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THE CCTC QUICK-REACTING GENERAL WAR GAMING SYSTEM (QUICK) PROGR--ETC(U)
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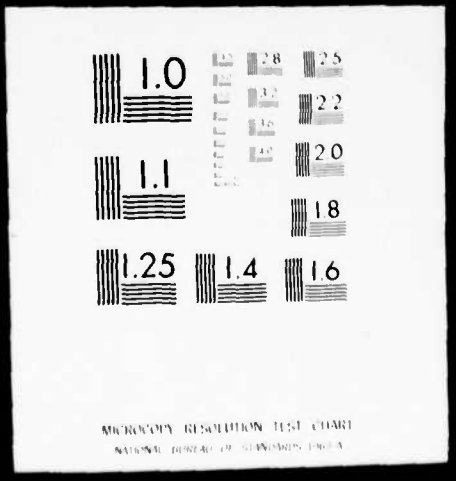

for all weapon groups) and so for efficiency the variable NOWPS(2) is used. While the expected value of $(ALERREST(1))^2$ is the same as $\sum_G (ALERREST(J))^2$, the variance of the latter version is much less, and it is therefore preferable as an estimator of the expected integration period, EXPINTPD and is:

$$EXPINTPD = NOWPS(1)(ALERREST(1)^2)*NTGTS)$$

To allow the possibility of using integration periods either longer or shorter than the theoretical EXPINTPD, a desired longest integration period DESINTPD is defined:

$$DESINTPD = EXPINTPD*RATIOINT$$

where RATIOINT is an adjustable input parameter. A low value allows higher sensitivity without oscillations in the values of the Lagrange multipliers but too low a value makes convergence to the correct stockpile sensitive to statistics of the target list. If the target list contains targets with heavy ballistic missile defenses or if a large fraction of the weapons are assigned by the fixed assignment capability, this parameter value should be increased (to 4.0 or above if necessary). If this period were used exactly in setting the rate of change of the target weight (i.e., WRATE = 1.0/DESINTPD), the WRATE would never become exactly zero as is required for a constant target weight. Obviously when the change in the target weight becomes small over a full pass, the WRATE should be allowed to go to zero. Therefore in:

$$WRATE = (1.0/DESINTPD) - (2.0/(NTGTS*RATIOINT))$$

the term (2./NTGTS*RATIOINT)) is subtracted, and if the resulting WRATE is negative it is set to zero. To avoid a situation where large errors cause the integration period to become ridiculously small, a limit that WRATE $\leq$ .07 is set.

Moreover, after the allocation is well under way, PROGRESS $\geq$ .5, the value of WRATE is not allowed to increase. In the program WTRATE(INTPRD) is used as a multiplier of the target weight; therefore we add 1.0 to WTRATE to obtain a suitable multiplier for the longest period NINTPRD.

The values for the three WTRATE variables are:

$$WTRATE(3) = 1 + WRATE$$

$$WTRATE(2) = 1 + WRATE + \frac{RINTPRD-1}{NTGTS} + RINTPRD$$

$$WTRATE(1) = 1 + WRATE + \frac{RINTPRD-1}{NTGTS} + 2.*RINTPRD$$

Input parameter RINTPRD is an approximate ratio between rate of change of target weights between different integration periods. An increase in

this parameter increases the sensitivity of the multiplier adjustment to recent target experience.

To restate, Lagrange multipliers are recomputed based on variable PROGRESS and after a specific number of targets have been processed. The adjustment is based on maintaining statistics of weapon allocation rates. The differences in true and observed rates, along with input sensitivity parameters, make up the formula for multiplier adjustment.

A.6 Derivation of Formula for Correlations in Weapon Delivery Probability

An exact calculation of the probability of target survival when it is subject to attack by correlated weapons is very lengthy. Both the conventional statistical analysis and the Bayesian incremental information approach have been examined. Both approaches for each time and hardness require the calculation component of the interaction terms between each weapon to be added with all possible combinations of the weapons already on the target. Thus the completely rigorous calculation would be impractical in a rapid response allocator. The method used here is based on an approximation derived from the properties of the log-gamma distribution.

When a group of weapons share a common failure risk the probability of success is likely to be either high or low for all weapons collectively. Thus the probability of success can itself be thought of as a random variable. For any chance value of this overall random variable there will exist the usual independent probabilities for individual weapons. However, on one trial the overall success probability for the group of weapons may be 90%, while in another trial it may be 50% depending on the particular success probability drawn for the trial.

The following mathematical model has been developed to deal with this type of problem. We assume that the probability of survival of a target with respect to the i^{th} weapon is itself a random variable S of the form

$$S_i = e^{-X_i}$$

where the X_i are random variables drawn from a known distribution.

If two weapons are involved, then the probability of survival with respect to both can be represented by the random variable S_T:

$$S_T = S_i S_j = e^{-(X_i + X_j)}$$

However, the random variables X_i and X_j may or may not be independent. If they are not independent then of course

$$\langle S_i S_j \rangle \quad \langle S_i \rangle \langle S_j \rangle$$

If the X_i are independently drawn from a known two-parameter family of distribution with a convolution property,* then the distribution of $X_i + X_j$ will of course be a member of the same distribution family. Moreover, since any probability distribution for the X_i implies a distribution for the corresponding S_i, the distribution for $S_i S_j$ can be calculated and the value for $\langle S_i S_j \rangle$ can be computed.

The gamma distribution given by:

$$P(X)dx = \frac{X^a e^{-X/b}}{b^{a+1}\Gamma(a+1)} dx \text{ for } X \geq 0$$

$$P(X) = 0 \text{ for } X \leq 0$$

is a well known two-parameter distribution with the required convolution property.

The gamma distribution is unique among convolving two-parameter distributions in that the expected value of e^{-X} is easily computed. This property is particularly important for QUICK since the damage function performs a computation of this value many times during the allocation. The expected value of e^{-X} is given by:

$$\langle e^{-X} \rangle = \int_0^\infty P(X) e^{-X} dX$$

which can be written

$$\langle S \rangle = \langle e^{-X} \rangle = \left(\frac{1}{b+1} \right)^{a+1}$$

This distribution is valid for $b > 0$ and $a > -1$. It has a mean $\mu = b(a+1)$ and a variance $\sigma^2 = b^2(a+1)$.

Since this distribution is completely defined by the mean and variance, the actual probability distribution of S can be computed at any time so long as a record of the mean and variance of the distribution is maintained. We now observe that:

$$a + 1 = \mu^2/\sigma^2$$

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* A probability distribution is said to "convolve" when the convolution of any two distributions in the family (i.e., the distribution of the sum of the two random variables) is itself a member of the same family.

and

$$b = \sigma^2/\mu$$

so the expected value of S can be written

$$\langle S \rangle = \left(\frac{1}{\frac{\sigma}{\mu} + 1} \right)^{\mu^2/\sigma^2}$$

or

$$-\ln \langle S \rangle = \frac{\mu^2}{\sigma^2} \ln \left(\frac{\sigma^2}{\mu} + 1 \right)$$

This distribution is sufficiently flexible to include almost any shape distribution of interest. For σ small the distribution in S approximates a gaussian centering on some specific survival probability. As the σ is increased the distribution widens, so that it can approximate a uniform probability from zero to one, or a sloping probability with more weight on zero or one. In the limit of very large σ the distribution consists essentially of spikes of different weight at zero and one.

If we were dealing with <u>independent</u> weapons we could calculate the parameters for the multiple weapon distribution from those for the single weapon distributions simply by making use of the additivity of the mean and the variance. Specifically the mean, μ_T, for the new distribution and the variance σ_T^2 would be given by:

$$\mu_T = \sum_i \mu_i$$

$$\sigma_T^2 = \sum_i \sigma_i^2$$

The expected value of target survivability S_T for the new distribution would then be obtainable through the equation:

$$- \ln \langle S_T \rangle = \frac{\mu_T^2}{\sigma_T^2} \ln \left[\left(\frac{\sigma_T^2}{\mu_T} \right) + 1 \right]$$

However, the variance is directly additive as above only if the weapons are really independent. To introduce the possibility of correlations we will write the variance as follows:

$$\sigma_T^2 = \sum_i \sum_j \sigma_i \Gamma_{ij} \sigma_j$$

where the quantity Γ_{ij} represents the correlation between the weapons. In the special case of uncorrelated weapons, $\Gamma_{ij} = 0$ for $i \neq j$ and 1 for $i = j$, which is identical with the previous form.

This approach of arbitrarily introducing the cross terms in this formulation to approximate the actual correlations is exact so long as the correlations are of such a form that the distribution of X remains a gamma distribution. To the extent that the actual correlations cause departures from the Γ distribution the approximation is in error. The correlation model thus amounts to the assumption that correlations can be adequately modeled without going outside the log-gamma distribution.

For implementation it seems appropriate to introduce an additional simplification. In the foregoing formuation the magnitude of the penalty for using correlated weapons will depend not only on the size of the correlation and the kill probability for the correlated weapons, but also on the shape of the distribution for the success probability for each weapon. This shape dependence introduces a complicating variable which undoubtedly exists, but for which it would not be easy to get data. It therefore seems desirable to eliminate this factor.

This can be done by standardizing on a single shape factor for all calculations of the effects of correlations. It is easiest to do this by considering only distributions with a very large σ, which are essentially spikes on zero and one. This choice tends to exaggerate the importance of correlations (and this fact should be borne in mind in assigning the correlations for the war game) but it significantly simplifies the data required, as well as the computation of the payoff.

