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It is the purpose of these notes to describe the theory of Hale and Kato for functional differential equations based on a space of initial data which satisfy some very reasonable axioms. ~~We also indicate~~ some recent results of Naito* showing how extensive the theory of linear systems can be developed in an abstract setting - in particular, the characterization of the spectrum of the infinitesimal generator together with the decomposition theory and exponential estimates of solutions.

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RETARDED EQUATIONS WITH INFINITE DELAYS*

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May, 1978

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RETARDED EQUATIONS WITH FINITE DELAYS

Jack K. Hale

1. Introduction. In retarded functional differential equations, the space of initial data is usually dictated by the form of the equation and the desired objectives. For finite delays, the most frequently occurring spaces are the continuous functions or $L^p \times \mathbb{R}^n$. In this case, the particular space for the initial data is not too important from the point of view of the qualitative theory since the function x_t representing the solution restricted to the interval $[t-r,t]$, $t \geq 0$, r the delay, becomes smoother as t increases.

If the delay is infinite, this is no longer the case since $x_t(\theta)$ for $\theta \leq -t$ coincides with the values of the initial function. This means that the qualitative behavior of the solution operator will depend upon the space of initial data. It thus becomes important to understand in an abstract manner those properties which develop the fundamental theory of existence, uniqueness, continuous dependence, continuation, etc. In addition, one needs to know some abstract properties which will imply something about global behavior of orbits: for example, when are bounded orbits precompact, when is stability in $\mathbb{R}^n$ equivalent to stability in the function space, etc.? In addition to yielding a better understanding of functional differential equations, an axiomatic development should eliminate duplication of effort. It should also be noted that the axiomatic approach is also useful with finite delays.

It is the purpose of these notes to describe the theory of Hale and Kato [1] for functional differential equations based on a space of initial data which satisfy some very reasonable axioms. We also indicate some recent results of Naito [2] showing how extensive the theory of linear systems can be developed in an abstract setting - in particular, the characterization of the spectrum of the infinitesimal generator together with the decomposition theory and exponential estimates of solutions.

2. Axioms for the phase space. In this section, we present the axioms for the phase space which seem to be convenient for a global qualitative theory of functional differential equations. By doing this, our axioms are easily phrased in terms of a translation semigroup. For the local theory, one can assume much less. A more critical and logical development of the axioms can be found in [1].

Let I be either a fixed finite interval $[-r,0]$ or the interval $(-\infty,0]$. Let $\hat{\mathscr{B}}$ be a linear space of functions mapping I into $\mathbb{R}^n$ with elements designated by $\hat{\phi},\hat{\psi},\ldots$, where $\hat{\phi} = \hat{\psi}$ means $\hat{\phi}(t) = \hat{\psi}(t)$, $t \in I$. Suppose there is a seminorm $|\cdot|_{\hat{\mathscr{B}}}$ on $\hat{\mathscr{B}}$ and suppose $\mathscr{B} = \hat{\mathscr{B}}/|\cdot|_{\hat{\mathscr{B}}}$ is a Banach space with $|\cdot|_{\mathscr{B}}$ naturally induced by $|\cdot|_{\hat{\mathscr{B}}}$. Elements of $\mathscr{B}$ are denoted by $\phi,\psi,\ldots$ and correspond to equivalence classes of $\hat{\mathscr{B}}$. For any $\phi \in \mathscr{B}$, corresponding elements in the equivalence classes are denoted by $\hat{\phi}$ and $\phi = \psi$ in $\mathscr{B}$ means $|\hat{\phi}-\hat{\psi}|_{\hat{\mathscr{B}}} = 0$ for all $\hat{\phi} \in \phi$, $\hat{\psi} \in \psi$.

To always distinguish between elements of the equivalence class and the equivalence class itself requires cumbersome notation. Therefore, in the following, we do not use a different symbol (except where confusion may arise) to distinguish these objects. In general, the symbol "^" is omitted with the reader being expected to make the distinction by context.

