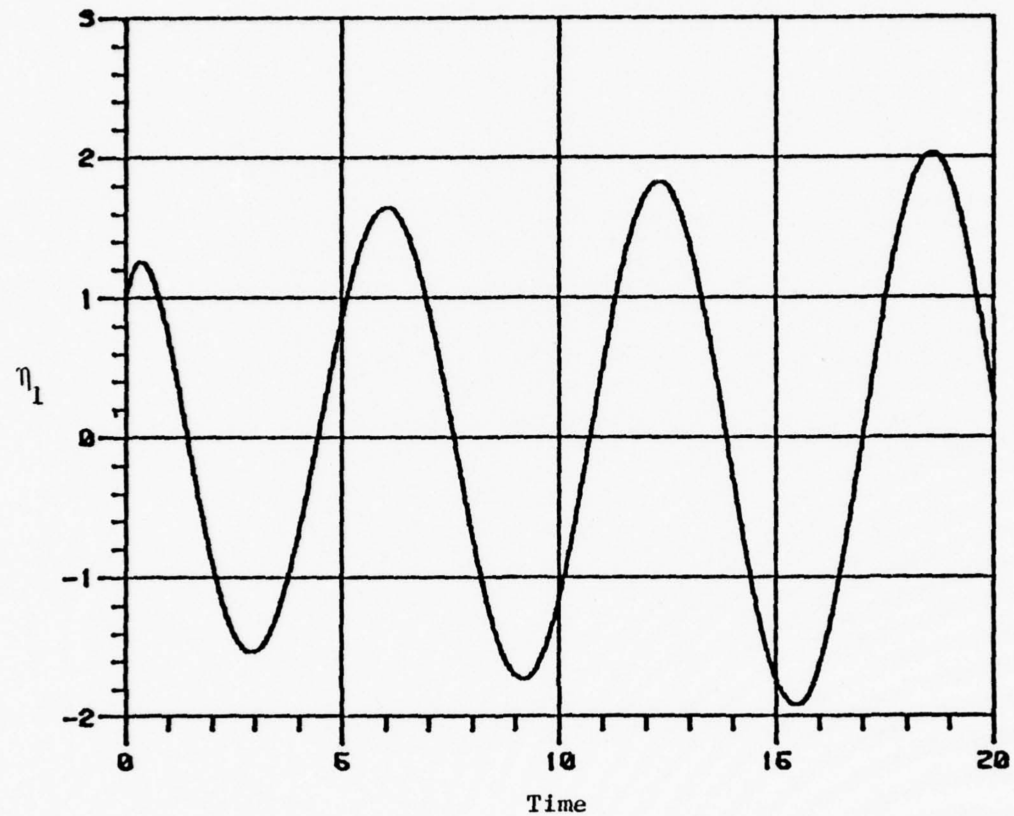

Figure 2.3. Solution for η_1 of equation (2.29) with μ = .7 and initial state (1,1).

We next find a bound for the stability of the full order system (2.27). From Lemma 2.3 we obtain

$$|\Phi_{22}(t,t_0)| \le K \left[1 + 2 \frac{\mu K\beta}{e^2(\alpha^2 - \mu K\beta)} \right] e^{-\sigma t/\mu} . \tag{2.31}$$

If we let $\alpha = \sigma = \gamma/2$ we obtain a value for $\hat{\mu}$ of .00442. For $\mu \in (0,\hat{\mu})$ we may use the bounds of (2.31) for $\Phi_{22}(t,t_0)$. Thus

$$K_2 = K \left[1 + \frac{2\mu K\beta}{e^2(\sigma^2 - \mu K\beta)} \right]$$

and $\sigma_2 = \sigma = .225$. From equation (2.28)

$$|\bar{x}(t)| \le |x_0| e^{-1.89(t-t_0)} .$$

which yields $K_1 = 1$ and $\sigma_1 = 1.89$.

Values for M_1, M_2 and M_3 are 1, 1.956 and 7.09 respectively. Substituting these values into

$$\mu \le \frac{\sigma_1\sigma_2}{\sigma_1 K_2 M_2 + K_1 M_1 K_2 M_3}$$

yields $\mu_1 = .00317$. Since $\mu_1 < \hat{\mu}$ we know from Corollary 2.5 that for $\mu \in (0,\mu_1)$ system (2.27) is uniformly asymptotically stable.

This example illustrates the use of Lemmas 2.2, 2.3, 2.4 and Corollary 2.5 in obtaining stability bounds of μ in system (2.27). The bounds K_2 and σ_2 are direct results of Lemma 2.2, thus making it unnecessary to determine the state transition matrix for the fast subsystem directly.

2.4 A Stability Assumption

In Chapter 5 the time-optimal control of nonlinear system (2.5) is treated where $x \in R^n$, $z \in R^m$, $u \in R^r$ and μ is greater than zero. The purpose of this section is to discuss the acceptability of stability assumption which is made in Chapter 5 concerning system (2.5).

We assume that the Jacobians $g_x(x,t)$ and $(B_2(x,t)u)_x$ and the partial derivatives $\frac{\partial g}{\partial t}$ and $\frac{\partial B_2}{\partial t}$ exist and are continuous with respect to x and t. The controls under consideration satisfy the constraint

$$u \in U = \{u: |u_i| \leq 1, \ i = 1,\ldots,r\}$$

and their components u_i are piecewise constant with a finite number p of discontinuities at the instants t_j where

$$t_0 < t_1 < t_2 < \ldots < t_p .$$

That is, at the instants t_j at least one of the components of u is switching from one constant value to another constant value. Suppose we choose an arbitrary control $\hat{u}(t)$ satisfying the above, and apply it to (2.5) to create the trajectory $\hat{x}(t), \hat{z}(t)$. It is assumed that for $\hat{x}(t)$ the inverse of $G(\hat{x},t)^{-1}$ exists for $t \in [t_0, t_p]$.

The stability assumption to be discussed in this section may be stated:

(i) The equilibrium $w = 0$ of the system

$$\mu\dot{w} = G(\hat{x}(t),t)w \tag{2.32}$$

is uniformly asymptotically stable, for μ small enough.

The approach used in the application of singular perturbation methods to approximate the solution to (2.5) with control input $\hat{u}(t)$ is to set $\mu = 0$ in (2.5b) to yield

$$\bar{z} = -G(\bar{x},t)^{-1}g(\bar{x},t) - G(\bar{x},t)^{-1}B_2(\bar{x},t)u \tag{2.33}$$

where the bar $(\bar{\cdot})$ denotes $\mu = 0$. Substituting (2.35) in (2.5a) yields the reduced order system

$$\begin{aligned}\dot{\bar{x}} &= f_0(\bar{x},t) + B_0(\bar{x},t)\bar{u} \\ &\triangleq f(\bar{x},t) - F(\bar{x},t)G(\bar{x},t)^{-1}g(\bar{x},t) \\ &\quad + (B_1(\bar{x},t) - F(\bar{x},t)G(\bar{x},t)^{-1}B_2(\bar{x},t))\bar{u} \, .\end{aligned} \tag{2.34}$$

One assumption [26] which will guarantee that the solution $\bar{x}(t)$ of (2.34) with input $\hat{u}(t)$ provides an approximation for $\hat{x}(t)$ is that the equilibrium $\bar{z}$ of system (2.5b) is uniformly asymptotically stable. We will show that the assumption (i) above is equivalent.

Consider $\hat{x}(t)$ and $\hat{u}(t)$ as inputs to (2.5b). Then (2.5b) may be treated as the linear time-varying system

$$\mu\dot{z} = \hat{G}(t)z + \hat{g}(t) + \hat{B}_2(t)\hat{u}(t) \tag{2.35}$$

which has the solution

$$\begin{aligned}z(t) &= \Phi_\mu(t,t_0)z_0 + \frac{1}{\mu}\int_{t_0}^{t} \Phi_\mu(t,\tau)\hat{g}(\tau)d\tau \\ &\quad + \frac{1}{\mu}\int_{t_0}^{t} \Phi_\mu(t,\tau)\hat{B}_2(\tau)\hat{u}(\tau)d\tau \, .\end{aligned} \tag{2.36}$$

Along any control interval (t_i, t_{i+1}) during which $u(t) = K$ where K is a vector of constants, we may integrate (2.36) by parts to yield

$$
\begin{aligned}
z(t) = {} & \Phi_\mu(t,t_i)z_i \\
& - \hat{G}(t)^{-1}\hat{g}(t) + \Phi_\mu(t,t_i)\hat{G}(t_i)^{-1}\hat{g}(t_i) \\
& + \int_{t_i}^{t} \Phi_\mu(t,\tau)[\hat{G}(\tau)^{-1}\hat{g}(\tau) + \hat{G}(\tau)^{-1}\dot{\hat{g}}(\tau)]d\tau \\
& + \hat{G}(t)^{-1}\hat{B}_2(t)K + \Phi_\mu(t,t_i)\hat{G}(t_i)^{-1}\hat{B}_2(t_i)K \\
& + \int_{t_i}^{t} \Phi_\mu(t,\tau)[\dot{\hat{G}}(\tau)^{-1}\hat{B}_2(\tau)K + \hat{G}(\tau)^{-1}\dot{\hat{B}}_2(\tau)K]d\tau
\end{aligned}
\tag{2.37}
$$

where z_i is the value of z at t_i. Since $\hat{g}(t) = g(\hat{x}(t),t)$, $\hat{B}_2(t) = B_2(\hat{x}(t),t)$ and $\hat{G}(t) = G(\hat{x}(t),t)$, (2.37) may be written

$$
\begin{aligned}
z(t) = {} & \Phi_\mu(t,t_i)[z_i + G(\hat{x}(t_i),t_i)^{-1}(g(\hat{x}(t_i),t_i) + B_2(\hat{x}(t_i),t_i)K)] \\
& - G(\hat{x}(t),t)^{-1}g(\hat{x}(t),t) - G(\hat{x}(t),t)^{-1}B_2(\hat{x}(t),t)K \\
& + \int_{t_i}^{t} \Phi_\mu(t,\tau)[\hat{G}(\tau)^{-1}(\hat{g}(\tau) + \hat{B}_2(\tau)K) \\
& + \hat{G}(\tau)^{-1}(\dot{\hat{g}}(\tau) + \hat{B}_2(\tau)K)]d\tau \, .
\end{aligned}
\tag{2.38}
$$

Suppose that $\Phi_\mu(t,t_i)$ satisfies

$$
|\Phi_\mu(t,t_i)| \leq Ke^{-\alpha\left(\frac{t-t_i}{\mu}\right)} , \tag{2.39}
$$

then the integral in equation (2.38) is $O(\mu)$ and the first term will exponentially decay with $O(\mu)$ time constant. This exponential decay is referred to as boundary layer phenomena and occurs at each switching of the control $\hat{u}(t)$. The remaining (middle) term in equation (2.38) is simply the equilibrium we wish to be uniformly asymptotically stable.

Assuming that (2.39) is satisfied is the same as the above assumption. It has been shown that under this assumption

$$z(t) = \bar{z}(t) + O(\mu) + \text{boundary layer terms} .$$

Therefore substituting $\bar{z}(t)$ into (5.1a) to yield the reduced order system will result in $\bar{x}(t)$ being an approximation of $x(t)$.

2.5 A Diagonalization Transformation

In this section system (2.1) and its adjoint system

$$\begin{aligned} \dot{p} &= - A_{11}(t)p - \frac{1}{\mu} A_{21}(t)q \\ \dot{q} &= - A_{12}(t)p - \frac{1}{\mu} A_{22}(t)q \end{aligned} \tag{2.40}$$

will be lower block diagonalized and upper block diagonalized respectively using the transformation defined by

$$\begin{bmatrix} \chi \\ \zeta \end{bmatrix} = \begin{bmatrix} I_n & -\mu F(t) \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{2.41a}$$

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ \mu F(t)' & I_m \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{2.41b}$$

This diagonalization will simplify the process of revealing the main results of the next chapter. When (2.41a) is applied to (2.1) and (2.41b) to (2.5) the result is

$$\dot{\chi} = (A_{11} - FA_{21})\chi + G\zeta + (B_1 - FB_2)u$$
$$\mu\dot{\zeta} = A_{21}\chi + (A_{22} + \mu A_{21}F)\zeta + B_2 u \qquad (2.42)$$

and

$$\dot{\xi} = -(A_{11} - FA_{21})'\xi - \frac{1}{\mu}A_{21}'\eta$$
$$\dot{\eta} = -G'\xi - \frac{1}{\mu}(A_{22} + \mu A_{21}F)'\eta \qquad (2.43)$$

where

$$G(t)' = A_{12}' - A_{22}'F' - \mu\dot{F}' + \mu F'A_{11}' - \mu F'A_{21}'F' \qquad (2.44)$$

and (ξ', η') are the adjoint variables corresponding to (χ', ζ'). In the work presented in this thesis the problem under consideration is that of time-optimal control. In order for this problem to have meaning the time required for the control must be finite. Therefore we only consider transforming systems (2.1) and (2.40) on the finite interval $[t_0, t_1]$.

Chang [34] has shown that under the assumption that the eigenvalues of $A_{22}(t)$ have real parts strictly less than zero, there exists a bounded solution $F(t,\mu)'$ to $G(t)' = 0$. He further shows that one solution $G(t)' = 0$ is

$$F(t,\mu)' = A_{22}'^{-1}A_{12}' + \mu W(t,\mu) \qquad (2.45)$$

where $W(t,\mu)$ can be shown to be bounded on $[t_0, t_1]$. Substituting (2.45) into (2.42) and (2.43) yields

$$\dot{x} = A_1(t)x + (B_0(t) - \mu W(t,\mu)B_2(t))u$$
$$\mu\dot{\zeta} = A_{21}(t)x + A_2(t)\zeta + B_2(t)u \tag{2.46}$$

and

$$\dot{\xi} = -A_1(t)\xi - \frac{1}{\mu}A_{21}(t)\eta$$
$$\dot{\eta} = -\frac{1}{\mu}A_2(t)\eta \tag{2.47}$$

where

$$A_1 \stackrel{\Delta}{=} A_0 - \mu W A_{21} ,$$
$$A_2 \stackrel{\Delta}{=} A_{22} + \mu A_{21}A_{12}A_{22}^{-1} + \mu^2 A_{21}W ,$$
$$W \stackrel{\Delta}{=} W(t,\mu) ,$$
$$B_0 = B_1 - A_{12}A_{22}^{-1}B_2 \tag{2.48}$$

and

$$A_0 \stackrel{\Delta}{=} A_{11} - A_{12}A_{22}^{-1}A_{21} .$$

Thus (2.46) is lower block diagonal and (2.47) is upper block diagonal.

2.6 Expansions of Two Integrals

In this section the two integrals

$$I_1 = \int_{t_0}^{t} \Phi_{22}(t,\tau)N(\tau)d\tau \qquad t \in [t_0,t_1] \tag{2.49}$$

and

$$I_2 = \int_{t_0}^{t} N(s)\Phi_{22}(s,t_0)ds \qquad t \in [t_0,t_1] \tag{2.50}$$

are expanded as series to $O(\mu^3)$ where $\Phi_{22}(t,t_0)$ is the state transition matrix of the fast subsystem (2.4). The following assumptions are made:

(i) The matrices $A_{22}(t)$ and $N(t)$ are continuous and have continuous derivatives on $[t_0,t_1]$ to any desired order.

(ii) The eigenvalues $\lambda_i(t)$ of $A_{22}(t)$ satisfy

$$\mathrm{Re}(\lambda_i(t)) < -\gamma < 0 \qquad t \in [t_0,t_1]$$

where γ is constant.

Lemma 2.6. Under (i) and (ii) the integral I_1 may be expanded as

$$\begin{aligned} I_1 = & -\mu A_{22}^{-1}(t)N(t) + \mu\Phi_{22}(t,t_0)A_{22}^{-1}(t_0)N(t_0) \\ & - \mu^2 A_{22}^{-1}(t)\dot{A}_{22}^{-1}(t)N(t) + \mu^2\Phi_{22}(t,t_0)A_{22}^{-1}(t_0) \\ & \times \dot{A}_{22}^{-1}(t_0)N(t_0) - \mu^2 A_{22}^{-2}(t)\dot{N}(t) \\ & + \mu^2\Phi_{22}(t,t_0)A_{22}^{-2}(t_0)\dot{N}(t_0) + O(\mu^3) \end{aligned} \tag{2.51}$$

for μ sufficiently small.

Proof of Lemma 2.6. The integral I_1 may be expressed

$$I_1 = \int_{t_0}^{t} \Phi_{22}(t,\tau)\,\frac{1}{\mu}\,A_{22}(\tau)\mu A_{22}^{-1}(\tau)N(\tau)d\tau\ .$$

Letting

$$dw = -\frac{1}{\mu}\,\Phi_{22}(t,\tau)A_{22}(\tau)d\tau$$

and

$$v = -\ \mu A_{22}^{-1}(\tau)N(\tau)$$

implies

$$w = \Phi_{22}(t,\tau)$$

and

$$dv = -\mu\dot{A}_{22}^{-1}(\tau)N(\tau)d\tau - \mu A_{22}^{-1}(\tau)\dot{N}(\tau)d\tau$$

which yields

$$I_1 = -\mu\Phi_{22}(t,\tau)A_{22}^{-1}(\tau)N(\tau)\Big|_{\tau=t_0}^{t} + \mu\int_{t_0}^{t} \Phi_{22}(t,\tau)\dot{A}_{22}^{-1}(\tau)N(\tau)d\tau$$

$$+ \mu\int_{t_0}^{t} \Phi_{22}(t,\tau)A_{22}^{-1}(\tau)\dot{N}(\tau)d\tau .$$

Noting that for μ sufficiently small system (2.4) is uniformly asymptotically stable by Lemma 2.3 and integrating these two integrals by parts yields

$$I_1 = -\mu\Phi_{22}(t,\tau)A_{22}^{-1}(\tau)N(\tau)\Big|_{\tau=t_0}^{t}$$

$$-\mu^2\Phi_{22}(t,\tau)A_{22}^{-1}(\tau)\dot{A}_{22}^{-1}(\tau)N(\tau)\Big|_{\tau=t_0}^{t}$$

$$-\mu^2\Phi_{22}(t,\tau)A_{22}^{-2}(\tau)\dot{N}(\tau)\Big|_{\tau=t_0}^{t} + O(\mu^3)$$

which yields (2.51) and Lemma 2.6 is proved.

Lemma 2.7. Under (i) and (ii) the integral I_2 may be expanded as

$$
\begin{aligned}
I_2 = {} & \mu N(t)A_{22}^{-1}(t)\Phi_{22}(t,t_0) - \mu N(t_0)A_{22}^{-1}(t_0) \\
& - \mu^2\dot{N}(t)A_{22}^{-2}(t)\Phi_{22}(t,t_0) + \mu^2\dot{N}(t_0)A_{22}^{-1}(t_0) \\
& - \mu^2 N(t)\dot{A}_{22}^{-1}(t)A_{22}^{-1}(t)\Phi_{22}(t,t_0) \\
& + \mu^2 N(t_0)\dot{A}_{22}^{-1}(t_0)A_{22}^{-1}(t_0) + O(\mu^3)
\end{aligned}
\tag{2.52}
$$

for μ sufficiently small.

The proof of Lemma 2.7 follows analogously to that of Lemma 2.6.

3. TIME-OPTIMAL CONTROL OF TIME-VARYING SYSTEMS

3.1 Introduction and Problem Statement

In this chapter we treat the time-optimal control of system (2.1) which for convenience is rewritten here

$$\begin{aligned} \dot{x} &= A_{11}(t)x + A_{12}(t)z + B_1(t)u \\ \mu\dot{z} &= A_{21}(t)x + A_{22}(t)z + B_2(t)u \end{aligned} \tag{3.1}$$

where $x \in R^n$, $z \in R^m$, $u \in R^r$ and μ is a scalar which is greater than zero. For all t, the matrices $A_{ij}(t)$, $i,j = 1,2$, are bounded with bounded derivatives and the eigenvalues of λ_i of $A_{22}(t)$ satisfy

$$Re(\lambda_i(t)) < -\gamma < 0 \ .$$

The main results of this chapter are the two time-scale property and the development of a near time-optimal control for (3.1). The necessary conditions of the minimum principle are applied to demonstrate the two time-scale property, that the time-optimal control of (3.1) is composed of an initial interval of control in a slow time-scale followed by an interval of control in a fast time-scale. Based on this property a near optimal control is proposed which is made of a time-optimal control of the reduced order system

$$\begin{aligned} \dot{\bar{x}} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})\bar{x} + (B_1 - A_{12}A_{22}^{-1}B_2)\bar{u} \\ &\triangleq A_0(t)\bar{x} + B_0(t)\bar{u} \end{aligned} \tag{3.2}$$

followed by a time-optimal control of the fast subsystem

$$\mu\dot{\tilde{z}} = A_{21}(t)x_F + A_{22}(t)\tilde{z} + B_2(t)\tilde{u} \ . \tag{3.3}$$

The advantage of this near optimal control is that it requires control computations only on reduced order systems.

