NORSAR

ROYAL NORWEGIAN COUNCIL FOR SCIENTIFIC AND INDUSTRIAL RESEARCH

Scientific Report No. 1-1977/78

STATISTICAL MODELS FOR SEISMIC MAGNITUDE

by

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Kjeller, March 1978

Sponsored by
Advanced Research Projects Agency
ARPA Order No. 2551

SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: F08606-78-C-0005	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (*and Subtitle*): (6) Statistical Models for Seismic Magnitude	5. TYPE OF REPORT & PERIOD COVERED: (9) Scientific Rep[redacted]t
	6. PERFORMING ORG. REPORT NUMBER: Sci. Rep. No. 1-77/78
7. AUTHOR(s): (10) Anders Christoffersson	8. CONTRACT OR GRANT NUMBER(s): (15) F08606-78-C-0005, ARPA Order-2551
9. PERFORMING ORGANIZATION NAME AND ADDRESS: NTNF/NORSAR, Post Box 51, N-2007 Kjeller, Norway	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NORSAR Phase 3
11. CONTROLLING OFFICE NAME AND ADDRESS: VELA Seismological Center, 312 Montgomery Street, Alexandria, Va. 22314	12. REPORT DATE: (11) Mar[redacted]78
	13. NUMBER OF PAGES: 55 pages
14. MONITORING AGENCY NAME & ADDRESS (*if different from Controlling Office*): (12) 59 p.	15. SECURITY CLASS. (*of this report*)
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (*of this Report*)

APPROVED FOR PUBLIC RELEASE, DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (*of the abstract entered in Block 20, if different from Report*)

(14) NORSAR-Scientific-2-77/78

18. SUPPLEMENTARY NOTES

19. KEY WORDS (*Continue on reverse side if necessary and identify by block number*)

Seismic magnitude, NORSAR, seismology

20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

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by a hypothetical network of 15 stations. It is found that network magnitudes computed by the conventional averaging technique are considerably biased, and that a maximum likelihood approach using instantaneous noise level estimates for non-detecting stations gives the most consistent magnitude estimates. Finally, the models are applied to evaluate the detection characteristics and associated seismicity as recorded by three VELA arrays (UBO, TFO, WMO).

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AFTAC Project Authorization No.	:	VELA VT/8702/B/PMP
ARPA Order No.	:	2551
Program Code No.	:	8F10
Name of Contractor	:	Royal Norwegian Council for Scientific and Industrial Research
Effective Date of Contract	:	1 October 1977
Contract Expiration Date	:	30 September 1978
Contract No.	:	F08606-78-C-0005
Project Manager	:	Nils Marås (02) 71 69 15
Title of Work	:	The Norwegian Seismic Array (NORSAR) Phase 3
Amount of Contract	:	$520,000
Contract Period Covered by the Report	:	1 October 1977 - 31 September 1978

The views and conclusions contained in this document are those of the authors and should not be interpreted as necessarily representing the official policies, either expressed or implied, of the Advanced Research Projects Agency, the Air Force Technical Applications Center, or the U.S. Government.

This research was supported in part by the Advanced Research Projects Agency of the Department of Defense and was monitored by AFTAC, Patrick AFB FL 32925, under contract no. F08606-78-C-0005.

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ABSTRACT

In this paper some statistical models in connection with seismic magnitude are presented. Two main situations are treated. The first deals with the estimation of magnitude for an event using a fixed network of stations and taking into account the detection and bias properties of the individual stations. The second treats the problem of estimating seismicity and detection and bias properties of individual stations. The models are applied to analyze the magnitude bias effects for an earthquake aftershock sequence from Japan, as recorded by a hypothetical network of 15 stations. It is found that network magnitudes computed by the conventional averaging technique are considerably biased, and that a maximum likelihood approach using instantaneous noise level estimates for non-detecting stations gives the most consistent magnitude estimates. Finally, the models are applied to evaluate the detection characteristics and associated seismicity as recorded by three VELA arrays (UBO, TFO, WMO).

1. INTRODUCTION

One of the main problems in estimating magnitude of seismic events from network data is that the events are not always detected by all stations. The standard procedure of estimating magnitude by averaging observed magnitude of recording stations gives estimates that are biased upwards. Methods to cope with this problem have been given by Herrin and Tucker (1972) who computed the expected error introduced by the averaging procedure and Ringdal (1976) who developed a maximum likelihood procedure. The maximum likelihood method given in this paper differs from Ringdal's approach in that it takes into account the probability that the event is detected by the network. This will have effect on the estimated magnitude for small events (events that are detected by a small number of stations). The maximum likelihood estimator for magnitude requires knowledge of the station detection parameters (threshold and slope of detection curve) and region - station bias.

In the second part of the paper, methods to estimate these parameters are develpoed. This is done under the assumption that the magnitude distribution in the source region can be approximated with the usual linear relationship between magnitude and logarithmic frequency.

The first method uses single station data for simultaneous estimation of the slope of the seismisity curve and the detection parameters of the station. This maximum likelihood estimator was, although not explicitly stated, given by Kelly and Lacoss (1969).

One interesting result is that the distribution of log (A/T) rather than magnitude at a station is independent of the focal locations of the events and scattering. The second method is developed for estimation of region - station bias and scattering standard deviation. Because of the complexity of the joint distribution in the general case detailed calculations is carried out only for two stations. By combining analysis made for different combinations of station pairs it is possible to estimate relative bias and scattering standard deviations for all stations in a network.

2. THEORY

2.1 Basic Assumptions

We assume that the amplitude of the signal generated by a seismic event in a specific region and arriving at a station can be modelled as

(1) $$A/T = e^{m} e^{Q} e^{B} e^{\varepsilon}$$

where

A/T	is amplitude over period
m	is the "true" magnitude of the event
Q	is the distance-depth correction
B	is the station-region bias, i.e. the average scattering effect due to inhomogeneities in the earth
ε	is the difference between total scattering and average scattering.

Taking logarithms of both sides in (1) we get

(2) $$\log (A/T) = m + Q + B + \varepsilon$$

or letting $y = \log (A/T)$ denote the station magnitude we get

(3) $$y = m + B + Q + \varepsilon$$

If we regard ε as a sum of effects from many travel path inhomogenities or scattering sources it can be shown (under general assumption) that the distribution of ε is Gaussian. The conditional distribution of y for given "true" magnitude m is then

(4) $$f(y/m) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp[-\frac{1}{2\sigma^2}(y-(B+Q+m))^2]$$

Here Q is assumed to be a known constant, and σ^2 is the variance of the scattering, i.e., of ε. We will further assume that the station has a detection curve giving the probability that the event is seen given y. We will here assume that this curve is of the form

$$\text{(5)} \qquad \Phi\left(\frac{y - G}{\gamma}\right) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}\,\gamma} \exp[-(z-G)^2/2\gamma^2]dz$$

where G can be interpreted as the average threshold and γ as the standard deviation of the threshold.

2.2 Distribution of Observed Log (A/T) at a Network of M Stations

Let N_i denote the subset of station log (A/T) for given true magnitude that are not seen at the i:th station. Then, define the variable a_i such that

$a_i = y_i$ if the event is seen at station i;

$a_i \in N_i$ if the event is not seen at station i.

Unless otherwise stated, the distance - depth corrections (Q_i) are regarded as known constants. The conditional distribution of a_i/m is given by

$$\text{(6)} \qquad h_i(a_i/m)\,da_i = \begin{cases} f_i(a_i/m)\Phi\left(\dfrac{a_i - G_i}{\gamma_i}\right) da_i & \text{for } a_i \notin N_i \\ \Phi\left(\dfrac{-(B_i + Q_i + m - G_i)}{\sqrt{\sigma_i^2 + \gamma_i^2}}\right) & \text{for } a_i \in N_i \end{cases}$$

If the scattering effects, $\gamma_1, \gamma_2, \ldots, \gamma_M$ are independent the joint distribution of $(a_1, a_2, \ldots, a_M/m)$ is

(7) $$h(a_1, a_2, \ldots, a_M/m)\,da_1 da_2 \ldots da_M = \prod_{i=1}^{M} h_i(a_i/m)\,da_i$$

If an event is declared as detected by the network whenever it is detected by at least one station, we obtain the detection probability for the network as

(8) $$P(\text{detect}/m) = 1 - \prod_{i=1}^{M} \Phi\left(\frac{-(B_i + Q_i + m - G_i)}{\sqrt{\sigma_i^2 + \gamma_i^2}}\right) = 1 - \prod_{i=1}^{M} P(a_i \in N_i/m)$$

The distribution of $(a_1, \ldots a_M/m)$ given that the event is detected by the network is then

(9) $$H(a_1, a_2, \ldots, a_M/m)\,da_1 \ldots da_M = \frac{h(a_1, \ldots, a_M/m)\,da_1 \ldots da_M}{1 - \prod_{i=1}^{M} P(a_i \in N_i/m)}$$

This is the conditional distribution of log (A/T) recorded at the network. If the distribution of true earthquake magnitudes is known we can derive the unconditional distribution of log (A/T) recorded at the network. Let $v(m)dm$ denote the distribution of earthquake magnitude in a region. The unconditional distribution of $a_1, \ldots, a_M$ is

(10) $$g(a_1, a_2, \ldots, a_M)\,da_1 \ldots da_M = \int_{-\infty}^{\infty} h(a_1, \ldots, a_M/m)\,da_1 \ldots da_M\, v(m)dm$$

The probability that all $a_i \in N_i$, i.e. the probability of not recording is

(11) $$P(a_1 \in N_1, \ldots, a_M \in N_M) = \int_{-\infty}^{\infty} \prod_{i=1}^{M} \Phi\left(\frac{-(m + B + Q_i - G)}{\sqrt{\sigma_i^2 + \gamma_i^2}}\right) v(m)dm$$

giving the unconditional distribution of log amplitude recorded at the network

(12) $$G(a_1,\dots,a_M)da_1 \dots da_M = \frac{g(a_1,\dots,a_M)da_1 \dots da_M}{1-P(a_1 \in N_1,\dots,a_M \in N_M)}$$

It is often assumed that there is a linear relation between magnitude and logarithmic frequency of earthquake occurrence. This corresponds to assuming that the distribution of earthquake magnitud is

(13) $$\begin{aligned} v(m)dm &= \beta \exp[\beta(m-m_0)]dm \quad &\text{for } m \geq m_0 \\ v(m)dm &= 0 \quad &\text{for } m < m_0 . \end{aligned}$$

Inserting this in (12) we get

(14) $$G(a_1,\dots,a_M)da_1 \dots da_M = \frac{\int_{m_0}^{\infty} h(a_1,\dots,a_n/m)da\dots da_M \exp(-\beta m)d}{\int_{m_0}^{\infty} (1 - \prod_{i=1}^{M} P(a_i \in N_i/m) \exp(-\beta m) dm}$$

The integrals in (14) exist for all m_0 and are convergent when $m_0 \to -\infty$.

In many cases it may be desirable to consider the distribution of station magnitudes instead of station log (A/T) . Letting m_i denote the observed magnitude at the i:th station we have

(15) $$m_i = a_i - Q_i$$

and the distribution of m_i/m is then

(16) $$h_i^*(m_i/m)\,dm_i = h_i(m_i+Q_i/m)\,dm_i = \begin{cases} f_i^*(m_i/m)\ \Phi\left(\dfrac{m_i + Q_i - G_i}{\gamma_i}\right) dm_i & \text{for } m_i \notin N_i \\ \Phi\left(\dfrac{-(B_i + Q_i + m - G_i)}{\sqrt{\sigma_i^2 + \gamma_i^2}}\right) & \text{for } m_i \in N_i \end{cases}$$

with $f_i^*(m_i/m) = f(m_i + Q_i/m)$.

From (16) we see that the distribution of m_i/m depends on the distance-depth correction. But this correction is introduced in order to obtain measures of event magnitudes that are independent of the focal locations. In order to avoid this dependence we can regard the distance-depth correction as a random variable normally distributed with expectation $\bar{Q}$ and standard-deviation σ_Q. We can then obtain the probabilities of detecting an event for given station magnitude. This curve takes the same form as (5). We get the detection curve

$$(17) \qquad \Phi\left(\frac{m_i - G_i^*}{\gamma_i^*}\right) \qquad \text{with} \quad G_i^* = G_i - \bar{Q}_i,\ \gamma_i^* = (\gamma_i^2 + \sigma_Q^2)^{1/2}$$

and the distribution of m_i/m as

$$(18) \qquad h_i^{**}(m_i/m) = \begin{cases} f_i^*(m_i/m)\ \Phi\left(\frac{m_i - G_i^*}{\gamma^*}\right) dm_i & \text{for } m_i \notin N_i \\ \Phi\left(\frac{-(B_i + m - G_i^*)}{\sqrt{\sigma_i^2 + \gamma_i^{*2}}}\right) & \text{for } m_i \in N_i \end{cases}$$

Using h_i^{**} instead of h_i in (9) and (14) then gives the conditional and unconditional distribution of observed station magnitudes.

Comments

We have here defined detection of an event as seen by at least one station. Using other possible definitions like seen by at least two stations will only change the denominators in (9) and (14), which in turn are derived from Eq. (8). In the following sections most results are derived using log (A/T) as a basis. The corresponding results in terms of magnitude are obtained by substituting magnitude for log (A/T) and setting all distance-depth corrections equal to zero. The station parameters G and γ are then interpreted as detection parameters in terms of magnitude and can be transformed back to log (A/T) basis in a way similar to that in Eq. (17).

3. ESTIMATION

The two distributions (9) and (14) are in general suited for two different estimation problems. The conditional case is useful mainly for estimation of magnitude of individual events while estimation of structural parameters such as seismicity, station bias, etc., is most conveniently done in the framework of the unconditional approach. In certain cases, the conditional approach can be used to estimate structural parameters and for those cases it has the advantage of not requiring any knowledge of prior distributions.