In the limit of large σ the quantity σ_i^2/μ_i approaches infinity while the quantity μ_i^2/σ_i^2 compensates to maintain the correct value of $-\ell n \langle S_i \rangle$

To illustrate the transition to this limit we let $b_i = \sigma_i^2/\mu_i$ and define

$$\beta_i = b_i/\ell n(b_i + 1)$$

Then

$$-\ell n \langle S_i \rangle = \mu_i/\beta_i$$

so:

$$\mu_i = \beta_i \left[-\ell n \langle S_i \rangle\right]$$

and

$$\sigma_i^2 = \mu_i b_i = b_i \beta_i \left[-\ell n \langle S_i \rangle\right]$$

The formula for obtaining the expected value of S_T can now be written

$$-\ell n \ \langle S_T \rangle = \frac{\mu_T^2}{\sigma_T^2} \ell n(b_T + 1)$$

and substituting,

$$\mu_T = \Sigma \mu_i \text{ and } \sigma_T^2 = \sum_{ij} \sigma_i \ \Gamma_{ij} \sigma_j$$

we obtain:

$$-\ell n \ \langle S_T \rangle = \frac{\left(\Sigma \beta_i \left[-\ell n \ \langle S_i \rangle \right] \right)^2 \left(\ell n(b_T + 1) \right)}{\sum_i \sum_j b_i \beta_i \left[-\ell n \ \langle S_i \rangle \right]^{1/2} \Gamma_{ij} \left[-\ell n \ \langle S_j \rangle \right]^{1/2}}$$

We now assign to all weapons the same value of b_i, so that all b_i are equal and all β_i are equal and we obtain:

$$-\ell n \ \langle S_T \rangle = \frac{\ell n(b_T + 1)}{\ell n(b_i + 1)} \frac{\left[\Sigma \ (-\ell n \langle S_i \rangle) \right]^2}{\Sigma \Sigma \ (-\ell n \langle S_i \rangle)^{1/2} \ \Gamma_{ij} \ (-\ell n \langle S_j \rangle)^{1/2}}$$

If we now let b_i approach infinity the ratio of the two logarithmic quantities will approach 1. Note that

$$b_T = \frac{\sigma_T^2}{\mu_T}, \qquad \text{so } b_T = \frac{\Sigma \Sigma \sigma_i \ \Gamma_{ij} \sigma_j}{\Sigma \mu_i}$$

It follows that $b_T \geq b_i$ and $b_T \leq \eta^2 \ b_i$, where η is the number of weapons. The limiting case $b_T = \eta^2 \ b_i$ occurs when all $\Gamma_{ij} = 1$ and all μ_i are equal. Therefore so long as $b_i \gg \eta^2$ the ratio of the logarithms will be essentially 1, and in the limit as b_i approaches infinity we obtain simply:

$$-\ell n \langle S_T \rangle = \frac{\left[\Sigma \ -\ell n \langle S_i \rangle \right]^2}{\Sigma \Sigma \left(-\ell n \langle S_i \rangle \right)^{1/2} \Gamma_{ij} \left(-\ell n \langle S_j \rangle \right)^{1/2}}$$

For compactness of notation let us identify the quantities

$$\mu_i = (-\ell n \langle S_i \rangle) \quad \text{and} \quad \mu_T = (-\ell n \langle S_T \rangle)$$

Then since $\Gamma_{ij} = 1$ if $i = j$ we obtain

$$\mu_T = \frac{\left[\sum_i \mu_i\right]^2}{\sum_i \mu_i + \sum_i \sum_{j=i} \left(\mu_i\right)^{1/2} \Gamma_{ij} \left(\mu_j\right)^{1/2}}$$

or equivalently

$$\mu_T = \frac{\left[\sum_i \mu_i\right]^2}{\sum_i \mu_i + \sum_i \sum_{j<i} \left(\mu_i\right)^{1/2} 2\,\Gamma_{ij} \left(\mu_j\right)^{1/2}}$$

This form has the basic properties desired. Notice there is only one interaction term between each pair of weapons. In addition, only two sums need to be maintained to compute μ_T. These are:

$$MU = \sum_i \mu_i$$

$$SIG = \sum_i \sum_{j<i} \left(\mu_i\right)^{1/2} 2\Gamma_{ij} \left(\mu_j\right)^{1/2}$$

From these the value μ_T is given simply:

$$\mu_T = (MU)^2 \; / \; (MU + SIG)$$

The addition of any new weapon adds one term to the MU sum, and several terms to the SIG sum.

The computation of the first sum is trivial; however, before the second one can be used it is necessary to provide a practical method of estimating Γ_{ij}.

We recall that the array RISK (A,G,J) was computed as an estimate of shared risk, and that:

$$RISK(A,G,J) = \sum_{L=1,5} SM(L)*SMAT(A,L)$$

For a particular weapon G and hardness component J, this relation might look as follows: (A is a weapon attribute index; L is a failure mode index.)

		SMAT(A,L)						
	A =	1	2	3	4	5	6	
L	SM(L)	All	Group	Reg	Class	Type	Alert	Independent Risk
1	-LOGF(DBL) = .20	.00	.10	.10	.10	.10	.40	.20
2	-LOGF(CC) = .00	.00	.10	.30	.10	.10	.30	.10
3	-LOGF(REL) = .05	.00	.05	.00	.10	.20	.00	.65
4	-LOGF(PEX) = .20	.00	.00	.10	.20	.20	.00	.50
5	-LOGF(STK) = .02	.00	.00	.00	.00	.00	.00	1.00
	RISK(A,G,J)	.000	.0225	.040	.065	.070	.08	.1925

Thus the SMAT array, a user input estimate of shared risk, is used simply to divide the five types of risk SM(L) between the independent weapon risk, and the six factors A that any two weapons might have in common. The total RISK over all A plus the independent risk is of course equal to the sum of SM(L). We are now interested in using the RISK array to derive reasonable values for the correlation coefficients Γ_{ij}.

The RISK array thus represents the amount of the risk for each weapon that is likely to be correlated with other weapons of the same class, type, etc.

The correlation coefficients should reflect the shared risk. If two weapons have only two attributes A in common then the shared risk should come only from these two common attributes. Moreover, the amount of risk that can be shared on the basis of one attribute cannot exceed the minimum risk associated with that attribute for either weapon. Therefore, to estimate the maximum risk, γ_{ij}, that can be shared by two weapons, i and j, we define:

$$\gamma_{ij} \text{ or } GAM(k,j) = \sum_{A} \delta\,(A_i,A_j)*\text{Min } RISK(A_i,G_i,J)RISK(A_j,G_j,J)$$

where $\delta = 0$ if $A_i \neq A_j$ and $\delta = 1$ if $A_i = A_j$.

The coefficients Γ_{ij} however must never exceed 1.0. Therefore it is appropriate to divide the shared risk GAM(i,j) by $\sum_{L}$ SM to obtain a normalized fraction guaranteed to be less than 1.0.

Thus the form of the second summation

$$SIG = \sum_{i} \sum_{j<i} 2(\mu_i)1/2\ \Gamma_{ij}(\mu_j)1/2$$

would become

$$SIG = \sum_{i} \sum_{j<i} 2(\mu_i)^{1/2}\ \frac{GAM(i,j)}{\sum_{L} SM}(\mu_j)^{1/2}$$

However, this form involves square roots which are inconvenient. Moreover, it represents an upper limit of correlation. We can reduce the size of the overestimate by using the largest (or maximum) $\sum_L SM$; i.e., using the least reliable weapon for normalization. In addition, we can simplify the form and provide for the removal of square roots if we also multiply by $(\mu_{min}/\mu_{max})^{1/2}$. (This is a factor less than 1.0 that has the effect of reducing slightly the assumed correlation between weapons of very different overall effectiveness.)

With these changes, the equation for SIG takes the form of

$$SIG = \sum_i \sum_{j<i} 2(\mu_i)^{1/2} \left| \frac{GAM(i,j)}{\text{Max} \sum_L SM} * \left(\frac{\mu min}{\mu max}\right)^{1/2} \right| (\mu_j)^{1/2}$$

The form in braces is still guaranteed to fall between zero and 1.0. It represents the actual form for Γ_{ij} used in the present version of the Allocator. This form has a computational advantage in that it simplifies the calculation of SIG. Assume that $\mu_i \leq \mu_j$. Then

$$\sum_L SM_i \geq \sum_L SM_j \qquad \text{and so}$$

$$SIG = \sum_i \sum_{j<i} 2(\mu_i)^{1/2} \left| \frac{GAM(i,j)}{\sum_L SM_i} * \left(\frac{\mu_i}{\mu_j}\right)^{1/2} \right| (\mu_j)^{1/2}$$

This reduces to:

$$SIG = \sum_i \sum_{j<i} 2*GAM(k,j)*\min_{i,j} \left| \frac{\mu_i}{\sum_L SM_i} \right|$$

This is the actual form used computationally. (For each weapon group G the quantity $\mu / \sum_L SM$ is identified in the FORTRAN as SSIG(G,J).)

The specific formula used for the terms in SIG is of heuristic origin and is obviously somewhat arbitrary. It is justified, in the final analysis, by the fact it is fairly simple and that it works. The resulting kill probabilities produce realistic cross targeting, and in cases where these probabilities can be compared with a rigorous statistical model of correlations, it produces a satisfactory approximation to the kill probability.