The time-optimal control problem treated here is that of finding the control $u^*(t)$ constrained

$$u \in U = \{u: |u_i| \leq 1, \ i = 1,\ldots,r\}$$

which steers the state $(x', z')'$ of (3.1) from the fixed initial state $(x_0', z_0')'$ at time t_0 to the fixed final state $(x_F', z_F')'$ in minimum time T^*. For notational purposes t_f^* is defined $t_f^* = T^* + t_0$. System (3.1) is assumed to satisfy the following condition in order to guarantee the existence of the time-optimal $u^*(t)$.

(i) There exist t_1 and μ_0 such that for all $\mu \in (0, \mu_0]$ there exists a control $u(t,\mu) \in U$ which steers (x', z') from $(x_0', z_0')'$ to $(x_F', z_F')'$ within time $t_1 - t_0$.

According to Theorem 17 of [35, p. 127], (i) guarantees the existence of $u^*(t)$ for $\mu \in (0, \mu_0]$. The normality of the control problem is assumed.

3.2 Necessary Conditions from the Minimum Principle

In this section the minimum principle is applied to the control problem. Then a diagonalization transformation and expansions in μ are used in the necessary conditions to reveal the two time-scale property. The Hamiltonian to be minimized is

$$\begin{aligned} H = 1 &+ p'A_{11}x + p'A_{12}z + p'B_1u \\ &+ \frac{1}{\mu} q'A_{21}x + \frac{1}{\mu} q'A_{22}z + \frac{1}{\mu} q'B_2u \end{aligned} \tag{3.4}$$

where

$$\dot{p} = - A_{11}'p - \frac{1}{\mu} A_{21}'q$$
$$\dot{q} = - A_{12}'p - \frac{1}{\mu} A_{22}'q \tag{3.5}$$

and

$$u^*(t) = - \mathrm{SGN}[B_1'p + \frac{1}{\mu} B_2'q] \ . \tag{3.6}$$

The normality of the control implies that

$$B_1'(t)p(t) + \frac{1}{\mu} B_2'(t)q(t) \triangleq S(t) = 0$$

only at isolated times t_j and not on a finite interval. Thus the switchings of the components of the optimal control will occur at the *instants* t_j *at which one of the components* s_i, $i = 1,\ldots,r$ of S satisfies

$$s_i(t_j) = 0 \ .$$

The instants t_j are referred to as the zeroes of S.

If we let $\Phi(t,t_0)$ be the state transition matrix of (3.1) satisfying

$$\frac{\partial}{\partial t} \Phi(t,t_0) = \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{\mu} A_{21} & \frac{1}{\mu} A_{22} \end{bmatrix} \Phi(t,t_0) \tag{3.7}$$

then (3.6) may be written

$$u^*(t) = - \mathrm{SGN} \left[[B_1' \vdots B_2'] \Phi'(t_f^*,t) \begin{bmatrix} p_F \\ \cdots \\ \frac{1}{\mu} q_F \end{bmatrix} \right] \tag{3.8}$$

where p_F and $\frac{1}{\mu} q_F$ are the values of $p(t)$ and $q(t)$ at t_f^*. A candidate for the optimal control could be found by determining t_f^*, p_F and q_F such that $u^*(t)$ steers $(x',z')'$ from $(x_0',z_0')'$ to $(x_F',z_F')'$. This is in general a difficult problem due to the stiffness which results from the presence of μ.

The minimum principle provides the additional necessary condition

$$H(t_f^*) = 0$$

which implies that as $\mu \to 0$, $\frac{1}{\mu} q_F$ remains bounded and therefore q_F is $O(\mu)$.

In order to reveal the two time-scale property the diagonalization transformation (2.41) is applied to (3.1) and (3.5) to yield

$$\begin{aligned} \dot{\chi} &= (A_{11} - FA_{21})\chi + G\zeta + (B_1 - FB_2)u \\ \mu\dot{\zeta} &= A_{21}\chi + (A_{22} + \mu A_{21}F)\zeta + B_2 u \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \dot{\xi} &= -(A_{11} - FA_{21})'\xi - \frac{1}{\mu} A_{21}'\eta \\ \dot{\eta} &= -G'\xi - \frac{1}{\mu}(A_{22} + \mu A_{21}F)'\eta \end{aligned} \tag{3.10}$$

where

$$G'(t) = A_{12}' - A_{22}'F' - \mu\dot{F}' + \mu F'A_{11}' - \mu F'A_{21}'F' . \tag{3.11}$$

In Section 2.4 it is shown that for μ sufficiently small and $t \in [t_0,t_1]$ a solution of

$$G(t)' = 0$$

is

$$F(t)' = A_{22}'^{-1}A_{21}' + \mu W(t,\mu)' \tag{3.12}$$

where $W \triangleq W(t,\mu)$ is bounded. Substitution of (3.12) into (3.9) and (3.10) yields

$$\dot{\chi} = A_1\chi + (B_0 - \mu W B_2)u \tag{3.13}$$

$$\mu\dot{\zeta} = A_{21}\chi + A_2\zeta + B_2 u$$

and

$$\dot{\xi} = -A_1'\xi - \frac{1}{\mu}A_{21}'\eta \tag{3.14}$$

$$\mu\dot{\eta} = -A_2'\eta ,$$

where $A_1 \triangleq A_0 - \mu W A_{21}$, $A_2 \triangleq A_{22} + \mu A_{21}A_{12}A_{22}^{-1} + \mu^2 A_{21}W$ and A_0 and B_0 are as defined in (3.2). Letting $\Phi_0(t,t_0)$ and $\Phi_{22}(t,t_0)$ be the state transition matrices satisfying

$$\frac{\partial}{\partial t}\Phi_0(t,t_0) = A_0(t)\Phi_0(t,t_0) \tag{3.15}$$

and

$$\frac{\partial}{\partial t}\Phi_{22}(t,t_0) = A_{22}(t)\Phi_{22}(t,t_0) \tag{3.16}$$

yields the approximate solution to (3.14)

$$\xi(t) = \Phi_0'(t_f^*,t)\xi_F - [\Phi_0'(t_f^*,t)A_{21}'(t_f^*)A_2'(t_f^*)^{-1}$$

$$- A_{21}'(t)A_2'(t)^{-1}\Phi_2'(t_f^*,t) + O(\mu)]\eta_F \tag{3.17}$$

$$\eta(t) = \Phi_{22}'(t_f^*,t)\eta_F + O(\mu) \tag{3.18}$$

where ξ_F and η_F are the values of ξ and η at t_f^*. Transformation (2.41) and (3.12) imply

$$p(t) = \xi(t) \tag{3.19}$$

$$q(t) = \eta(t) - \mu A_{22}'(t)^{-1}A_{12}'(t)p(t) + O(\mu^2)$$

and

$$\xi_F = p_F$$
$$\eta_F = q_F + \mu A_{22}'(t_f^*)^{-1} A_{12}'(t_f^*) p_F + O(\mu^2) \ . \tag{3.20}$$

The substitution of (3.17), (3.18) and (3.20) into (3.19) yields

$$p(t) = \Phi_0'(t_f^*,t)(p_F - A_{21}'(t_f^*)A_2'(t_f^*)^{-1} q_F) + A_{21}'(t)A_2'(t)^{-1}\Phi_{22}'(t_f^*,t)q_F + O(\mu) \tag{3.21}$$

$$\begin{aligned} q(t) = {} & \Phi_{22}'(t_f^*,t)(q_F + \mu A_{22}'(t_f^*)^{-1} A_{12}'(t_f^*) p_F) \\ & - \mu A_{22}'(t)^{-1} A_{12}'(t) \Phi_0'(t_f^*,t)(p_F - A_{21}'(t_f^*)A_2'(t_f^*)^{-1} q_F) \\ & - \mu A_{22}'(t)^{-1} A_{12}'(t) A_{21}'(t) A_2'(t)^{-1} \Phi_{22}'(t_f^*,t) q_F \\ & + O(\mu^2) \ . \end{aligned} \tag{3.22}$$

Finally substituting (3.21) and (3.22) into $u^*(t)$, (3.6), yields

$$\begin{aligned} u^*(t) = - \, \mathrm{SGN}[& B_0'(t)\Phi_0'(t_f^*,t)(p_F - A_{21}'(t_f^*)A_2'(t_f^*)^{-1} q_F) \\ & + B_0'(t)A_{21}'(t)A_2'(t)^{-1}\Phi_{22}'(t_f^*,t)q_F \\ & + B_2'(t)\Phi_{22}'(t_f^*,t)[\tfrac{1}{\mu} q_F + A_{22}'(t_f^*)^{-1} A_{12}(t_f^*) p_F \\ & + O(\mu)] \ . \end{aligned} \tag{3.23}$$

Lemma 2.1 implies that for μ small enough $\Phi_{22}(t,t_0)$ is *bounded by* a decaying exponential with $O(\mu)$ time constant. This implies that $\Phi_{22}'(t_f^*,t)$ is exponentially decaying in reverse time.

Lemma 3.1. The time-optimal control $u^*(t)$ is composed of an initial interval of control in a slow time-scale followed by an interval of control in a fast time-scale. In particular, for any $\epsilon > 0$ there exists $\bar{\mu}$ and τ such that for $\mu \in [0,\bar{\mu}]$, $u^*(t)$ satisfies

$$u^*(t) = \begin{cases} -\mathrm{SGN}[B_0'(t)\Phi_0'(t_f^*,t)p_F + O(\mu) + O(\epsilon)] \\ \qquad\qquad\qquad t \in [t_0, t_f^*-\tau) \\ -\mathrm{SGN}[B_0'(t)p_F + B_2'(t)\Phi_{22}'(t_f^*,t) \\ \qquad \times (\frac{1}{\mu} q_F + A_{22}'(t_f^*)^{-1} A_{12}'(t_f^*)p_F + O(\mu)] \\ \qquad\qquad\qquad t \in [t_f^*-\tau, t_f^*] \end{cases} \tag{3.24}$$

where $\tau = O(\mu)$.

<u>Proof of Lemma 3.1</u>. By Lemma 2.1 there exists, for μ small enough, a constant K such that

$$|\Phi_{22}'(t_f^*,t)| < Ke^{\sigma\left(\frac{t-t_f^*}{\mu}\right)} \quad \text{for } t < t_f^* \ .$$

Letting $Ke^{-\frac{\sigma\tau}{\mu}} = \epsilon$ implies $\tau = -\frac{\mu}{\sigma}\ln(\frac{\epsilon}{K})$ where ϵ is chosen such that $\epsilon/K < 1$. Thus $\tau = O(\mu)$ and for $t < t_f^*-\tau$,

$$|\Phi_{22}'(t_f^*,t)| < \epsilon \ . \tag{3.25}$$

Recalling that $q_F = O(\mu)$ and noting that $\tau = O(\mu)$ implies that

$$\Phi_0'(t_f^*,t) = I + O(\mu) \qquad t \in [t_f^*-\tau, t_f^*]$$

and hence we have (3.24). For small enough ϵ and μ, the switchings on the slow interval $[t_0,t_f^*-\tau]$ are primarily dependent on $\Phi_0'(t_f^*,t)$ and the switchings on the fast interval $[t_f^*-\tau,t_f^*]$ of length $O(\mu)$, are primarily dependent on B_0p_F and $\Phi_{22}'(t_f^*,t)$. Thus the switchings on the initial interval are in a slow time-scale and the switchings on the final interval are in a fast time-scale and Lemma 3.1 is proved.

Since τ is $O(\mu)$, $x(t_f^*-\tau)$ is $O(\mu)$ from x_F. Thus the two time-scale property implies that the control on the slow control interval steers x to $O(\mu)$ from x_F. The fast switchings in $[t_f^*-\tau, t_f^*]$ steer z to z_F and x the last $O(\mu)$ distance to x_F. Based on this argument the near-optimal control is proposed in the next section.

3.3 Slow and Fast Control

The reduced control problem is that of finding the control $\bar{u}^*(t) \in U$ which steers the state $\bar{x}$ of the reduced order system (3.2) from x_0 at time t_0 to x_F in minimum time $\bar{T}^*$. For notational purposes $\bar{t}_f^* = \bar{T}^* + t_0$. Since setting μ to 0 in (3.1) to yield (3.2) implies that z may be steered instantaneously, the reduced problem does not include the steering of z.

The minimum principle yields the following necessary conditions for the reduced control problem. The Hamiltonian to be minimized for $u \in U$ is

$$H = 1 + \bar{p}'A_0\bar{x} + \bar{p}'B_0\bar{u} \qquad (3.26)$$

where $\bar{p}$ satisfies

$$\dot{\bar{p}} = - A_0(t)\bar{p} \qquad (3.27)$$

and

$$\bar{u}^*(t) = - \mathrm{SGN}[B_0'\bar{p}(t)] \; . \qquad (3.28)$$

The solution of (3.27) is

$$\bar{p}(t) = - \Phi_0'(\bar{t}_f^*, t)\bar{p}_F \qquad (3.29)$$

which when substituted into (3.28) yields

$$\overline{u}^*(t) = - \mathrm{SGN}[B_0'\Phi_0'(\overline{t}_f^*,t)\overline{p}_F] \ . \tag{3.30}$$

We let $\hat{x}(t),\hat{z}(t)$ be the result of applying $\overline{u}^*(t)$ to the full order system (3.1) and $\overline{x}^*(t)$ be the optimal trajectory for the reduced order system (3.2).

<u>Lemma 3.2</u>. Let $\overline{u}^*(t)$ be applied to (3.1). Then

$$\hat{x}(t) = \overline{x}^*(t) + O(\mu) \tag{3.31}$$

for $t \in [t_0,\overline{T}^*]$.

<u>Proof of Lemma 3.2</u>. Let $e(t) = X(t) - \overline{x}(t)$, then

$$\dot{e} = A_0(t)e - \mu W A_{21}(t)X - \mu W B_2 u$$

where it is assumed that $e(t_0) = 0$. The error $e(t)$ satisfies

$$e(t) = - \mu \int_{t_0}^{t} \Phi_0(t,\tau)W(\tau,\mu)[A_{21}(\tau)X(\tau) + B_2(\tau)u(\tau)]d\tau \ .$$

On $[t_0,\overline{t}_f^*]X(t)$, $A_{21}(t)$ and $B_2(t)$ are bounded and $u \in U$. Therefore $e(t) = O(\mu)$. Lemma 2.1 implies that there exists $\hat{\mu}$ such that for $\mu \in (0,\hat{\mu})$ $z(t)$ is bounded on $[t_0,\overline{t}_f^*]$ and thus transformation (2.41) implies

$$\hat{x}(t) = X(t) + O(\mu) \ .$$

Finally this implies that

$$\hat{x}(t) = \overline{x}^* + O(\mu)$$

and Lemma 3.2 is proved.

The slow state control problem is that of finding the control $u_s^*(t) \in U$ which steers x of (3.1) from x_0 to x_F in minimum time T_s^*. For this problem the initial value of z is z_0 and its final value is free which implies that $q_F = 0$. Thus equation (3.24) implies

$$u_s^*(t) = \begin{cases} -\mathrm{SGN}[B_0'\Phi_0'(t_f^*,t)p_F + O(\mu) + O(\epsilon)] \\ \qquad\qquad t \in [t_0, t_f^* - \tau] \\ -\mathrm{SGN}[B_0'p_F + B_2'\Phi_{22}'(t_f^*,t)A_{22}'(t_f^*)^{-1}A_{12}'(t_f^*)p_F \\ \qquad + O(\mu)] \qquad t \in [t_f^* - \tau, t_f^*] \end{cases} \tag{3.32}$$

and $u_s^*(t)$ also possesses the two time-scale property. On the slow interval $[t_0, t_f^* - \tau)$ the switchings of $u_s^*(t)$ are primarily dependent on $B_0'\Phi_0'(t_f^*,t)p_F$. As $\mu \to 0$, $t_f^* \to \bar{t}^*$ and by (3.31) $\hat{x}(t) \to \bar{x}^*(t)$. Therefore as $\mu \to 0$, $p_F \to \bar{p}_F$.

Lemma 3.3. Let the zeroes of $B_0'(t)\Phi_0'(\bar{t}_f,t)\bar{p}_F$ of the reduced order control (3.30) be simple. Then for small enough μ the switching times of $u_s^*(t)$ on $[t_0, t_f^* - \tau)$ are $O(\mu^a)$ different from those of $\bar{u}^*(t)$. Furthermore the minimum time T_s^* for the slow control problem satisfied

$$T_s^* = \bar{T}^* + O(\mu^a) \tag{3.33}$$

where $a > 0$ is constant.

Proof of Lemma 3.3. The fact that the zeroes of $B_0'(t)\Phi_0'(\bar{t}_f,t)\bar{p}_F$ are simple implies that the control sequence of $\bar{u}^*(t)$ is, for μ small enough, the same as the control sequence of $u_s^*(t)$ on $[t_0, t_f^* - \tau)$. The choice of the parameter ϵ is dependent on the reverse time exponential decay of $\Phi_{22}'(t_f^*,t)$ and hence satisfies

$$\lim_{\mu \to 0} \epsilon = 0 .$$

The $O(\epsilon)$ and $O(\mu)$ terms of (3.32) shift the switching times of $\bar{u}^*(t)$ by some μ-dependent variations. Since these variations go to zero as $\mu \to 0$ there exists $b > 0$ such that these variations are $O(\mu^b)$. For a finite number of switching instants t_j of $\bar{u}^*(t)$

$$T_s^* = \bar{T}^* + O(\mu^b) + \tau$$

where $\tau = O(\mu)$. Thus there exists $a > 0$ such that

$$T_s^* = \bar{T}^* + O(\mu^a)$$

and Lemma 3.3 is proved.

Lemma 3.3 suggests that by varying the switchings of $\bar{u}^*(t)$ by some method such as an iterative method [6] or switching sensitivities [22] and adding some fast switchings on the τ interval the control $u_s^*(t)$ may be found. Thus $\bar{u}^*(t)$ is a near-optimal control for the slow control problem in the sense that it steers x to $O(\mu)$ from x_F and requires the near-optimal time $\bar{T}^*$.

The point $z(\bar{t}_f^*)$ of the fast state z at $\bar{t}_f^*$ after the application of $\bar{u}^*(t)$ to the full order system will be some finite distance from z_F for μ small enough, by the stability Lemma 2.1. Thus the time-optimal control $u_f^*(t)$ which steers the state of (3.1) from $z(\bar{t}_f^*)$ to z_F will require $O(\mu)$ time. For any bounded control on an $O(\mu)$ interval the slow state x will only be moved an $O(\mu)$ distance.

Let $\tilde{u}_f^*(t)$ be the control which time-optimally steers z of the fast subsystem (3.3), where x_F is the final slow state, from $z(\bar{t}_f^*)$ to z_F in time $\tilde{T}_f^*$.

Lemma 3.4. The control

$$\tilde{u}^*(t) = \begin{cases} \bar{u}^*(t) & t \in [t_0, \bar{t}_f^*) \\ \tilde{u}_f^*(t) & t \in [\bar{t}_f^*, \bar{t}_f^* + \tilde{T}_f^*] \end{cases} \qquad (3.34)$$

is a near time-optimal control for (3.1) in the sense that it steers $(x',z')'$ to a point which is $O(\mu)$ from $(x_F', z_F')'$. Furthermore

$$T^* = \bar{T}^* + O(\mu) + O(\mu^a) . \qquad (3.35)$$

Proof of Lemma 3.4. As discussed above $\tilde{T}_f^* = O(\mu)$ and therefore

$$x(\bar{t}_f^* + \tilde{T}_f^*) = x_F + O(\mu) .$$

Since x will be $O(\mu)$ from x_F on $[\bar{t}_f^*, t_f^* + T_f^*)$, $u_f^*(t)$ will steer z from $z(\bar{t}_f^*)$ to $O(\mu)$ from z_F. The control $u^*(t)$ requires time $\bar{T}^* + \tilde{T}_F^*$ and therefore by equation (3.33)

$$T_s^* = \bar{T}^* + O(\mu) + O(\mu^a) .$$

The time T^* for time-optimally steering both x and z must be greater than T_s^*. Since it is possible to steer the fast state in $O(\mu)$ time, equation (3.35) is satisfied and Lemma 3.4 is proved.