We shall here address both approaches and begin with the conditional.

3.1 Conditional Maximum Likelihood

Consider first the case of one station and one event. In this case the distribution of observed log (A/T)=a for given true magnitude m is

$$(19)\quad H(a/m)da = f(a/m)\,\Phi\,\frac{(a-G)}{\gamma}da \Big/ \Phi\left(\frac{B+Q+m-G}{\sqrt{\sigma^2+\gamma^2}}\right)$$

or explicit

$$(20)\quad H(a/m)da = \frac{1}{\sqrt{2\pi}\,\sigma}\exp[-(a-(B+Q+m))^2/2\sigma^2]$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}\exp(-t^2/2)dt \Big/ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp(-t^2/2)\,dt$$

with $y = (a-G)/\gamma$

$$x = (B+Q+m-G)/\sqrt{\sigma^2+\gamma^2}$$

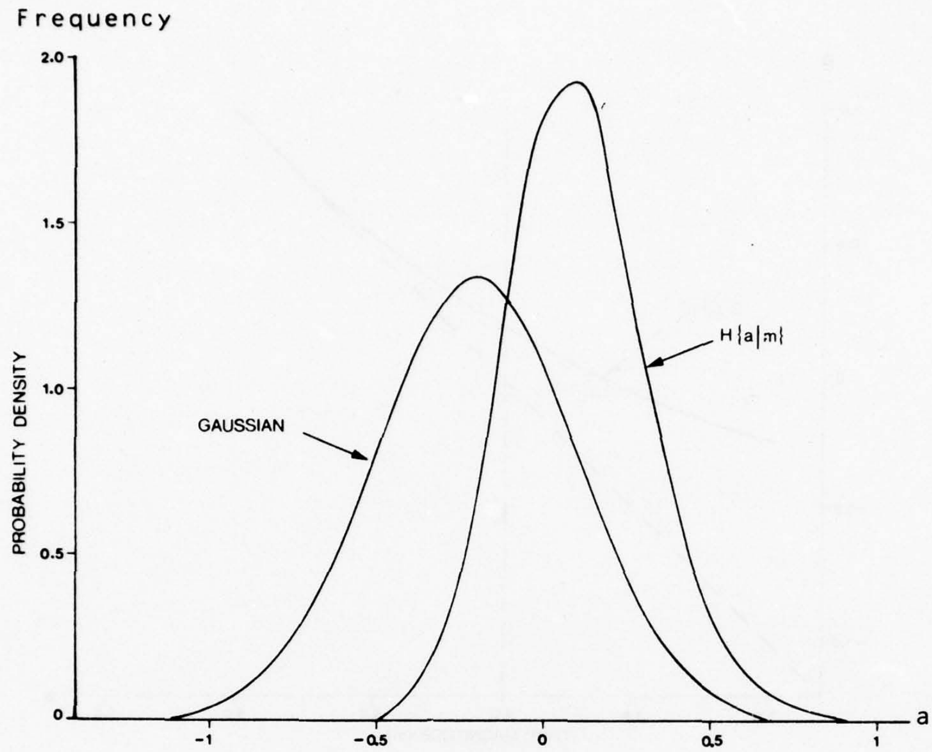

Fig. 1 Frequency distribution H(a/m) for m = 3.8, G = 0.0, B+Q = -4.0, σ = 0.3, γ = 0.2 and the corresponding Gaussian distribution with expectation -0.2 and standard deviation 0.3.

The shape of the distribution is shown in Fig. 1 together with the corresponding Gaussian distribution. The Gaussian distribution corresponds to the case where the station has a probability of not detecting an event. We see from the figure that the effect of nondetection is substantial. It is shown in Appendix 1 that

$$(21) \qquad E(a/m) = B + Q + m + \frac{\sigma^2}{\sqrt{\sigma^2+\gamma^2}} \; Z(x)$$

with x as before and

$$Z(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \Bigg/ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt$$

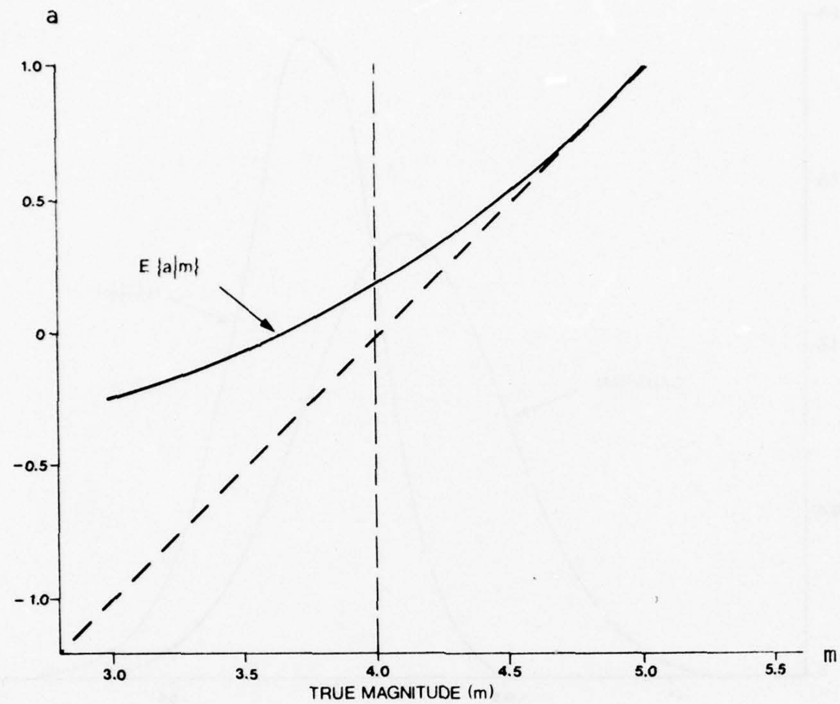

Fig. 2 Expected log (A/T) at a station for given true magnitude. Station parameters used are G = 0.0, B+Q = -4.0, $\sigma = 0.3$, $\gamma = 0.2$.

Figure 2 shows E(a/m) for B+Q = -0.4, G = .0, $\sigma = 0.3$, $\gamma = 0.2$. As can be seen from the figure, the bias is large for events of smaller magnitude and also quite large for events around the detection threshold.

In this case it is readily shown that the maximum likelihood estimate corresponds to solving the following equation for m

$$a = B + Q + m + \frac{\sigma^2}{\sqrt{\sigma^2 + \gamma^2}} Z(x) \tag{22}$$

In the case of a network consisting of several station, one might estimate the magnitude for each station that detected the event according to the above approach and then average these values to obtain an estimate of the true magnitude. However, if the stations that did not see the event were operating properly, the information that some of the stations did not see the event is useful information and should be utilized in the estimating procedure. This was first done by Ringdal (1976) who considered the case with $\gamma = 0$, i.e., the stations detect the event as soon as the 'observed magnitude' m_i is larger than G. His results will differ from those given here as he included in the sample space the cases where the event was not seen at any of the stations. This will have effect on the estimated magnitude for smaller events but for larger events and/or large number of stations the differences will be small. The reason for this is that the probability of not seeing an event tends to zero as the true magnitude increases and/or the number of stations increases.

Suppose that we have a network of M stations recording an event. Suppose further that the event is seen by at least one station. The corresponding log likelihood is

$$(23) \quad \log L = \sum_{i=1}^{M} \log h_i(a_i/m) - \log\left(1 - \prod_{i=1}^{M} \Phi\left(\frac{-(B_i + Q_i + m - G_i)}{\sqrt{\sigma_i^2 + \gamma_i^2}}\right)\right) .$$

Comments

We can see here that the likelihood is not a product of 'individual likelihoods'. If we had considered the case with events seen at all stations or a specific station, the likelihood (19) would have been a product of 'individual likelihoods'.

It is shown in Appendix 1 that the likelihood estimator for magnitude is consistent and its asymptotic standard error is given.

Can we use the conditional approach to estimate the parameters B_i, G_i, σ_i and γ_i and magnitude simultaneously? To try to answer this question we assume that we have a sample of N events such that each event is seen and recorded by at least one station. In terms of "true" log (A/T)-Q arriving at the stations we have the following model

$$y_{ij} = B_i + m_j + \varepsilon_{ij} \quad i=1,2,\ldots,M; \quad j=1,2,\ldots,N \tag{24}$$

If we had had the case where every event was always seen at every station this would have been the standard model for analysis of variance of a two way classification. In order to make the model identified we have to impose a normalizing condition on the B_i:s (or m_j:s). The reason for this is that these unknowns are identified only up to an additive constant. We will here adopt the condition $\sum_{i=1}^{M} B_i = 0$, that is, the average station bias is set equal to zero.

In the analysis of variance model with fixed effects the likelihood does not have a maximum if we treat the σ_i:s as unknown. This will clearly also be the case when we have missing observations, i.e. some stations fail to report an event bacause of the detection properties of these stations. However, even if we treated the σ_i:s as known constants there are difficulties because the number of unknowns increases with sample size. That is, as the number of events increases the number of unknown event magnitudes increases. Therefore, the standard methods for investigating the properties of the maximum likelihood estimator do not apply. The reason for this is that the estimator for m_j is not sufficient independently of B_i, G_i and γ_i. And as we cannot obtain a simple relation expressing the estimator for m_j in terms of data and the parameters B_i, G_i and γ_i we cannot use the approach used by Christoffersson (1970) in connection with estimating factor loading in the case of missing observations.

So, at least for the time being, we have to regard the conditional approach as a heuristic one for estimating the station parameters.

For the case of just estimating the magnitude, the likelihood estimator seems to be fairly insensitive to moderate changes in the present parameters. However, it is important that the stations that do not report an event have been operating properly at that time. Otherwise, there will be large biases in the estimated magnitudes. The asymptotic standard errors, on the other hand, are directly related to the present level of scattering, i.e., on the σ_i values and also to some extent on the γ_i:s. Tables 1 and 2 and Fig. 3 show an application for a simulated network consisting of 15 stations to an aftershock from Japan. Here all σ_i are preset to 0.3 and all γ_i to 0.2. In the figure there are some outliers. The first, event no. 2,

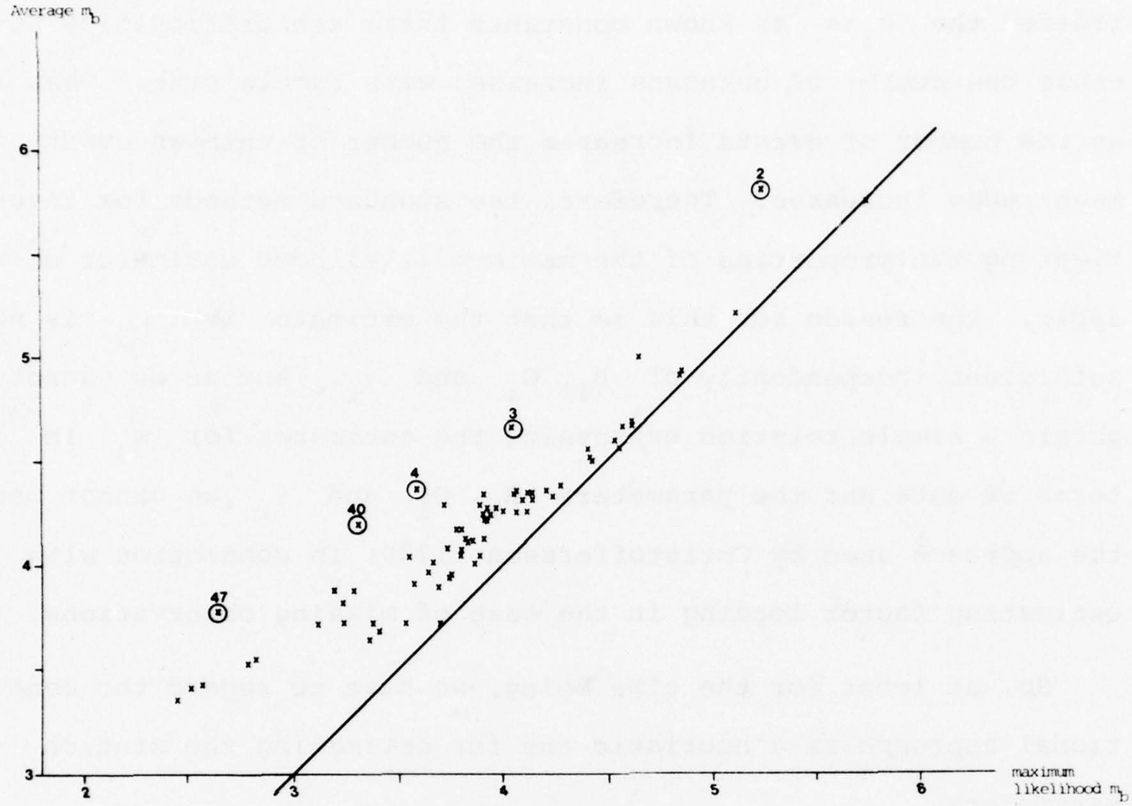

Fig. 3 Average m_b and maximum likelihood m_b for a simulated network of 15 stations. Aftershock consisting of 72 events.

illustrates the importance of the earlier mentioned condition that the stations have to be in operation and/or that no extremely high noise levels or interfering events are present because this event is so large that it should have been seen by all the stations. Therefore it is probably best to have a variable threshold, for example related to the noise level in the time interval preceding the signal, as suggested by Ringdal (1976). The other outliers, events 3, 4, 40 and 47, are small events seen by 3, 1, 1 and 1 stations, respectively, and reflect the bias in the conventional method for magnitude estimation.