The first axiom is

Axiom (α_0'): There is a constant K such that $|\hat{\phi}(0)| \leq K|\hat{\phi}|_{\hat{\mathscr{B}}}$ for every $\hat{\phi} \in \hat{\mathscr{B}}$.

This axiom implies that $\hat{\phi}(0) = \hat{\psi}(0)$ for every $\hat{\phi} \in \phi$, $\hat{\psi} \in \phi$. Therefore, for every equivalence class ϕ there is associated a unique $\phi(0)$ and Axiom (α_0') can be rewritten as

Axiom (α_0): There is a constant K such that $|\phi(0)| \leq K|\phi|_{\mathscr{B}}$ for all $\phi \in \mathscr{B}$.

For any $A \geq 0$, if $x\colon I_A \to \mathbb{R}^n$ is a given function and $t \in [0,A]$, define $x_t\colon I \to \mathbb{R}^n$ by

$$x_t(\theta) = x(t+\theta), \quad \theta \in I.$$

For any $\hat{\phi} \in \hat{\mathscr{B}}$, let $F_A(\hat{\phi})$ be the set of functions $\hat{x}\colon I \cup [0,A] \to \mathbb{R}^n$ such that $\hat{x}_0 = \hat{\phi}$, $\hat{x}$ is continuous on $[0,A]$ and define

$$F_A = \bigcup \{F_A(\hat{\phi}),\ \hat{\phi} \in \hat{\mathscr{B}}\}.$$

Axiom (α_1): $x_t \in \mathscr{B}$ for all $x \in F_A$, $t \in [0,A]$.

For the next axioms, we need two seminorms in $\mathscr{B}$, one corresponding to the restriction of ϕ to $I\setminus[-\beta,0]$ and the other to the restriction of ϕ to $[-\beta,0]$ for $-\beta \in I$. More specifically, for any $-\beta \in I$, let

$$|\phi|_\beta = \inf_{\hat{\eta}\in\hat{\mathscr{B}}} \{\inf_{\hat{\psi}\in\hat{\mathscr{B}}} \{|\hat{\psi}|_{\hat{\mathscr{B}}} : \hat{\psi}(\theta) = \hat{\eta}(\theta),\ \theta \in I\setminus[-\beta,0]\} : \eta = \phi\}$$

$$|\phi|_{(\beta)} = \inf_{\hat{\eta}\in\hat{\mathscr{B}}} \{\inf_{\hat{\psi}\in\hat{\mathscr{B}}} \{|\hat{\psi}|_{\hat{\mathscr{B}}} : \hat{\psi}(\theta) = \hat{\eta}(\theta),\ \theta \in [-\beta,0]\} : \eta = \phi\}$$

From axiom (α_1), for any $\phi \in \mathscr{B}$, $-\beta \in I$, let

$$(\tau^\beta\phi)(\theta) = \begin{cases} \phi(\theta+\beta), & \theta \in I\setminus[-\beta,0] \\ \phi(0), & \theta \in [-\beta,0]. \end{cases}$$

Axiom (α_2): If $\phi = \psi$ in $\mathscr{B}$, then $|\tau^\beta\phi - \tau^\beta\psi|_\beta = 0$.

Axiom (α_3): $|\phi|_{\mathscr{B}} \leq |\phi|_\beta + |\phi|_{(\beta)}$ for any $-\beta \in I$ and all $\phi \in \mathscr{B}$.

Axiom (α_4): There is a constant K such that $|\phi|_{(\beta)} \leq K \sup_{[-\beta,0]}|\phi(\theta)|$ for any $-\beta \in I$ and all $\phi \in \mathscr{B}$.

Axiom (α_5): All constant functions belong to $\mathscr{B}$.