As pointed out in the introduction, Section 3.1, the near optimal control is computed as the time-optimal controls for two reduced order systems (3.2) and (3.3). As developed here, this control requires the knowledge of $z(\bar{t}_f^*)$ which may be found from integration of system (3.1) with the control $\bar{u}^*(t)$. Singular perturbation techniques are well developed for such differential equations and imply that for μ small enough

$$z(\bar{t}_f^*) = \bar{z}(\bar{t}_f^*) + O(\mu) \tag{3.36}$$

where

$$\bar{z} = - A_{22}^{-1}A_{21}\bar{x} - A_{22}^{-1}B_2\bar{u} \ . \tag{3.37}$$

Thus the use of $\bar{z}(\bar{t}_f^*)$ instead of $z(\bar{t}_f^*)$ will introduce *another* $O(\mu)$ error in z when it is not possible to compute $z(\bar{t}_f^*)$.

The following lemma makes it possible to replace the assumption of the normality of the time-optimal control for the full order system (3.1) by the assumption of the normality of the time-optimal controls for the reduced order system (3.2) and the fast system (3.3).

Lemma 3.5. Assume that the reduced order control $\bar{u}^*(t)$ and the fast control $\tilde{u}_f^*(t)$ are normal. Then for small enough μ the control $u^*(t)$ is normal.

Proof of Lemma 3.5. The minimum principle implies that

$$\tilde{u}_f^*(t) = - \text{SGN}[\frac{1}{\mu} B_2'(t)\Phi_{22}'(\bar{t}_f^* + \tilde{T}_f^*,t)q_F] \ . \tag{3.38}$$

Let $F(t) = B_2'(t)\Phi_{22}'(\bar{t}_f + \tilde{T}_f^*,t)$ and $S(t) = B_0'(t)\Phi_0'(\bar{t}_f^*,t)$. The normality assumption on $\bar{u}^*(t)$ and $\tilde{u}_f^*(t)$ implies that the components $f_i(t)$ and $s_i(t)$, $i = 1,\ldots,r$ of F(t) and S(t) satisfy

$$f_i(t) = 0$$

and

$$s_i(t) = 0$$

at *isolated instants of time* t_j. *By equation* (3.24) for $u^*(t)$ the normality of $\bar{u}^*(t)$ implies the normality of $u^*(t)$ for $t \in [t_0,t_f^* - \tau)$ and the normality of $u_s^*(t)$ implies the normality of $u^*(t)$ for $t \in [t_f^* - \tau,t_f^*]$ and therefore Lemma 3.5 is proved.

4. THE INTERMEDIATE POINT ALGORITHM

4.1 Introduction

This chapter presents the intermediate point algorithm which exploits the two time-scale property developed in the last chapter. As pointed out there, the time-optimal control can be viewed as being made of two parts. The first part is primarily concerned with the control of slow states while the second part is primarily concerned with the control of fast states. Corresponding to the two parts of the control, the optimal trajectory is divided into two parts. The point lying at the intersection of the slow and fast part of the optimal trajectory is called the intermediate point. The intermediate point algorithm consists of iterations for the intermediate point based on computations for lower order systems. From the intermediate point the slow and fast parts of the control are calculated.

The system for which the algorithm is developed is

$$\dot{x} = A_1 x + B_1 u \tag{4.1a}$$

$$\mu\dot{z} = A_2 z + B_2 u \tag{4.1b}$$

where $x \in R^n$, $z \in R^m$, $u \in R^r$ and $\mu > 0$. In [3] a transformation is presented which transforms linear time-invariant singularly perturbed systems to the form of (4.1). The eigenvalues λ_i of A_2 satisfy

$$\mathrm{Re}(\lambda_i) < 0 \ , \tag{4.2}$$

u is constrained

$$u \in U = \{u: \ |u_i| \leq 1, \ i = 1,\ldots,r\} \tag{4.3}$$

and (4.1) is completely controllable from each component u_i of control. This controllability implies the normality of the problem treated here which is that of finding the control $u^*(t)$ which steers $(x',z')'$ of (4.1) from $(x_0',z_0')'$ at $t_0 = 0$ to $(x_F',z_F')'$ in minimum time T^*. For small μ (4.1a) is referred to as the slow subsystem, (4.1b) as the fast subsystem and x and z as the corresponding slow and fast states.

Systems (4.1a) and (4.1b) are equivalent to the systems referred to as the reduced order and fast subsystems in the last chapter. Consequently Lemma 3.4 implies that a near-optimal control is

$$\hat{u}(t) = \begin{matrix} \hat{u}_s(t) & t \in [0,T_s) \\ \hat{u}_f(t) & t \in [T_s,\hat{T}) \end{matrix} \tag{4.4}$$

where $\hat{u}_s(t)$ time-optimally steers x from x_0 to x_F and $\hat{u}_f(t)$ time-optimally steers z from $z(T_s)$ to z_F. This control is near optimal in the sense that it steers z to z_F and x to $O(\mu)$ from x_F and $\hat{T} = T^* + O(\mu)$.

4.2 The Intermediate Point Algorithm

Since every part of a time-optimal trajectory for (4.1) is an optimal trajectory, the time-optimal control $u^*(t)$ can be viewed as a concatenation of two optimal controls

$$u^*(t) = \begin{matrix} u_s^*(t) & t \in [0,t_I) \\ u_f^*(t) & t \in [t_I,T^*) \end{matrix} \tag{4.5}$$

where t_I is some intermediate time when the state x is at an intermediate point $x(t_I)$. The interval $[t_I,T^*]$ is expected to be considerably shorter than $[0,t_I]$. The controls $u_s^*(t)$ and $u_f^*(t)$ are time-optimal to and from the intermediate point. We denote the slow and fast parts of the intermediate by x_I and z_I, that is, $x(t_I) = x_I$ and $z(t_I) = z_I$.

The name of the intermediate point algorithm comes from the fact that this algorithm iterates for an intermediate point, which lies on the optimal trajectory. In each iteration the nth iterate for the intermediate point is denoted by (x_I^n, z_I^n) and the corresponding iterate for the intermediate time is denoted t_I^n.

The first guess for the intermediate point is provided by the near optimal control (4.4). The intermediate time is defined

$$t_I^0 = T_s \ .$$

The slow intermediate point is defined

$$x_I^0 = x_F \ .$$

The controls $\hat{u}_s(t)$ and $\hat{u}_f(t)$ are thus first guesses $u_s^0(t)$, $u_f^0(t)$ for $u_s^*(t)$ and $u_f^*(t)$ respectively. As the next guess x_I^1 for x_I, we define the point which will be steered to x_F by $u_f^0(t)$. In general, if we have an nth guess $u_f^n(t)$ for $u_f^*(t)$, then x_I^{n+1} which is transferred to x_F by $u_f^n(t)$ is

$$x_I^{n+1} = e^{A_1(t_I^n - T^n)} x_F - \int_{t_I^n}^{T^n} e^{A_1(t_I^n - \tau)} B_1 u_f^n(\tau) d\tau \tag{4.6}$$

where $T^n - t_I^n$ is the minimum time in which $u_f^n(t)$ steers z from z_I^n to z_F. On the other hand z_I^n is the point to which $u_s^n(t)$ steers z to z_0, while optimally steering the slow state x from x_0 to x_I^n in time t_I^n, that is,

$$z_I^n = e^{A_2\left(\frac{t_I^n}{\mu}\right)} z_0 + \frac{1}{\mu} \int_0^{t_I^n} e^{A_2\left(\frac{t_I^n - \tau}{\mu}\right)} B_2 u_s^n(\tau) d\tau \ . \tag{4.7}$$

Expressions (4.6) and (4.7) for x_I^n and z_I^n and the definitions of $u_s^*(t)$ and $u_f^n(t)$ constitute the intermediate point algorithm. The nth iteration consists of the following four steps:

<u>Step 1</u>. Find $u_s^n(t)$ which steers x of (4.1a) from x_0 to x_I^n in minimum time t_I^n.

<u>Step 2</u>. Evaluate z_I^n from (4.7).

<u>Step 3</u>. Find $u_f^n(t)$ which steers z of (4.1b) from z_I^n to z_f in minimum time $T^n - t_I^n$.

<u>Step 4</u>. Knowing $u_f^n(t)$, T^n and t_I^n from the n-1 step, evaluate x_I^{n+1} from (4.6).

The procedure initialized with $x_I^0 = x_F$. It terminates when

$$|x_I^{n+1} - x_I^n| < \epsilon \tag{4.8}$$

where ϵ is a preassigned scalar such as $\epsilon = 10^{-6}$. Then $u_f^n(t)$ will steer $(x',z')'$ from $x_I^{n\prime}, z_I^{n\prime})'$ to $(x_F', z_F')'$. Thus the control

$$u^n(t) = \begin{cases} u_s^n(t) & t \in [0, t_I^n) \\ u_f^n(t) & t \in [t_I^n, T^n] \end{cases} \tag{4.9}$$

will steer $(x',z')'$ from $(x_0', z_0')'$ to $(x_F', z_F')'$. A test for the optimality of this control is given in Lemma 4.1.

The convergence properties of this algorithm have not been treated with any rigor. What is known is that for a number of examples which are presented in [5] the convergence was quite rapid, sometimes in less than ten iterations. The idea which led to the development of the algorithm which may contain the seed for a convergence proof is simple. For μ small enough the last interval of constant control of $u_s^n(t)$ will be long enough so that z_I^n will be arbitrarily close to its equilibrium value at the end of the interval. In this case making an $O(\mu)$ change in x_I^n to yield x_I^{n+1} will not appreciably change z_I^n and therefore $u_f^n(t)$ will change only a small amount. The condition that remains to be discovered is that under which the small changes in $u_f^n(t)$ are in the proper direction for convergence.

The computational efficiency of the algorithm depends on Steps 1 and 3 which require the calculation of time optimal controls for the reduced order systems (4.1a) and (4.1b). For second order subsystems phase plane techniques can be used. Thus for fourth order systems with second order subsystems the algorithm can be easily implemented. For higher order systems one of two approaches can be used. The first is to apply other computational methods for time-optimal control such as those in [15,22]. The other approach is to see if the eigenvalues of the system are such that the system may be broken down into more than two subsystems. The advantage of the intermediate point algorithm is that for systems with smaller μ the convergence should be better. Thus the order of computations that are needed are reduced and the difficulties characteristic of stiff systems are removed since the stiffer the systems, the faster the convergence. This indeed is the reason for the development of singular perturbation methods for control theory.

As in most optimization procedures, a control to which the procedure has converged and the corresponding final time T, must be tested for optimality. When this algorithm has converged the control steers the state of (4.1) from the initial state to the final state to within the desired accuracy. The next step is to see if the necessary conditions of the maximum principle are satisfied. Let the rows of B_1' and B_2' be denoted by $b_1^{(j)\prime}$ and $b_2^{(j)\prime}$, $j = 1,\ldots,m$. Then from equation (3.6) the components of $u^*(t)$ are

$$u_j^*(t) = -\operatorname{sgn}[b_1^{(j)\prime}p(t) + \frac{1}{\mu} b_2^{(j)\prime}q(t)] \ . \tag{4.10}$$

As the kth switching instant t_j^k of any component $u_j^*(t)$

$$b_1^{(j)\prime}p(t_j^k) + \frac{1}{\mu} b_2^{(j)\prime}q(t_j^k) = 0 \ . \tag{4.11}$$

This implies

$$b_1^{(j)\prime} e^{-A_1'(T^*-t_j^k)} p_F + \frac{1}{\mu} b_2^{(j)\prime} e^{-A_2'\left(\frac{T^*-t_j^k}{\mu}\right)} q_F \ . \tag{4.12}$$

As the final time the Hamiltonian satisfies

$$H(T^*) = 0$$

which implies

$$H = 1 + p_F'A_1x_F + p_FB_1u + \frac{1}{\mu} q_F'A_2z_F + \frac{1}{\mu} q_F'B_2u = 0 \ . \tag{4.13}$$

These are the conditions of the minimum principle that remain to be satisfied which leads to the following conclusion.

<u>Lemma 4.1</u>. Suppose the p_F and q_F can be found such that (4.13) is satisfied and such that for each switching instant t_j^n of each component of control, (4.12) is satisfied for the switching instants and final time of the control computed by the intermediate point algorithm. Then the control computed from the intermediate point algorithm satisfies the necessary conditions of the minimum principle and is therefore a candidate for the optimal control.

From an engineering standpoint we may be satisfied if the control $u_f^n(t)$ requires a time considerably shorter than $u_s^n(t)$, since, as was discussed in the last chapter $u_s^0(t)$ should require a time $O(\mu)$ different from the minimum time needed to steer x from x_0 to x_F. Thus if with an additional short interval of control added, x and z can both be steered to their final values (x_F, z_F), the control if not optimal, may satisfy our needs anyway. In this statement it has been assumed that $u_s^n(t)$ requires approximately the same amount of time as $u_s^0(t)$ since x_I^n should be $O(\mu)$ from x_F.

4.3 Example

System (4.14) represents a generator driving a pump which pumps water into a reservoir. Water flows from this reservoir into a second reservoir at a rate proportional to x_1, a small change in water depth around the nominal value $\bar{x}_1$. Water also flows out of the second reservoir at a rate proportional to x_2, a small change in water depth around the nominal depth $\bar{x}_2$.

The reservoir depths x_1 and x_2, the angular velocity z_1 of the motor shaft and the armature current z_2 are the state variables chosen to model this system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & k_p & 0 \\ k_1 & -k_2 & 0 & 0 \\ 0 & 0 & \frac{1}{T_m}\cdot\frac{-1}{1+\beta} & \frac{\beta}{1+\beta} \\ 0 & 0 & -1/T_e & 1/T_e \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/T_e \end{bmatrix} u \qquad (4.14)$$

The parameter μ is inversely dependent on the friction in the motor and pump. The control u is the normalized armature voltage.

Substituting possible values into (4.14) yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1.5 & 0 & 0 & 0 \\ 1.5 & -2 & 0 & 0 \\ 0 & 0 & -4.54 & .091 \\ 0 & 0 & 100 & -100 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \end{bmatrix} u \, . \tag{4.15}$$

The initial state chosen for this example is

$$\begin{bmatrix} x_{10} \\ x_{20} \\ z_{10} \\ z_{20} \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \\ 5 \\ 45 \end{bmatrix} \, . \tag{4.16}$$

The problem is to steer the state of this system $[x_1 \; x_2 \; x_3 \; x_4]'$ from the initial state to the origin in minimum time. In order to accomplish this a linear transformation is applied to (4.15). The system eigenvalues are -1.5, -2.0, -4.448 and -100.0952. From the values of these eigenvalues the matrix of eigenvectors is

$$P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 1 & -.6127 & -.0152 \\ 0 & 0 & -2.9848 & -98.595 \\ 0 & 0 & -2.9803 & 103,530 \end{bmatrix} \tag{4.17}$$

and its inverse is

$$P^{-1} = \begin{bmatrix} 1 & 0 & .33472 & .309\times10^{-3} \\ -3 & 1 & -1.2092 & -.112\times10^{-2} \\ 0 & 0 & -.33471 & -.318\times10^{-3} \\ 0 & 0 & -.963\times10^{-5} & -.965\times10^{-5} \end{bmatrix} \, . \tag{4.18}$$

A normalization matrix is

$$T = \begin{bmatrix} +.02087 & 0 & 0 & 0 \\ 0 & -.05684 & 0 & 0 \\ 0 & 0 & -.007255 & 0 \\ 0 & 0 & 0 & -.00000964 \end{bmatrix} . \tag{4.19}$$

Applying the transformation $(x',z') = PT(X',\zeta')'$ to (4.15) yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\zeta}_3 \\ \dot{\zeta}_4 \end{bmatrix} = \begin{bmatrix} -1.5 & 0 & 0 & 0 \\ 0 & -2.0 & 0 & 0 \\ 0 & 0 & -4.448 & 0 \\ 0 & 0 & 0 & -100.0952 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix} - \begin{bmatrix} -1.5 \\ -2 \\ -4.448 \\ -100.0952 \end{bmatrix} u \tag{4.20}$$

and the corresponding transformed initial state

$$\begin{bmatrix} x_{10} \\ x_{20} \\ \zeta_{30} \\ \zeta_{40} \end{bmatrix} = \begin{bmatrix} -.2294 \\ -2.2168 \\ .13455\times10^{-4} \\ .64924\times10^{-7} \end{bmatrix} . \tag{4.21}$$

After ten iterations of the intermediate point algorithm the control

$$u = \begin{cases} -1 & t \in [0,\ 1.0058) \\ +1 & t \in [1.005,\ 1.5829) \\ -1 & t \in [1{,}5829,\ 1.7281) \\ +1 & t \in [1.7281,\ 1.7350) \end{cases} \tag{4.22}$$

was found which steers the state to a final point

$$\begin{bmatrix} x_{1F} \\ x_{2F} \\ \zeta_{3F} \\ \zeta_{4F} \end{bmatrix} = \begin{bmatrix} -.7\times10^{-8} \\ -.9\times10^{-8} \\ -.5\times10^{-16} \\ -.2\times10^{-14} \end{bmatrix} . \tag{4.23}$$

This is equivalent to the final state

$$\begin{bmatrix} X_{1F} \\ X_{2F} \\ Z_{1F} \\ Z_{2F} \end{bmatrix} = \begin{bmatrix} 1.46\times10^{-10} \\ -5.1156\times10^{-10} \\ 1.082\times10^{-18} \\ -1.99\times10^{-15} \end{bmatrix} . \tag{4.24}$$

This example demonstrates the use of a linear transformation in conjunction with the intermediate point algorithm to find the time-optimal control for a system not in the form of system (4.1).

5. TIME-OPTIMAL CONTROL OF A CLASS OF NONLINEAR SYSTEMS

5.1 Introduction and Problem Statement

In this chapter the problem treated is that of finding the control $u^*(t)$ which time-optimally steers the state $(x',z')'$ of the nonlinear system

$$\dot{x} = f(x,t) + F(x,t)z + B_1(x,t)u \tag{5.1a}$$

$$\mu\dot{z} = g(x,t) + G(x,t)z + B_2(x,t)u \tag{5.1b}$$

from an initial fixed point $(x_0',z_0')'$ at time t_0 to a final fixed point $(x_F',z_F')'$. Here $x \in R^n$, $z \in R^m$, $\mu > 0$ and $u \in R^r$ is constrained

$$u \in U = \{u: |u_i| \leq 1, \ i = 1,\ldots,r\} . \tag{5.2}$$

The vectors $f(x,t)$ and $g(x,t)$ and the matrices $F(x,t)$, $G(x,t)$, $B_1(x,t)$ and $B_2(x,t)$ have bounded derivatives with respect to t and x. For piecewise constant controls $\hat{u}(t) \in U$ with a finite number of discontinuities and the corresponding trajectories $\hat{x}(t)$ and $\hat{z}(t)$, the inverse $G(\hat{x}(t),t)^{-1}$ exists. Also the homogeneous system

$$\mu\dot{w} = G(\hat{x}(t),t)w \tag{5.3}$$

has $w = 0$ as a uniformly asymptotically stable equilibrium as discussed in Section 2.5.

The existence and normality of the time-optimal control of system (5.1) is assumed.

The chapter is organized as follows. In Section 5.2 the necessary conditions for the time-optimal control $u^*(t)$ of (5.1) are stated. From these conditions $u^*(t)$ *is shown to consist of two parts*. The first part is made of switchings in a "slow" time-scale while the second part is made of switchings in a "fast" time-scale. This is referred to as a two time-scale property. The reduced order system is defined in Section 5.3. From the time-optimal control of the reduced order system a near time-optimal control is proposed and the sense in which it is near optimal demonstrated. An example is presented in Section 5.4.