TABLE 1

ESTIMATES OF STATION BIAS B_i, AND STATION DETECTION THRESHOLD G_i

Station	B_i	Q_i	G_i
LAO	0.07	3.7	-0.1
MBC	0.29	3.9	+0.2
NAO	0.00	3.7	0.0
RES	0.38	3.7	+0.5
HFS	0.00	3.7	+0.1
UBO	-0.16	3.7	+0.1
KBL	0.09	3.9	+0.2
FFC	-0.03	3.7	+0.6
BLC	0.20	3.7	+0.9
ALE	0.05	3.7	+0.8
FBC	0.05	3.7	+0.8
CHG	-0.39	3.7	+0.3
COL	-0.21	3.8	+0.4
FCC	-0.15	3.7	+0.7
YKC	-0.21	3.7	+0.7

TABLE 2

EVENT	LAO	MBL	NAO	RES	HFS	UBO	KBL	FFC	BLC	ALE	FBC	CHG	COL	FCC	YKL	Average m_b	Max. Likelihood m_b	Standard Error
	STATION																	
1	3.52	-	-	-	-	-	-	-	-	-	-	3.92	-	-	-	3.72	3.55	0.157
2	5.88	-	5.92	6.08	5.99	5.70	-	-	5.85	5.91	5.63	-	5.20	-	-	5.80	5.25	0.078
3	-	-	4.69	-	4.75	-	-	-	-	-	-	-	4.55	-	-	4.66	4.06	0.101
4	-	-	-	-	-	-	-	-	-	-	-	-	4.37	-	-	4.37	3.60	0.148
5	3.82	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.82	3.24	0.244
6	5.10	5.18	4.66	5.28	4.97	4.92	5.09	4.98	5.20	4.92	4.79	-	4.47	4.76	4.67	4.93	4.87	0.079
7	3.73	-	4.05	4.25	-	-	-	-	-	-	-	-	-	-	-	4.01	3.67	0.138
8	4.92	5.19	4.76	5.21	4.97	4.99	5.00	4.88	5.03	4.86	4.99	-	4.50	4.74	4.80	4.92	4.86	0.079
9	4.00	4.47	-	4.28	4.37	-	4.25	-	-	-	-	-	-	-	-	4.27	3.94	0.019
10	4.53	4.77	4.58	4.74	4.61	3.90	4.56	4.66	-	4.67	4.49	-	4.28	4.37	-	4.51	4.43	0.085
11	-	-	4.05	4.62	4.00	3.92	4.57	4.26	4.65	-	-	-	-	4.42	-	4.31	4.11	0.097
12	3.88	4.45	4.09	4.50	4.26	-	4.40	-	-	-	-	-	-	-	-	4.26	4.01	0.104
13	3.55	-	-	3.88	-	-	-	-	-	-	-	-	-	-	-	3.71	3.25	0.240
14	4.13	3.90	4.00	4.41	4.00	3.80	4.03	-	-	-	-	-	-	-	-	4.04	3.90	0.112
15	4.06	4.57	4.16	4.72	4.26	3.70	-	4.33	-	-	-	-	-	-	-	4.26	4.08	0.099
16	3.80	-	-	-	-	-	3.66	-	-	-	-	-	-	-	-	3.73	3.39	0.192
17	3.71	-	-	-	4.00	4.02	-	-	-	-	-	-	-	-	-	3.91	3.70	0.133
18	3.72	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.72	3.12	0.306
19	3.71	-	4.08	4.50	4.42	-	4.35	-	-	-	-	-	-	-	-	4.21	3.94	0.109
20	3.95	4.38	4.01	4.30	3.90	-	-	-	-	-	-	-	-	-	-	4.11	3.86	0.116
21	3.55	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.55	2.83	0.619
22	3.53	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.53	2.79	0.698
23	3.76	4.23	4.05	-	4.23	-	-	-	-	-	-	-	-	-	-	4.07	3.81	0.120
24	3.88	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.88	3.30	0.221
25	3.78	4.30	-	-	-	-	-	-	-	-	-	-	-	-	-	4.04	3.56	0.154
26	4.56	5.83	4.86	-	4.89	-	5.44	-	-	4.72	-	4.39	5.31	-	-	5.00	4.67	0.081
27	4.05	4.59	4.01	4.54	4.22	-	-	-	-	-	-	-	-	-	-	4.28	3.98	0.106
28	4.83	4.90	4.55	4.77	4.58	4.49	4.69	4.98	-	5.12	4.82	4.20	4.17	-	4.77	4.68	4.64	0.081
29	3.90	4.28	-	4.47	4.02	-	-	-	-	-	-	-	-	-	-	4.17	3.81	0.121
30	4.12	4.20	4.30	4.45	4.02	-	-	-	-	-	-	-	-	-	-	4.22	3.93	0.109
31	3.38	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.38	2.45	2.354
32	4.44	4.90	4.50	4.90	4.48	4.10	4.89	4.47	4.87	4.48	4.78	4.35	4.24	4.74	4.46	4.57	4.57	0.082
33	4.09	4.50	4.38	4.72	4.01	-	4.46	-	-	-	-	-	-	-	-	4.36	4.08	0.099
34	4.34	4.45	4.03	4.37	-	4.44	4.25	4.47	4.42	-	-	-	-	-	-	4.35	4.14	0.096
35	3.81	-	3.79	-	-	3.90	-	-	-	-	-	4.54	-	-	-	4.01	3.87	0.115
36	5.83	6.17	5.73	6.06	5.63	5.68	5.58	5.96	6.07	5.86	5.74	5.19	5.28	5.77	5.77	5.75	5.76	0.078

TABLE 2 (cont.)

EVENT	LAO	KLB	NAO	RES	HFS	UBO	KBL	FFC	BLC	ALE	FBC	CHG	COL	FCC	YKC	Average m_b	Max. Likelihood m_b	Standard Error
	STATION																	
37	4.47	4.62	4.25	4.50	4.07	4.10	4.18	4.41	4.50	4.50	-	-	-	-	-	4.36	4.22	0.092
38	4.15	4.38	-	4.57	-	4.07	-	-	-	-	-	-	-	-	-	4.29	3.90	0.112
39	4.37	4.47	4.43	4.64	4.05	4.20	-	-	-	-	-	4.08	-	-	-	4.32	4.16	0.095
40	-	4.20	-	-	-	-	-	-	-	-	-	-	-	-	-	4.20	3.32	0.214
41	-	4.20	3.95	4.35	3.83	-	4.30	-	-	-	-	-	-	-	-	4.13	3.83	0.118
42	-	4.57	4.33	4.62	4.13	4.10	4.69	-	-	-	-	4.04	-	-	-	4.35	4.16	0.095
43	-	4.40	-	4.40	-	-	4.08	-	-	-	-	-	-	-	-	4.29	3.73	0 129
44	3.41	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.41	2.52	1.763
45	-	-	-	-	-	-	3.88	-	-	-	-	-	-	-	-	3.88	3.20	0.260
46	4.08	4.36	-	4.55	3.90	4.01	3.89	-	-	-	-	-	-	-	-	4.13	3.92	0.111
47	-	-	-	3.78	-	-	-	-	-	-	-	-	-	-	-	3.78	2.65	1.009
48	3.93	3.95	-	-	-	3.92	-	-	-	-	-	-	4.02	-	-	3.95	3.76	0.127
49	3.45	-	3.80	-	3.77	3.55	-	-	-	-	-	4.04	-	-	-	3.72	3.71	0.131
50	4.04	4.65	4.18	4.68	4.20	3.87	4.09	4.50	4.68	-	4.62	4.16	-	-	-	4.33	4.25	0.091
51	4.13	4.47	4.07	-	3.83	-	-	-	-	-	-	-	-	-	-	4.12	3.85	0.116
52	4.24	-	4.23	4.20	-	4.25	-	-	-	-	-	-	4.85	-	-	4.35	4.07	0.100
53	4.12	-	4.41	4.60	4.59	3.95	4.48	4.28	4.58	4.50	4.52	4.20	-	-	-	4.38	4.29	0.089
54	4.69	4.79	4.39	4.89	3.93	4.34	4.47	4.54	4.75	4.66	4.71	3.85	-	4.47	-	4.50	4.44	0.085
55	4.37	4.71	4.77	4.71	4.79	4.38	4.93	4.62	4.84	4.70	4.79	4.54	-	4.64	4.57	4.67	4.63	0.081
56	3.91	4.23	-	4.59	3.93	-	4.29	-	4.50	-	-	-	-	-	-	4.24	3.94	0.109
57	4.75	4.97	4.54	5.02	4.71	4.24	4.55	4.64	4.80	4.61	4.82	-	-	4.60	4.41	4.67	4.58	0.082
58	3.56	4.08	-	4.12	-	4.02	-	-	-	-	-	-	-	-	-	3.94	3.67	0.138
59	3.43	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.43	2.57	1.473
60	3.98	4.45	4.08	-	-	-	-	-	-	-	-	-	-	-	-	4.17	3.79	0.123
61	4.71	5.02	4.55	5.04	4.19	4.46	4.43	4.63	4.80	4.66	4.60	-	4.24	4.47	4.56	4.60	4.55	0.082
62	4.31	-	-	4.57	-	4.27	-	-	-	-	-	3.81	-	-	-	4.24	3.94	0.109
63	4.13	4.22	4.29	-	4.61	4.41	4.60	-	-	-	-	3.53	-	-	-	4.26	4.13	0.096
64	-	5.42	5.15	5.56	5.13	4.82	5.51	5.14	5.58	5.42	5.39	5.00	4.47	5.34	5.03	5.21	5.13	0.078
65	3.78	-	3.60	-	-	-	-	-	-	-	-	-	-	-	-	3.69	3.42	0.183
66	-	-	3.88	4.15	3.70	-	-	-	-	-	-	-	-	-	-	3.91	3.59	0.150
67	3.50	3.44	3.55	4.08	-	-	-	-	-	-	-	-	-	-	-	3.64	3.37	0.198
68	4.03	4.29	-	-	-	3.92	-	-	-	-	-	-	-	-	-	4.08	3.74	0.128
69	4.28	4.40	3.91	4.76	-	-	-	-	-	-	-	-	-	-	-	4.34	3.92	0.110
70	4.04	4.28	3.78	4.42	3.75	-	-	-	-	-	-	-	-	-	-	4.05	3.81	0.120
71	4.32	4.05	3.32	-	-	-	-	-	-	-	-	-	-	-	4.19	3.97	3.75	0.127
72	4.67	4.68	4.36	-	4.45	4.33	4.73	4.49	4.98	4.50	4.56	-	-	4.39	-	4.56	4.42	0.085

3.2 Unconditional Maximum Likelihood Estimation

Consider first t[illegible] [illegible]se with just one station. From Eq. (14) we get, letting $m_0 \to -\infty$

$$(25)\quad G(a)da = \int_{-\infty}^{\infty} h(a/m)da \exp(-\beta m)\, dm \Big/ \int_{-\infty}^{\infty} [1-P(a\epsilon N/m)] \exp(-\beta m)\, dm$$

Now

$$(26)\quad \int_{-\infty}^{\infty} h(a/m) \exp(-\beta m)\, dm = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp[-(a-(B+Q+m))^2/2\sigma^2]$$

$$\exp(-\beta m)\ \Phi\left(\frac{a-G}{\gamma}\right) dm = \text{const.} \exp(-\beta a)\ \Phi\left(\frac{a-G}{\gamma}\right)$$

$$(27)\quad \int_{-\infty}^{\infty} [1-P(a\ N/m)]\exp(-\beta m)\, dm = \int_{-\infty}^{\infty} \Phi\left(\frac{B+Q+m-G}{\sqrt{\sigma^2+\gamma^2}}\right) \exp(-\beta m)\, dm =$$

$$= \text{const.}$$

Thus, the distribution of seen events at the station is proportional to

$$(28)\quad \exp(-\beta a)\ \Phi\left(\frac{a-G}{\gamma}\right)$$

Evaluating the proportionality constant (see Appendix 2) we find that the distribution is

$$(29)\quad G(a)da = \exp(\beta G-\gamma^2\beta^2/2) \exp(-\beta a)\ \Phi\left(\frac{a-G}{\gamma}\right) da$$

We note that this distribution is independent of the scattering standard deviation, the region-station bias and distance-depth correction, i.e., the observed distribution depends only on the seismicity parameter β and the station detection parameters G and γ.

Turning to the likelihood estimator we have for a sample of N independent observations, $a_1, a_2, \ldots, a_N$,

$$\log L = N \log \beta + N \beta G - \frac{1}{2} N \gamma^2 \beta^2 - \beta \sum_{i=1}^{N} a_i + \sum_{i=1}^{N} \log \Phi \left(\frac{a_i - G}{\gamma} \right) \quad (30)$$

The derivatives of $\log L$ with respect to β, G, and γ are

$$\frac{\delta \log L}{\delta \beta} = \frac{N}{\beta} + N G - N \gamma^2 \beta - \sum_{i=1}^{N} a_i \quad (31)$$

$$\frac{\delta \log L}{\delta G} = N \beta - \frac{1}{\gamma} \sum_{i=1}^{N} \frac{\Phi'(y_i)}{\Phi(y_i)} \quad (32)$$

$$\frac{\delta \log L}{\delta \gamma} = N \beta^2 \gamma - \frac{1}{\gamma} \sum_{i=1}^{N} y_i \frac{\Phi'(y_i)}{\Phi(y_i)} \quad (33)$$

where $y_i = \left(\frac{a_i - G}{\gamma} \right)$

The likelihood estimator is defined as the solution to

$$\frac{\delta \log L}{\delta \beta} = \frac{\delta \log L}{\delta G} = \frac{\delta \log L}{\delta \gamma} = 0 \quad (34)$$

which corresponds to the maximum of $\log L$.

Although not explicitly stated these likelihood equations (eq. 31-33) were obtained by Kelly and Lacoss (1969) who in addition were estimating the total number of earthquakes in a given time period. The likelihood estimator for β considered by Aki (1965) can be obtained as a special case from the above equations by putting $\gamma=0$ and $G=\min(a_1, \ldots, a_N)$.