In summary, the mathematics is as follows:*

For a single weapon let

SSK = single shot kill probability, and let

SSS = single shot target survival probability

then SSK is given by

$$-\text{LOGF(SSK)} = \sum_{L} \text{SM(L)}$$

As usual, SSS = 1.0 - SSK, and we define μ_i or MUP for group G_i relative to hardness component J as:

$$\underline{\text{MUP(G,J)} = -\text{LOGF(SSS)}}$$

We also define SSIG(G,J) as:

$$\underline{\text{SSIG(G,J)} = \text{LOGF(SSS)/LOGF(SSK)} = \text{MUP(G,J)} / \sum_{L} \text{SM(L)}}$$

Finally we define RISK(A,G,J) as:

$$\underline{\text{RISK(A,G,J)} = \sum_{L=1,5} \text{SM(L)*SMAT(A,L)}}$$

The preceding three arrays (underlined for emphasis) are the main input for the estimation of kill probabilities.

The target survivability relative to multiple weapons S_T is given by

$$S_T = e^{-\mu T}$$

where $\mu T = (MU)^2 / (MU + SIG)$

and where $MU = \sum_i \mu_i = \sum_i MUP(G_1,J)$

and $SIG = \sum_i \sum_{j<i} 2(\mu_i)^{1/2} \Gamma_{ij} (\mu_j)^{1/2}$

The individual terms in SIG for specific i and j can be thought of as:

$$DSIG(i,j) = 2(\mu_i)^{1/2} \Gamma_{i,j} (\mu_j)^{1/2}$$

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* The displayed mathematics for the calculation of MUP are for the exponential damage law. The derivation of the quantity, MUP, required for use of the square root damage law is discussed in the Derivation of Square Root Damage Function section of this chapter and are not of any importance in this discussion of correlation effects.

which we identify computationally as

$$DSIG(k,j) = 2*GAM(i,j)*\underset{k=i,j}{Min}\left\{SSIG(G_k,J)\right\}$$

where GAM(i,j), the maximum risk shared by i and j, is estimated as

$$GAM(i,j) = \sum_{A} \delta(A_i,A_j)*Min\left\{RISK(A_i,G_i,J),\ RISK(A_i,G_j,J)\right\}$$

where δ, the Kroniker δ, is 0 if $A_i \neq A_j$, and 1 if $A_i = A_j$.

The simple form used for DSIG above implies that Γ_{ij} has the form:

$$\Gamma_{i,j} = \frac{GAM(i,j)}{\underset{i,j}{Max}\left[\sum_{L} SM(L)\right]} * \left(\frac{\mu Min}{\mu Max}\right)^{1/2}$$

however, this form never enters explicitly into the calculations.

To combine this treatment for the analysis of weapon correlations with the preceding treatment of time dependent target values we simply use the S_T evaluated above to supply the S(NI,J) required in the formula

$$VT = \sum_{J=1}^{J=M} \sum_{NI=0}^{NI=NN} [V(NI,J) - V(NI+1,J)] * S(NI,J)$$

The weapons to be included in the evaluation S_T for any NI are of course those on the target up to and including the time NI.

This, of course, requires that separate sums for MU and SIG be maintained for each relevant time interval, NI, and each hardness component J. Thus these variables are actually two dimensional arrays MU(NI,J) and SIG(NI,J). Moreover, every potential payoff estimate (both for each weapon that might be added, and for each that might be deleted) requires a separate complete set of sums.

Derivation of Damage Functions

A Universal Damage Function: Consider the situation for which the lethal radius and CEP of a single weapon are small compared to the target dimensions. This case becomes quite pertinent under any of the following circumstances:

Very large cities

Targets whose uncertainty of location is larger than the area of influence of a weapon

Employment of large numbers of small weapons (e.g., cluster warheads)

Hardening which reduces effective weapon radius below target size (e.g., blast shelters for urban population).

In such a situation, where the value density of the target does not vary significantly over the area of effect of a single weapon, one can usefully employ the concept of weapon density (weapons targeted per unit area) and seek the weapon density as a function of value density which optimizes the total target destruction for a given total number of weapons.

Before such an optimization can be effected, however, it is necessary to obtain the relationship between the weapon density applied to a subregion, expressed for convenience as the fraction of the original value surviving. In the most general case, this function can vary with position in the target, reflecting the possibility of varying degrees of vulnerability over the target.

We introduce the following notation:

X	Position within target (x, y coordinates)
$\omega(X)$	Density of weapons targeted in vicinity of X (number/unit area)
V(X)	Target value density in vicinity of X (value/unit area)
$F(\omega)$	Fraction of destruction produced by weapon density ω, in the absence of hardening
$\mu(X)$	Vulnerability (hardening) factor ($0 \leq \mu \leq 1$) expressed as effective degradation of weapon density
W	Total number of weapons intended against target.

The total payoff for a given weapon density distribution is then given by:

$$H = \int_A VF(\mu\omega)\,dA \qquad (1)$$

where the integration is understood to be over the whole target area, and dA is the area element.

Similarly, the total number of planned weapons is given by:

$$W = \int_A \omega\,dA$$

We seek now the weapon density distribution which maximizes the payoff for a given W. Introducing a Lagrange multiplier ≤ 0, and applying the generalized method described above,* we seek the weapon density function which maximizes the unconstrained Lagrangian.

$$L = H - \lambda W \tag{3}$$

This is equivalent to maximizing:

$$L = \int_A [VF(\omega\mu) - \lambda\omega]\,dA \tag{4}$$

The density function ω_λ which maximizes this Lagrangian for a given λ is obtained simply by maximizing the expression inside the integral at each point (see cell problem discussion in Everett's paper, appendix C). The optimum density at any point is therefore a solution of:

$$\underset{\omega}{MAX} = \{VF(\mu\omega) - \lambda\omega\} \tag{5}$$

For the case where F is monotone increasing, concave (diminishing returns), and differentiable, an internal maximum of (5) can be sought by zeroing its derivative:

$$\frac{d}{d\omega}[VF(\mu\omega) - \lambda\omega] = VF'(\mu\omega_\lambda)\ \mu - \omega = 0 \tag{6}$$

Letting $G = (F')^{-1}$ stand for the inverse function of the derivative of F leads to:

$$\omega_\lambda = \frac{1}{\mu} G\ \frac{\lambda}{V\mu} \tag{7}$$

Equation (7) gives the internal maximization of (5). To complete the solution we must account for the constraint $\omega^* \geq 0$ (negative densities are not allowed). Thus the optimum is given by (5) only if $\omega_\lambda^* \geq 0$ and if $VF(\mu\omega) - \lambda\omega \geq 0$, since otherwise (5) is maximized by $\omega = 0$. The complete solution can therefore be stated:

$$\omega\lambda = \begin{cases} \frac{1}{\mu} G\ \frac{\lambda}{V\mu} & \text{if } \omega_\lambda \geq 0 \text{ and } VF(\mu\omega) - \lambda\omega \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

(This solution is also valid even if F is not concave -- a situation in which G may be multivalued -- provided that one uses that value of $G(\lambda/V\mu)$ for which $VF(\mu\omega) - \lambda\omega$ is a maximum.)

Observe that the optimum density given by (8) is a function only of V and μ, and is explicitly independent of position. If we can further assume that the vulnerability μ is a function only of the value density V and is otherwise independent of position,* then we can simplify the formulation and solution somewhat. In this case, all pertinent target characteristics are summarized by two functions:

$A(V)$ = total area of those areas whose value density is greater than V

$\mu(V)$ = vulnerability factor as a function of value density

The optimum weapon density ω_λ given by (8) becomes then a function only of the value density V:

$$\omega_\lambda(V) = \begin{cases} \frac{1}{\mu(V)} G\left(\frac{\lambda}{V\mu(V)}\right) & \text{if } \omega_\lambda \geq 0 \text{ and } VF(\mu\omega) - \lambda\omega \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

and the total payoff and total weapons are given in the simple form of Stieltjes integrals:

$$H_\lambda = -\int_0^\infty VF(\omega_\lambda \mu(V))dA(V)$$

$$W_\lambda = -\int_0^\infty \omega_\lambda \, dA(V) \tag{10}$$

This completes the general optimization of weapon density. For explicit solutions we require specific functions for the target value distribution function $A(V)$, the destruction function $F(\omega)$, and the vulnerability distribution $\mu(V)$. We shall now consider several pertinent cases.

*Which seems generally quite plausible, and is in any case certainly true if the variation of μ arises from optimization of shelter deployment, for example.

Locally Random Impact Model: When the CEP is not significantly smaller than the lethal radius, or when the delivery probability of individual weapons is low, the situation over any homogeneous part of the target can be closely approximated by regarding the weapons as having been dropped uniformly at random over that part.

Consider, therefore, a region of area A (large compared to the lethal area of a single weapon) into which N weapons each with lethal area πR_K^2 and delivery probability P are delivered uniformly and independently at random. The probability that any given point in the region will survive one weapon is:

$$S(1) = 1 - \frac{P\pi R_K^2}{A} \tag{11}$$

and, since weapon arrivals are independent events, the probability of surviving N is:

$$S(N) = \left(1 - \frac{P\pi R_K^2}{A}\right)^N \tag{12}$$

Introducing the parameters K and ω:

$$K = P\pi R_K^2 = \text{expected lethal area of one weapon}$$

$$\omega = N/A \quad = \text{weapon density} \tag{13}$$

allows (12) to be written as:

$$S(\omega) = \left(1 - \frac{K\omega}{N}\right)^N \tag{14}$$

This gives for the destruction function:

$$F_N(\omega) = 1 - S(\omega) = 1 - \left(1 - \frac{K\omega}{N}\right)^N \tag{15}$$

Equation (15) still contains an extra parameter, N, which is the number of weapons in the area A used to derive (12)--presumed large compared to the effects of a single weapon and small compared to the total target size. We are currently interested in the limit as this area A becomes infinite compared to the effects of a single weapon, hence in the limit as $N \longrightarrow \infty$:

$$F_\infty(\omega) = \lim_{N \longrightarrow \infty} F_N(\omega) = 1 - e^{-K\omega} \tag{16}$$

which becomes our final destruction function for the locally random impact model.