5.2 Necessary Conditions

In this section the necessary conditions of the minimum principle [36] are applied to the time-optimal control of (5.1). The Hamiltonian is

$$H = 1 + p'f + p'Fz + p'B_1u + \frac{1}{\mu} q'g + \frac{1}{\mu} q'Gz + \frac{1}{\mu} q'B_2u \tag{5.4}$$

where the costates p and q, which correspond to x and z respectively, satisfy

$$\dot{p} = - f_x'p - (Fz)_x'p - (B_1u)_x'p - \frac{1}{\mu} g_x' q - \frac{1}{\mu} (Gz)_x'q - \frac{1}{\mu} (B_2u)_x'q \, , \tag{5.5a}$$

$$\dot{q} = - F'p - \frac{1}{\mu} G'q \, . \tag{5.5b}$$

Here the Jacobian of a vector is indicated by the subscript. The control which minimizes the Hamiltonian is

$$u^*(t) = - \mathrm{SGN}[B_1'p + \frac{1}{\mu} B_2'q] \, . \tag{5.6}$$

The normality of the time-optimal control implies that

$$B_1'p + \frac{1}{\mu} B_2'q = S(x,z,t,\mu) = 0$$

only at isolated times t_j and not on a finite interval. Thus the switchings of the optimal control will be determined by the instants t_j at which one of the components s_i, $i = 1,\dots,r$ of S satisfies

$$s_i(x,z,t_j,\mu) = 0 \ .$$

The instants t_j are referred to as the zeroes of S.

For the sake of simplicity let

$$\begin{aligned}
A_{11}(x,z,t) &= f_x(x,t)' + (F(x,t)z)_x' + (B_1(x,t)u)_x', \\
A_{12}(x,z,t) &= g_x(x,t)' + (G(x,t)z)_x' + (B_2(x,t)u)_x', \\
A_{21}(x,t) &= F(x,t)', \\
A_{22}(x,t) &= G(x,t)' \ .
\end{aligned} \tag{5.7}$$

Substituting (5.7) into (5.5) yields

$$\begin{aligned}
\dot{p} &= - A_{11}p - \frac{1}{\mu} A_{12}q \\
\dot{q} &= - A_{21}p - \frac{1}{\mu} A_{22}q \ .
\end{aligned} \tag{5.8}$$

We wish to find approximate solutions for p and q to substitute into (5.6) to reveal the two time-scale property. Therefore we apply the transformation

$$\begin{aligned}
\xi &= p \\
\eta &= \frac{1}{\mu} q + A_{22}^{-1}A_{21}p
\end{aligned} \tag{5.9}$$

to (5.8) to yield

$$\begin{aligned}
\dot{\xi} &= - A_0\xi - A_{12}\eta \\
\dot{\eta} &= M\xi - \frac{1}{\mu} A_2\eta
\end{aligned} \tag{5.10}$$

where

$$A_0(x,z,t) = A_{11} - A_{12}A_{22}^{-1}A_{21} \tag{5.11a}$$

$$M(x,z,t) = \dot{A}_{22}^{-1}A_{21} + A_{22}^{-1}A_{21} - A_{22}^{-1}A_{21}A_{11} + A_{22}^{-1}A_{21}A_{12}A_{22}^{-1}A_{21} \tag{5.11b}$$

and

$$A_2(x,z,t,\mu) = A_{22} + \mu A_{22}^{-1}A_{21}A_{12} \ . \tag{5.11c}$$

Consider any candidate $\hat{u}(t)$ for the optimal control and the corresponding trajectory $\hat{x}(t),\hat{z}(t)$. Substituting $\hat{u}(t)$, $\hat{x}(t)$ and $\hat{z}(t)$ into A_0, A_{12}, M and A_2 results in the time-varying system

$$\begin{aligned} \dot{\xi} &= - \hat{A}_0(t)\xi - \hat{A}_{12}(t)\eta \\ \dot{\eta} &= \hat{M}(t)\xi - \frac{1}{\mu}\hat{A}_2(t)\eta \ , \end{aligned} \tag{5.12}$$

and the solution to (5.12) is

$$\xi(t) = \Phi_0'(t_f,t)\xi_F - \int_{t_f}^{t} \Phi_0'(\tau,t)\hat{A}_{12}(\tau)\eta(\tau)d\tau \tag{5.13a}$$

$$\eta(t) = \Phi_2'(t_f,t)\eta_F + \int_{t_f}^{t} \Phi_2'(\tau,t)\hat{M}(\tau)\xi(\tau)d\tau \tag{5.13b}$$

where ξ_F and η_F are the values of ξ and η at the final time t_f and the state transition matrices Φ_0 and Φ_2 satisfy

$$\frac{\partial}{\partial t}\Phi_0'(t_f,t) = - \hat{A}_0(t)\Phi_0'(t_f,t) \tag{5.14a}$$

and

$$\frac{\partial}{\partial t}\Phi_2'(t_f,t) = - \hat{A}_2(t)\Phi_2'(t_f,t) \ . \tag{5.14b}$$

Since $\hat{M}(\tau)$ and $\xi(\tau)$ are bounded for $t \in [t_0, t_f]$, Lemma 2.6 implies that

$$\eta(t) = \Phi_2'(t_f, t)\eta_f + O(\mu) . \tag{5.15}$$

Substituting (5.15) into (5.13a) yields

$$\xi(t) = \Phi_0'(t_f, t)\xi_F - \int_{t_f}^{t} \Phi_0 \, (\tau, t)\hat{A}_{12}(\tau)\Phi_2'(t_f, \tau)\eta_f d\tau + O(\mu) \tag{5.16}$$

from which, by Lemma 2.7, we find

$$\xi(t) = \Phi_0(t_f, t)'\xi_F + O(\mu) . \tag{5.17}$$

Rather than derive approximate solutions for p and q from (5.9), (5.17) and (5.15) it is simpler to apply transformation (5.9) to (5.6) and substitute (5.17) and (5.15) into the result to yield

$$u^*(t) = - \mathrm{SGN}[B_0^{*\prime}\Phi_0'(t_f^*, t)\xi_F + B_2^{*\prime}\Phi_2'(t_f^*, t)\eta_F + O(\mu)] \tag{5.18}$$

where $B_0' = B_1' - B_2'A_{22}'^{-1}A_{21}'$, the * denotes that B_0' and B_2' are evaluated along the optimal trajectory and t_f^* is the optimal final time. Analysis of (5.18) yields the two time-scale property.

<u>Lemma 5.1</u>. The time-optimal control $u^*(t)$ is composed of an initial interval of switchings in a slow time-scale followed by an interval of switchings in a fast time-scale. Furthermore $u^*(t)$ satisfies

$$u^*(t) = \begin{cases} -\mathrm{SGN}[B_0^{*\prime}\, \Phi_0'(t_f^*, t)\xi_F + O(\mu) + O(\epsilon)] & t \in [t_0, t_f^* - \tau) \\ -\mathrm{SGN}[B_0^{*\prime}\, \xi_F + B_2^{*\prime}\, \Phi_\mu(t_f^*, t)\eta_F + O(\mu)] & t \in [t_f^* - \tau, t_f^*) \end{cases} \tag{5.19}$$

where $\Phi_\mu(t_0, t)$ is the state transition matrix of a system which is uniformly asymptotically stable and ϵ may be chosen arbitrarily small.

Proof of Lemma 5.1. The proof proceeds as follows. First it is shown that

$$\Phi_2'(t_f^*,t) = \Phi_\mu'(t_f^*,t) + O(\mu) \ . \tag{5.20}$$

Then based on (5.20) the two time-scale property is revealed.

Consider the homogeneous system

$$\mu\dot{h} = -(A_{22} + \mu A_{22}^{-1}A_{21}A_{12})h \tag{5.21}$$

with final state η_F at time t_f. The solution to (5.21) is

$$h(t) = \Phi_2'(t_f,t)\eta_F \ . \tag{5.22}$$

Since (5.3) is uniformly asymptotically stable the homogeneous system

$$\mu\dot{y} = A_{22}(t)y = -G(\hat{x}(t),t)y \tag{5.23}$$

with state transition matrix $\Phi_\mu'(t_f,t)$, is uniformly asymptotically stable in reverse time. In terms of Φ_μ, (5.22) may be approximated

$$h(t) = \Phi_\mu'(t_f,t)\eta_F + \mu W(t_f,t)\eta_F \ . \tag{5.24}$$

Substituting (5.22) and (5.24) into (5.18) yields

$$u^*(t) = -\,\mathrm{SGN}[B_0^{*\prime}\Phi_0'(t_f^*,t)\xi_F + B_2^{*\prime}\Phi_\mu(t_f^*,t)'\eta_F + O(\mu)] \ . \tag{5.25}$$

For any ε, there exists a μ_0 and a τ such that if $\mu \in (0,\mu_0]$

$$|\Phi_\mu'(t_f,t)| < \varepsilon \quad \text{for} \quad t \in [t_0,t_f-\tau] \ . \tag{5.26}$$

Furthermore $\tau = O(\mu)$ which implies

$$\Phi_0'(t_f,t) = I + O(\mu) \qquad t \in [t_f-\tau,t_f] \ . \tag{5.27}$$

Equations (5.25), (5.26) and (5.27) thus imply (5.19). The initial or slow control interval is $[t_0, t_f^* - \tau)$ and the final $O(\mu)$ interval, called the fast control interval is $[t_f^* - \tau, t_f^*]$. Since τ is $O(\mu)$, (5.19) implies that the switchings in the slow control interval are primarily dependent on $B_0^{*\prime}\Phi_0{}'(t_f,t)\xi_F$ for μ small enough. Similarly in the fast control interval the switchings are primarily dependent on

$$B_0^{*\prime}\xi_F + B_2^{*\prime}\Phi_\mu(t_f^*,t)'\eta_F \; . \tag{5.28}$$

Lemma 5.1 is proved.

One additional fact can be discerned from equation (5.19) for the control. After the completion of the slow control interval the slow state x will be within $O(\mu)$ of x_F since τ is $O(\mu)$. This agrees with our intuition since in a system in which some states can be steered much more rapidly than others, it makes sense that the time-optimal control should first concentrate on steering the slow states near to their final state and then steer the fast states rapidly to their final states while also steering the slow states the last small distance. Based on this understanding of (5.19) the near-optimal control will be proposed in the next section.

5.3 Slow and Fast Control

The reduced order system is defined and the near-optimal control is proposed in this section. Setting μ to zero in (5.16) yields the reduced order system

$$
\begin{aligned}
\dot{\bar{x}} &= f_0(\bar{x},t) + B_0(\bar{x},t)\bar{u} \\
&= f(\bar{x},t) - F(\bar{x},t)G(\bar{x},t)^{-1}g(\bar{x},t) \\
&\quad + [B_1(\bar{x},t) - F(\bar{x},t)G(\bar{x},t)^{-1}B_2(\bar{x},t)]\bar{u} \ .
\end{aligned} \tag{5.29}
$$

The reduced order time-optimal problem for this system is that of finding the control $\bar{u}^*(t)$ which steers the state $\bar{x}(t)$ of (5.29) from the initial point x_0 at time t_0 to the final point x_F in minimum time $\bar{T}^*$.

The minimum principle provides the following necessary conditions for the reduced problem:

$$H = 1 + \bar{p}'f_0(\bar{x},t) + \bar{p}'B_0(\bar{x},t)\bar{u} \ , \tag{5.30}$$

$$\dot{\bar{p}} = - f_0(\bar{x},t)_{\bar{x}}'\bar{p} - (B_0(\bar{x},t)\bar{u})_{\bar{x}}'\bar{p} \ , \tag{5.31}$$

$$\bar{u}^*(t) = - \mathrm{SGN}[B_0'(\bar{x},t)\bar{p}(t)] \ . \tag{5.32}$$

Along an optimal trajectory $\bar{x}^*(t)$ system (5.31) is written

$$\dot{\bar{p}} = - \bar{A}_0(t)\bar{p} \tag{5.33}$$

which has the solution

$$\bar{p} = \bar{\Phi}_0'(\bar{t}_f^*,t)\bar{p}_F \ . \tag{5.34}$$

Substituting (5.34) into (5.32) yields

$$\bar{u}^*(t) = - \mathrm{SGN}[B_0'(\bar{x}^*,t)\bar{\Phi}_0'(\bar{t}_f^*,t)\bar{p}_F] \tag{5.35}$$

where $\bar{t}_f^*$ is the optimal final time.

We consider the slow control problem of finding the control $u_s^*(t)$ which steers x of (5.1) from x_0 at time t_0 to x_F in minimum time T_s^* and show that $\bar{u}^*(t)$ is a near-optimal control for this problem.

Lemma 5.2. Let $\hat{x}(t), \hat{z}(t)$ be the trajectory corresponding to the application of $\overline{u}^*(t)$ to system (5.1) and $\overline{x}^*(t)$ be the optimal trajectory for the reduced order system. Then

$$\hat{x}(t) = \overline{x}^*(t) + O(\mu) \ . \tag{5.36}$$

Proof of Lemma 5.2. Let $e(t) = x(t) - \overline{x}(t)$. The trajectory $\hat{x}(t), \hat{z}(t)$ satisfies

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x},t) + F(\hat{x},t)z + B_1(\hat{x},t)\overline{u}^* \\ \mu\dot{\hat{z}} &= g(\hat{x},t) + G(\hat{x},t)z + B_2(\hat{x},t)\overline{u}^* \ . \end{aligned} \tag{5.37}$$

Let t_i be the instants at which any of the components of $\overline{u}^*(t)$ switches. Then on any interval $[t_i, t_{i+1})$, equation (2.38) implies that

$$\begin{aligned} \hat{z}(t) = \Phi_\mu(t,t_i)[z_i &+ G(\hat{x}(t_i),t_i)^{-1} g(\hat{x}(t_i),t_i) \\ &+ B_2(\hat{x}(t_i),t_i)\overline{u}^*(t_i)] - G(\hat{x},t)^{-1} g(\hat{x},t) \\ &- G(\hat{x},t)^{-1} B_2(\hat{x},t)\overline{u}^*(t) + O(\mu) \ . \end{aligned} \tag{5.38}$$

Thus on $[t_i, t_{i+1})$

$$\begin{aligned} \dot{\hat{x}} = f_0(\hat{x},t) &+ B_0(\hat{x},t)\overline{u}^* + O(\mu) \\ &+ F(\hat{x},t)\Phi_\mu(t,t_i)W(t_i) \end{aligned} \tag{5.39}$$

where

$$\begin{aligned} W(t_i) = z_i &+ G(\hat{x}(t_i),t_i)^{-1} g(\hat{x}(t_i),t_i) \\ &+ B_2(\hat{x}(t_i),t_i)\overline{u}^*(t_i) \ . \end{aligned} \tag{5.40}$$

Let $x_a(t)$ be the point for all t satisfying the Taylor theorem for

$$f_0(\hat{x},t) + B_0(\hat{x},t)\overline{u}^* = f_0(\overline{x},t) + B_0(\overline{x},t)\overline{u}^* + (f_0(x_a(t),t) + B_0(x_a(t),t)\overline{u}^*)_x e \ , \tag{5.41}$$

then

$$\begin{aligned} \dot{e} &= (f_0(x_a(t),t) + B_0(x_a(t),t)\overline{u}^*)_x e + O(\mu) \\ &\quad + F(\hat{x},t)\Phi_\mu(t,t_i)W(t_i) \\ &\triangleq A(t)e + O(\mu) + \hat{F}(t)\Phi_\mu(t,t_i)W(t_i) \ . \end{aligned} \tag{5.42}$$

The solution to (5.42) is

$$e(t) = \Phi_e(t,t_i)e(t_i) + O(\mu) + \int_{t_i}^{t} \Phi_e(t,\tau)\hat{F}(\tau)\Phi_\mu(\tau,t_i)W(t_i)d\tau \tag{5.43}$$

and thus by Lemma 2.2

$$e(t) = \Phi_e(t,t_i)e(t_i) + O(\mu) \ . \tag{5.44}$$

Since $e(t_0) = 0$, $e(t_1) = O(\mu)$. If $e(t_i) = O(\mu)$, then $e(t_{i+1}) = O(\mu)$ by (5.44) and thus since there are a finite number of t_i

$$e(t) = O(\mu) \qquad t \in [t_0,\overline{t}_f^*]$$

and therefore

$$\hat{x}(t) = \overline{x}^*(t) + O(\mu)$$

and Lemma 5.2 is proved.

For the slow control of (5.1) the final value z_F is free and therefore $q_F = 0$. Thus transformation (5.9) implies that

$$\begin{aligned} \xi_F &= p_F \\ \eta_F &= A_{22}^{-1}(t_f^*)A_{21}(t_f^*)p \end{aligned} \tag{5.45}$$

where A_{22}^{-1} and A_{21} are evaluated along the optimal trajectory. Substituting (5.45) into (5.25) yields

$$\begin{aligned} u^*(t) = - \mathrm{SGN}[&B_0^{*\prime}\Phi_0{}'(t_f^*,t)p_F \\ &+ B_2^{*\prime}\Phi_\mu{}'(t_f^*,t)A_{22}^{-1}(t_f^*)A_{21}(t_f^*)p_F + O(\mu)] \end{aligned} \tag{5.46}$$

which implies that $u_s^*(t)$ also possesses the two time-scale property. On an initial interval $[t_0, t_0+T_s^*-\tau]$ the switchings are primarily dependent on the term $B_0^{*\prime}\Phi_0{}'(t_f^*,t)p_F$. As μ goes to zero the optimal trajectory $x_s^*(t)$ corresponding to $u_s^*(t)$ converges to $\bar{x}^*(t)$ and $u_s^*(t)$ converges to $\bar{u}^*(t)$. Therefore $B_0^{*\prime}\Phi_0{}'(t_f^*,t)p_F$ converges to $B_0{}'(\bar{x}^*,t)\bar{\Phi}_0{}'(\bar{t}_f,t)p_F$ of equation (5.35). The fast switchings of $u_s^*(t)$ dependent on $B_2^{*\prime}\Phi_\mu{}'(t_f^*,t)$ takes place on the $O(\mu)$ interval $[T_s^*+t_0-\tau, T_s^*+t_0]$.

<u>Lemma 5.3</u>. Let the zeroes of $B_0{}'(\bar{x}^*,t)\bar{\Phi}_0{}'(t_f^*,t)p_F$ of (5.32) be simple. Then there exists a μ^* such that for $\mu \in (0,\mu^*]$ the switching times of $u_s^*(t)$ on $[t_0, t_0+T_s^*-\tau]$ are $O(\mu^a)$ different from these of $\bar{u}^*(t)$. Furthermore the minimum time T_s^* for the slow control problem satisfies

$$T_s^* = \bar{T}^* + O(\mu^a)$$

where $\bar{T}^*$ is the minimum time for the reduced order problem and $a > 0$ is constant.

Proof of Lemma 5.3. The fact that the zeroes of $B_0'(\overline{x}^*,t)\overline{\Phi}_0'(\overline{t}_f^*,t)p_F$ are simple implies that the control sequence of

$$u_s^*(t) = - \mathrm{SGN}[B_0'(\overline{x}^*,t)\overline{\Phi}_0'(\overline{t}_f^*,t)p_F + O(\mu) + O(\epsilon)] \quad t \in [t_0,t_0+T_s^*-\tau] \tag{5.47}$$

is, for μ small enough, the same as that of $\overline{u}^*(t)$. Here the choice of ϵ is dependent on the reverse time exponential decay of $\Phi_\mu'(t_f^*,t)$ and the difference

$$B_0^{*\prime}\Phi_0'(t_f^*,t)p_F - B_0'(\overline{x}^*,t)\overline{\Phi}_0'(t_f^*,t)'p_F$$

and clearly satisfies

$$\lim_{\mu \to 0} \epsilon = 0 .$$

The $O(\epsilon)$ and $O(\mu)$ terms in (5.47) shift the switching times by some variations dependent on μ. Since these variations go to zero as $\mu \to 0$, they are $O(\mu^a)$ dependent where $a > 0$.

Finally since a finite number of switchings are varying by $O(\mu^a)$ and $\tau = O(\mu)$

$$T_s^* = \overline{T}^* + O(\mu^a)$$

and Lemma 5.3 is proved.

The point here is not the particular value of a but rather that for $\mu \in (0,\mu^*]$ it is possible, by varying the switchings of $\overline{u}^*(t)$ and adding some fast switchings to find the optimal control. This might be done either by an iterative method such as the one presented here or by switching sensitivities [22].