To illustrate the above method, data in the distance range 30^{o}-60^{o} observed at the stations UBO, TFO and WMO have been analyzed. The results in terms of base 10 logarithms are shown in Figs. 4, 5 and 6. From the figures we see that the model fits the data reasonably well. However, for UBO (Fig. 4) there is statistically significant deviation between model and data. On the other hand, the number of observations is large (4562) so that the probability of detecting small deviations is large. The maximum deviation between data and model is 0.02. Whether this deviation is of any practical significance will depend on the situation in which the model is applied. For example, in seismic risk studies this deviation may be of very great importance.

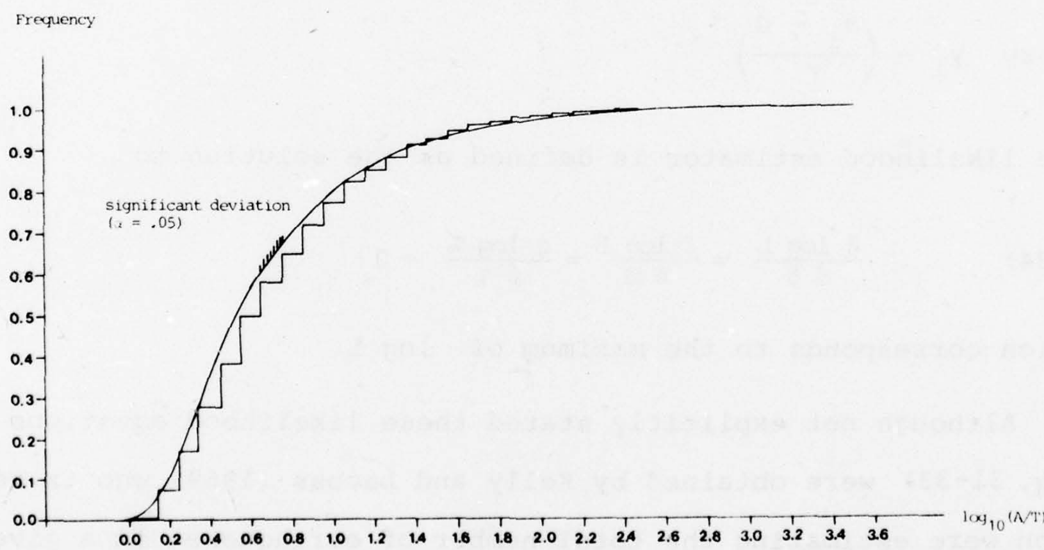

Fig. 4 UBO distance 30^{o}-90^{o}. 4562 events. Observed and theoretical $\log_{10}(A/T)$ cumulative distributions. $\beta = 0.85$ (.017), $\sigma = 0.19$ (.010), $\gamma = 0.11$ (.007). Standard errors within parenthesis.

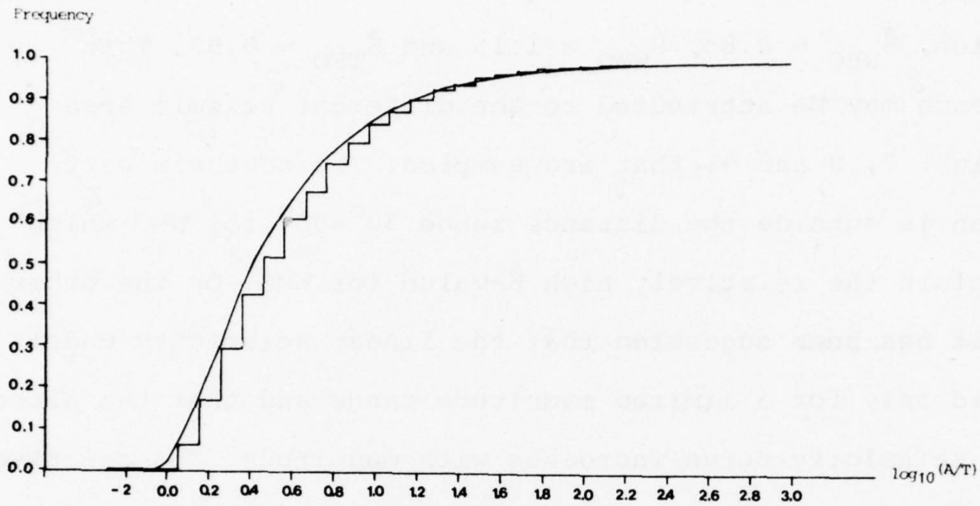

Fig. 5 TFO distance 30°-90°. 3725 events. Observed and theoretical $\log_{10}$ (A/T) cumulative distributions. $\beta = 0.93$ (.019), $\sigma = 0.09$ (.008), $\gamma = 0.10$ (.005). Standard error within parenthesis.

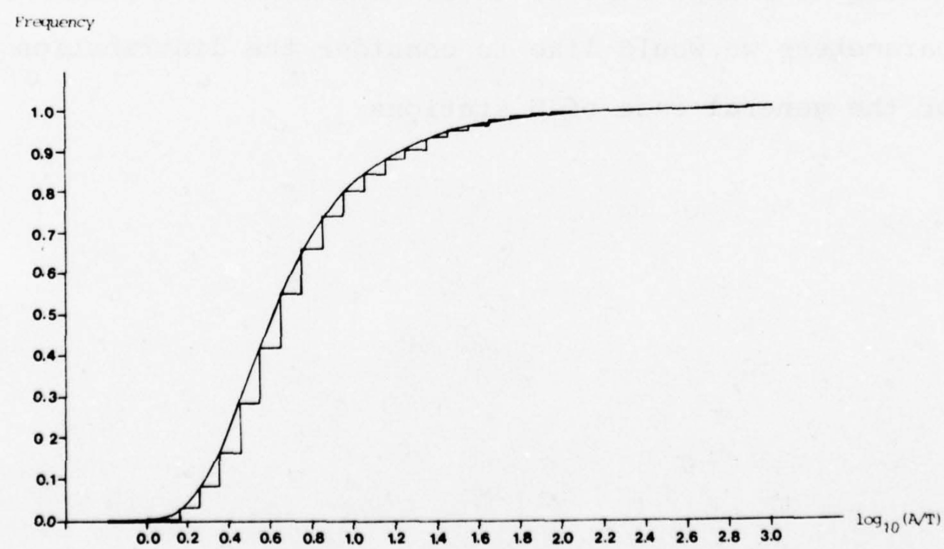

Fig. 6 WMO distance 30°-90°. 2321 events. Observed and theoretical $\log_{10}$ (A/T) cumulative distributions. $\beta = 1.16$ (.035), $\sigma = 0.40$ (.015), $\gamma = 0.16$ (.007). Standard error within parenthesis.

Looking at the estimated β-values we see a rather large variation, $\hat{\beta}_{UBO} = 0.85$, $\hat{\beta}_{WMO} = 1.15$ and $\hat{\beta}_{TFO} = 0.93$. This difference may be attributed to the different seismic area (see Figs. 7, 8 and 9) that are sampled. The southern part of Japan is outside the distance range 30^o-90^o for WMO which may explain the relatively high $\hat{\beta}$-value for WMO. On the other hand, it has been suggested that the linear seismicity model is valid only for a limited magnitude range and that the slope of the seismicity curve increases with magnitude. The relatively high detection threshold and β for WMO is supporting this latter hypothesis.

Whereas the magnitude distribution for one station can be used to obtain reliable estimates of the detection parameters G and γ no information about the station bias B and the scattering σ can be obtained as the distribution of observed magnitude at a station is independent of these parameters. To estimate these parameters we would like to consider the distribution (14) for the general case of M stations.

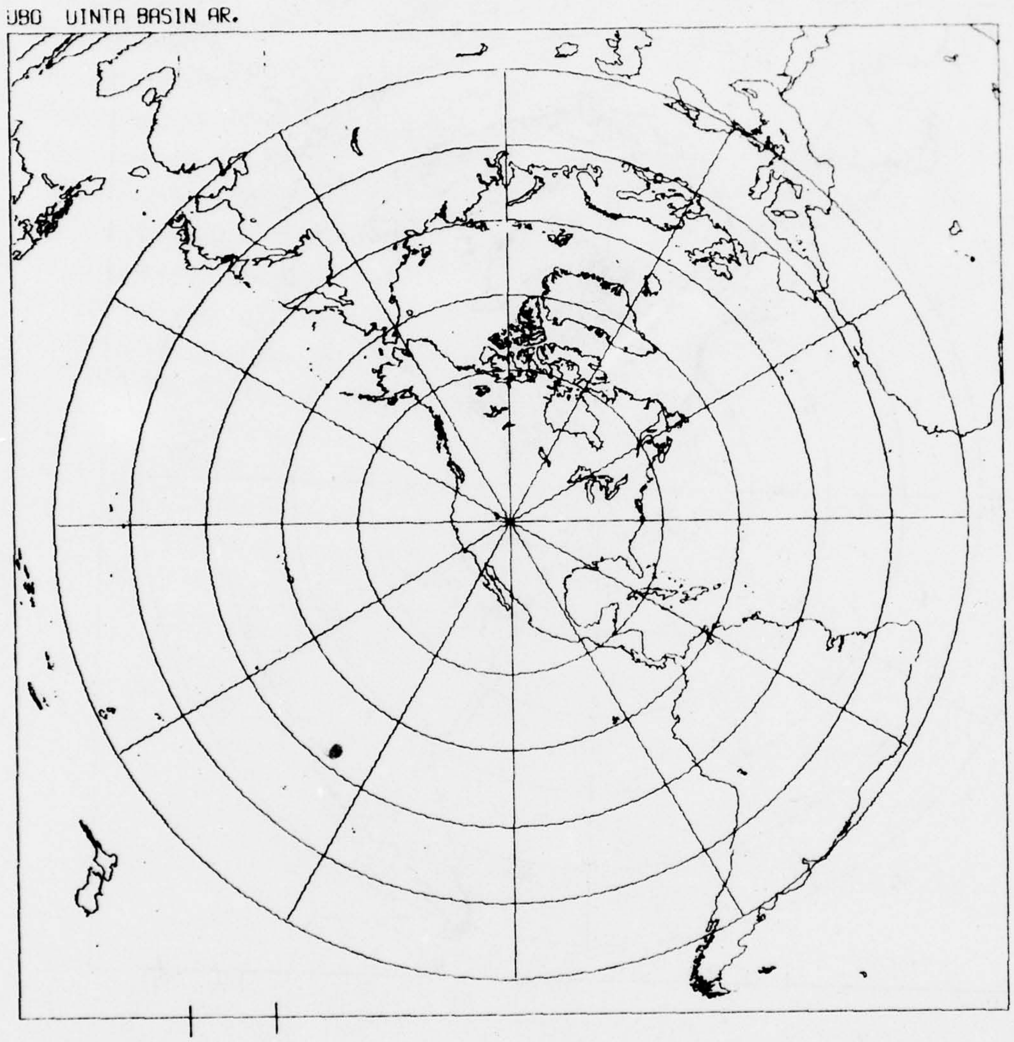

Fig. 7 Azimuthal projection with center at UBO. The distance curves from origin are 30°, 45°, 60°, 75°, 90°. Azimuths are 0, 30, 60, 90 a.s.o. degrees.

WMO WICHITA MTS.

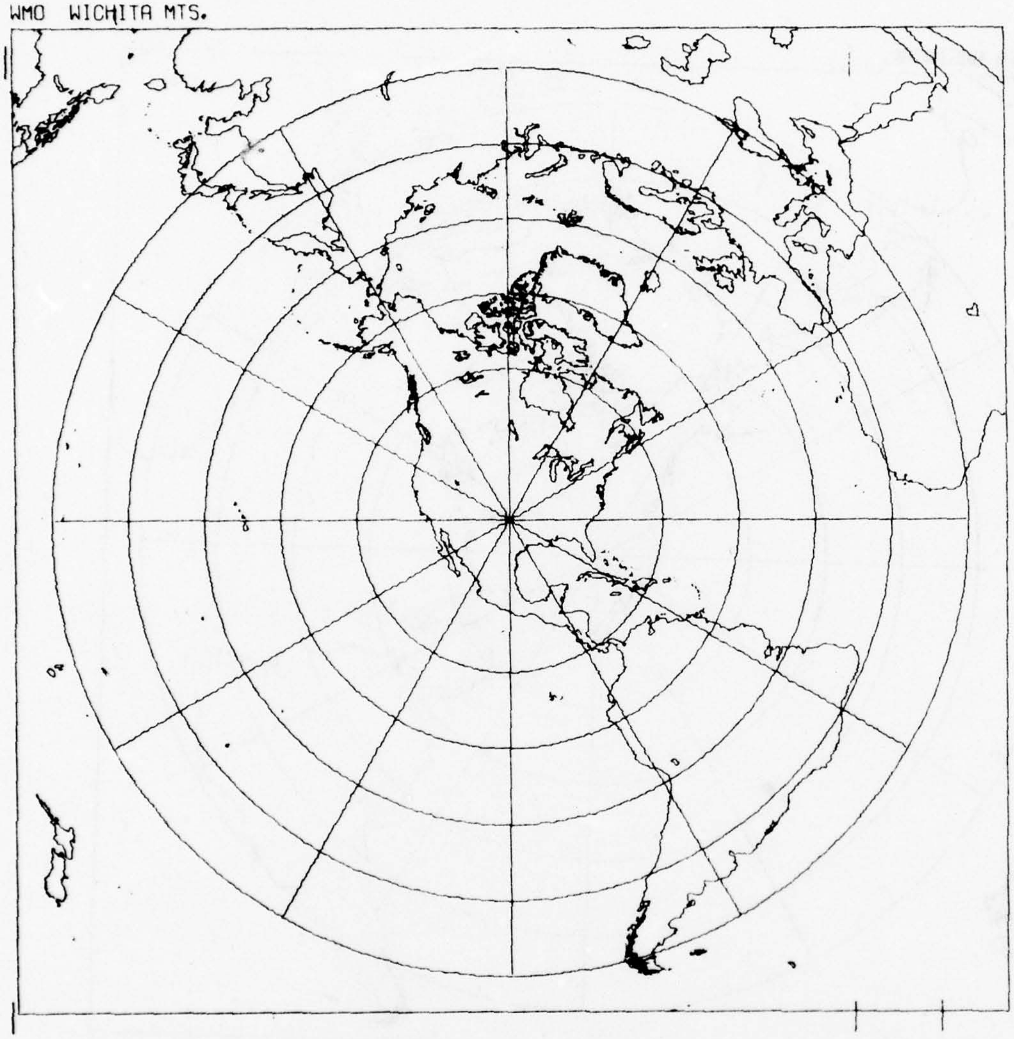

Fig. 8 Azimuthal projection with center at WMO. The distance curves from origin are 30°, 45°, 60°, 75°, 90°. Azimuths are 0, 30, 60, 90 a.s.o. degrees.