Introduce the following definitions

$$S(t): \mathscr{B} \to \mathscr{B}, \ t \geq 0$$

$$[S(t)\phi](\theta) = \begin{cases} \phi(0) & \theta \in [-t,0] \\ \phi(t+\theta) & \theta \in I\setminus[-t,0] \end{cases}$$

$$\mathscr{B}_0 = \{\phi \in \mathscr{B}: \phi(0) = 0\}$$

$$S_0(t): \mathscr{B}_0 \to \mathscr{B}_0, \ S_0(t) = S(t)|\ \mathscr{B}_0.$$

The linear operator $S(t)$ satisfies the property that $S(0) = I$, the identity operator, $S(t+\tau) = S(t)S(\tau)$, $t, \tau \geq 0$; that is, $\{S(t), t \geq 0\}$ is a semigroup of linear operators on $\mathscr{B}$. The final axioms are

<u>Axiom (α_6):</u> $S(t)$ <u>is strongly continuous for</u> $t \geq 0$.

<u>Axiom (α_7):</u> <u>There is a</u> $t_0 > 0$ <u>such that</u> $|S_0(t_0)| < 1$.

For a more complete description of the α-limits sets of orbits of functional differential equations, the following axiom is convenient.

<u>Axiom (α_8):</u> <u>If</u> $\{\phi^k\}$ <u>converges to</u> ϕ <u>uniformly on compact subsets of</u> I <u>and if</u> $\{\phi^k\}$ <u>is a Cauchy sequence in</u> $\mathscr{B}$, <u>then</u> $\phi \in \mathscr{B}$ <u>and</u> $\phi^k \to \phi$ <u>in</u> $\mathscr{B}$.

<u>Example 2.1</u>. (<u>Spaces</u> $L_p(\mu) \times \mathbb{R}^n$). Suppose $g: I \to [0,\infty)$, $G: I \to [0,\infty)$ are continuous

$$g(t+s) \leq G(t)g(s), \quad t,s \in I.$$

If g is a nondecreasing function, one can satisfy this condition with $G(t) = 1$ for all $t \in I$. If $g(t) = \exp(\lambda t)$, then we can take $G(t) = \exp(\lambda t)$. Let

$$\mathscr{B} = \{\phi: I \to \mathbb{R}^n, \text{ measurable, } |\phi| < \infty\}$$

$$|\phi|_{\mathscr{B}} = \{|\phi(0)|^p + \int_I g(\theta)|\phi(\theta)|^p d\theta\}^{1/p}.$$

Axioms (α_0)-(α_8) are satisfied if

$$\int_I g(\theta)d\theta < \infty$$

$$G(\beta_0) < 1 \quad \text{for some} \quad \beta_0 \in I.$$

If these conditions are satisfied then $\mathscr{B}$ is isomorphic to $L^p(\mu_g) \times \mathbb{R}^n$ where μ_g is the measure induced by the function g,

$$\mu_g(E) = \int_E g(\theta)d\theta, \qquad E \subset I.$$

In this example,

$$|S(t)| = \sup_{s\varepsilon I}\left[\frac{g(s-t)}{g(s)}\right]^{1/p} \leq [G(-t)]^{1/p}.$$

Example 2.2. (Spaces of continuous functions C_γ). For any $\gamma \varepsilon \mathbb{R}$, let

$$\mathscr{B} = C_\gamma \overset{\text{def}}{=} \{\phi\colon I \to \mathbb{R}^n, \text{ continuous, } e^{\gamma\theta}\phi(\theta) \to \text{a limit as } \theta \to -\infty\}$$

$$|\phi|_{C_\gamma} = \sup_{\theta\varepsilon I} e^{\gamma\theta}|\phi(\theta)|.$$

Axioms (α_0)-(α_8) are satisfied if $\gamma > 0$. Also, $|S(t)| \leq e^{-\gamma t}$.