From Lemmas 5.2 and 5.3 it is clear that $\bar{u}^*(t)$ is a near-optimal control for the slow control problem, in the sense that it steers x(t) to within $O(\mu)$ of x_F in near minimum time $\bar{T}^*$.

We now begin the task of finding a near optimal control $\tilde{u}^*(t)$ for the time-optimal control of (x',z') to $(x_F',z_F')'$. Suppose that $\bar{u}^*(t)$ is applied to (5.1); then $z(\bar{t}_f^*)$ will be within some bounded region surrounding z_F. Since $z(\bar{t}_f^*)$ is some finite distance from z_F and z is controlled in a fast $(\frac{1}{\mu})$ time scale it is possible to steer z from $z(\bar{t}_f^*)$ to z_F by some fast switchings of $O(\mu)$ duration. To this end we define the fast system

$$\mu\dot{\tilde{z}} = g(x_F,t) + G(x_F,t)\tilde{z} + B_2(x_F,t)\tilde{u} \tag{5.48}$$

where x_F is the fixed final point for x. System (5.48) is linear time-varying and we consider the problem of finding the control $u_f^*(t)$ which time-optimally steers z from $z(\bar{t}_f^*)$ to z_F. Due to the presence of μ in (5.48) the time T_f^* required for $u_f^*(t)$ is $O(\mu)$. Suppose that $\bar{u}^*(t)$ is applied to (5.1) to drive x to $O(\mu)$ from x_F. Applying $u_f^*(t)$ for an additional $O(\mu)$ time will leave x $O(\mu)$ from x_F.

<u>Lemma 5.4</u>. The control

$$\tilde{u}^*(t) = \begin{cases} \bar{u}^*(t) & t \in [t_0,\bar{t}_f^*) \\ \tilde{u}_f^*(t) & t \in [\bar{t}_f^*,\bar{t}_f^*+\tilde{T}_f^*) \end{cases} \tag{5.49}$$

is near optimal in the sense that it steers $(x',z')'$ from $(x_0',z_0')'$ to $O(\mu)$ from $(x_F',z_F')'$ in near minimum time $\bar{T}^* + T_f^* = T^* + O(\mu^a)$.

<u>Proof of Lemma 5.4</u>. It has already been shown that $u^*(t)$ steers x to $O(\mu)$ from x_F. The next step is to show that it steers z to $O(\mu)$ from z_F.

On the interval $[\bar{t}_f^*, \bar{t}_f^* + \tilde{T}_f^*]$

$$x(t) = x_F + O(\mu) \; . \tag{5.50}$$

Let $e(t) = z(t) - \tilde{z}(t)$. Then

$$\begin{aligned} \dot{e} &= \dot{z} - \dot{\tilde{z}} \\ &= g(x,t) - g(x_F,t) + [G(x,t) - G(x_F,t)]z \\ &\quad + [B_2(x,t) - B_2(x_F,t)]\tilde{u} \; . \end{aligned} \tag{5.51}$$

By an analogous argument to that in the proof of Lemma 5.3 and (5.50) it can be shown that e(t) is $O(\mu)$. Since $\tilde{u}_f^*(t)$ steers z to z_F, $\tilde{u}^*(t)$ steers z to $O(\mu)$ from z_F.

Finally $T_f^* = O(\mu)$ and therefore $\bar{T}_f + T_f^* = T_s + O(\mu^a)$. The time T_s^* for the slow control problem satisfies

$$T_s^* \leq T^*$$

and thus

$$\bar{T}^* + \tilde{T}_f^* = T^* + O(\mu^a) \; .$$

This completes the proof of Lemma 5.4.

5.4 <u>Example</u>

This example illustrates the result of applying the near optimal control $\tilde{u}^*(t)$ to the control of the system

$$
\begin{aligned}
\dot{x} &= -\sin x + \frac{1}{2} z_1 + \frac{1}{2} z_2 \\
\mu\dot{z}_1 &= -z_1 + u \\
\mu\dot{z}_2 &= -2z_2 + 2u
\end{aligned}
\tag{5.52}
$$

where $\mu = 0.1$. Let the initial and final states be $(\pi/2, 0.7, -0.8)$ and $(0,0,0)$ respectively. Setting $\mu = 0$ yields the reduced order system

$$\dot{\bar{x}} = -\sin \bar{x} + \bar{u} \; . \tag{5.53}$$

The reduced time-optimal control

$$\bar{u}^*(t) = -1 \qquad t \in [0,1) \tag{5.54}$$

steers $\bar{x}$ from $\pi/2$ to 0. When this control is applied to (5.52), (x,z_1,z_2) is steered to $(.0679, -1, -1)$. This is acceptable if we are not interested in steering z. For instance if z represents actuator dynamics the final position of the actuator may not be important. But suppose we are interested in steering z to zero. Then we find the control

$$\tilde{u}_f^*(t) = \begin{cases} +1 & t \in [1, 1.1098) \\ -1 & t \in [1.1098, 1.139) \end{cases}$$

which steers (z_1,z_2) from $(-1, -1)$ to $(0, 0)$. When $\bar{u}^*(t)$ followed by $\tilde{u}_f^*(t)$ is applied to (5.52) the state (x,z_1,z_2) is steered to (0.658, 0,00, 0,00). Thus x is steered to within 5% of its final state, and z is steered to 0. The near-optimal control is calculated entirely on reduced order systems.

6. NONLINEAR EXAMPLES

6.1 Introduction

In this chapter two nonlinear examples are presented. One of these examples does not fit into the form of system (5.1) and therefore the theory developed in Chapter 5 is not directly applicable. However, due to the uniform asymptotic stability of the homogeneous part of the fast subsystems, both of these examples have the two time-scale property. The purpose of this chapter is to demonstrate ways to treat particular problems based on the two time-scale property.

In each example the controls are calculated for reduced order time-optimal problems. Then iterative schemes are applied to calculate optimal controls. The first example is a magnetic suspension system with both state and control constraints. As a result of the form of the time-optimal control for the reduced system, a singular arc in the time-optimal control for the full order system is proposed. The second example is more complicated and in order to find the time-optimal control for this system a nested iterative scheme is developed.

In the discussion of each example an effort is made to show the thinking used in the development of the results. The general approach is always to first find the time-optimal control for the reduced order system. This time-optimal control steers the slow state x of the full order system to a point which is $O(\mu)$ from its final desired point. The first iterative method is designed to perturb this control in such a way that the resultant control steers x to its final desired final point. Then the control which steers the state of the fast subsystem to its final desired state in minimum time is found. This control steers the state z of the full order system to a point $O(\mu)$ from its final desired point. Once again an iterative method is applied to change this control so that it steers z to its final state. Then the intermediate point algorithm is applied to find the time-optimal control which steers $(x',z')'$ to its final desired point.

6.2 Magnetic Suspension System

The system treated here is nonlinear in the state z. Thus this system is not of the form of (5.1). From the two time-scale property it is conjectured that the time-optimal trajectory has a singular arc. Also a state constraint becomes a control constraint in the reduced problem.

The problem is to find the control $u^*(t)$ which time-optimally steers the state of the system

$$\begin{aligned} \dot{x}_1 &= 10x_2 \\ \dot{x}_2 &= -20(z)^2/x_1 + 10 \\ L\dot{z} &= -Rz + u \end{aligned} \tag{6.1}$$

from an initial point (x_{10},x_{20},z_0) to a final point (x_{1F},x_{2F},z_F) subject to the constraints $u \leq Umax$ and $|z| \leq 1$. Also it is assumed that $0.1 \leq x_1 \leq 1$ and $|x_2| \leq 1$. System (6.1) is one possible model of a magnetic suspension system consisting of an electro-magnet which is suspending an iron ball beneath it. The states x_1 and x_2 are the position and velocity of the ball, z is the current through the magnet and u is the voltage input. The parameters L and R represent the inductance and resistance of the magnet.

In attempting to control such a system an engineering assumption that might be made is that the current z can be changed instantaneously from one value to another. This corresponds to setting L = 0 and thus L takes the place of μ in this system. The parameter L is set to zero. Then $(z)^2 = v$, constrained $0 \leq v \leq 1$, can be treated as control for the reduced order system

$$\begin{aligned} \dot{\bar{x}}_1 &= 10\bar{x}_2 \\ \dot{\bar{x}}_2 &= -20v/\bar{x}_1 + 10 . \end{aligned} \qquad (6.2)$$

The reduced problem is to find $v^*(t)$ which steers $(\bar{x}_1, \bar{x}_2)$ from the initial point (x_{10}, x_{20}) to the final point (x_{1F}, x_{2F}) in minimum time.

The slow control problem for (6.1) is to find the control $u_s^*(t)$ which time-optimally steers (x_1, x_2) from (x_{10}, x_{20}) to (x_{1F}, x_{2F}) with the final value of z free. This control is found in two steps. First, a near optimal control is found based on $v^*(t)$. Then this control is iteratively adjusted to $u_s^*(t)$. Finally the time-optimal $u^*(t)$ is found via the intermediate point algorithm. For this system, the value z_F is the amount of current which will hold the ball in the position x_{1F} with the velocity $x_{2F} = 0$. Thus x_{1F}, x_{2F}, z_F is a nominal point around which a regulator can be designed.

The Hamiltonian for the reduced control problem is

$$H = 1 + 10\bar{p}_1\bar{x}_2 - 20\bar{p}_2 v/\bar{x}_1 + 10\bar{p}_2 \qquad (6.3)$$

where $\bar{p}_1$ and $\bar{p}_2$ are the adjoint variables corresponding to the states $\bar{x}_1$ and $\bar{x}_2$ and satisfy

$$\begin{aligned} \dot{\bar{p}}_1 &= -20v\bar{p}_2/(\bar{x}_1)^2 \\ \dot{\bar{p}}_2 &= -10\bar{p}_1 . \end{aligned} \qquad (6.4)$$

By the minimum principle

$$v^*(t) = \begin{cases} 0 & \text{if } 20\bar{p}_2/\bar{x}_1 < 0 \\ 1 & \text{if } 20\bar{p}_2/\bar{x}_1 > 0 \end{cases} \tag{6.5}$$

and H = 0 along the optimal trajectory. It can be shown that the reduced problem is normal and therefore no singular arc arises. Thus v = 0 or v = 1. A phase plane study shows that the possible control sequences in the region of interest are {1,0}, {0,1}, {0} and {1}. The reduced system time-optimal trajectory from $(x_{10},x_{20}) = (0.3,0,0)$ to $(x_{1F},x_{2F}) = (0.6,0.0)$ is plotted in Figure 6.1 and the corresponding time-optimal control is

$$v^*(t) = \begin{cases} 0 & t \in [0, .0658) \\ 1 & t \in [.0658, .0920) \end{cases} .$$

The next step is to find the slow control $u_s^*(t)$. Suppose that the control sequence which drives $(\bar{x}_1,\bar{x}_2)$ from (x_{10},x_{20}) to (x_{1F},x_{2F}) is $\{v_1,v_2\}$. For L sufficiently small a near optimal control is the following:

1. Apply $u_1 = \pm$ Umax to time-optimally steer z to $z^2 = v_1$ at $t = t_1$.
2. Apply u_2 = R or 0 to hold z on $z^2 = v_1$ until (x_1,x_2) hits the switching curve for the reduced system at $t = t_2$.
3. Apply $u_3 = \pm$ Umax to time-optimally steer z to $z^2 = v_2$ at $t = t_3$.
4. Apply u_4 = R or 0 to hold on $z^2 = v_2$ until $x_2 = x_{2F}$ at $t = t_4$.

At $t = t_4$, $x_2 = x_{2F}$ and x_1 is O(L) from x_{1F} at the position x_{1R} (x_1 reached). For smaller L the distance $|x_{1R}-x_{1F}|$ is smaller. The near

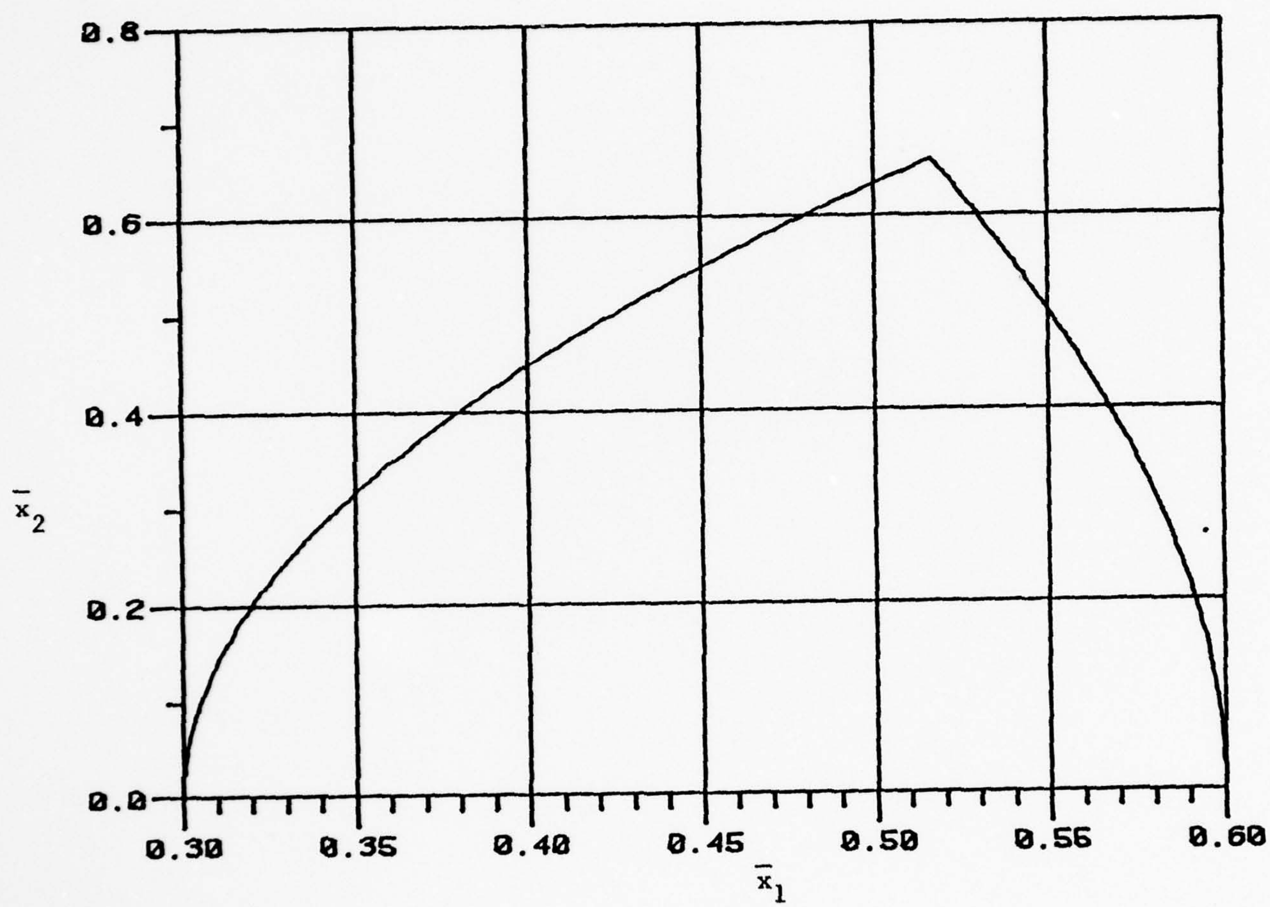

Figure 6.1. Phase plane time-optimal trajectory ($\bar{x}_1$-$\bar{x}_2$) for the reduced order system (6.2).

optimal phase plane trajectory for the points $(x_{10}, x_{20}) = (0.3, 0.0)$ and $(x_{1F}, x_{2F}) = (0.6, 0.0)$ is plotted in Figure 6.2 for $R = 1$ and $L = .05$. In Figure 6.3 the corresponding z trajectory is plotted versus time. Note that for these values of R and L, $x_{1R} = 0.68$.

The conjecture is that by varying t_i, $i = 1, \ldots, 4$, with the sequence $\{u_1, u_2, u_3, u_4\}$ the control $u_s^*(t)$ will be found. If this is true the time-optimal trajectory will contain a singular arc as either u_2 or $u_4 = 0$ on a finite interval. On this interval z is also zero. When u and $z_3 = 0$ the equation for $x_2(t)$ is

$$x_2(t) = 10t + K \tag{6.6}$$

where K is a constant. The minimum principle provides the necessary conditions

$$H = 1 + p_1 10x_2 - 20p_2(z)^2/x_1 + 10p_2 - Rqz/L + qu/L = 0 \tag{6.7}$$

where the adjoint variables satisfy

$$\begin{aligned} \dot{p}_1 &= -20p_2(z)^2/(x_1)^2 \\ \dot{p}_2 &= -10p_1 \\ \dot{q} &= 40zp_2/x_1 + Rq/L \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} u &< 0 \quad \text{if } q > 0 \\ u &> 0 \quad \text{if } q < 0 \,. \end{aligned} \tag{6.9}$$

Thus for u to be 0 on a finite interval, q must be 0 on that interval. This implies

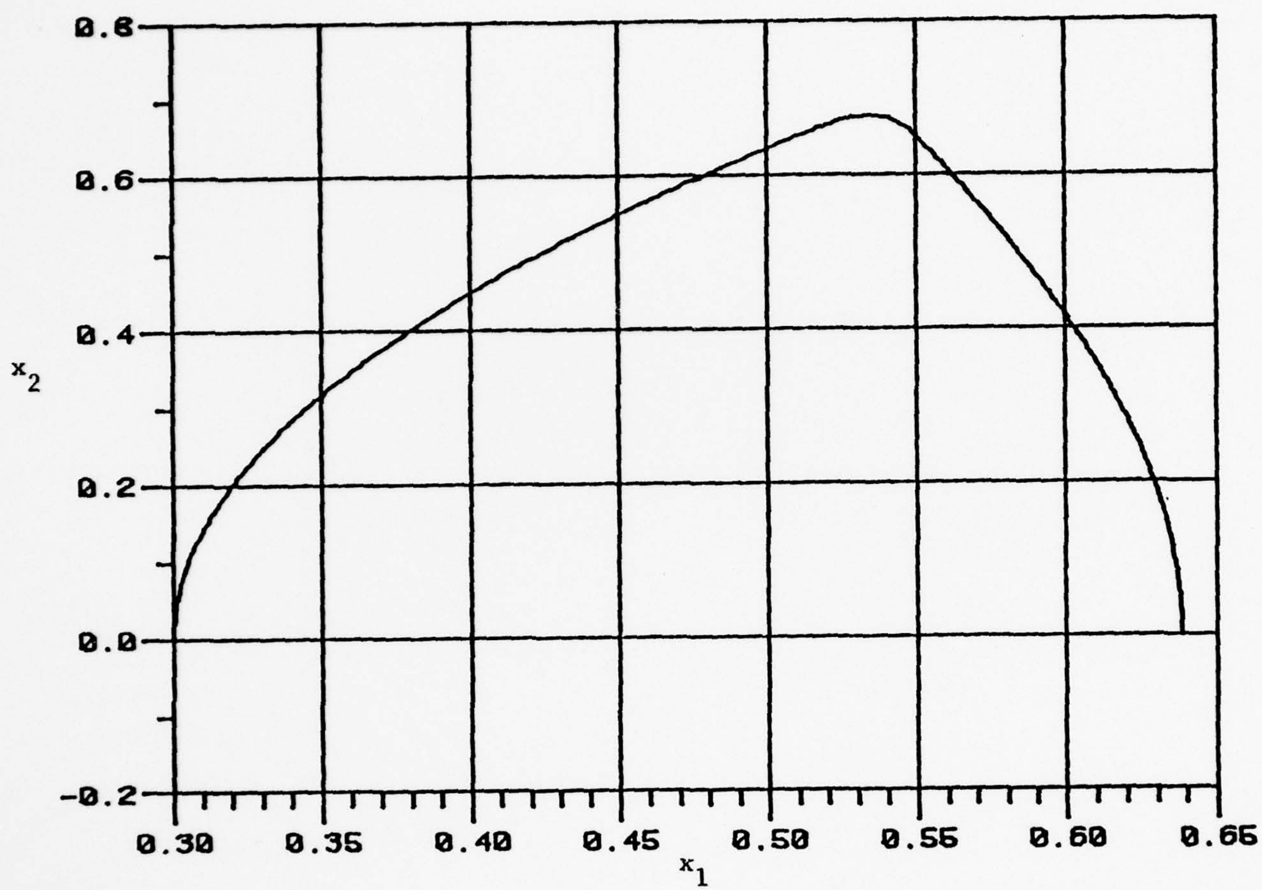

Figure 6.2. The x_1 versus x_2 trajectory for (6.1) corresponding to the slow near optimal control.