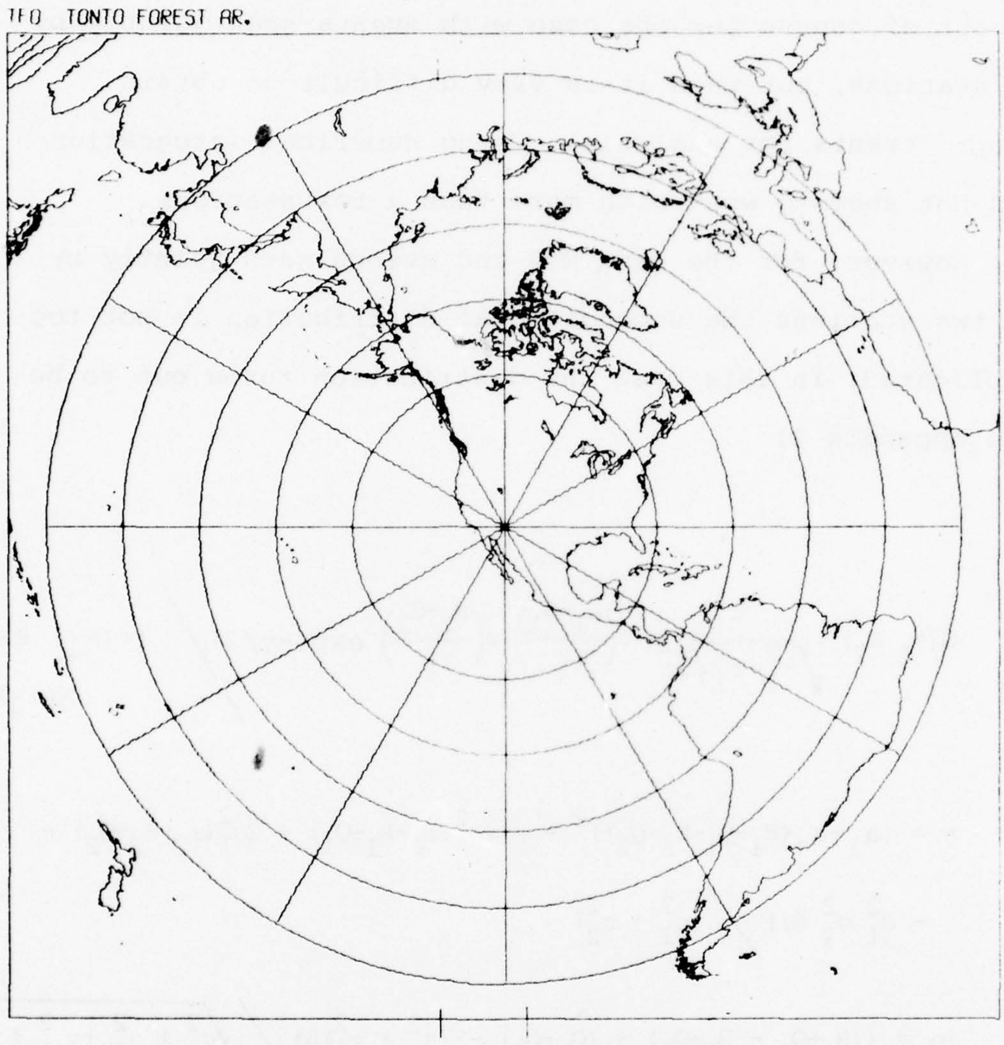

Fig. 9 Azimuthal projection with center at TFO. The distance curves from origin are 30°, 45°, 60°, 75°, 90°. Azimuths are 0, 30, 60, 90 a.s.o. degrees.

Unfortunately, evaluation of the integrals in (14) for the general case leads to very complicated expressions which do not seem to be useful from a practical point of view (except of course for the case with events seen jointly by all stations, but then it is very difficult to obtain enough events for analysis). Using numerical integration does not seem to work with more than a few stations.

However, for the case M=2 and events seen jointly by the two stations the unconditional distribution is not too complicated. In this case the distribution turns out to be (see Appendix 2)

$$(35)\qquad G(a_1\ a_2) = \frac{1}{\sqrt{\sigma_1^2+\sigma_1^2}\sqrt{2\pi}}\,\beta\,\Phi\!\left(\frac{a_1-G_1}{\gamma_1}\right)\Phi\!\left(\frac{a_2-G_2}{\gamma_2}\right)\exp(-y/2) \Big/ \left(\Phi(\alpha_1)\exp(\alpha_2) + \Phi(\alpha_3)\exp(\alpha_4)\right)$$

where

$$(36)\qquad y = (a_1-a_1-(B_1+Q_1-B_2-Q_2))^2 + \beta(2\sigma_2^2(a_1-B_1-Q_1) + 2\sigma_1^2(a_2-B_2-Q_2) - \sigma_1^2\,\sigma_2^2\,\beta)) \Big/ (\sigma_1^2+\sigma_2^2)$$

$$(37)\qquad \alpha_1 = ((B_1+Q_1-B_2-Q_2) - (G_1-G_2) - (\sigma_2^2+\gamma_2^2)\beta) \Big/ \sqrt{\sigma_1^2+\sigma_2^2+\gamma_1^2+\gamma_2^2}$$

$$(38)\qquad \alpha_2 = \beta(B_2+Q_2-G_2) + \beta^2(\sigma_2^2+\gamma_2^2)\ /\ 2$$

$$(39)\qquad \alpha_3 = ((B_2+Q_2-B_1-Q_1) - (G_2-G_1) - (\sigma_1^2+\gamma_1^2)\beta) \Big/ \sqrt{\sigma_1^2+\sigma_2^2+\gamma_1^2+\gamma_2^2}$$

$$(40)\qquad \alpha_4 = \beta(B_1+Q_1-G_1) + \beta^2(\sigma_1^2+\gamma_1^2)\ /\ 2$$

When comparing two stations it is a common practice to plot the magnitude at one station for given magnitude at the other. It is shown in Appendix 2 that the conditional distribution of log (A/T) at station 2 for given log (A/T) at station 1 is

$$(41)\quad g(a_2/a_1) = \Phi\left(\frac{a_2-G_2}{\gamma_2}\right) \frac{1}{\sqrt{2\pi}\,\sigma} \exp[-(a_2-a_1-(Q_2-Q_1)-B)^2/2\sigma^2] \Big/ \Phi\left(\frac{a_1+B+(Q_2-Q_1)-G_2}{\sqrt{\sigma^2+\gamma_2^2}}\right)$$

where

$$(42)\quad B = B_2 - B_1 - \beta\,\sigma_1^2$$

$$(43)\quad \sigma^2 = \sigma_1^2 + \sigma_2^2$$

The parameters that are identified are B, G_2, σ^2 and γ_2^2, and these can thus be estimated. The corresponding likelihood estimator is given in Appendix 2.

This distribution is of the same type as the distribution of observed log (A/T) for given true magnitude (eq. (20)) so the first two moments of (41) are

$$(44)\quad E(a_2/a_1) = B+(Q_2-Q_1)+a_1+\frac{\sigma^2}{\sqrt{\sigma^2+\gamma_2^2}}\,\frac{\Phi'(x)}{\Phi(x)}$$

and

$$(45)\quad D^2(a_2/a_1) = \sigma^2 - \frac{\sigma^4}{(\sigma^2+\gamma_2^2)}\,\frac{\Phi'(x)}{\Phi(x)}\left(x + \frac{\Phi'(x)}{\Phi(x)}\right)$$

where B and σ^2 given by (42), (43) and

$$(46) \qquad x = \frac{a_1+B+(\Omega_2-\Omega_1)-G_2}{\sqrt{\sigma^2+\gamma_2^2}}$$

If we had also considered events not seen at station 2 (but seen at station 1) we would have obtained the following distribution

$$(47) \quad g^*(a_2/a_1)da_2 = \begin{cases} \dfrac{1}{\sqrt{2\pi}\ \sigma} \exp[-(a_2-a_1-(\Omega_1-\Omega_2)-B)/2\sigma^2]\,\Phi\left(\dfrac{a_2-G_2}{\gamma}\right) da_2 & \text{if seen at st. 2} \\ \Phi\left(\dfrac{-(a_1+B(\Omega_2-\Omega_1)-G_2)}{\sqrt{\sigma^2+\gamma_2^2}}\right) & \text{if not seen at st. 2} \end{cases}$$

B and σ are given by (43). This distribution is of the same type as (6).

Ringdal (1975) based a likelihood estimator for the detection parameters on the probabilities of detecting an event at station 2 for given observed magnitude at station 1, i.e., essentially on Eq. (47). However with this latter approach it is not possible to separate the effects of scattering (σ^2) and slope of the detection curve (γ_2).

To illustrate the method based on the distribution (41) the same data as for the single station case has been used. The analysis has been made for both log (A/T) and magnitude data. Letting "*" denote the parameters in the model with magnitude data, the relation to parameters with log (A/T) data is

$$(48) \qquad B^* = B \; ; \; G_2 = G_2^* + \bar{Q}_2 ; \; \sigma^* = \sigma \; ; \; \gamma_2^* = \sqrt{\gamma_2^2 + S_{Q2}^2}$$

where $\bar{Q}$ is the average distance-deepth correction and S_{Q2}^2 is the corresponding variance.

TABLE 3

LOG (A/T) DATA

Station	Ref.station	B	G	S	γ
UBO	WMO	-.04 (.01)	.07 (.01)	.37 (.01)	.05 (.01)
	TFO	.15 (.01)	.11 (.02)	.37 (.01)	.08 (.02)
WMO	UBO	-.34 (.02)	.37 (.02)	.37 (.01)	.17 (.01)
	TFO	-.07 (.02)	.39 (.02)	.35 (.01)	.18 (.01)
TFO	UBO	-.44 (.01)	.03 (.01)	.37 (.01)	.08 (.01)
	WMO	-.29 (.01)	.01* (.02)	.36 (.01)	.05 (.02)

*Lower limit for G.
Standard error within brackets.

TABLE 4

MAGNITUDE DATA

Station	Ref.station	B	G	S	γ	$\bar{Q}$	S_Q
UBO	WMO	.14 (.01)	3.88 (.03)	.36 (.01)	.24 (.02)	3.71	.16
	TFO	-.05 (.01)	3.82 (.03)	.37 (.01)	.21 (.02)	3.72	.17
WMO	UBO	-.32 (.02)	4.17 (.04)	.37 (.01)	.24 (.01)	3.75	.17
	TFO	-.06 (.02)	4.20 (.04)	.35 (.01)	.26 (.02)	3.74	.16
TFO	UBO	-.42 (.01)	3.77 (.03)	.36 (.01)	.22 (.01)	3.71	.17
	WMO	-.26 (.01)	3.67 (.03)	.35 (.01)	.20 (.03)	3.71	.15

Standard errors within brackets.

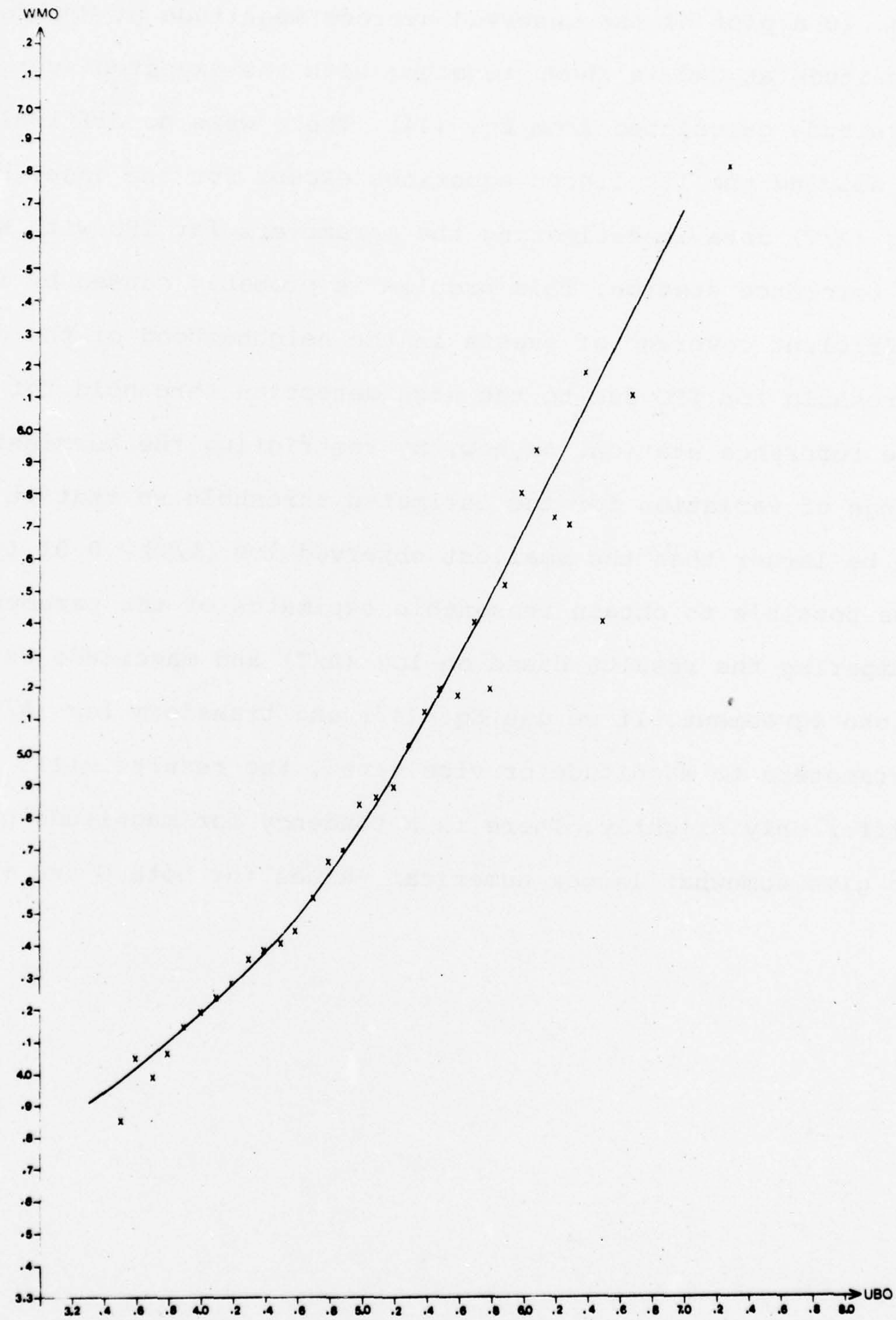

Fig. 10 Average observed magnitude at WMO for given observed magnitude at UBO for 1958 events in the distance range 30°-90°. β= -0.32 (.02), σ=4.17 (.04), S=0.37 (.01), γ=0.24 (.01), $\bar{Q}$=3.75, S_Q=0.17.