"Perfect" Weapon Model: At the other extreme from the locally random impact model is the hypothetical situation where the weapons have zero CEP, delivery probability of unity, and completely destroy a hexagonal region of area K with no damage outside the region.

This situation closely resembles the case of "cookie-cutter" weapons of zero CEP and unit delivery probability, and deviates from the latter only when the area covered is so densely packed that the "cookie-cutter" circles begin to overlap--which does not occur until the fractional coverage exceeds $\pi/(2\sqrt{3})$ or about .91.

For such "perfect" weapons the destruction fraction is given by:

$$F = \begin{cases} K\omega & \omega < 1/K \\ 1 & \omega \geq 1/K \end{cases} \tag{17}$$

Intermediate Cases: We have considered two extremes, locally random impact, and perfect weapons. For actual situations, the targeting will not be random, but some optimum pattern of DGZs.

As the CEP becomes larger than the lethal radius, or the delivery probability becomes small, the situation--even though based on a pattern of DGZs--approaches a situation described by the random impact moded. On the other hand, for high delivery probability and small CEP, the situation begins to approach the "perfect" weapon case--particularly as the weapon effect radius becomes sharp (close to "cookie-cutter"--e.g., the conventional σ_{20} model).

Returning to the destruction function given by (15) containing the extra parameter N (from which the random model was obtained by letting $N \longrightarrow \infty$), we observe the remarkable fact that for N=1, this function is precisely the damage function (17).

Since this function contains, for the extreme values of N, the two limits we have considered, it seems reasonable to suppose that any actual intermediate case could be adequately approximated by this function for some intermediate value of N.

We shall accordingly adopt this general function as our destruction function, subject to subsequent empirical verification.

The general law therefore becomes:

$$F_N(\omega) = \begin{cases} 1 - \left(1 - \frac{K\omega}{N}\right)^N & \omega < \frac{N}{K} \\ 1 & \omega \geq \frac{N}{K} \end{cases} \tag{18}$$

For purposes of determining the optimum distribution of weapon density over a target of varying value density we wish to employ Eq. (9), for which we require the function $G = (F')^{-1}$. Accordingly,

$$F'_N(\omega) = \frac{d}{d\omega} F_N(\omega) = \begin{cases} K\left(1 - \frac{K\omega}{N}\right)^{N-1} & \omega < \frac{N}{K} \\ 0 & \omega \geq \frac{N}{K} \end{cases} \tag{19}$$

for which the inverse function is easily determined to be:

$$G_N(X) = \frac{N}{K}\left[1 - \left(\frac{X}{K}\right)^{1/(N-1)}\right] \tag{20}$$

Thus from (9), the optimum weapon density is given by:

$$\omega_\lambda(V) = \begin{cases} \frac{1}{\mu(V)} \frac{N}{K}\left[1 - \left(\frac{\lambda}{KV\mu(V)}\right)\frac{1}{N-1}\right] & \frac{\lambda}{KV\mu} < 1 \\ 0 & \frac{\lambda}{KV\mu} \geq 1 \end{cases} \tag{21}$$

and for which the destruction fraction is easily calculated:

$$F_N(\omega_\lambda\ \mu) = \begin{cases} 1 - \left(\dfrac{\lambda}{KV\mu(V)}\right)^{N/N-1} & \dfrac{\lambda}{KV\mu} < 1 \\ \\ 1 & \dfrac{\lambda}{KV\mu} \geq 1 \end{cases} \tag{22}$$

This completes the general treatment for arbitrary target value distributions.

Gaussian Targets: A particularly important special case is that of a Gaussian target, for which the value density distribution is given by:

$$V(x,y) = \frac{1}{2\pi\sigma^2}\, e^{-r^2/2\sigma^2} \tag{23}$$

(The total value is here normalized to unity.) From (23) we determine the relationship between radius and value to be:

$$r^2(V) = -2\sigma^2\ \ell n(2\pi\sigma^2 V) \tag{24}$$

and hence the cumulative area distribution function to be:

$$A(V) = \pi r^2(V) = -2\pi\sigma^2\ \ell n(2\pi\sigma^2 V) \quad \text{for } V \leq \frac{1}{2\pi\sigma^2} \tag{25}$$

and the differential element is:

$$dA(V) = -\frac{2\pi\sigma^2}{V}\, dV \tag{26}$$

Solution For Constant Vulnerability: Combining Eq. (10) with (26) and (22), and letting $\mu = 1$:

$$H_\lambda = \int_{\lambda/K}^{1/(2\pi\sigma^2)} V\left[1 - \left(\frac{\lambda}{KV}\right)^{\frac{N}{N-1}}\right]\left(\frac{2\pi\sigma^2}{V}\right)dV$$

$$= 1 - \frac{2\pi\sigma^2\lambda}{K} - (N-1)\left[\left(\frac{2\pi\sigma^2\lambda}{K}\right)^{\frac{N}{N-1}} - \frac{2\pi\sigma^2\lambda}{K}\right] \qquad (27)$$

Transforming the Lagrange multiplier λ to a new multiplier β:

$$\beta = \left[\frac{2\pi\sigma^2\lambda}{K}\right]^{1/(N-1)} \qquad (28)$$

we can rewrite (27) as:

$$H_\beta = 1 - \beta^{N-1}\left[1 + (N-1)\cdot(1-\beta)\right] \qquad (29)$$

The total numbar of weapons as given by (10), (21), and (26):

$$W_\lambda = \int_{\lambda/K}^{1/(2\pi\sigma^2)} \frac{N}{K}\left[1 - \left(\frac{\lambda}{KV}\right)^{\frac{1}{N-1}}\right]\left(\frac{2\pi\sigma^2}{V}\right)dV \qquad (30)$$

leads, in terms of β, to:

$$W_\beta = \frac{N(N-1)2\pi\sigma^2}{K}\left[\beta - \ell n(\beta - 1)\right] \qquad (31)$$

In order to permit explicit exhibition of payoff as a function of number of weapons, it is necessary to define a new function, γ, which is the inverse of

$$y - \ell ny - 1 = x \qquad (32)$$

that is, $y = \gamma(x)$. It is defined for all nonnegative arguments, with values on the interval zero-one. With this function, (29) and (31) can be rewritten, in terms of surviving value:

$$S = \beta^{N-1}\left[1 + (N-1)(1-\beta)\right]$$
$$\beta = T\left(\frac{KW}{2\pi\sigma^2 N(N-1)}\right) \tag{33}$$

Equations (33) summarize the relationship between surviving fraction, S, and number of weapons targeted, W, for Gaussian targets, and with a model parameter N, which can range from 1 to ∞.

The two limiting forms of (33), corresponding to $N = 1$ and $N \to \infty$ are interesting and important, and are easily shown to be:

$$S_1 = \exp(-KW/2\pi\sigma^2)$$
$$S_\infty = \left(1 + \frac{\sqrt{KW}}{\pi\sigma^2}\right)\exp\left(-\frac{\sqrt{KW}}{\pi\sigma^2}\right)$$

These are often termed the power law (or exponential law) and the square root law, respectively.

Derivation of Kill Probability Function

A variety of kill probability functions are in general use. The "normal model" employs a function of the form:

$$P_K(r) = e^{-r^2/2\sigma_K^2} \tag{34}$$

The "cookie-cutter" model employs a discontinuous function:

$$P_K(r) = \begin{cases} 1 & R_K \geq r \geq 0 \\ 0 & r > R_K \end{cases} \tag{35}$$

where R_K is the so-called "lethal radius." The relation between R_K and σ_K is obtained by equating lethal areas

$$\pi R_K^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2\sigma_K^2} \, r dr d\theta \tag{36}$$

leading to the relation

$$\sigma_K^2 = .5 R_K^2 \tag{37}$$

Other functions have often been used and, indeed, it has occasionally been found convenient to employ a generalized kill function of the form:

$$G_K(r) = e^{-K} \sum_{j=0}^{W-1} \frac{K_j}{j!} \tag{38}$$

where

$$K = \frac{Wr^2}{a^2}$$

Again, we can equate lethal areas to relate a with R_K:

$$\pi R_K^2 = \int_0^{2\pi} \int_0^{\infty} G_K(r) \, r dr d\theta \tag{39}$$

so that

$$R_K^2 = a^2 \text{ for all } W \tag{40}$$

The parameter W serves to alter the shape of this kill probability curve. Thus, $G_K(r)$ reduces to the normal curve for W = 1 and the cookie-cutter for $W \longrightarrow \infty$. Standard kill curves, such as the σ_{20} and σ_{30} curves of AFM 200-8, representing, respectively, ground burst and optimal air burst blast damage probabilities as a function of distance, can readily be approximated. W = 6 approximates closely the σ_{20} curve, and W = 3 approximates the σ_{30} curve.