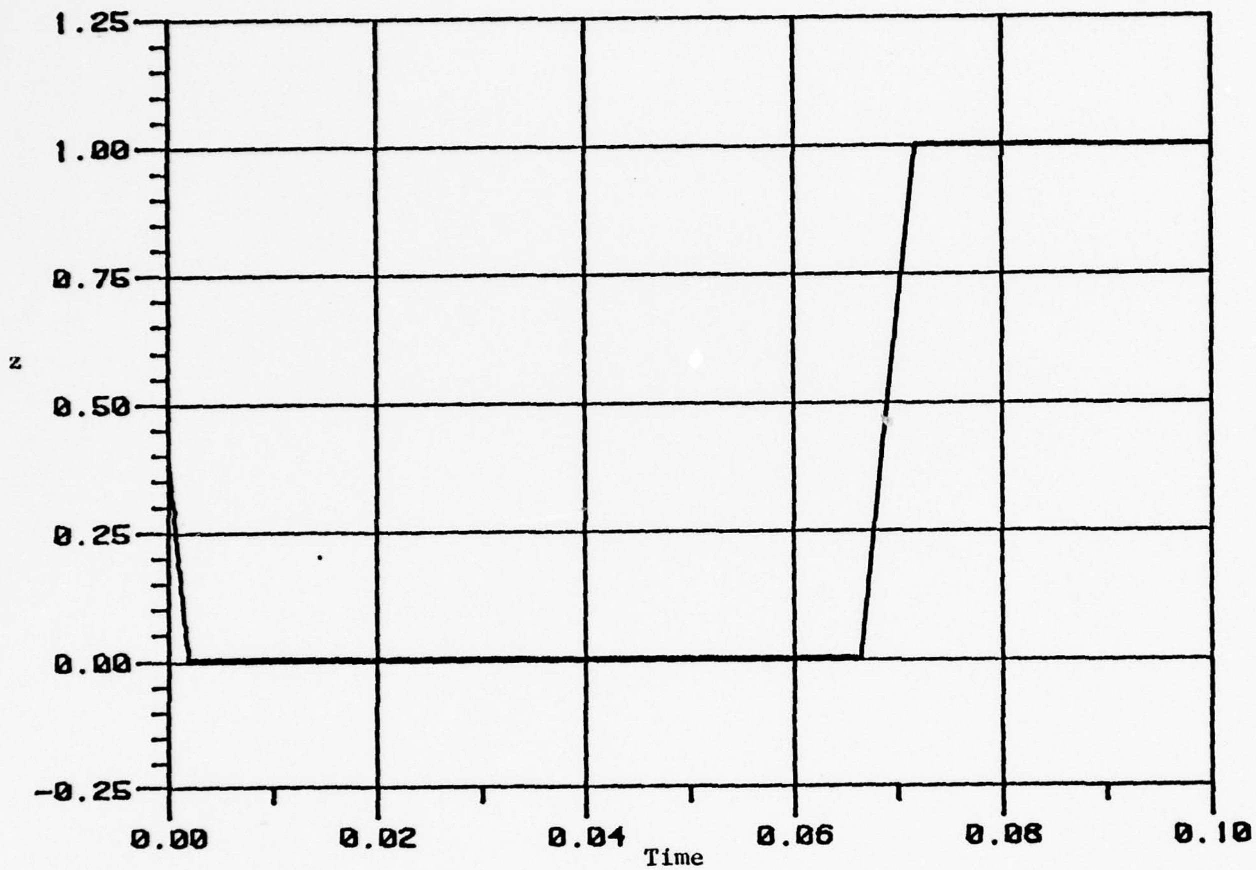

Figure 6.3. The z versus t trajectory for (6.1) corresponding to the slow near optimal control.

$$\dot{q} = 0$$

$$\Rightarrow z = 0 \tag{6.10}$$

$$\Rightarrow H = 1 + p_1 10x_2 + 10p_2 = 0$$

and

$$\Rightarrow \dot{p}_1 = 0$$

$$\dot{p}_2 = -10p_1$$

$$\dot{q} = 0$$

$$\Rightarrow p_1 = K_1,\ p_2 = -10K_1 t + K_2 \tag{6.11}$$

where K_1 and K_2 are constant. Equations (6.10) and (6.11) imply

$$\begin{aligned} x_2 &= \frac{-1-10p_2}{10p_1} = \frac{-1}{10K_1} + \frac{100K_1 t}{10K_1} - \frac{10K_2}{K_1} \\ &= 10t + \frac{-1-10K_2}{10K_1} \ . \end{aligned} \tag{6.12}$$

Thus if in (6.6)

$$K = \frac{-1-10K_2}{10K_1} \tag{6.13}$$

the singular arc satisfies the necessary conditions of the minimum principle.

The near optimal control steers (x_1,x_2) from (x_{10},x_{20}) to (x_{1R},x_{2F}). There must be a point (x_{1G},x_{2F}) such that a near optimal control based on $v^*(t)$ steering $(\bar{x}_1,\bar{x}_2)$ to this point would steer (x_1,x_2) to (x_{1F},x_{2F}). An iterative method for x_{1G} is as follows:

1. Initialize with $x_{1G}^{(0)} = x_{1F}$.

2. Find the near optimal control $\bar{u}_s^{(n)}(t)$ to $(x_{1G}^{(n)}, x_{2F})$.

   Let the position reached be $x_{1R}^{(n)}$.

3. $x_{1G}^{(n+1)} = x_{1G}^{(n)} - \alpha(x_{1R}^{(n)} - x_{1F})$.

4. If $|x_{1R}^{(n)} - x_{1F}| \leq 0.01$, stop.

5. Let $n \leftarrow n+1$ and go to step 2.

Experimentally $\alpha = 0.6$ provides convergence within fifteen iterations. In this way $u_s^*(t)$ is found. In Figure 6.4 the phase plane trajectory corresponding to $u_s^*(t)$ for $(x_{10}, x_{20}) = (0.3, 0.0)$ and $(x_{1F}, x_{2F}) = (0.6, 0.0)$ is plotted and in Figure 6.5, the corresponding z versus time trajectory.

Suppose that $u_s^*(t)$ may be found, as above, to steer (x_1, x_2) from (x_{10}, x_{20}) to any intermediate point (x_{1I}, x_{2I}). Then we may apply the following intermediate point algorithm to find $u^*(t)$ which steers (x_1, x_2, z) from (x_{10}, x_{20}, z_0) to (x_{1F}, x_{2F}, z_F).

1. Initialize with $x_{1I}^{(0)} = x_{1F}, x_{2I}^{(0)} = x_{2F}$, $n = 0$.

2. Find $u_s^{(n)}(t)$ to steer (x_1, x_2) to $(x_{1I}^{(n)}, x_{2I}^{(n)})$ in time $T_s^{(n)}$.

3. Find $z(T_s^{(n)})$.

4. Find $u_f^{(n)}$ to steer z_3 from $z_3(T_s^{(u)})$ to x_{3F} in time $T_f^{(n)}$.

5. Find $x_1(T_s^{(n)} + T_f^{(n)})$, $x_2(T_s^{(n)} + T_f^{(n)})$.

6. If $|x_1(T_s^{(n)} + T_f^{(n)}) - x_{1F}| + |x_2(T_s^{(n)} + T_f^{(n)}) - x_{2F}| \leq \epsilon$ where ϵ is a predetermined allowable error, stop.

7. Integrate (6.1) in reverse time with $u_f^{(n)}$ to find $x_{1I}^{(n+1)}, x_{2I}^{(n+1)}$. That is, $u_f^{(n)}$ steers (x_1, x_2) from $(x_{1I}^{(n+1)}, x_{2I}^{(n+1)})$ to (x_{1F}, x_{2F}).

8. Let $n \leftarrow n+1$ and go to step 2.

After the algorithm converges a control has been found which steers (x_1, x_2, z) from (x_{10}, x_{20}, z_0) to (x_{1F}, x_{2F}) in time $T_s^{(n)} + T_f^{(n)}$. For the

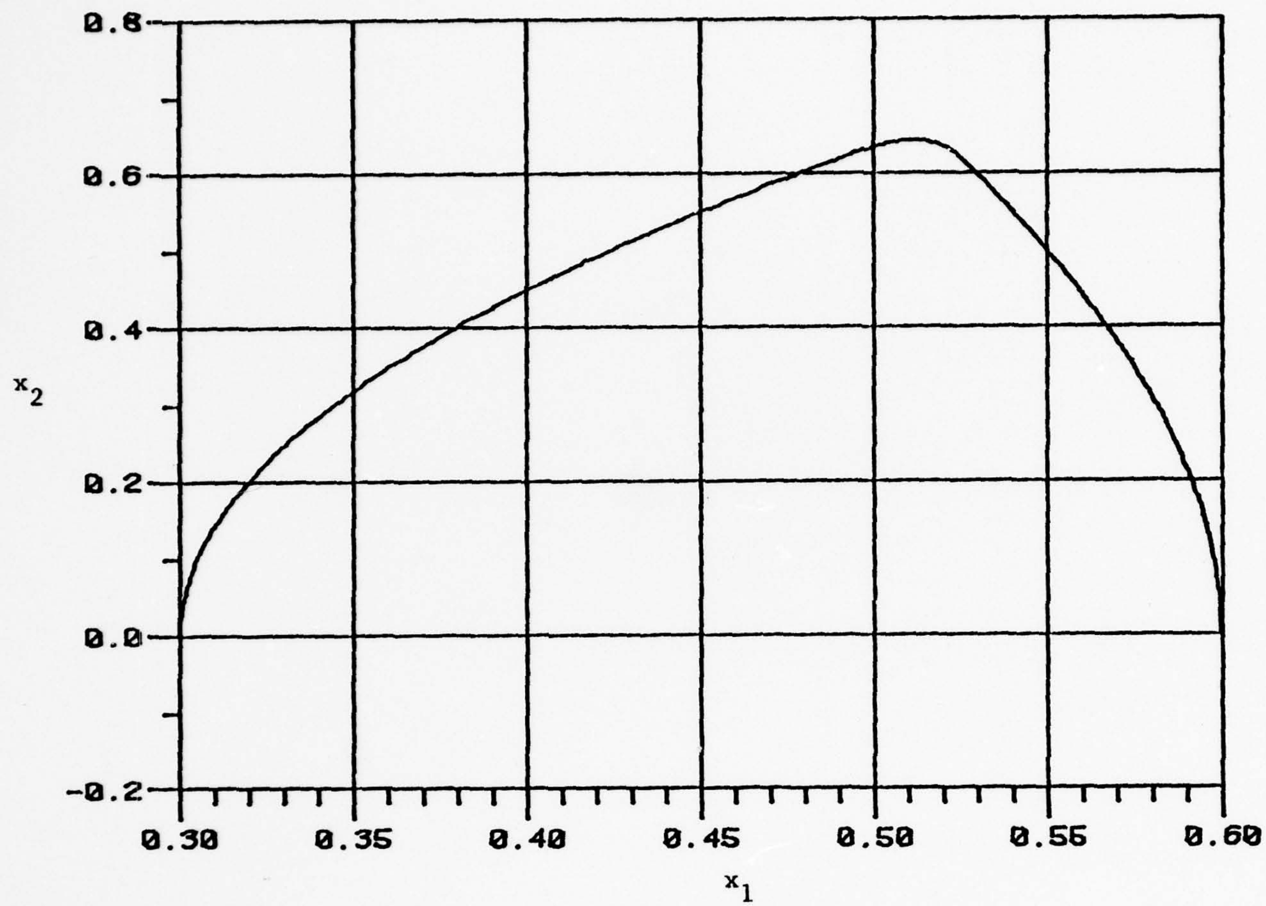

Figure 6.4. The x_1 versus x_2 trajectory of (6.1) corresponding to $u_s^*(t)$.

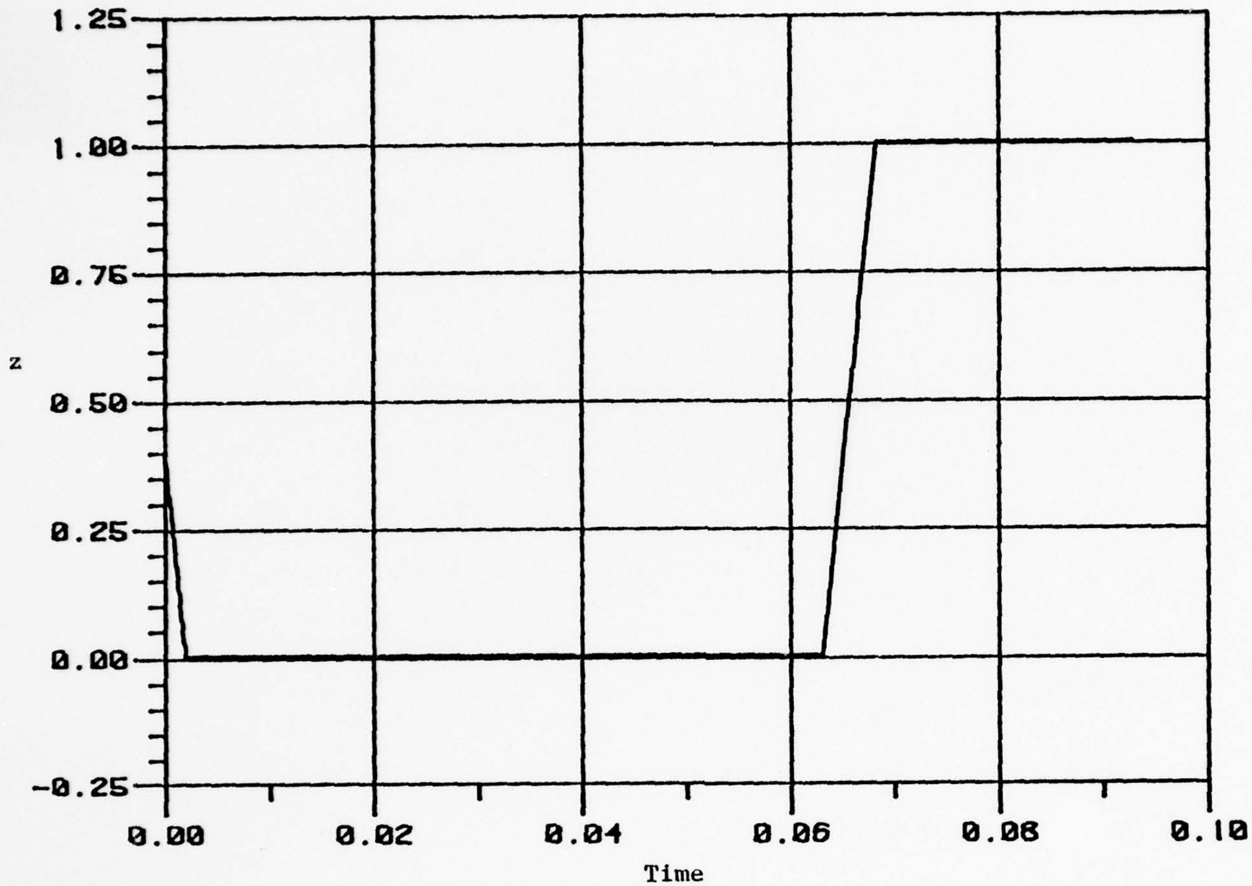

Figure 6.5 The z versus t trajectory of (6.1) corresponding to $u_s^*(t)$.

problem of steering from (0.3,0.0,.387) to (0.6,0.0,.548) with $L = .05$ and $R = 1$, the intermediate point algorithm converges to drive (x_1,x_2) to 10^{-2} from (x_{1F},x_{1F}). The times required for this are $T_s^{(n)} = .902$, $T_f^{(n)} = .002$. This is proposed to be time-optimal as $T_f^{(n)}$ is 2% of $T_s^{(n)}$ and $T_s^{(n)}$ is 7% larger than $\overline{T}^*$ the time required for $v^*(t)$. In Figure 6.6 the x_1-x_2 plus phase plane trajectory is plotted and in Figure 6.7 the trajectory z versus t is plotted. For systems such as (6.1) the smaller L the closer $T_s^{(n)} + T_f^{(n)}$ is into $\overline{T}^*$. Thus there exists L^0 such that for $L \in (0,L^0)$ the control provided by the intermediate point algorithm is optimal.

6.3 An Example of Nested Iteratives

The system treated in this example is

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{2}(z_2 - x_2)e^{(x_2)^2} \\
\mu\dot{z}_1 &= -z_1 + u \\
\mu\dot{z}_2 &= -2z_2 + 2x_2 + 2u
\end{aligned}
\tag{6.14}
$$

where the control us is constrained

$$|u| \leq 1 \tag{6.15}$$

and μ is a small parameter greater than zero. The problem under consideration is that of finding the control $u^*(t)$ which time-optimally steers the state (x_1,x_2,z_1,z_2) from a fixed initial point $(x_{10},x_{20},z_{10},z_{20})$ to the origin.

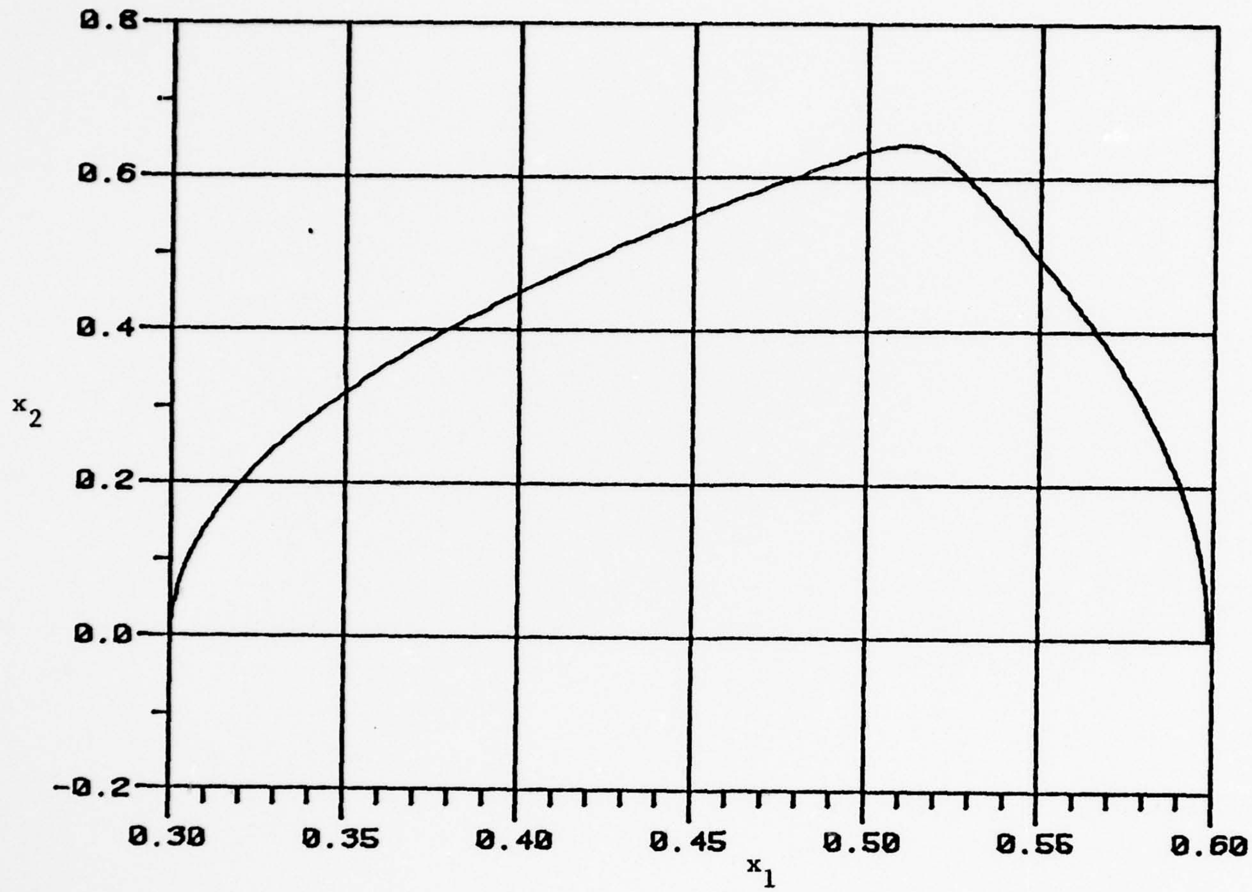

Figure 6.6. The x_1 versus x_2 trajectory corresponding to the control calculated by the intermediate point algorithm.