The results of the analysis are shown in Tables 3 and 4. In Fig. 10 a plot of the observed average magnitude at WMO for given magnitude at UBO is shown together with the expected average magnitude calculated from Eq. (44). There were no difficulties in solving the likelihood equations except for one case; using log (A/T) data in estimating the parameters for TFO with WMO as reference station. This problem is probably caused by insufficient coverage of events in the neighborhood of the detection threshold for TFO due to the high detection threshold for the reference station. Anyhow, by restricting the permissible range of variation for the estimated threshold so that it had to be larger than the smallest observed log (A/T) + 0.01 it was possible to obtain reasonable estimates of the parameters. Comparing the results based on log (A/T) and magnitude we find close agreement. If we use Eq. (47) and transform log (A/T) parameters to magnitude or vice versa, the results will differ only slightly. There is a tendency for magnitude data to give somewhat larger numerical values for both G and γ.

In the distribution (41) it is neighter possible to identify the individual scattering variances nor the region-station biases But if we can assume that the parameters are at least approximatly constant over the different regions we can combine the results for all possible combinations of station pairs and obtain estimates of scattering variance and region station bias. Using log (A/T) data we have for the different combinations (see eq. 43).

$$\sigma^2_{UBO} + \sigma^2_{WMO} = .37^2$$

$$\sigma^2_{UBO} + \sigma^2_{TFO} = .37^2$$

(49)
$$\sigma^2_{UBO} + \sigma^2_{WMO} = .37^2$$

$$\sigma^2_{WMO} + \sigma^2_{TFO} = .35^2$$

$$\sigma^2_{UBO} + \sigma^2_{TFO} = .37^2$$

$$\sigma^2_{WMO} + \sigma^2_{TFO} = .36^2$$

Solving this system by least squares we obtain the following estimates of the scattering standard deviations: $\hat{\sigma}_{UBO} = .27$; $\hat{\sigma}_{WMO} = .25$ and $\hat{\sigma}_{TFO} = .25$ Similarily, for magnitude data we get $\hat{\sigma}_{UBO} = .27$; $\hat{\sigma}_{WMO} = .25$ and $\hat{\sigma}_{TFO} = .24$

Turning to the region-station bias we obtain from eq. (43) using log (A/T) data

$$B_{UBO} - B_{WMO} - \beta\ \sigma^2_{UBO} = -.04$$

$$B_{UBO} - B_{TFO} - \beta\ \sigma^2_{UBO} = .15$$

$$B_{WMO} - B_{UBO} - \beta\ \sigma^2_{WMO} = -.34$$

(50)
$$B_{WMO} - B_{TFO} - \beta\ \sigma^2_{WMO} = -.07$$

$$B_{TFO} - B_{UBO} - \beta\ \sigma^2_{TFO} = -.44$$

$$B_{TFO} - B_{WMO} - \beta\ \sigma^2_{TFO} = -.29$$

We first note that the region-station coefficients can only be determined up to an unknown additive constant. We therefore adopt the normalizing condition $B_{TFO}= 0$. If in addition we substitude the estimated scattering variances obtained from (49) for σ^2_{UBO}, σ^2_{WMO} and σ^2_{TFO} we can solve (50) for B_{UBO}, B_{WMO} and β by least squares. This gives $\hat{B}_{UBO}= .31$; $\hat{B}_{WMO}= .14$ and $\hat{\beta}= 2.78$, (in base 10 logarithms $\hat{\beta}= 1.20$). This estimate of β must be regarded as very unreliable compared to the estimates of B_{UBO} and B_{WMO} because the diagonal element in the inverse of the moment matrix obtained when solving (50) is more than 100 times the elements corresponding to B_{UBO} and B_{WMO}. Using magnitude data we obtain $\hat{B}_{UBO}= .28$; $\hat{B}_{WMO}= .12$ and $\hat{\beta}= 2.53$ (in base 10 logarithms $\hat{\beta}= 1.10$). The reliability of this $\hat{\beta}$-value is of the same order as for log (A/T) data.

It must be noted that the results concerning bias and scattering are based on the assumption that the parameters are the same for all sampled regions and that the analysis of single station data indicated that the slopes (β) may not be equal. This difficulty can be overcome by using data from <u>one specific</u> region. However, in this paper, data is used only to illustrate the statistical methods. And for this purpose data is considered to be of sufficient quality.

ACKNOWLEDGEMENT

The research work presented in this paper was initiated while the author was visiting the Applied Seismology Group, Lincoln Laboratory, M.I.T. I would like to express my thanks to M. Chinnery and R.T. Lacoss for their valuable discussions and suggestions. I also wish to thank R. Needham and D. Newton for data and computer support. Finally I would like to thank E.S. Husebye and F. Ringdal at NORSAR for many valuable discussions in connection with this work.

This research was supported in part by the Advanced Research Projects Agency of the U.S. Department of Defence and was monitored by AFTAC, Patrick AFB, FL 32925, under Contract No. F08606-77-C-0001.

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Appendix 1

The conditional distribution of observed log(A/T) at a station is given by eq. (19) as

(A1.1) $$H(a/m) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-(a-(B+G+m))^2/2\sigma^2]\ \Phi(y) \Big/ \Phi(x)$$

where $y = (a-G)/\gamma$; $x = (B+Q+m-G)/\sqrt{\sigma^2+\gamma^2}$

and $$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt$$

To obtain the expectation of observed magnitude for given true magnitude we first evaluate

(A1.2) $$F = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (a-(B+Q+m)) \exp[-(a-(B+Q+m))^2/2\sigma^2]\ \Phi(\frac{a-G}{\gamma})\, da$$

Put $u = a - (B+Q+m)$ and $G_o = B+Q+m-G$

Thus

(A1.3) $$F = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} u \exp(-u^2/2\sigma^2)\ \Phi(\frac{u+G_o}{\gamma})\, du$$

(A1.4) $$\frac{d}{\partial u} \exp(-u^2/2\sigma^2) = -u/\sigma^2 \exp(-u^2/2\sigma^2)$$

Thus

(A1.5) $$F = \left[\frac{-\sigma}{\sqrt{2\pi}} \exp(-u^2/2\sigma^2)\ \Phi(\frac{u+G_o}{\gamma})\right]_{-\infty}^{\infty}$$

$$+ \int_{-\infty}^{\infty} \frac{\sigma}{\sqrt{2\pi}} \exp(-u^2/2\sigma^2) \frac{1}{\sqrt{2\pi}\,\gamma} \exp[-(u+G_o)^2/2\gamma^2]\, du$$

$$=0 + \int_{-\infty}^{\infty} \frac{\sigma}{2\pi\gamma} \exp[-(u^2\gamma^2+(u+G_o)^2\sigma^2)/2\sigma^2\gamma^2]\, du$$

Now

(A1.6) $$u^2\gamma^2 + (u+G)^2\sigma^2 = \frac{\sigma^2+\gamma^2}{\sigma^2\gamma^2}(u+\frac{G_o\sigma^2}{\sigma^2+\gamma^2}) + \frac{G_o^2}{\sigma^2}\gamma^2$$

We then find

(A1.7) $$F = \exp[-G_o^2/(\sigma^2+\gamma^2)2] \int_{\infty}^{\infty} \frac{\sigma}{2\pi\gamma} \exp[-(\sigma^2+\gamma^2)(u+G_o\sigma^2/(\sigma^2+\gamma^2))^2/2\sigma^2\gamma^2]\,du$$

and finally

(A1.8) $$F = \frac{\sigma^2}{\sqrt{\sigma^2+\gamma^2}\,\sqrt{2\pi}} \exp[-G_o^2/2(\sigma^2+\gamma^2)]$$

Then

(A1.9) $$E(u/m) = F/\Phi(x)$$

and

(A1.10) $$E(a\ /m) = B+Q+m+F/\Phi(x) = B+Q+m+\frac{\sigma^2}{\sqrt{\sigma^2+\gamma^2}} \frac{\Phi'(x)}{\Phi(x)}$$

$$\text{where } x = \frac{B+Q+m-G}{\sqrt{\sigma^2+\gamma^2}}$$

To obtain the variance of observed magnitude for given true magni-tude we first evaluate

(A1.11) $$F = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \frac{1}{\sigma^2} \left(\frac{u^2}{\sigma^2} - 1\right) \exp(-u^2/2\sigma^2)\ \Phi\left(\frac{u+G_o}{\gamma}\right)\ du =$$

$$= \left[\frac{-u}{\sqrt{2\pi\sigma^3}} \exp(-u^2/2\sigma^2)\ \Phi\left(\frac{u+G_o}{\gamma}\right)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{u}{2\pi\sigma^3\gamma} \exp(-u^2/2\sigma^2)\ \exp[-(u+G_o)^2/2\gamma^2]du$$

$$= \int_{-\infty}^{\infty} \frac{u}{2\pi\sigma^3\gamma} \exp(-u^2/2\sigma^2)\ \exp[-(u+G_o)^2/2\gamma^2]\ du =$$

$$= \frac{-G_o \exp[-G_o^2/2(\sigma^2+\gamma^2)]}{\sqrt{2\pi}\,\sqrt{\sigma^2+\gamma^2}\,(\sigma^2+\gamma^2)} = \frac{-x}{(\sigma^2+\gamma^2)}\ \Phi'(x) \text{ with } x \text{ as before.}$$

This gives

(A1.12) $$E(u^2/m) = \sigma^2 - \frac{\sigma^4\, x\, \Phi'(x)}{(\sigma^2+\gamma^2)\Phi(x)}$$

and

(A1.13) $$D^2(u/m) = E(u^2/m) - E^2(u/m) = \sigma^2 - \frac{\sigma^4}{(\sigma^2+\gamma^2)}\frac{\Phi'(x)}{\Phi(x)}\left(x + \frac{\Phi'(x)}{\Phi(x)}\right)$$

Thus, the variance in the observed distribution is

(A1.14) $$D^2(a/m) = \sigma^2 - \frac{\sigma^4}{(\sigma^2+\gamma^2)}\frac{\Phi'(x)}{\Phi(x)}\left(x + \frac{\Phi'(x)}{\Phi(x)}\right)$$

It can be easily shown that

(A1.15) $$\lim_{x\to\infty} \frac{\Phi'(x)}{\Phi(x)}\left(x + \frac{\Phi'(x)}{\Phi(x)}\right) = 0$$

and

(A1.16) $$\lim_{x\to-\infty} \frac{\Phi'(x)}{\Phi(x)}\left(x + \frac{\Phi'(x)}{\Phi(x)}\right) = 1$$

This gives

(A1.17) $$\lim_{m\to\infty} D^2(a/m) = \sigma^2$$

and

(A1.18) $$\lim_{m\to-\infty} D^2(a/m) = \sigma^2\left(1 - \frac{1}{1+\gamma^2/\sigma^2}\right)$$

Turning to the likelihood estimator we have

(A1.19) $$\log L = \log H(a/M) =$$

$$= \text{const.} - (a-(B+Q+m))^2/2\sigma^2 - \log \Phi(x)$$

(A1.20) $$\frac{\partial \log L}{\partial m} = (a-(B+Q+m))/\sigma^2 - \frac{1}{\sqrt{\sigma^2+\gamma^2}}\frac{\Phi'(x)}{\Phi(x)}$$

Solving $\frac{\partial \text{ Log } L}{\partial m} = 0$ then leads to solving

(A1.21) $$a = B+Q+m+\frac{\sigma^2}{\sqrt{\sigma^2+\gamma^2}}\frac{\Phi'(x)}{\Phi(x)}$$

or letting $z(x) = \frac{\Phi'(x)}{\Phi(x)}$

$$(A1.22) \quad a = B+Q+m+ \frac{\sigma^2}{\sqrt{\sigma^2+\gamma^2}} z(x)$$

Turning to the case of several stations, the likelihood is from (eq. 19)

$$(A1.23) \quad L(m) = \prod_{i=1}^{M} h_i(a_i/m)/(1-\prod_{i=1}^{M} P(a_i \varepsilon N_i/m) =$$

$$= L^*(m)/P^*(m)$$

with

$$(A1.24) \quad L^*(m) = \prod_{i=1}^{M} (h_i(a_i/m)$$

$$(A1.25) \quad P^*(m) = 1 - \prod_{i=1}^{M} P(a_i \varepsilon N_i/m)$$

$$(A1.26) \quad h_i(a_i/m) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma_i} \exp[-(a_i-(B_i+Q_i+m))^2/2\sigma_i^2] \, \Phi(\frac{a_i-G_i}{\gamma_i}) & \text{if the event is seen} \\ \Phi(\frac{-(m+B_i+Q_i-G_i)}{\sqrt{\sigma_i^2+\gamma_i^2}}) & \text{if the event is not seen} \end{cases}$$

and

$$(A1.27) \quad P(a_i \varepsilon N_i/m) = \Phi(\frac{-(m+B_i+Q_i-G_i)}{\sqrt{\sigma_i^2+\gamma_i^2}})$$

It was mentioned earlier that this likelihood is not a product of independent likelihoods, so the usual asymptotic results for maximum likelihood do not apply directly. However, by showing that the maximum for $L(m)$ is the same as for $L^*(m)$ when M tends to infinity we can apply the asymptotic formulaes to L*(m) which is a regular likelihood.