3. Functional differential equations. Suppose $\mathscr{B}$ is a space of functions satisfying Axioms (α_0)-(α_7), Ω is an open set in $\mathbb{R} \times \mathscr{B}$, $f\colon \Omega \to \mathbb{R}^n$ is continuous and consider the retarded functional differential equation

$$\dot{x}(t) = f(t,x_t), \tag{3.1}$$

where $x_t(\theta) = x(t+\theta)$, $\theta \varepsilon I$. For any $(\sigma,\phi) \varepsilon \Omega$, a solution $x = x(\sigma,\phi,f)$ through (σ,ϕ) is a function defined on an interval $I_\sigma \cup [\sigma,\sigma\ \alpha]$, $I_\sigma = \{\zeta \varepsilon \mathbb{R}, \zeta = \theta + \sigma, \theta \varepsilon I\}$, $\alpha > 0$, such that $x_\sigma = \phi$, x satisfies (3.1) on $[\sigma,\sigma+\alpha]$.

The following results have been proved by Hale and Kato [1].

Theorem 3.1. (Existence) For any $(\sigma,\phi) \in \Omega$, there is a solution of Equation (3.1) through (σ,ϕ).

Theorem 3.2. (Uniqueness) If $f(t,\phi)$ is Lipschitzian in ϕ on Ω, then the solution through $(\sigma,\phi) \in \Omega$ is unique and there is a continuous function $K(t)$ such that

$$|x_t(\sigma,\phi) - x_t(\sigma,\psi)| \leq K(t-\sigma)|\phi-\psi| \quad , \quad t \geq \sigma.$$

Theorem 3.3. (Continuation) If x is a noncontinuable solution of (3.1) on $I_\sigma \cup [\sigma,\delta]$ and f takes closed bounded sets of Ω into bounded sets, then, for any closed bounded set W in Ω, there is a sequence $t_k \to \delta^-$ such that $(t_k, x_{t_k}) \notin W$. If, in addition, there is an $r_0 > 0$, $k > 0$, such that

$$\sup_{[-r_0,0]}|\phi(\theta)| \leq k|\phi|_{\mathscr{B}}$$

then there is a t_W such that $(t,x_t) \notin W$ for $t_W \leq t < \delta$.

Theorem 3.4. (Continuous dependence) Suppose $f = f_\lambda$ in (3.1) depends continuously upon a parameter λ in a Banach space. If the solution $x(\sigma,\phi,\lambda)$ of (3.1) through (σ,ϕ) is unique, then $x(\sigma,\phi,\lambda)$ is continuous in (σ,ϕ,λ). If, in addition, $f_\lambda(t,\phi)$ is continuously differentiable in (ϕ,λ), then $x(\sigma,\phi,\lambda)$ is continuously differentiable in (ϕ,λ).

For the above results, not all of the axioms are needed as may be seen from [1]. Also, there has been interest in studying functional differential equations in spaces like $L^p \times \mathbb{R}^n$ when the right-hand side of (3.1) is not a function on L^p; for example, it may be a differential difference equation. It is only a minor technical problem to adapt the proofs in [1] to this case by considering the corresponding integral equation. The majority of the changes are notational and therefore will not be discussed.

Some additional results from [1] which are more global in nature will now be given.

Theorem 3.5. *If* f *is independent of* t *and is completely continuous, then the* ω*-limit set of any solution* $x(t)$ *of* (3.1) *bounded for* $t \geq 0$, *is nonempty, compact, connected. If, in addition,* (α_8) *is satisfied, the* ω*-limit set is invariant.*

Theorem 3.6. *A solution* u *of* (3.1) *is (uniformly)(asymptotically) stable in* $\mathcal{B}$ *if and only if it is (uniformly)(asymptotically) stable in* $\mathbb{R}^n$.

It is also proved in [1] that for autonomous and periodic systems, the solution $x = 0$ of (3.1) is uniformly stable and asymptotically stable if and only if it is uniformly asymptotically stable.