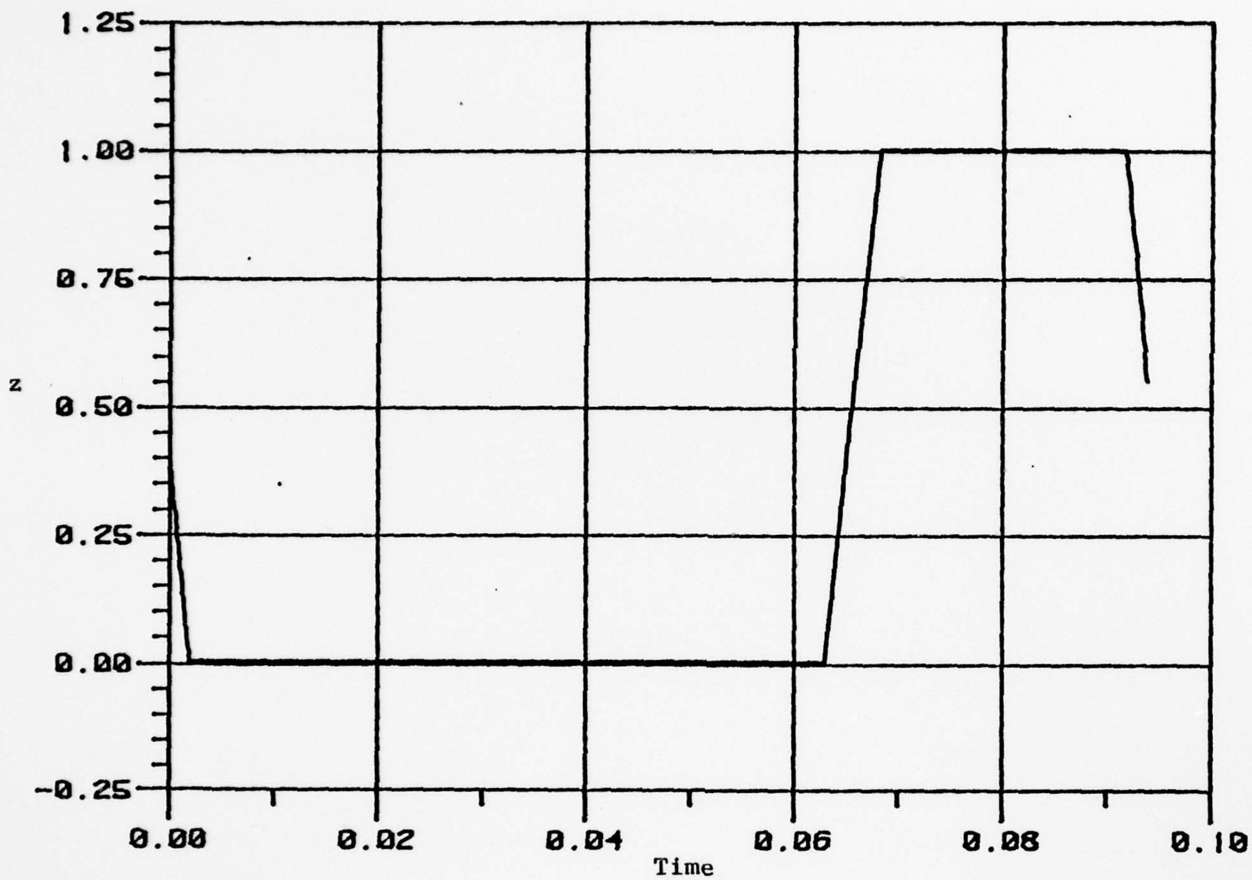

Figure 6.7. The z versus t trajectory corresponding to the control calculated by the intermediate point algorithm.

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Setting μ to zero yields the reduced order system

$$\dot{\bar{x}}_1 = \bar{x}_2$$
$$\dot{\bar{x}}_2 = \frac{1}{2}\bar{u}e^{(\bar{x}_2)^2} \quad . \tag{6.16}$$

The solution to the problem of finding the control $\bar{u}^*(t)$ which time-optimally steers $(\bar{x}_1,\bar{x}_2)$ from (x_{10},x_{20}) to the origin, in time $\bar{T}^*$, is in [37]. There it is shown that for (x_{10},z_{20}) in the region G,

$$(x_{10},x_{20}) \in G = \{(x_{10},x_{20}): |x_{10}| < 1 + e^{-(x_{20})^2}\} \tag{6.17}$$

the optimal control exists and for (x_{10},x_{20}) not in the region G an ϵ-optimal control [36] exists. For the problems considered for (6.14) and (6.16) it is assumed that $(x_{10},x_{20}) \in G$.

Since the final desired point is the origin, $x_{2F} = 0$, the fast subsystem is

$$\mu\dot{\tilde{z}}_1 = -\tilde{z}_1 + \tilde{u}$$
$$\mu\dot{\tilde{z}}_2 = -2\tilde{z}_2 + 2\tilde{u} \quad . \tag{6.18}$$

The near optimal control for system (6.14) is

$$\tilde{u}(t) = \begin{cases} \bar{u}^*(t) & t \in [0,\bar{T}^*) \\ \tilde{u}_f^*(t) & t \in [\bar{T}^*,\bar{T}^*+\tilde{T}_f^*) \end{cases} \tag{6.19}$$

where $\tilde{u}_f^*(t)$ steers (z_1,z_2) from $(z_1(\bar{T}^*),z_2(\bar{T}^*))$ to the origin in minimum time $\tilde{T}_f^*$. For the initial point $(x_{10},x_{20},z_{10},z_{20}) = (1,1,-1,-1)$ and $\mu = .4$, the near optimal control is

$$\tilde{u}(t) = \begin{cases} -1 & t \in [0,315) \\ 1 & t \in [3.15,4.80) \\ -1 & t \in [4.80,5.24) \\ 1 & t \in [5.24,5.30) \end{cases} \tag{6.20}$$

which steers (x_1,x_2,z_1,z_2) of (6.14) to (-.29,-.25,0,0,-.2). The phase plane trajectory of x_1 versus x_2 is plotted in Figure 6.8. The corresponding trajectories z_1 versus t and z_2 versus t are plotted in Figures 6.9 and 6.10. For $\mu = .1$ and the same initial point the near optimal control

$$\tilde{u}(t) = \begin{cases} -1 & t \in [0,3.15) \\ 1 & t \in [3.15,4.79) \\ -1 & t \in [4.79,4.90) \\ 1 & t \in [4.90,4.93) \end{cases} \tag{6.21}$$

steers (x_1,x_2,z_1,z_2) to (-.06,-.07,0.0,-.056) and thus as expected the near optimal control steers the state to a point closer to the origin.

In order to improve on the near optimal control we make the conjecture that the optimal control may be found by perturbing the switching times of u(t). This is done in two steps. Let $u_s^*(t)$ be the control which time-optimally steers (x_1,x_2) of (6.14) from (x_{10},x_{20}) to the origin in time T_s^* and let $u_f^*(t)$ be the control which time-optimally steers (z_1,z_2) from $(z_1(T_s^*),z_2(T_s^*))$ to the origin in time T^*.

The controls $u_s^*(t)$ and $u_f^*(t)$ are found by an iterative method which is essentially the same as the one described in the magnetic suspension example. Here this method will be described in terms of finding the control $u_s^*(t)$. The fast control $u_f^*(t)$ is found analogously.

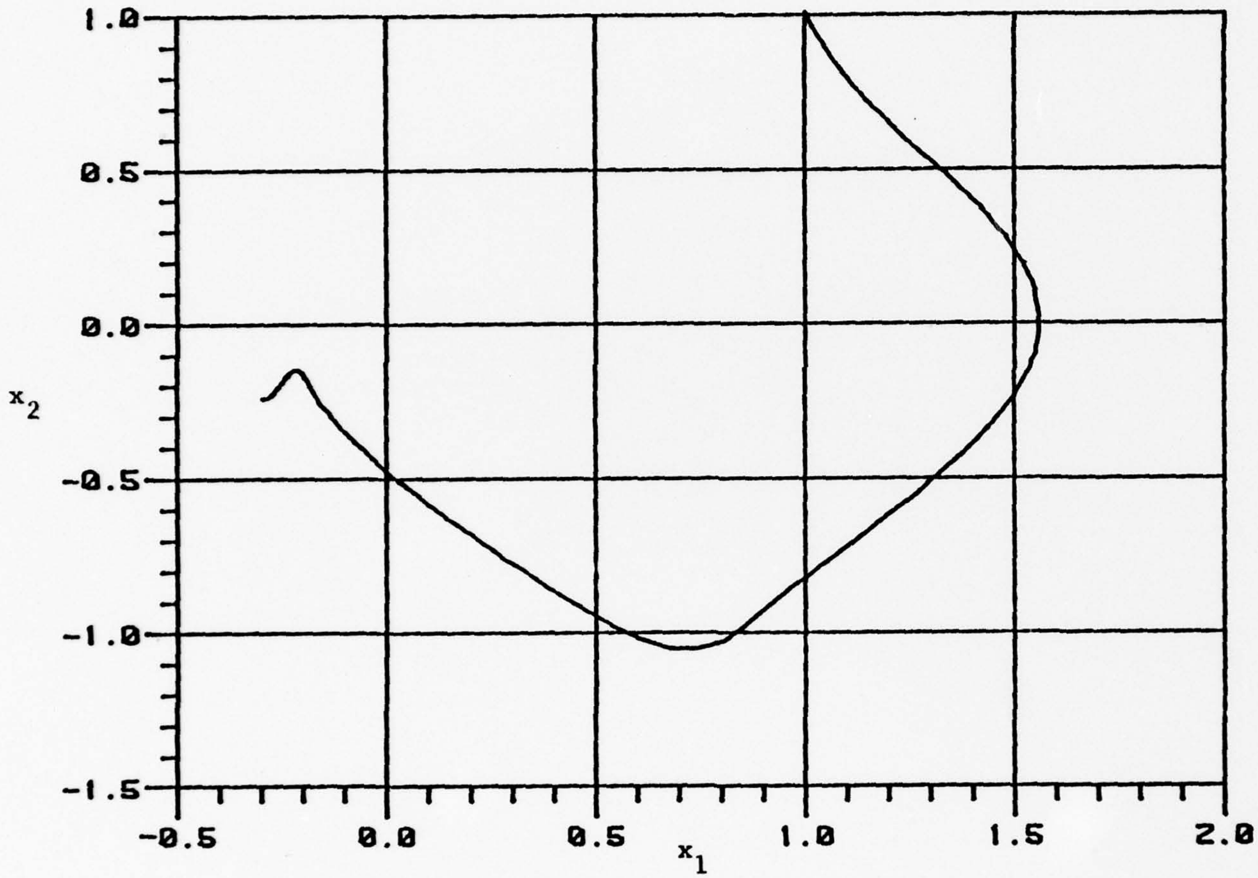

Figure 6.8. The x_1 versus x_2 trajectory corresponding to control (6.20).

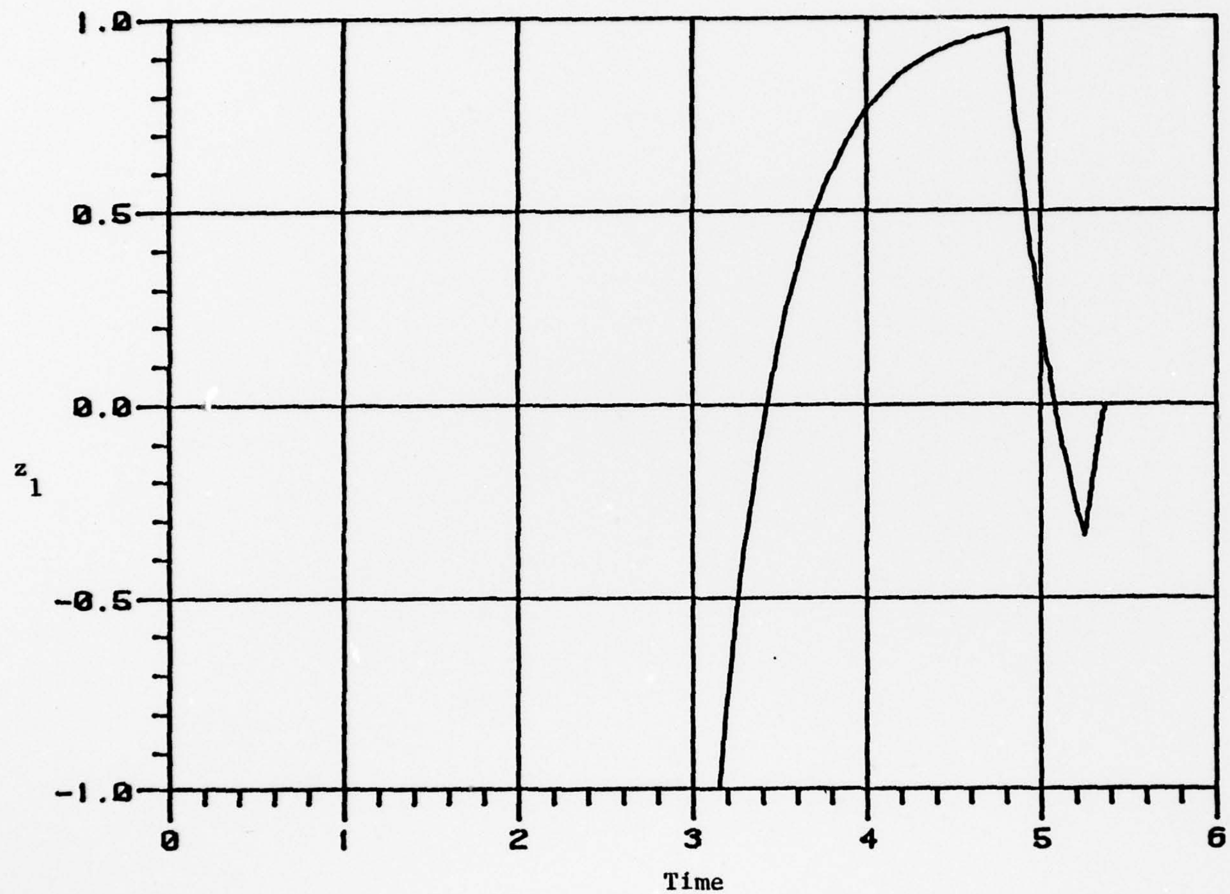

Figure 6.9. The z_1 versus t trajectory corresponding to control (6.20).

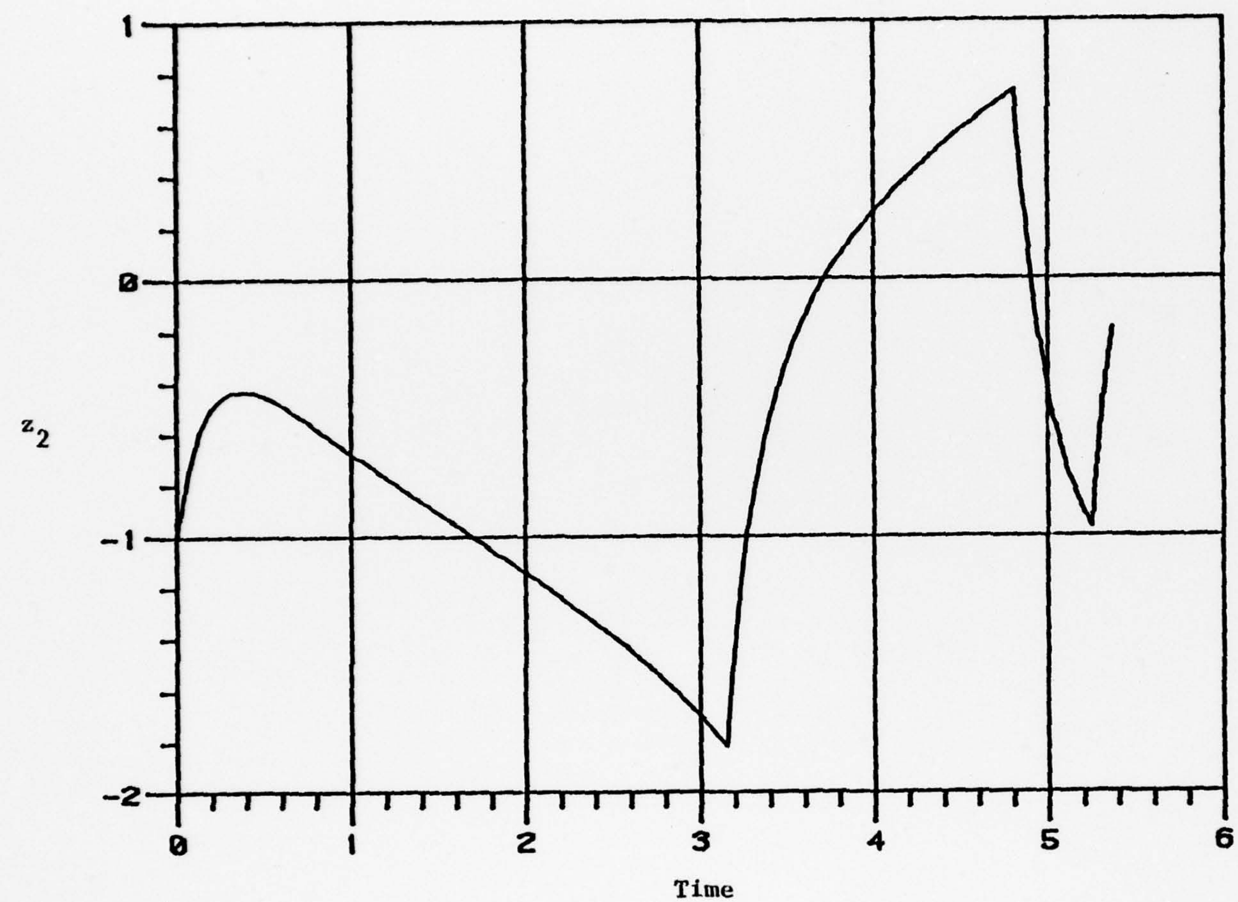

Figure 6.10. The z_2 versus t trajectory corresponding to control (6.20).

There must be a point (x_{1G},x_{2G}) such that the control $\bar{u}^*(t)$ designed to steer $(\bar{x}_1,\bar{x}_2)$ from (x_{10},x_{20}) to (x_{1G},x_{2G}) will steer (x_1,x_2) to any desired point (x_{1F},x_{2F}). Thus $\bar{u}^*(t)$ for the final point (x_{1G},x_{2G}) is in fact $u_s^*(t)$. Let (x_{1R},x_{2R}) be the point to which $\bar{u}^*(t)$ steers (x_1,x_2) of (6.14). The following is an iterative method for (x_{1G},x_{2G}) and $u_s^*(t)$.

1. Initialize with $(x_{1G}{}^{(0)},x_{2G}{}^{(0)}) = (x_{1F},x_{2F})$.
2. Find the control $\bar{u}^{(n)}t$ which steers $(\bar{x}_1,\bar{x}_2)$ to $(x_{1G}{}^{(n)},x_{2G}{}^{(n)})$ in minimum time $\bar{T}^{(n)}$. Let the point reached by (x_1,x_2) when $\bar{u}^{(n)}(t)$ is applied to (6.14) be $(x_{1R}{}^{(n)},x_{2R}{}^{(n)})$.
3. $x_{1G}{}^{(n+1)} = x_{1G}{}^{(n)} - \alpha(x_{1R}{}^{(n)} - x_{1F})$,
   $x_{2G}{}^{(n+1)} = x_{2G}{}^{(n)} - \alpha(x_{2R}{}^{(n)} - x_{2F})$.
4. If $|x_{1R}{}^{(n)} - x_{1F}| + |x_{2R}{}^{(n)} - x_{2F}| \leq 0.01$, stop.
5. Let $n \leftarrow n+1$ and go to step 2.

After this method has converged $\bar{u}^{(n)}(t)$ should be $u_s^*(t)$. As in the magnetic suspension the value $\alpha = 0.6$ provides convergence in less than 15 steps for the initial points which were treated experimentally.