We have

(A1.28) $\log L(m) = \log L^*(m) - \log P^*(m)$

Provided $B_i, Q_i, G_i, \sigma_i, \gamma_i$ are well behaved for all i, that is, are such that each station has a non zero probability to see the event it follows that log L(m) tends to log L*(m) for every finite m. Specifically assume that there exists a $\delta > 0$ that

(A1.29) $\delta \leq \Phi\left(\frac{-(a_i + B_i + Q_i - G_i)}{\sqrt{\sigma_i^2 + \gamma_i^2}}\right) \leq 1 - \delta$ for all i

(A1.30) $1 - (1-\delta)^M \leq P^*(m_o) \leq 1$

And as M tends to infinity we have

(A1.31) $\lim_{M \to \infty} P^*(m_o) = 1$

It then follows directly that

(A1.32) $\lim_{M \to \infty} \log P^*(m) = 0$

(A1.33) $\lim_{M \to \infty} \frac{\partial^n \log P^*(m)}{\partial m^n} = 0$ for all finite m and n=1,2,3....

From this it follows that the asymptotic properties of the estimator obtained by maximizing L(m) is the same as that obtained by maximizing L*(m). As the latter is a regular maximum likelihood estimator we have, letting m_o denote the true magnitude of the event and m the likelihood estimator, that $\sqrt{M}\,(m - m_o)$ tends to a normally distributed variable with expectation zero and variance

$$(A1.34) \quad D^2(\sqrt{M}\,(m-m_o)) = [(\frac{1}{M} E(\frac{\partial \log L^*(m)}{\partial m} / m=m_o)^2)]^{-1}$$

Now

$$(A1.35) \quad \frac{1}{M} E[(\frac{\partial \log L^*(m)}{\partial m} / m=m_o)^2] = \frac{1}{M} \sum_{i=1}^{M} E[(\frac{\partial \log h_i(a_i/m)}{\partial m} / m=m_o)^2]$$

$$(A1.36) \quad \frac{\partial \log h_i(a_i/m)}{\partial m} / m=m_o = \begin{cases} \dfrac{a_i-(B_i+Q_i+m_o)}{\sigma_i^2} & \text{if the event is seen} \\[2ex] \dfrac{-1}{\sqrt{\sigma_i^2+\gamma_i^2}} \dfrac{\Phi'(\frac{-(m_o+B_i+Q_i-G_i)}{\sqrt{\sigma_i^2+\gamma_i^2}})}{\Phi(\frac{-(m_o+B_i+Q_i-G_i)}{\sqrt{\sigma_i^2+\gamma_i^2}})} & \text{if the event is not seen} \end{cases}$$

or letting

$$(A1.37) \quad u_i = a_i - (B_i+Q_i+m_o)$$

$$(A1.38) \quad x_i = (m_o+B_i+Q_i+G_i)/\sqrt{\sigma_i^2+\gamma_i^2}$$

we have

$$(A1.39) \quad \frac{\partial \log h_i(a_i/m)}{\partial m} / m=m_o = \begin{cases} u_i/\sigma_i^2 \\ -\dfrac{1}{\sqrt{\sigma_i^2+\gamma_i^2}} \dfrac{\Phi'(x_i)}{\Phi(-x_i)} \end{cases}$$

and

$$(A1.40) \quad E((\frac{\partial \log h_i(a_i/m)}{\partial m} / m=m_o)^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \frac{u_i^2}{\sigma^4} \exp(-u^2/2\sigma_i^2)\, \Phi(\frac{u_i+G_o}{\gamma_i})\, du_i + $$

$$+ \frac{1}{\sigma_i^2+\gamma_i^2} (\frac{\Phi'(x_i)}{\Phi(-x_i)})^2 \quad \Phi(-x_i)$$

From eq. (A1.11) it follows that

(A1.41) $$E\left(\left(\frac{\partial \log h_i(a_i/m)}{\partial m} \;/\; m=m_o\right)^2\right) =$$

$$= \Phi(x_i)\,\frac{1}{\sigma_i^2} + \frac{\Phi'(x_i)}{\sigma_i^2+\gamma_i^2}\left(\frac{\Phi'(x_i)}{\Phi(-x_i)} - x_i\right) = b_i$$

And finally

(A1.42) $$D^2(\sqrt{M}(m-m_o)) = 1/\left(\frac{1}{M}\sum_{i=1}^{M} b_i\right)$$

Appendix 2

The unconditional distribution of observed log(A/T) at a station is from eq. (23) proportional to

(A2.1) $$G^*(a) = \exp(-\beta a)\ \Phi(\frac{a-G}{\gamma})$$

As the integral of the distribution equals 1 we can determine the proportionality constant by evaluating

(A2.2) $$\int_{-\infty}^{\infty} \exp(-\beta a)\ \Phi(\frac{a-G}{\gamma})\ da = [\frac{-1}{\beta} \exp(-\beta a)\ \Phi(\frac{a-G}{\gamma})]_{-\infty}^{\infty}$$

$$+ \frac{1}{\beta} \int_{-\infty}^{\infty} \exp(-\beta a)\ \frac{1}{\sqrt{2\pi}\ \gamma}\ \exp[-(a-G)^2/2\gamma^2]\ da$$

$$= 0 + \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{}{\sqrt{2\pi\gamma}}\ \exp[-(a-(G-2\beta\gamma))^2/2\gamma^2]\ \exp[-\beta G+\beta^2\gamma^2/2]\ da$$

$$= \frac{1}{\beta}\ \exp(-\beta G+\beta^2\gamma^2/2)$$

Thus, the distribution of observed log (A/T) is

(A2.3) $$G(a) = \beta\ \exp(\beta G-\beta^2\gamma^2/2)\ \exp(-\beta a)\ \Phi(\frac{a-G}{\gamma})$$

Suppose we have a sample of N independent events recorded at the station. The likelihood estimator of the parameters β, G, γ are obtained by maximizing the logarithm of the likelihood. We have, letting a_i denote the observed log (A/T).

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(A2.4) $$\log L(\beta,G,\gamma) = \sum_{i=1}^{N} \log G(a_i) =$$

$$= N \log \beta + N\beta G - \frac{1}{2} N\gamma^2\beta^2 - \beta \sum_{i=1}^{N} a_i$$

$$+ \sum_{i=1}^{N} \log \Phi\left(\frac{a_i - G}{\gamma}\right)$$

The first order derivatives are, letting $y_i = (a_i - G)/\gamma$

(A2.5) $$\frac{\partial \log L(\beta,G,\gamma)}{\partial\beta} = \frac{N}{\beta} + NG - N\gamma^2\beta - \sum_{i=1}^{N} a_i$$

(A2.6) $$\frac{\partial \log L(\beta,G,\gamma)}{\partial G} = N\beta - \frac{1}{\gamma} \sum_{i=1}^{N} \frac{\Phi'(y_i)}{\Phi(y_i)}$$

(A2.7) $$\frac{\partial \log L(\beta,G,\gamma)}{\partial\gamma} = -N\beta^2\gamma - \frac{1}{\gamma} \sum_{i=1}^{N} y_i \frac{\Phi'(y_i)}{\Phi(y_i)}$$

Turning to the second order derivatives we have

(A2.8) $$\frac{\partial^2 \log L(\beta,G,\gamma)}{\partial\beta^2} = -\frac{N}{\beta^2} - N\gamma^2$$

(A2.9) $$\frac{\partial^2 \log L(\beta,G,\gamma)}{\partial\beta\partial G} = N$$

(A2.10) $$\frac{\partial^2 \log L(\beta,G,\gamma)}{\partial\beta\partial\gamma} = -2N\beta\gamma$$

(A2.11) $$\frac{\partial^2 \log L(\beta,G,\gamma)}{\partial G^2} = -\frac{1}{\gamma^2} \sum_{i=1}^{N} \frac{\Phi'(y_i)}{\Phi(y_i)} \left(y_i + \frac{\Phi'(y_i)}{\Phi(y_i)}\right)$$

(A2.12) $$\frac{\partial^2 \log L(\beta,G,\gamma)}{\partial G\partial\gamma} = \frac{1}{\gamma^2} \sum_{i=1}^{N} \frac{\Phi'(y_i)}{\Phi(y_i)} \left(1 - y_i^2 - y_i \frac{\Phi'(y_i)}{\Phi(y_i)}\right)$$

$$\text{(A2.13)} \quad \frac{\partial^2 \log L(\beta,G,\gamma)}{\partial\gamma^2} = -N\beta^2 + \frac{1}{\gamma^2}\sum_{i=1}^{N} \frac{y_i\Phi'(y_i)}{\Phi(y_i)}\left(2-y_i^2-y_i\frac{\Phi'(y_i)}{\Phi(y_i)}\right)$$

Put $\theta = (\beta,G,\gamma)$. Let $\hat\theta = (\hat\beta, \hat G, \hat\gamma)$ denote the likelihood estimator and $\theta_o = (\beta_o, G_o, \gamma_o)$ the true parameter set. As we here have a regular likelihood estimator it follows that the limiting distribution of $\sqrt{N}\,(\hat\theta-\theta_o)$ is normal with expectation zero and covariance matrix

$$\text{(A2.14)} \quad D^2(\sqrt{N}(\hat\theta-\theta_o)) = \left(-\frac{1}{N} E\left(\frac{\partial^2 \log L(\theta)}{\partial\theta^2}/\theta=\theta_o\right)\right)^{-1}$$

where $\frac{\partial^2 \log L}{\partial\theta^2}$ given by eqs. (A2.8)-(A2.13). For large samples we may use $\frac{1}{N}\left(\frac{\partial^2 \log L(\theta)}{\partial\theta^2}/\theta=\hat\theta\right)$ computed from the data instead of $E\left(\frac{\partial^2 \log L(\theta)}{\partial\theta^2}/\theta=\theta^o\right)$ because

$$\text{(A2.15)} \quad \underset{N\to\infty}{\text{plim}} \ \frac{1}{N}\left(\frac{\partial^2 \log L(\theta)}{\partial\theta^2}/\theta=\hat\theta\right) = E\left(\frac{\partial^2 \log L(\theta)}{\partial\theta^2}/\theta=\theta_o\right)$$

Next, consider the case of events seen jointly by two stations. The joint distribution of log (A/T):s a_1 and a_2 is

$$\text{(A2.16)} \quad G(a_1 a_2) = \int_{-\infty}^{\infty} h_1(a_1/m)\, h_2(a_2/m) \exp(-\beta m)\, dm$$

$$\int_{-\infty}^{\infty} \Phi\left(\frac{m+B_1+Q_1-G_1}{\sqrt{\sigma_1^2+\gamma_1^2}}\right)\Phi\left(\frac{m+B_2+Q_2-G_2}{\sqrt{\sigma_2^2+\gamma_2^2}}\right) \exp(-\beta m)dm$$

with

$$\text{A(2.17)} \quad h_i(a_i/m) = \frac{1}{\sqrt{2\pi}\sigma_i}\Phi\left(\frac{a_i-G_i}{\gamma_i}\right)\exp[-(a_i-B_i-Q_i-m)^2/2\sigma_i^2] \qquad \text{for } i = a, 2$$

Put $u = a_1 - B_1 - Q_1$ $v = a_2 - B_2 - Q_2$

$a = a_1 - G_1$ $b = a_2 - G_2$

Then

(A2.18) $$F_1 = \int_{-\infty}^{\infty} h_1(a_1/m)h_2(a_2/m)\exp(-\beta m)\,dm =$$

$$= \Phi\left(\frac{a}{\gamma_1}\right)\Phi\left(\frac{b}{\gamma_2}\right)\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_1\sigma_2}\exp[-((u-m)^2/2\sigma_1^2+(v-m)^2/2\sigma_2^2)]\exp(-\beta m)\,dm$$

Now

(A2.19) $$-\frac{1}{2}((u-m)^2/\sigma_1^2+(v-m)^2/\sigma_2^2)-\beta m =$$

$$= -\frac{1}{2}\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\left(m-\frac{1}{\sigma_1^2+\sigma_2^2}(\sigma_2^2u+\sigma_1^2v-\sigma_1^2\sigma_2^2\beta)\right)^2 -$$

$$-\frac{1}{2(\sigma_1^2+\sigma_2^2)}((u-v)^2+\beta(2\sigma_2^2u+2\sigma_1^2v-\sigma_1^2\sigma_2^2\beta)$$