From the point of view of this paper, a more important property proved in [1] concerns the solution operator for (3.1). Suppose $f: \mathbb{R} \times \mathcal{B} \to \mathbb{R}^n$ and for every $(\sigma,\phi) \in \mathbb{R} \times \mathcal{B}$, there exists a unique solution $x(\sigma,\phi)$ of (3.1) through (σ,ϕ) defined on $I_\sigma \cup [\sigma,\infty)$. The solution operator $T(t,\sigma)$ is defined by

$$
(3.2) \qquad \begin{aligned} &T(t,\sigma): \quad \mathscr{B} \to \mathscr{B}, \quad t \geq \sigma \\ &T(t,\sigma)\phi = x_t(\sigma,\phi). \end{aligned}
$$

Let $\alpha(A)$ be the <u>Kuratowskii measure of noncompactness</u> of a bounded set $A \subset \mathscr{B}$,

$$
\alpha(A) = \inf\{d > 0: \ A \text{ has a finite cover of diameter less than } d\}.
$$

A map $T: \mathscr{B} \to \mathscr{B}$ is said to be a <u>conditional α-contraction</u> if T is continuous, there is a constant $k \in [0,1)$ such that $\alpha(TA) \leq k\alpha(A)$ for every bounded set $A \subset \mathscr{B}$ for which TA is bounded. If T is a conditional α-contraction and takes bounded sets into bounded sets, then T is called an <u>α-contraction.</u> A family of mappings $U(t,\sigma): \mathscr{B} \to \mathscr{B}$, $t \geq \sigma$, is called <u>conditionally completely continuous</u> if $U(t,\sigma)\phi$ is continuous in (t,σ,ϕ) and, for any bounded set $A \subset \mathscr{B}$, there is a compact set $A^* \subset \mathscr{B}$ such that $U(\tau,\sigma)\phi \in A$ for $\tau \in [\sigma,t]$ implies $U(t,\sigma)\phi \in A^*$. If $U(t,\sigma)$ is conditionally completely continuous and for any bounded set $A \subset \mathscr{B}$ and any compact set $J \subset [\sigma,\infty)$, there is a bounded set $A_0 \subset \mathscr{B}$ such that $U(\tau,\sigma)A \subset A_0$ for every $\tau \in J$, then $U(t,\sigma)$, $t \geq \sigma$, is completely continuous in the usual sense. If $T: \mathscr{B} \to \mathscr{B}$ is linear and continuous, $\alpha(T) = \inf\{k: \alpha(TA) \leq k\alpha(A)$ for all bounded sets $A \subset \mathscr{B}\}$.

Theorem 3.7 (*Representation of solution operator*). *If* $f\colon \mathbb{R} \times \mathscr{B} \to \mathbb{R}^n$ *is completely continuous, then* $U(t,\sigma)\colon \mathscr{B} \to \mathscr{B}$ *defined by*

$$T(t,\sigma)\phi = S(t-\sigma)\phi + U(t,\sigma)\phi, \qquad t \geq \sigma,$$

is conditionally completely continuous for $t \geq \sigma$. *Furthermore, for any bounded set* $A \subset \mathscr{B}$ *for which* $T(s,\sigma)A$ *is bounded uniformly for* $\sigma \leq s \leq t$, *we have*

$$(T(t,\sigma)A) = \alpha(S(t-\sigma))\alpha(A) = \alpha(S_0(t-\sigma))\alpha(A) \leq |S_0(t-\sigma)|\alpha(A).$$

In particular, for t_0 *defined in Axiom* (α_7), $T(\sigma+t_0,\sigma)$ *is an* α*-contraction*.

4. *Linear autonomous equations*. For linear autonomous equations, an extensive theory can be developed without imposing many hypotheses on the space-especially Axiom (α_7) about $|S_0(t_0)| < 1$. On the other hand, Axiom (α_8) seems to play a more important role than in the previous section. Consequently, in this section we develop a theory of linear systems using only the axioms that seem necessary and follow closely the paper of Naito [2] but in less generality. We restate the hypotheses explicitly in the form that is needed.