Integration of a kill probability function over appropriate density functions allows the representation of such factors as delivery error, geodetic error, extended targets, etc.

Assume an extended target with the Gaussian normal value distribution as follows:

$$V(r) = \frac{1}{2\pi\sigma_{Tgt}^2} e^{-r^2/2\sigma_{Tgt}^2} \tag{41}$$

$V(r)$ = value per unit area at distance r from center

σ_{Tgt} = standard deviation of value distribution

Clearly:

$$1.0 = \frac{1}{2\pi\sigma_{Tgt}^2} \int_o^\infty e^{-r^2/2\sigma_{Tgt}^2}\, dr \tag{41}$$

Define a radius, R95, such at 95% of the value of the target is contained within this distance of the target center. (This R95 is the target radius used in the QUICK system.)

Then
$$\int_o^{R95} e^{-r^2/2\sigma_{Tgt}^2}\, dr = .95 \int_o^\infty e^{-r^2/2\sigma_{Tgt}^2}\, dr \tag{43}$$

Solving this equation for σ_{Tgt} in terms of R95, we get:

$$\sigma_{Tgt} = R_{95}/2.448$$

Assume a CEP, the radius of a circle with center at an aiming point which will contain 50% of the centers of impact of weapons aimed at the aiming point. Assuming a circular normal (Gaussian) distribution of the aiming errors:

$$p(r) = \frac{r}{\sigma_{CEP}^2} e^{-r^2/2\sigma_{CEP}^2}$$

where

$p(r)$ = probability aiming error is r

σ_{CEP} = standard deviation of aiming errors

By definition of CEP

$$\int_0^{CEP} p(r)\, dr = 0.5$$

Solving for σ_{CEP} in terms CEP

$$\sigma_{CEP} = .8493 * CEP$$

Assume a weapon is aimed at the center of the target. From the nature of the Gaussian distribution we can define a standard deviation $\sigma_D^2 = \sigma_{CEP}^2 + \sigma_{Tgt}^2$ such that the circular normal distribution characterized by σ_D^2 is the convolution of the distributions characterized by σ_{CEP}^2 and σ_{Tgt}^2.

Therefore, if

$P_K(W)$ = probability of target kill

W = kill function parameter

$G_K(r)$ = kill function from Eq. (38)

then

$$P_K(W) = \frac{1}{2\pi\sigma_D^2} \int_0^{\infty} \int_0^{2\pi} \exp\left[-\frac{r^2}{2\sigma_D^2}\right] G_K(r)\, r\, d\theta \qquad (46)$$

Evaluating the integrals

$$P_K(W) = 1 - \left(\frac{2WX^2}{1 + 2WX^2} \right)^W \tag{47}$$

where $X = \sigma_D / R_K$

or

$$P_K(W) = 1 - \left(\frac{\sigma_D^2}{\sigma_D^2 + \frac{1}{2W} R_K^2} \right)^W$$

which is the function used in QUICK.

APPENDIX B

OPTIMIZATION OF DGZs FOR COMPLEX TARGETS

Module ALOCOUT is responsible for selecting optimum desired ground zeros DGZs) for weapons allocated to complex targets. The complex target may contain several component target elements, each with specific coordinates, hardness, and some given time dependence of value. To place this diverse target element information on a commensurate basis for efficient DGZ selection, each target component of the complex is represented as a series of simple point value elements. Complex elements with more than one hardness component generate more than one such target element, and area targets generate several elements, spread over the area of the target, to represent a value spread over the area. A (DGZ) Desired Ground Zero Selector then uses the data to select optimum aim points within the target complex.

The selection of DGZs is a two-step process. First, the prescribed warheads are assigned initial coordinates through a "lay-down" process in which each successive warhead is targeted directly against the target element where the highest payoff is achieved, taking into account collateral damage to all other target elements. Second, a general-purpose function optimizer, *FINDMIN*, calculates the derivatives of the payoff as a function of x and y coordinates of each weapon and adjusts the coordinates to minimize the surviving target value. FINDMIN terminates either after a maximum number of iterations (which can be specified by the analyst) or after it finds that it can no longer make significant improvements in the payoff.

The mathematical representation used is as follows.

The weapons allocated to a complex target are to be placed in a manner which attempts to minimize the total escaping target value. To simplify discussion, the notation below is introduced. A second subscript, j, referencing the j^{th} target element, is used when needed.

$\bar{V}_j$ = value of j^{th} target element remaining immediately following arrival of the i^{th} weapon

S_j = probability of survival of j^{th} target element associated with weapon i

E_j = amount of value of j^{th} target element that "escapes" between arrival of weapons i - 1 and i

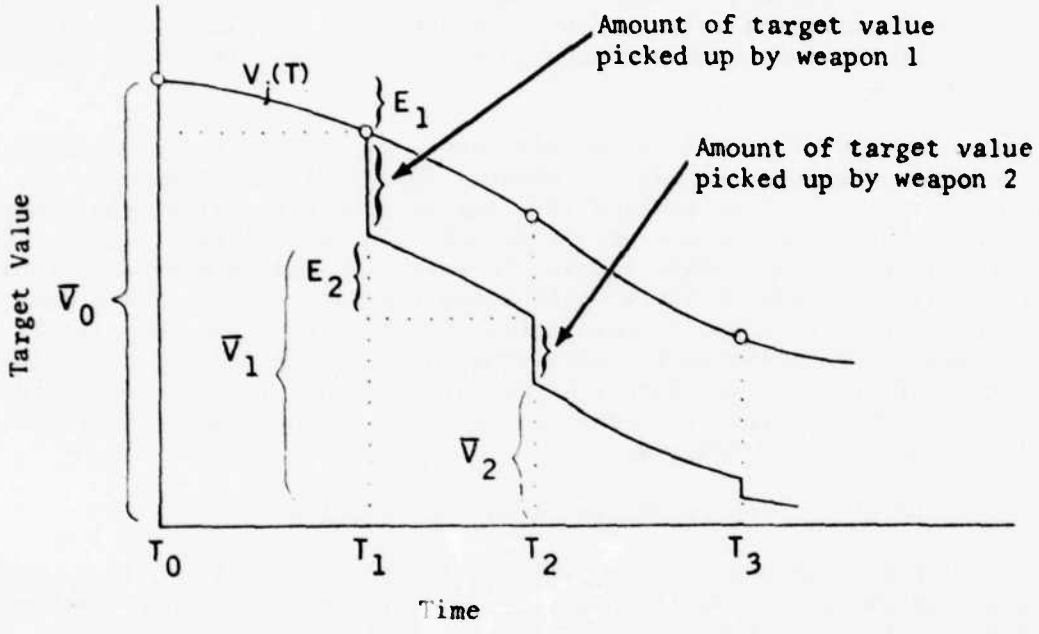
T_i = time of arrival of weapon i (T_0 is an initial time when the full target value is applied) ($T_i \leq T_{i+1}$ all i)

$V_j(T_i)$ = value of j^{th} target, at time T_i

N = number of weapons

NT = number of targets

The following sketch illustrates the treatment of the time-dependent values of the j^{th} target.



From this sketch, the following relationships should be apparent. The equations immediately below refer to a single target (j), but for simplicity the j subscript is omitted.

$$\bar{V}_i = V(T_i)S_i\bar{V}_{i-1}/V(T_{i-1}) \qquad (i = 1,2, \ldots, N)$$

$$E_i = \bar{V}_{i-1}\left[1 - V(T_i)/V(T_{i-1})\right] \quad (i = 1,2, \ldots, N+1)$$

From the previous equations,

$$\bar{V}_i = \left[\prod_{k=1}^{i} S_k\right] V(T_i) \quad \text{and} \quad E_i = \left[\prod_{k=1}^{i} S_k\right]\left[V(T_{i-1}) - V(T_i)\right]$$

(For i = 1, the product $\left(\prod_{k=1}^{i-1} S_k\right)$ is understood = 1. Also $V(T_{N+1}) = 0$.)

The total escaping value associated with target j is

$$\sum_{i=1}^{N+1} E_{ij} = \sum_{i=1}^{N+1}\left(\left[\prod_{k=1}^{i-1} S_{kj}\right]\left[V_j(T_{i-1}) - V_j(T_i)\right]\right)$$

The value on target j which escapes after arrival of weapon i is given by

$$\sum_{p=i+1}^{N+1} E_{pj}$$

The effective value of target j associated with weapon i defined by

$$F_{ij} = \left(\sum_{p=i+1}^{N+1} E_{pj}\right) \Big/ S_{ij}$$

This value is introduced for computational efficiency and may be thought of as the total value available for weapon i, the effect of all other weapons having been taken into account.

The marginal value picked up on target j due to weapon i is given by

$$F_{ij}(1 - S_{ij})$$

where S_{ij} is a function of, among other things, the position of weapon i. For a fixed weapon configuration, weapon i can be moved from (x,y) to (x',y') and the marginal escaped value is given by:

$$\sum_{j=1}^{NT} F_{ij}(S_{ij} - S'_{ij})$$

To establish an initial weapon configuration, a lay-down is performed as follows. Initially, set $S_{ij} = 1$ for all i, j. Denote by S^j_{ik} the survival probability of the k^{th} target, relative to the i^{th} weapon, when this weapon is placed on the j^{th} target. Now the i^{th} weapon is placed on that target, j, which yields a maximum value for the expression

$$\sum_{k=1}^{NT} F_{ik}(S_{ik} - S^j_{ik})$$

The S_{ik} are now set to equal to S^j_{ik} (k = 1,2,..., NT) the F_{ik} (all i, k) are redetermined, i is increased by one, and the process repeated until all weapons have been allocated.