And we find

(A2.20) $$F_1 = \Phi\left(\frac{a}{\gamma_1}\right)\Phi\left(\frac{b}{\gamma_2}\right)\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\sigma_1^2+\sigma_2^2}}\exp[-((u-v)^2+\beta(2\sigma_2^2u+2\sigma_1^2v-\sigma_1^2\sigma_2^2\beta))/2(\sigma_1^2+\sigma_2^2)]$$

Next

(A2.21) $$F_2 = \int_{-\infty}^{\infty}\Phi\left(\frac{m+B_1+Q_1-G_1}{\sqrt{\sigma_1^2+\gamma_1^2}}\right)\Phi\left(\frac{m+B_2+Q_2-G_2}{\sqrt{\sigma_2^2+\gamma_2^2}}\right)\exp(-\beta m)\,dm$$

Put $-Q_1-B_1+G_1 = c$; $-Q_2-B_2+G_2 = d$; $s_1 = \sqrt{\sigma_1^2+\gamma_1^2}$; $s_2 = \sqrt{\sigma_2^2+\gamma_2^2}$; $s = \sqrt{\sigma_1^2+\sigma_2^2+\gamma_1^2+\gamma_2^2}$

Then

(A2.22) $$F_2 = \int_{\infty}^{\infty}\Phi\left(\frac{m-c}{s_1}\right)\Phi\left(\frac{m-d}{s_2}\right)\exp(-\beta m)\,dm =$$

$$= -\frac{1}{\beta}\left[\Phi\left(\frac{m-c}{s_1}\right)\Phi\left(\frac{m-d}{s_2}\right)\exp(-\beta m)\right]_{-\infty}^{\infty} +$$

$$+ \frac{1}{\beta} \int_{-\infty}^{\infty} \Phi\left(\frac{m-c}{s_1}\right) \frac{1}{\sqrt{2\pi}\, s_2} \exp[-(m-d)^2/2s_2^2] \exp(-\beta m)\, dm$$

$$+ \frac{1}{\beta} \int_{-\infty}^{\infty} \Phi\left(\frac{m-d}{s_2}\right) \frac{1}{\sqrt{2\pi}\, s_1} \exp[-(m-c)^2/2s_1^2] \exp(-\beta m)\, dm$$

(A2.23) $$-(m-d)^2/2s_2^2 - \beta m = -(m-(d-s_2^2\beta))^2/2s_2^2 - d\beta + \frac{1}{2}\beta^2 s_2^2$$

Similarily

(A2.24) $$-(m-c)^2/2s_1^2 - \beta m = -(m-(c-s_1\beta))^2/2s_1^2 - c\beta + \frac{1}{2}\beta^2 s_1^2$$

So

(A2.25) $$F_2 = \exp(-\beta d+\beta^2 s_2^2/2)\, \frac{1}{\beta}\, \Phi\left(\frac{e-s_2\ -c}{\sqrt{s_1^2+s_2^2}}\right) \Phi\, \frac{1}{\beta} \left(\frac{c-s_1\ -d}{\sqrt{s_1^2+s_2^2}}\right) \exp(-\beta c+\beta^2 s_1^2/2)$$

And finally

(A2.26) $$G(a_1 a_2) = \frac{\beta \dfrac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}\, \Phi\left(\dfrac{a_1-G_1}{\gamma_1}\right)\Phi\left(\dfrac{a_2-G_2}{\gamma_2}\right)\exp(-y/2)}{(\Phi(\alpha_1)\exp(\alpha_2) + \Phi(\alpha_3)\exp(\alpha_4))}$$

where

(A2.27) $$y = ((a_1-a_2-(B_1+Q_1-B_2-Q_2))^2+\beta(2\sigma_2^2(a_1-B_1-Q_1)+2\sigma_1^2(a_2-B_2-Q_2)-\sigma_1^2\sigma_2^2\beta))/(\sigma_1^2+\sigma_2^2)$$

(A2.28) $$\alpha_1 = ((B_1+Q_1-B_2-Q_2)-(G_1-G_2)-(\sigma_2^2+\gamma_2^2)\beta)\Big/\sqrt{\sigma_1^2+\sigma_2^2+\gamma_1^2+\gamma_2^2}$$

(A2.29) $$\alpha_2 = \beta(B_2+Q_2-G_2)+\beta^2(\sigma_2^2+\gamma_2^2)/2$$

(A2.30) $$\alpha_3 = ((B_2+Q_2-B_1-Q_1)-(G_2-G_1)-(\sigma_1^2+\gamma_1^2)\beta)\Big/\sqrt{(\sigma_1^2+\sigma_2^2+\gamma_1^2+\gamma_2^2)}$$

(A2.31) $$\alpha_4 = \beta(B_1+Q_1-G_1)+\beta^2(\sigma_1^2+\gamma_1^2)/2$$

The marginal distribution of a_1, given that the event is seen at both stations is

(A2.32) $g(a_1) = \int_{-\infty}^{\infty} G(a_1 a_2)\, da_2 =$

$$= C_o \int_{-\infty}^{\infty} \Phi\left(\frac{a_2 - G_2}{\gamma_2}\right) \exp[-((a_1 - a_2(B_1 + Q_1 - B_2 - Q_2))^2 - 2\beta\sigma_1^2(a_2 - B_2 - Q_2))/2(\sigma_1^2 + \sigma_2^2)]\, da_2$$

$$= C_o \exp[-(\beta^2\sigma_1^4 - 2\beta^2\sigma_1^4(a_1 - B_1 - Q_1))/2(\sigma_1^2 + \sigma_2^2)]\, \sqrt{2\pi}\, \sqrt{\sigma_1^2 + \sigma_2^2}$$

$$\Phi\left(\left(\frac{B_2 + Q_2 + a_1 - B_1 - Q_1 - \beta\sigma_1^2 - G_2}{\sqrt{\sigma_1^2 + \sigma_2^2 + \gamma_2^2}}\right)\right)$$

From this we can obtain the conditional distribution of a_2 given a_1 for events seen at both stations. We get

(A2.33) $g(a_2/a_1) = G(a_1 a_2)/g(a_1) =$

$$= \frac{\Phi\left(\frac{a_2 - G_2}{\gamma_2}\right) \frac{1}{\sqrt{2\pi}\sigma} \exp[-(a_2 - a_1 - B)^2/2\sigma^2]}{\Phi\left(\frac{a_1 + B - G_2}{\sqrt{\sigma^2 + \gamma_2^2}}\right)}$$

where $B = B_2 + Q_2 - B_1 - Q_1 - \beta\sigma_1^2$ and

$\sigma^2 = \sigma_1^2 + \sigma_2^2$

This distribution is of the same type as the distribution of observed log (A/T) for given true magnitude eq. (A1.1) so the first two moments of $g(a_2/a_1)$ are given by eq. (A1.10) and eq. (A1.14). On the other hand, had we also considered the events not seen at station 2 given that the events were recorded at station 1 it is readily seen that we obtain the following distribution

(A2.34)

$$g(a_2/a_1)da_2 = \begin{cases} \Phi\left(\frac{a_2 - G_2}{\gamma_2}\right) \frac{1}{\sqrt{2\Pi}\,\sigma} \exp[-(a_2 - (a_1 + (B_2 + Q_2 - B_1 - Q_1 - \sigma_1^2\beta))^2/2\sigma^2]\, da_2 & \text{if seen at station 2} \\ \Phi\left(\frac{-(a_1 + (B_2 + Q_2 - B_1 - Q_1 - \sigma_1^2\beta)) - G_2}{\sqrt{\sigma^2 + \gamma_2^2}}\right) & \text{if not seen at station 2} \end{cases}$$

The log likelihood based on (A2.33) for a sample of N independent observations is except for a constant

$$\text{(A2.35)} \log L = \sum_{i}^{N} -N\log\gamma - (a_{2i} - Q_{2i} - (a_{1i} - Q_{1i}) - B)^2/2\sigma^2 + \log\Phi\left(\frac{a_{2i}-G_2}{\gamma_2}\right)$$

$$- \log\Phi\left(\frac{a_{1i}+B+Q_{2i}-Q_{1i}-G_2}{\sqrt{\sigma^2+\gamma_2^2}}\right)$$

where $B = (B_2 - B_1 - \beta\sigma_1^2)$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$

Putting $G = G_2$ and $\gamma = \gamma_2$, $Q_i = Q_{2i} - Q_{1i}$

$$\text{(A2.36)} \qquad x_i = (B + Q_i + a_{1i} - G)/\sqrt{\sigma^2+\gamma^2}$$

$$\text{(A2.37)} \qquad y_i = (a_{2i} - G)/\gamma$$

and

$$\text{(A2.38)} \qquad z(x) = \frac{\Phi'(x)}{\Phi(x)}$$

The first order derivatives are

$$\text{(A2.39)} \qquad \frac{\partial \log L}{\partial B} = \sum_{i=1}^{N} (a_{2i} - B - Q_i - a_{1i})/\sigma^2 - \frac{1}{\sqrt{\sigma^2+\gamma^2}} z(x_i)$$

$$\text{(A2.40)} \qquad \frac{\partial \log L}{\partial G} = -\sum_{i-1}^{N} z(y_i)/\gamma + \frac{1}{\sqrt{\sigma^2+\gamma^2}} z(x_i)$$

$$\text{(A2.41)} \qquad \frac{\partial \log L}{\partial \sigma} = \frac{-N}{\sigma} + \sum_{i=1}^{N} (a_{2i} - B - Q_i - a_{1i})^2/\sigma^3 + \frac{\sigma x_i}{(\sigma^2+\gamma^2)} z(x_i)$$

$$\text{(A2.42)} \qquad \frac{\partial \log L}{\partial \gamma} = \sum_{i-1}^{N} - y_i z(y_i)/\gamma + \frac{\gamma\ x_i}{(\sigma^2+\gamma^2)} z(x_i)$$

The likelihood estimator is then obtained by solving

$$\frac{\partial \log L}{\partial \beta} = \frac{\partial \log L}{\partial G} = \frac{\partial \log L}{\partial \sigma} = \frac{\partial \log L}{\partial \gamma} = 0$$

Let $\theta_0' = (B,G,\sigma,\gamma)$ denote the true parameter set and $\hat{\theta} = (\hat{B},\hat{G},\hat{\sigma},\hat{\gamma})$ denote the likelihood estimate. It then follows directly from the general properties of the maximum likelihood method that $\sqrt{N}(\theta_0-\hat{\theta})$ is asymptotically normally distributed with expectation zero and covariance matris given by

$$\left(-\frac{1}{N}\,E\left(\frac{\partial^2 \log L}{\partial\theta^2/\theta=\theta_0}\right)\right)^{-1}$$

which may be estimated by

$$\left(-\frac{1}{N}\;\frac{\partial^2 \log L}{\partial\theta^2/\theta=\hat{\theta}}\right)^{-1}$$

Turning to these second order derivatives we find after some calculation

$$\text{(A2.43)}\qquad \frac{\partial^2 \log L}{\partial B^2} = \frac{N}{\sigma^2} + \frac{1}{\sigma^2+\gamma^2}\sum_{i=1}^{N} z(x_i)(x_i+z(x_i))$$

$$\text{(A2.44)}\qquad \frac{\partial^2 \log L}{\partial G^2} = \sum_{i=1}^{N} -z(y_i)(y_i+z(y_i))/\gamma^2 + \frac{1}{\sigma^2+\gamma^2}\,z(x_i)(x+z(x_i))$$

$$\text{(A2.45)}\qquad \frac{\partial^2 \log L}{\partial \sigma^2} = \frac{N}{\sigma^2} + \sum_{i=1}^{N} -(a_{2i}-B-Q_i-a_{2i})^2\cdot 3/\sigma^4 +$$

$$+\frac{x\;z(x\;)}{(\sigma^2+\gamma_2^2)^2}\;[\sigma^2(x_i(x_i+z(x_i))-2)+\gamma^2]$$

$$(A2.46)\quad \frac{\partial^2 \log L}{\partial \gamma^2} = \sum_{i=1}^{N} \frac{y_i z(y_i)}{\gamma^2} (2-y_i(y_i+z(y_i)) +$$

$$+ \frac{x_i z(x_i)}{(\sigma^2+\gamma^2)^2} [\gamma^2(x_i(x_i+z(x_i))-2) + \sigma^2]$$

$$(A2.47)\quad \frac{\partial^2 \log L}{\partial B \partial G} = \sum_{i=1}^{N} - \frac{z(x_i)}{\sigma^2+\gamma^2} (x_i + z(x_i))$$

$$(A2.48)\quad \frac{\partial^2 \log L}{\partial B \partial \sigma} = \sum_{i=1}^{N} - (a_{2i}-B-Q_i-a_{1i})2/\sigma^3 -$$

$$- \frac{\sigma z(x_i)}{(\sigma^2+\gamma^2)3/2} (x_i(x_i+z(x_i)-1)$$

$$(A2.49)\quad \frac{\partial^2 \log L}{\partial B \partial \gamma} = \sum_{i=1}^{N} - \frac{\gamma z(x_i)}{(\sigma^2+\gamma^2)3/2} (x_i(x_i+z(x_i))-1)$$

$$(A2.50)\quad \frac{\partial^2 \log L}{\partial G \partial \sigma} = \sum_{i=1}^{N} \frac{\sigma}{(\sigma^2+\gamma^2)3/2} (x_i(x_i+z(x_i)-1)$$

$$(A2.51)\quad \frac{\partial^2 \log L}{\partial G \partial \gamma} = \sum_{i=1}^{N} - \frac{z(y_i)}{\gamma^2} (y_i(y_i+z(y_i)-1) +$$

$$+ \frac{\gamma z(x_i)}{(\sigma^2+\gamma^2)3/2} (x_i(x_i+z(x_i))-1)$$

and finally

$$(A2.52)\quad \frac{\partial^2 \log L}{\partial \sigma \partial \gamma} = \sum_{i=1}^{N} - \frac{\gamma\sigma x_i z(x_i)}{(\sigma^2+\gamma^2)^2} (3-x_i(x_i+2(x_i)))$$