Axiom (β_0): *There is a constant* K *such that* $|\phi(0)| \leq K|\phi|$ *for all* $\phi \in \mathscr{B}$.

Axiom (β_1): $x_t \in \mathscr{B}$ for all $x \in F_A$, $t \in [0,A]$, x_t is continuous in t.

Axiom (β_2): For any $x \in F_\infty$, $t \in [0,\infty)$,

$$|x_t|_{\mathscr{B}} \quad K(t)\sup_{0 \le s \le t}|x(s)| + M(t)|x_0|$$

where K, M are continuous, $M(t+s) \le M(t)M(s)$.

Axiom (β_3): If $\{\phi^k\}$ converges to ϕ uniformly on compact sets of I and if $\{\phi^k\}$ is a Cauchy sequence in $\mathscr{B}$, then $\phi \in \mathscr{B}$ and $\phi^k \to \phi$ in $\mathscr{B}$.

Axioms (α_0-α_6) and (α_8) imply Axioms (β_0-β_3).

Suppose $L: \mathscr{B} \to \mathbb{R}^n$ is a continuous linear operator and consider the autonomous linear equation

$$\dot{x}(t) = Lx_t.$$

Since this equation is autonomous, we take the initial time to be zero. The results in [1] or [2] imply that the solution operator $T(t)$, $t \ge 0$, defined by $T(t)\phi = x_t(\phi)$ for $\phi \in \mathscr{B}$, is a strongly continuous semigroup of bounded linear operators on $\mathscr{B}$. Let A be the infinitesimal generator of $T(t)$.

The specific form of the infinitesimal generator is not known. However, it is surprising how much of the general theory of linear systems is independent of this fact. In this section, we state some results on the spectrum $\sigma(A)$ of A, the point spectrum $P_\sigma(A)$ of A and the resolvent set $\rho(A)$ of A. The first observation concerns the point spectrum (see [1]).

Theorem 4.1. *If* A *is the infinitesimal generator of* $T(t)$, *then* $P_\sigma(A)$ *is the set of* λ *for which there exists a* $b \neq 0$, $b \in \mathbb{C}^n$, *such that* $e^{\lambda \cdot}b \in \mathscr{B}$ *and*

$$\text{(4.1)} \qquad \begin{aligned} &\det \Delta(\lambda) = 0, \\ &\Delta(\lambda) = \lambda I - L(e^{\lambda \cdot} I). \end{aligned}$$

If $S(t)$ is the semigroup defined in Section 2 by the differential equation $\dot{x}(t) = 0$ in $\mathscr{B}$ and

$$\text{(4.2)} \qquad T(t)\phi = S(t)\phi + U(t)\phi, \quad \phi \in \mathscr{B},$$

then the Representation Theorem 3.7 says that $U(t)$ is completely continuous and

$$\text{(4.3)} \qquad \alpha(T(t)) = \alpha(S(t)) \leq |S(t)|$$

where α is the Kuratowskii measure of noncompactness and, for a bounded linear operator T,

$$(4.4) \qquad \alpha(T) = \inf\{k : \alpha(TB) \le k\alpha(B) \text{ for all bounded sets } B \subseteq \mathscr{B}\}$$

For any bounded linear operator T, let $r_e(T)$ be the smallest closed disk in the complex plane with center zero which contains the essential spectrum of T. It is shown in [3] that $r_e(T) = \lim_{n\to\infty}\alpha(T^n)^{1/n}$. Since $\alpha(T(t)) = \alpha(S(t))$ it also follows from [3] that $r_e(T(t)) = r_e(S(t))$. To estimate $\alpha(S(t))$, observe that

$$\alpha(S(t+\tau)) = \alpha(S(t)S(\tau)) \le \alpha(S(t))\alpha(S(\tau))$$

for all $t, \tau \ge 0$ and so there is a $\beta \in [-\infty, +\infty)$ such that

$$(4.5) \qquad \beta = \lim_{t\to\infty} \frac{\log \alpha(S(t))}{t} = \inf_{t>0} \frac{\log \alpha(S(t))}{t}$$