This weapon configuration can now be input as the initial position to a "hill climber" routine, based on a steepest descent algorithm, which attempts to optimize further by replacing the discrete set of possible weapon positions with the two-dimensional continuum. The function to be minimized is:

$$\sum_{j=1}^{NT} \sum_{i=1}^{N+1} E_{ij}$$

Processing by the optimizer will be terminated either when the optimum has been achieved or when a specified number of iterations have been completed. In either case, to insure that the local optimum obtained cannot be further improved, the value of removing, in sequence, each of the weapons from its final location and placing it on one of the target points is explored. If the results obtained by this method are better than those achieved with the previous configuration, this new assignment will be used as an initial one for a second utilization of subroutine FINDMIN. If not, the results of the first use of subroutine FINDMIN will be kept.

APPENDIX C

GENERALIZED LAGRANGE MULTIPLIER METHOD FOR SOLVING PROBLEMS OF OPTIMUM ALLOCATION OF RESOURCES

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The usefulness of Lagrange multipliers for optimization in the presence of constraints is not limited to differentiable functions. They can be applied to problems of maximizing an arbitrary real valued objective function over any set whatever, subject to bounds on the values of any other finite collection of real valued functions defined on the same set. While the use of the Lagrange multipliers does not guarantee that a solution will necessarily be found for all problems, it is 'fail-safe' in the sense that any solution found by their use is a true solution. Since the method is so simple compared to other available methods it is often worth trying first, and succeeds in a surprising fraction of cases. They are particularly well suited to the solution of problems of allocating limited resources among a set of independent activities.

IN MOST textbook treatments, Lagrange multipliers are introduced in a context of differentiable functions, and are used to produce constrained stationary points. Their validity or usefulness often appears to be connected with differentiation of the functions to be optimized. Many typical operations-research problems, however, involve discontinuous or nondifferentiable functions (integral valued functions, for example), which must be optimized subject to constraints.

We shall show that with a different viewpoint the use of Lagrange multipliers constitutes a technique whose goal is *maximization* (rather than location of stationary points) of a function with constraints, and that in this light there are no restrictions (such as continuity or differentiability) on the functions to be maximized. Indeed, the domain of the function to be maximized can be any set (of any cardinal number) whatever.

The basic theorems upon which the techniques to be presented depend are quite simple and elementary, and it seems likely that some of them may have been employed previously. However, their generality and applicability do not seem to be well understood at present (to operations analysts at least). The presentation will consequently place primary emphasis on the implications and applications of the basic theorems, as well as

discussion of a number of techniques for extending the usefulness of the methods.

FORMULATION

FOR CLARITY of presentation, we shall develop the subject in a language of problems concerning the optimal allocation of resources. Other applications of the theorems will suggest themselves.

Let us suppose that there is a set S (completely arbitrary) that is interpreted as the set of possible strategies or actions. Defined on this strategy set is a real valued function H, called a *payoff function*. $H(x)$ is interpreted as the payoff (or utility) which accrues from employing the strategy $x \in S$. In addition, there are n real valued functions $C^k (k = 1 \cdots n)$ defined on S, which are called *Resource* functions. The interpretation of these functions is that employment of the strategy $x \in S$ will require the expenditure of an amount $C^k(x)$ of the kth resource.

The problem to be solved is the maximization of the payoff subject to given constraints $c^k, k = 1 \cdots n$, on each resource; i.e., to find

$$\max_{x \in S} H(x)$$

subject to $C^k(x) \leq c^k$, all k.

A particular subclass of this general problem with wide application is what will be called a *cell problem* (or separable problem) in which there are a number, m, of independent areas into which the resources may be committed, and for which the over-all payoff that accrues is simply the sum of the payoffs that accrue from each independent venture (cell). In this type of problem we have as before, for each cell, a strategy S_i, a payoff function H_i defined on S_i, and n resource functions C_i^k defined on S_i. $H_i(x_i)$ is the payoff in the ith cell for employing strategy $x_i \in S_i$, and for each k, $C_i^k(x_i)$ is the amount of the kth resource expended in the ith cell by employing strategy x_i in that cell. In this case the problem to be solved is to find a strategy set, one element for each cell, which maximizes the total payoff subject to constraints c^k on the total resources expended; i.e.,

$$\max_{\substack{\text{all choices of } \{x_i\} \\ x_i \in S_i}} \sum_{i=1}^{i=m} H_i(x_i)$$

subject to $$\sum_{i=1}^{i=m} C_i^k(x_i) \leq c^k \quad \text{for all } k.$$

This type of problem is simply a subclass of the previous general problem since it can be translated to the previous problem by the following identifications:

$$S = \prod_{i=1}^{i=m} S_i \text{ (direct product set)},$$

[where a strategy $x \in S$ consists of an ordered m-tuple $(x_1, \cdots, x_m)$ of strategies, one for each S_i]

$$H(x) = \sum_{i=1}^{i=m} H_i(x_i),$$

$$C^k(x) = \sum_{i=1}^{i=m} C_i^k(x_i), \quad \text{all } k$$

MAIN THEOREM AND SOME OF ITS IMPLICATIONS

We now present the main theorem concerning the use of Lagrange multipliers, and discuss its meaning and implications. The proof will be supplied in a later section.

Theorem 1

1. λ^k, $k = 1, n$ *are nonnegative real numbers,*
2. $x^* \in S$ *maximizes the function*

$$H(x) - \sum_{k=1}^{k=n} \lambda^k C^k(x) \text{ over all } x \in S,$$

→3. x^* *maximizes* $H(x)$ *over all those* $x \in S$ *such that* $C^k \leq C^k(x^*)$ *for all* k.

Discussion

This theorem says, for any choice of nonnegative λ^k, $k = 1, n$, that if an *unconstrained* maximum of the new (Lagrangian) function

$$H(x) - \sum_{k=1}^{k=n} \lambda^k C^k(x)$$

can be found (were x^*, say, is a strategy which produces the maximum), then this solution is a solution to that *constrained* maximization problem whose constraints are, in fact, the amount of each resource expended in achieving the unconstrained solution. Thus if x^* produced the unconstrained maximum, and required resources $C^k(x^*)$, then x^* itself produces the greatest payoff which can be achieved without using more of any resource than x^* does.

According to Theorem 1, one can simply choose an arbitrary set of nonnegative λ's, find an unconstrained maximum of the modified function, $H(x) - \sum_{k=1}^{k=n} \lambda^k C^k(x)$, and one has as a result a solution to a constrained problem. Notice, however, that the particular constrained problem which is solved is not known in advance, but arises in the course of solution and is, in fact, the problem whose constraints equal the resources expended by the strategy that solved the unconstrained problem.

In general, different choices of the λ^k's lead to different resource levels, and it may be necessary to adjust them by trial and error to achieve any given set of constraints stated in advance.

However, it is noteworthy that in most operations-research work one is not simply interested in achieving the optimum payoff for some given resource levels, but rather in exploring the entire range of what can be

obtained as a function of the resource commitments. In this case it matters little whether *this* function is produced by solving a spectrum of problems with constraints stated in advance, or by simply sweeping through the λ^k's to solve a spectrum of problems whose constraint levels are produced in the course of solution. The method when applicable is therefore quite efficient if the whole spectrum of constraints is to be investigated. Even in the case where only a single constraint set is of interest the use of this method, and adjustment of the λ^k's until the constraint set is achieved, is often more efficient than alternative procedures.

A *limitation* of the Lagrange multiplier method arises from the fact that it does not guarantee that an answer can be found in every case. It simply asserts that if an answer can be found it will indeed be optimum.

In cases where multiple constraints are involved that are not completely independent it may not be possible to simultaneously utilize all resources to the full allowance of the constraints. This can happen if the utilization of one resource requires the utilization of others, or equivalently in cases where some constraints may involve various combinations of others. These cases are analogous to problems in linear programming where certain constraints prove to be irrelevant in the optimum solution.

In such cases one might actually find the optimum solution but be unable to establish the optimality of the result because of incompletely utilized resources. Nevertheless, there is a large class of allocation problems in which the constraints really are independent (i.e., the resources can be consumed independently in the region of interest). In such cases solutions can usually be obtained that give consumption values adequately close to the constraint values. The existence of optimum solutions that can be found by this method actually depends upon an approximate concavity requirement in the region of the solution that will be discussed more carefully later.

At this point we wish to remind the reader of the generality of Theorem 1. *There are no restrictions whatever on the nature of the strategy set* S, *nor on the functions* H *and* C^k *other than real-valuedness.* The strategy set may therefore be a discrete finite set, or an infinite set of any cardinality. Furthermore, the payoff function and the resource functions can take on negative as well as positive values. [$C^k(x)$ negative may be interpreted as *production* rather than expenditure of the kth resource.]