In this manner an improved near optimal control is found

$$\tilde{u}^*(t) = \begin{cases} u_s^*(t) \\ u_f^*(t) \quad . \end{cases} \tag{6.22}$$

For the initial condition (-1,2,-1,-1) considered above and $\mu = 0.4$,

$$\tilde{u}^*(t) = \begin{cases} -1 & t \in [0,3.06) \\ 1 & t \in [3.06,5.00) \\ -1 & t \in [5.00,5.45) \\ 1 & t \in [5.45,5.57) \end{cases} \tag{6.23}$$

and steers (x_1,x_2,z_1,z_2) to (.032,.076,0.0,.009). This is an improved control over control (6.20) since it steers the state of (6.14) closer to the origin.

The intermediate point algorithm can now be applied since it is possible to find $u_s^*(t)$ to steer (x_1,x_2) to points (x_{1F},x_{2F}) and $u_f^*(t)$ to steer (z_1,z_2) to the origin. Thus, even though (6.14) is not block diagonalized, controls may be found to steer the slow and fast states to desired points from controls calculated for reduced order systems.

After six iterations of the intermediate point algorithm the control

$$u^{(6)}(t) = \begin{cases} -1 & t \in [0,3.105) \\ 1 & t \in [3.105,5.046) \\ -1 & t \in [5.046,5.253) \\ 1 & t \in [5.253,5.304) \end{cases} \tag{6.24}$$

steers (x_1,x_2,z_1,z_2) from the above initial point to $(.0047,.037,-.3\times 10^{-7},$ $.0008)$ for $\mu = 2$. The intermediate point algorithm is applied in nested iterations. That is, in each iteration of the algorithm these are iterations for $u_s^{(n)}(t)$ and $u_f^{(n)}(t)$. For $u^*(t)$, in Figure 6.11, the x_1 versus x_2 trajectory is plotted and in Figures 6.12 and 6.13, the z_1 versus t and z_2 versus t trajectories are plotted.

For this example the intermediate point algorithm does not provide any particular improvement over the near optimal control $\tilde{u}^*(t)$. The control is calculated as the concatenation of a time-optimal control for the reduced order system and a time-optimal control for the fast subsystem. The calculation of the time-optimal control for the reduced order system requires Runga Kutta integration for the switching times and therefore, since the intermediate point algorithm requires six iterations, it

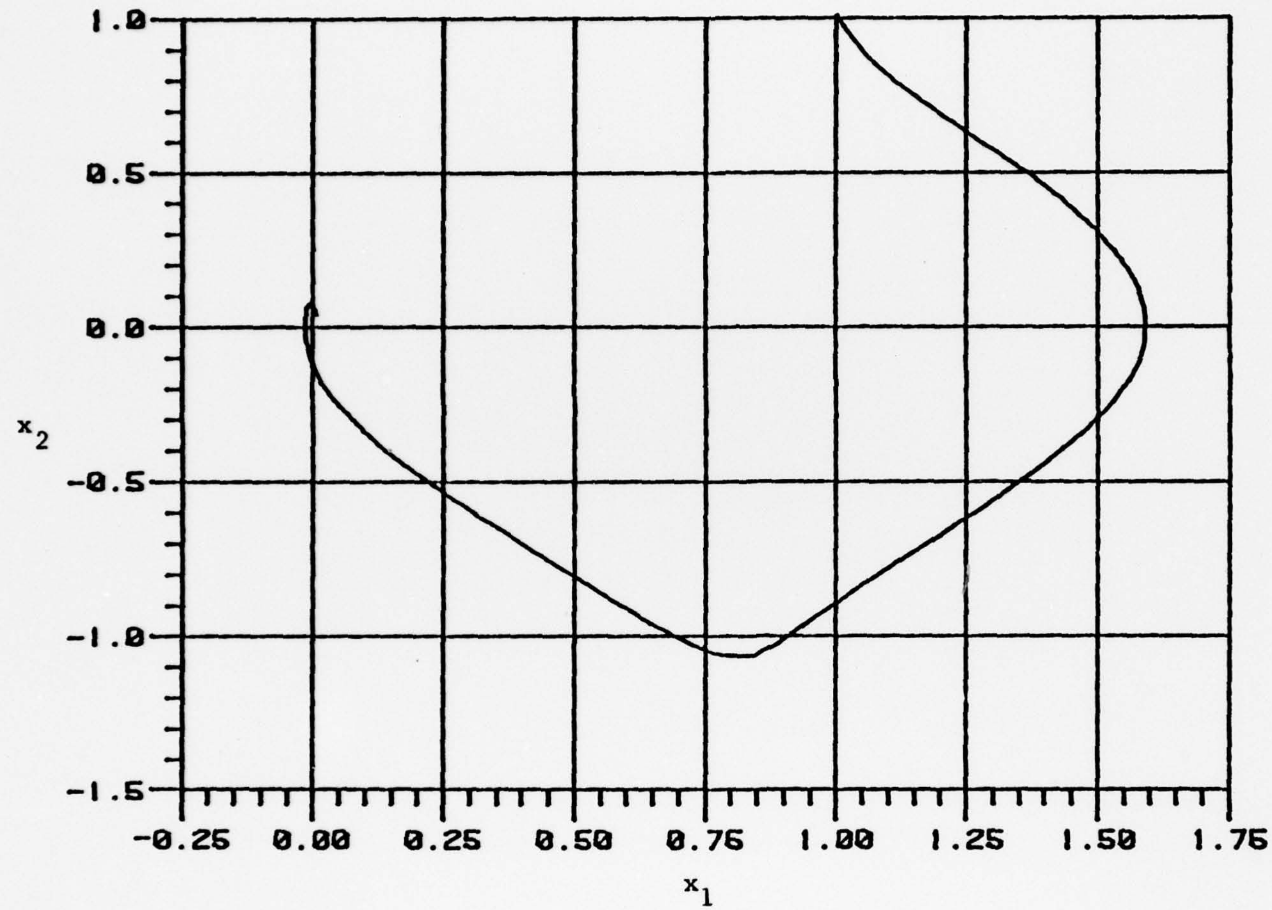

Figure 6.11. The x_1 versus x_2 trajectory corresponding to control (6.24).

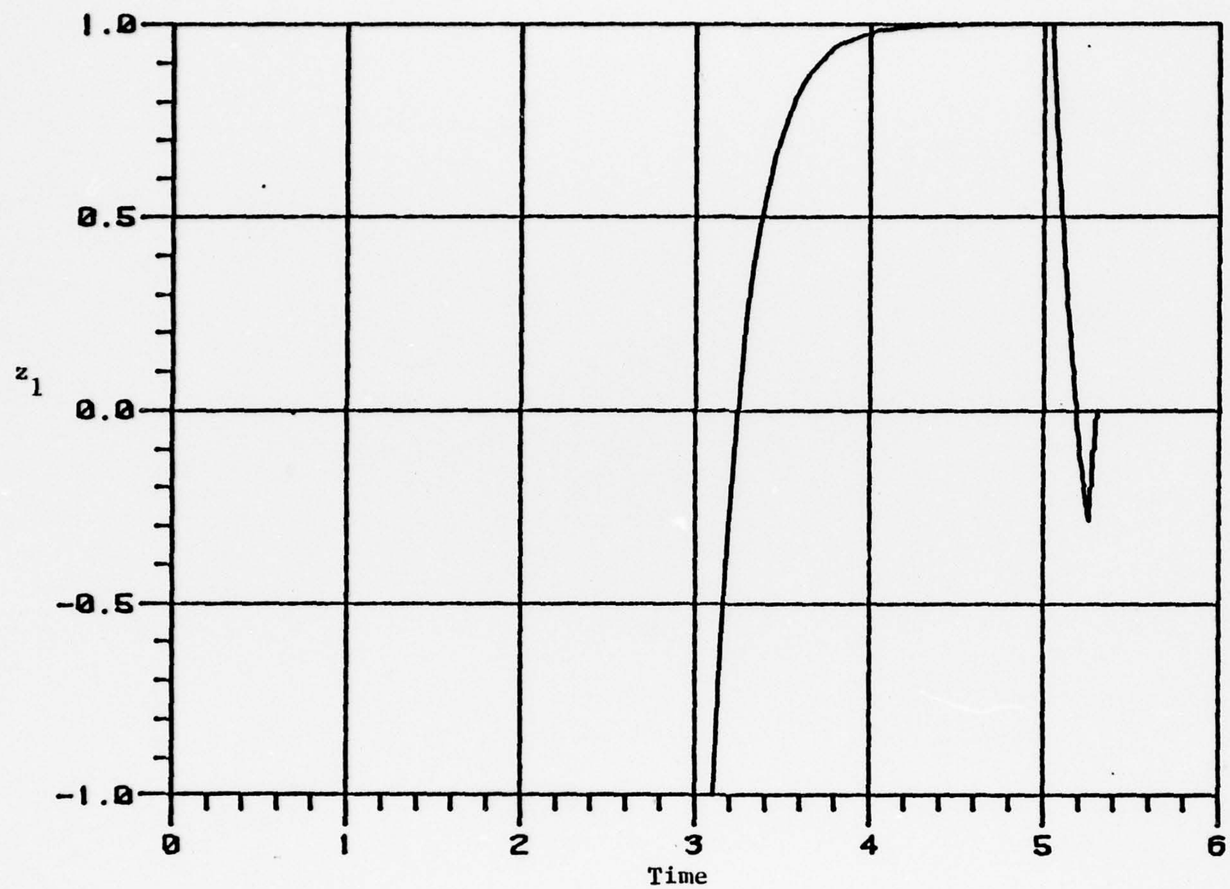

Figure 6.12. The z_1 versus t trajectory corresponding to control (6.24).

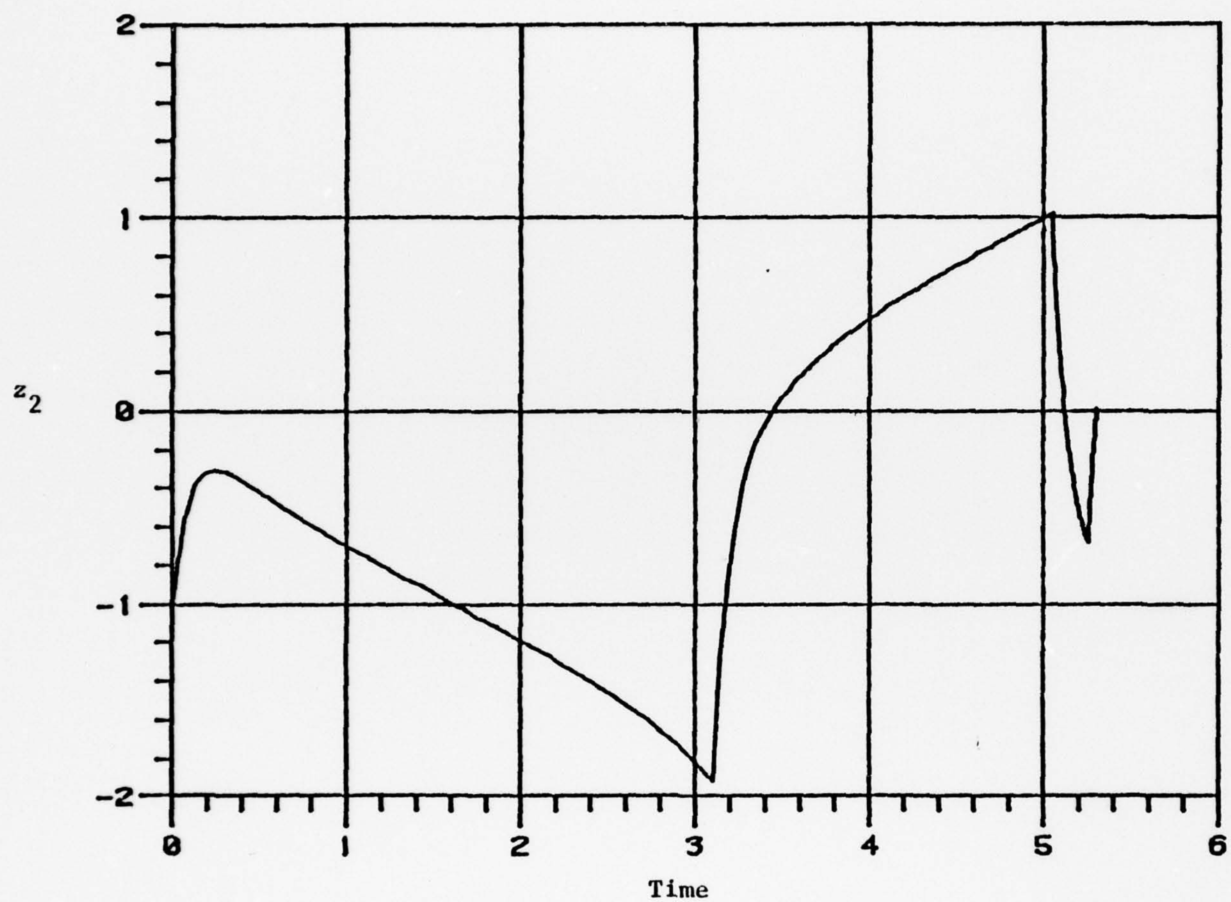

Figure 6.13. The z_2 versus t trajectory corresponding to control (6.24).

requires six times as much integration. With this integration the slow control cannot be calculated to steer the x states to much closer than 10^{-2} of the final point. For the same initial state and $\mu = 0.2$, the x_1-x_2 trajectory corresponding to

$$\tilde{u}^*(t) = \begin{cases} -1 & t \in [0,3.099) \\ 1 & t \in [3.099,4.898) \\ -1 & t \in [4.898,5.123) \\ 1 & t \in [5.123,5.183) \end{cases} \tag{6.25}$$

is plotted in Figure 6.14. The final error is approximately the same as that in Figure 6.11. The trajectory in Figure 6.11 is apt to be closer to the shape of the optimal x_1-x_2 trajectory. However $\tilde{u}^*(t)$ requires less time and is therefore a better control if our criterion is that of steering the state close to the final point. This is a case where the optimal control is not necessarily the best control to use. This is particularly true since the models for real systems are not perfectly accurate anyway.

Thus by using the two time-scale property and the near optimal controls, iterative methods can be developed in order to steer the state of a full order system using controls calculated from reduced order systems.

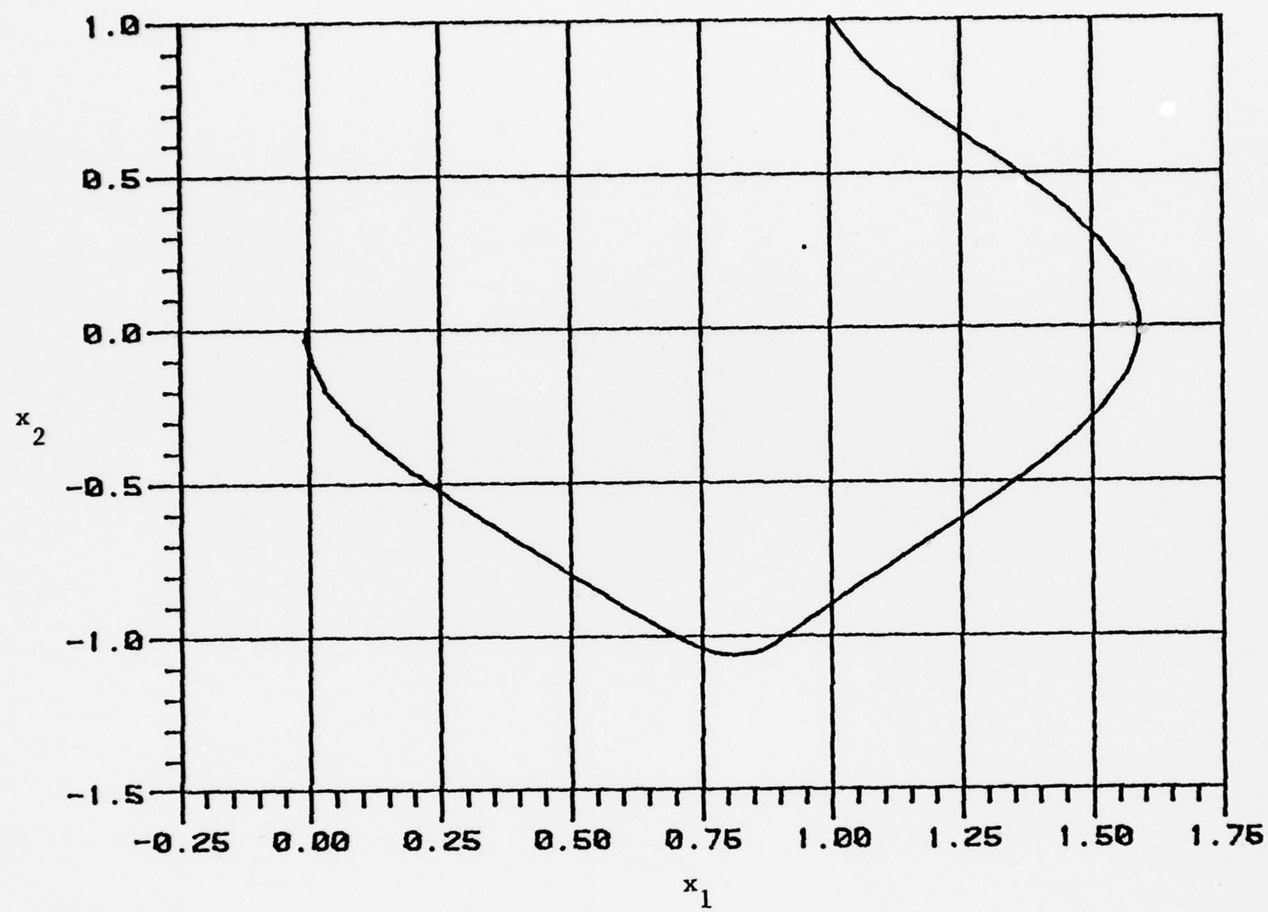

Figure 6.14. The x_1 versus x_2 trajectory corresponding to *control* (6.25).

7. CONCLUSION

In this thesis we first developed stability bounds for μ in the linear time-varying system

$$\begin{aligned} \dot{x} &= A_{11}(t)x + A_{12}(t)z + B_1(t)u \\ \mu\dot{z} &= A_{21}(t)x + A_{22}(t)z + B_2(t)u \quad . \end{aligned} \tag{7.1}$$

That is, for μ within these bounds (under approximate assumptions on (7.1)) the uniform asymptotic stability of the full order system and fast subsystem are guaranteed. Then the problem of steering the state of (7.1) from a fixed initial point to a fixed final point in minimum time was treated. The two time-scale property for this problem was revealed by expanding the necessary conditions of the minimum principle in the singular perturbation parameter μ. In a similar manner the two time-scale property of the time-optimal control of

$$\begin{aligned} \dot{x} &= f(x,t) + F(x,t)z + B_1(x,t)u \\ \mu\dot{z} &= g(x,t) + G(x,t)z + B_2(x,t)u \end{aligned} \tag{7.2}$$

was revealed. This property implies that the time-optimal control has switchings in a slow time-scale on an initial interval and then switchings in a fast time-scale on the final interval.

On the basis of the two time-scale property a near time-optimal control was developed for (7.1) and (7.2). These near optimal controls were applied to the two nonlinear examples and some iterative methods were developed to improve the near-optimal control.

Thus the main results were the development of bounds for μ in (7.1), the revelation of the two time-scale property for (7.1) and (7.2), the development of near optimal controls and the calculation of controls for some nonlinear examples.

There are several possible directions for future research. The classes of singularly perturbed systems which possess the two time-scale property might be extended. The solution of control problems for singularly perturbed systems with state constraints requires further study. Such a study would have as its purpose the discovery of properties of constrained control problems on the basis of the nature of the solution for near optimal controls calculated for reduced order systems. An example of this is in the magnetic suspension system presented in the last section in which a singular arc is proposed based on the time-optimal control of the reduced order system. For linear time-invariant singularly perturbed systems it has been shown [29] that the time-optimal feedback control for the full order system. This result could be extended to linear time-varying and some classes of nonlinear systems. Finally, research could be made in order to develop iterative methods designed to take advantage of the time-scale decomposition which is characteristic of the solution to other optimal control problems for singularly perturbed systems.

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