Thus,

$$(4.6) \qquad r_e(T(t)) = r_e(S(t)) = e^{\beta t}, \qquad t > 0.$$

If μ is in $\sigma(T(t))$ and $|\lambda| > \exp(\beta t)$, then μ is a normal eigenvalue of $T(t)$ and $\mu = \exp(\lambda t)$ for some $\lambda \in P_\sigma(A)$. Thus, one obtains (see [2]).

<u>Theorem 4.2</u>. <u>The spectral radius</u> $r_\sigma(T(t))$ <u>of</u> $T(t)$ <u>is given by</u>

$$r_\sigma(T(t)) = e^{t\alpha}, \qquad t \ge 0$$

where

$$\alpha = \alpha_L = \max\{\beta, \sup\{\text{Re } \lambda : \lambda \in P_\sigma(A)\}\}.$$

Also, for any $\varepsilon > 0$, *there is a* $c(\varepsilon) > 0$ *such that*

$$|T(t)| \leq c(\varepsilon)e^{(\alpha+\varepsilon)t}, \qquad t \geq 0.$$

A more difficult result from [2] is the following one.

Theorem 4.3. *Any point* λ *such that* Re $\lambda > \beta$ *is a normal point of* A; *that is,* λ *does not lie in the essential spectrum of* A.

With this result, for any $\lambda \in \sigma(A)$, Re $\lambda > \beta$, the space may be decomposed as

$$\mathscr{B} = \mathfrak{N}(A-\lambda I)^k \oplus \mathscr{R}(A-\lambda I)^k$$

for some integer k, where $\mathfrak{N}(A-\lambda I)^k$ is of finite dimension, invariant under A and $\mathscr{R}(A-\lambda I)^k$ is a closed subspace of $\mathscr{B}$. The subspace $\mathfrak{N}(A-\lambda I)^k$ can be explicitly computed (see [1]).

A more explicit representation of this decomposition is needed, but is unavailable at the present time. More specific information is needed about A and its adjoint A^*. A step in the right direction is the following theorem of Naito [2] generalizing a corresponding theorem of Stech [4].

Theorem 4.4. The domain $\mathcal{D}(A^*)$ of A^* is independent of L,

$$\mathcal{D}(A^*) = \mathcal{D}(B^*)$$

where B is the infinitesimal generator of S(t).

It remains to give the variation of constants formula. Let

$$\mu = \lim_{t\to\infty} \frac{\log M(t)}{t} = \inf_{t>0} \frac{\log M(t)}{t}$$

$c > \max(\alpha_L, \mu)$, where α_L is defined in Theorem 4.2, and define

$$X(t) = \begin{cases} \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} e^{\lambda t}\Delta(\lambda)^{-1} d\lambda \,, & t > 0 \\ I \,, & t = 0 \,. \end{cases}$$

The matrix X(t) is continuous and is called the fundamental matrix for Equation (4.1).

Theorem 4.4. [2]. If f is a continuous function from $[0,\infty)$ to $\mathbb{R}^n$, then the solution $x(t,\phi,f)$, $x_0(\cdot,\phi,f) = \phi$, of the equation

$$\dot{x}(t) = Lx_t + f(t)$$

is given by

$$x(t,\phi,f) = x(t,\phi,0) + \int_0^t X(t-\tau)f(\tau)d\tau \ , \qquad t \geq 0$$

$$x(t,\phi,0) = \phi(0) + \int_0^t X(t-\tau)L(S(\tau)\phi)d\tau \ , \qquad t \geq 0$$

where $S(t)$ is the semigroup generated by the equation $\dot{x}(t) = 0$ in $\mathscr{B}$.

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