Application to Cell Problem

One of the most important applications of Theorem 1 is in the solution of cell problems. As shown in the Formulation Section, these problems are a subclass of the general problem to which Theorem 1 is applicable. In this case, maximizing the unconstrained Lagrangian function

$$H(x) - \sum_{k=1}^{k=m} \lambda^k C^k(x)$$

is equivalent to finding

$$\max_{x_1 \in S_1, \ldots, x_n \in S_n} [\sum_{i=1}^{n} H_i(x_i)] - \sum_{k=1}^{k} \lambda^k [\sum_{i=1}^{n} C_i^k(x_i)],$$

which (interchanging summation order) is the same as:

$$\max_{x_1 \in S_1, \ldots, x_n \in S_n} \sum_{i=1}^{n} [H_i(x_i) - \sum_{k=1}^{k} \lambda^k C_i^k(x_i)].$$

But, since the choices x_i may be made independently in each cell as a consequence of $S = \prod_{i=1}^{n} S_i$, the sum is obviously maximized by simply maximizing

$$H_i(x_i) - \sum_{k=1}^{k} \lambda^k C_i^k(x_i)$$

in each cell independently of strategy choices in other cells, and summing the payoffs and resources expended for each cell (for the strategy that maximized the Lagrangian for that cell) to get the total payoff and resource expenditures. Theorem 1 then assures us that the result of this process is a solution to the over-all constrained problem with constraints equal to the total resources expended by the strategy produced by this procedure.

Observe that there is no possibility that just a local maximum to the over-all problem has been obtained. If the Lagrangian in each cell has been correctly maximized (i.e., is not itself merely locally maximized), then theorem 1 guarantees that the result is a *global* maximum to the over-all problem.

Theorem 1 says nothing about the manner in which one obtains the maxima of the unconstrained Lagrangian functions, but simply asserts that if one can find them, then one can also have maxima of a problem with constraints. The Lagrange multipliers therefore are not a way in themselves of finding maxima, but a technique for converting optimization problems with constrained resources into unconstrained maximization problems.

This conversion is especially crucial for cell problems with constraints on total resource expenditures, where the conversion to unconstrained maximization of the Lagrangian function uncouples what was an essentially combinatorial problem (because of the interaction of choices in each cell through total resource constraints) *into* a vastly simpler problem involving independent strategy selections in each cell.

The present treatment of Lagrange multipliers was motivated, in fact, by a cell problem involving continuous, differentiable payoff functions, the solution of which was attempted by a classical Lagrange multiplier approach. In this case, the resulting (transcendental) equations had in many circumstances a multiplicity of solutions, and the embarrassing problem arose as to which of several solutions to select for each cell. It appeared as though it might be necessary to try all combinations of choices of solutions—an impossible task in this case which involved several hun-

dred cells. As a result of this difficulty, a closer look was taken at the role of Lagrange multipliers, and the present treatment is the result. The original problem of multiple solutions is, of course, easily solved by simply selecting that solution in each cell which gives the largest value for the Lagrangian.

It is the recognition that the objective is to maximize the Lagrangian, by whatever means, not to zero its derivative, which is decisive. In many cases it *is* expeditious to maximize the Lagrangian by finding zeroes of its derivative. One can then easily select a final value by testing each solution (if there is more than one) to find which gives the largest (global) maximum. This procedure automatically excludes any solutions that correspond to minima or saddle values, and also facilitates taking into account any boundary conditions (such as nonnegative resource constraints) by testing the boundary cases as well.†

In other cases (particularly cases of nonnumerical strategies, or discrete strategy sets such as integers), the Lagrangian may best be maximized by trial and error procedures, or even direct computer scanning of all possibilities.

Another possibility is illustrated by cases wherein resources may be applied only in integral numbers. Often in such cases one can define a continuous differentiable payoff function that attains its correct value on the integers. A useful trick applicable to many such cases is to maximize analytically the Lagrangian based upon the continuous function, and then test the integer on each side of the solution, selecting the one that maximizes the Lagrangian.

PROOF OF MAIN THEOREM

THE PROOF of the main theorem presented and discussed in the previous section is quite elementary and direct:

Proof of Main Theorem. By assumptions (1) and (2) of Theorem 1, λ^k, $k=1\cdots n$, are nonnegative real numbers, and $x^0 \epsilon S$ maximizes

$$H(x)-\sum_{k=1}^{k=n}\lambda^k C^k(x)$$

over all $x \epsilon S$ (the x^0 producing the maximum may very well not be unique—all that we require is that x^0 be *some* element that maximizes the Lagrangian). This means that, for all $x \epsilon S$,

$$H(x^0)-\sum_{k=1}^{k=n}\lambda^k C^k(x^0) \geq H(x)-\sum_{k=1}^{k=n}\lambda^k C^k(x),$$

† This type of constraint (e.g., nonnegativity of resources), which holds independently for each cell rather than over-all as with total resources, is handled by simply restricting the strategy set for the cell appropriately. The Lagrange multipliers are reserved for over-all constraints.

and hence, that

$$H(x^*) \geqq H(x) + \sum_{k=1}^{k=n} \lambda^k [C^k(x^*) - C^k(x)]$$

for all $x \epsilon S$. But if the latter inequality is true for all $x \epsilon S$, it is necessarily true for any subset of S, and hence true on that subset S^* of S for which the resources never exceed the resources $C^k(x^*)$. Notationally: $x \epsilon S^* \Leftrightarrow$ for all k, $C^k(x) \leqq C^k(x^*)$. However, on the subset S^* the term

$$\sum_{k=1}^{k=n} \lambda^k [C^k(x^*) - C^k(x)]$$

is nonnegative by definition of the subset and the nonnegativity of the λ^k's, hence our inequality reduces to $H(x^*) \geqq H(x)$ for all $x \epsilon S^*$, and the theorem is proved.

LAMBDA THEOREM

THEOREM 2

1. Let $\{\lambda_1^k\}, \{\lambda_2^k | k = 1 \cdots n$ *be two sets of* λ^k*'s that produce solutions* x_1^* *and* x_2^**, respectively. Furthermore, assume that the resource expenditures of these two solutions differ in only the jth resource.*

$$C^k(x_1^*) = C^k(x_2^*) \text{ for } k \neq j$$

and that $C^j(x_1^*) > C^j(x_2^*)$.

2. Then: $\lambda_2^j \geqq [H(x_1^*) - H(x_2^*)]/[C^j(x_1^*) - C^j(x_2^*)] \geqq \lambda_1^j$.

This theorem states that, given two optimum solutions produced by Lagrange multipliers for which only one resource expenditure differs, the ratio of the change in optimum payoff to the change in that resource expenditure is bounded between the two multipliers that correspond to the changed resource.

Thus the Lagrange multipliers, which were introduced in order to constrain the resource expenditures, in fact give some information concerning the effect of relaxing the constraints.

In particular, if the set of solutions produced by Lagrange multipliers results in an optimum payoff that is a differentiable function of the resources expended at some point, then it follows from Theorem 2 that the λ^k's at this point are in fact the partial derivatives (or total derivative in case of one resource) of the optimum payoff with respect to each resource (all other resources kept constant):

$$[\partial H^*/\partial C^j]_{\substack{C_k \text{ constant} \\ k \neq j}} = \lambda^j.$$

Proof. The proof of Theorem 2 is also quite elementary. By hypothesis x_1^* is the solution produced by $\{\lambda_1^k\}$, hence x_1^* maximizes the Lagrangian for $\{\lambda_1^k\}$, which implies:

$$H(x_1^*) \geqq H(x) + \lambda_1^j [C^j(x_1^*) - C^j(x)] + \sum_{k \neq j} \lambda_1^k [C^k(x_1^*) - C^k(x)]$$

holds for all $x \in \delta$, and hence in particular holds for x_2°. But since by hypothesis $C^k(x_1^\circ) = C^k(x_2^\circ)$ for $k \neq j$, we can deduce that

$$H(x_1^\circ) \geqq H(x_2^\circ) + \lambda_i^j [C^j(x_1^\circ) - C^j(x_2^\circ)],$$

which, since by hypothesis $C^j(x_1^\circ) > C^j(x_2^\circ)$, implies that:

$$[H(x_1^\circ) - H(x_2^\circ)]/[C^j(x_1^\circ) - C^j(x_2^\circ)] \geqq \lambda_i^j,$$

which proves one side of the conclusion of Theorem 2. Interchanging the roles of x_1° and x_2° [and observing the reversal of the sign of

$$C^j(x_1^\circ) - C^j(x_2^\circ)]$$

produces the other side of the inequality to complete the proof of Theorem 2.

An obvious consequence of Theorem 2 is the fact that, if all but one resource level is held constant, the resource that changes is a monotone *decreasing* function of its associated multiplier. This fact indicates the direction to make changes when employing a trial and error method of adjusting the multipliers in order to achieve some given constraints on the resources.

The Lambda Theorem also suggests a potentially useful technique for choosing a starting set of multipliers for such a trial-and-error method of achieving given constraint levels in a cell problem. Beginning with any reasonably good allocation of the given resources, one can often calculate easily what the effect on the payoff is for a small additional increment of each resource, optimally placed within the cells. The differential payoff divided by the increment of resource is then taken as the starting λ for that resource. The λ's are then adjusted by trial and error until the Lagrange solution corresponds to the given constraints, producing the optimum allocation.

THE EPSILON THEOREM

A NATURAL question with respect to the practical application of the Lagrange method concerns its stability—supposing that as a result of methods of calculation or approximation one cannot precisely maximize the Lagrangian, but can only guarantee to achieve a value close to the maximum. Such a solution can very well be at a drastically different resource level and payoff than that which actually achieves the maximum, and yet produce a value of the Lagrangian very near to the maximum. For the method to be practical, it is required that in this situation a solution that nearly maximizes the Lagrangian must be a solution that also nearly maximizes the payoff for the resource levels that *it itself* produces (which may be quite different than those of the solution that actually

maximizes the Lagrangian). Only in such a circumstance would it be safe to assert that the solutions produced by any nonexact procedures (such as numerical computation with finite accuracy, or methods based upon approximations) were in fact approximately optimal solutions to the constrained problem. Such required assurance of insensitivity is supplied by the following ('epsilon') theorem.

THEOREM 3

1. $\hat{x}$ comes within ϵ of maximizing the Lagrangian, i.e., for all $x \epsilon \delta$:

$$H(\hat{x}) - \sum \lambda^k C^k(\hat{x}) > H(x) - \sum \lambda^k C^k(x) - \epsilon.$$

→2. $\hat{x}$ is a solution of the constrained problem with constraints $c^k = C^k(\hat{x})$ that is itself within ϵ of the maximum for those constraints.

The proof of this theorem, which is a simple extension of Theorem 1, exactly parallels the proof of Theorem 1 (with an added ϵ) and will not be repeated.

ADDITIONAL REMARKS, CONCLUSIONS, AND COMPUTATIONAL PLOYS

Gaps or Inaccessible Regions

Theorem 1 assures us that any maximum of the Lagrangian necessarily is a solution of the constrained maximum problem for constraints equal to the resource levels expended in maximizing the Lagrangian.

The Lagrange multiplier method therefore generates a mapping of the space of lambda vectors (components λ^k, $k = 1, \cdots, n$) into the space of constraint vectors (components c^k, $k = 1 \cdots n$). There is no a priori guarantee, however, that this mapping is onto—for a given problem there may be inaccessible regions (called *gaps*) consisting of constraint vectors that are not generated by any λ vectors. Optimum payoffs for constraints inside such inaccessible regions can therefore not be discovered by straightforward application of the Lagrange multiplier method, and must hence be sought by other means.

The basic cause of an inaccessible region is nonconcavity in the function of optimum payoff vs. resource constraints (convexities in the envelope of the set of achievable payoff points in the space of payoff vs. constraint levels). This possibility, and several methods for dealing with it, will now be investigated.

Before beginning this investigation, however, we wish to point out that even though the Lagrange multiplier method is not certain to obtain the desired solutions in all cases, any solutions that it does yield are guaranteed by Theorem 1 to be true solutions. The procedure is therefore 'fail-safe,' a very reassuring property. It has been our experience over the last several years, which includes application of this method to a variety

of production and military allocation problems, that the method has been extremely successful, and nearly always has directly yielded all solutions of interest. The few situations in which the direct method failed were readily solved by simple modifications to the procedure, some of which will now be mentioned.

Source of Gaps

Consider the $(n+1)$ dimensional space of payoff vs. resource expenditures. This space will be called PR space for brevity. Every strategy $x \epsilon S$ maps into a point in this space corresponding to $H(x), C^k(x) (k = 1 \cdots n)$. The entire problem is therefore represented by this set of accessible points in PR space. The problem of finding the maximum of H subject to constraints c^k, $k = 1 \cdots n$, is simply the problem of selecting that point of our set in PR space of maximum H that is contained in the subspace of PR space where the resources are bounded by the c^k's. The set of all such points (corresponding to all sets of values in the c^k's) will be called the *envelope*, and constitutes the entire set of solutions for all possible constraint levels.

Consider now any solution x^* produced by a set of Lagrange multipliers (λ^k). By definition x^* maximizes the Lagrangian; consequently we have that

$$H(x^*) - \sum \lambda^k C^k(x^*) \geq H(x) - \sum \lambda^k C^k(x)$$

for all $x \epsilon S$. Rearranging terms slightly, we have:

$$H(x) \leq H(x^*) - \sum \lambda^k C^k(x^*) + \sum \lambda^k C^k(x)$$

for all $x \epsilon S$. If we consider now the hyperplane in PR space defined by $H = \sum \lambda^k C^k + \alpha$ where $\alpha = H(x^*) - \sum \lambda^k C^k(x^*)$, we see that, because of the previous inequality, none of the accessible points in PR space lies above this hyperplane, and at least one point, $H(x^*), C^k(x^*)\ k = 1 \cdots n$, lies on it.

Each solution produced by Lagrange multipliers therefore defines a bounding hyperplane that is tangent to the set of accessible points in PR space at the point corresponding to the solution (hence tangent to the envelope), and which constitutes an upper bound to the entire set of accessible points. It is clear that, since no such tangent bounding hyperplanes exist in regions where the envelope of accessible points in PR space is not concave, the Lagrange multiplier method cannot produce solutions in such a region. Conversely, for any point on the envelope (solution) where a tangent bounding hyperplane *does* exist (envelope concave at the point), it is obvious that there exists a set of multipliers (namely the slopes of the hyperplane) for which the strategy corresponding to the point in question maximizes the Lagrangian.

Thus the Lagrange method will succeed in producing all solutions that correspond to concave regions of the envelope (optimized payoff vs. constraint level), and fail in all nonconcave regions.

A fortunate feature of cell problems with many cells is the fact that, even though there may be large convexities in the envelope in the PR space for each cell, the result of over-all optimization is an envelope in the PR space for the total problem in which the convexities are vastly reduced in significance.† This property is the major reason for the general success of the Lagrange method in solving cell problems.

Some Methods for Handling Gaps

Despite the general success of Lagrange multipliers (at least for the problems we have encountered), occasions may arise where gaps occur in regions of critical interest. Under such circumstances there are several useful techniques that can be attempted before abandoning the procedure altogether.

First, all solutions that can be obtained outside the gaps contribute a good deal of information and can be used to bound the solution in the gap region. As was previously shown, each solution that can be obtained by Lagrange multipliers defines a bounding hyperplane that gives an upper bound to the maximum payoff at all points, and hence inside the gap as well. For any point inside a gap, therefore, an upper bound can be obtained by finding the minimum payoff for that point over the set of bounding hyperplanes corresponding to the solutions that one could calculate.

On the other hand, every solution that can be obtained that has the property that none of its resource expenditures exceeds the resources of a point in a gap for which one is seeking bounds, obviously constitutes a *lower bound* to the optimum payoff at the point in question, and the maximum of these lower bounds can be selected as a lower bound to the payoff in question. Thus the set of solutions that *can* be obtained by Lagrange multipliers can be used to obtain bounds on the optimum payoff for inaccessible regions.

There is another technique that is often successful in reducing gaps in instances where the bounds one can compute leave too large a region of uncertainty, and where the gap is caused by degeneracy in which a number of cells have gaps corresponding to the *same* multiplier. A gap is char-

† In fact, the gap structure for the over all problem obviously simply reflects faithfully the gap structure in the individual cells, with each gap in a cell corresponding to a given multiplier value occurring with the same magnitude (same jump in payoff and resources) in the over-all optimization at precisely the same multiplier value. Only degeneracies in which several cells have gaps corresponding to the same multiplier can cause a larger gap in the over-all problem, and such degeneracy is easily removed by techniques to be discussed in the following section.

acterized by the behavior that, as the λ's are continuously varied, there are abrupt discontinuities in the resource levels generated. These discontinuities can often be filled in cell problems by the following technique.

Given two sets of λ's, $(\lambda_1^k), (\lambda_2^k)$, which are very close, but for which the generated resource levels markedly differ, one can make a *mixed* calculation in a cell problem using the set (λ_1^k) in some cells and the set (λ_2^k) in the others. If the two sets of λ's are close together, maximizing the Lagrangian in any cell for one set will necessarily result in a solution that nearly maximizes the Lagrangian for the other set, hence by the Epsilon Theorem will yield a result that is guaranteed to be nearly optimum.

Somewhat more generally, one can simply exploit the Epsilon Theorem directly in a cell problem, working with a given set of λ's but deliberately modifying the choices in some or all cells in a way which moves in the direction of the desired expenditure of resources. By summing the deviations from maximum of the Lagrangian in each cell (epsilons) in which the strategies are so modified, a bound on the error of the result is obtained (which can be kept quite small in most cases by judicious choice of deviations). This appears to be a quite powerful strategem.

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The Program Maintenance Manual consists of four volumes which facilitate maintenance of the war gaming system. This volume, Volume III, provides the programmer/analyst with a technical description of the purpose, functions, general procedures, and programming techniques applicable to the modules and subroutines

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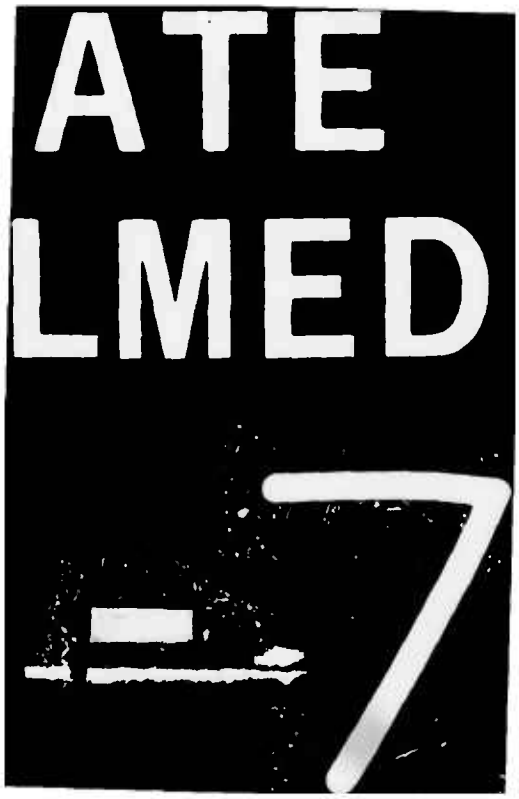
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