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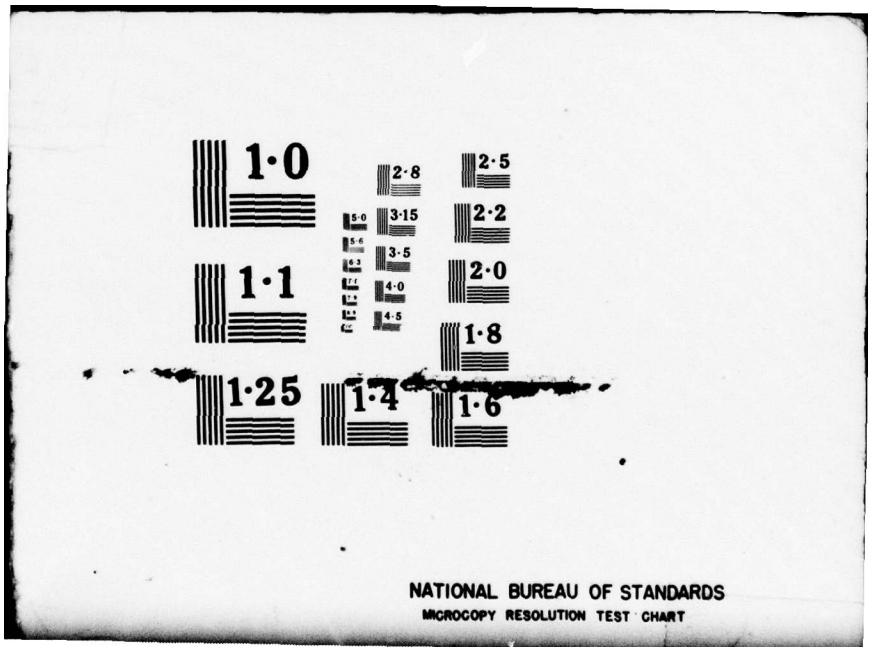
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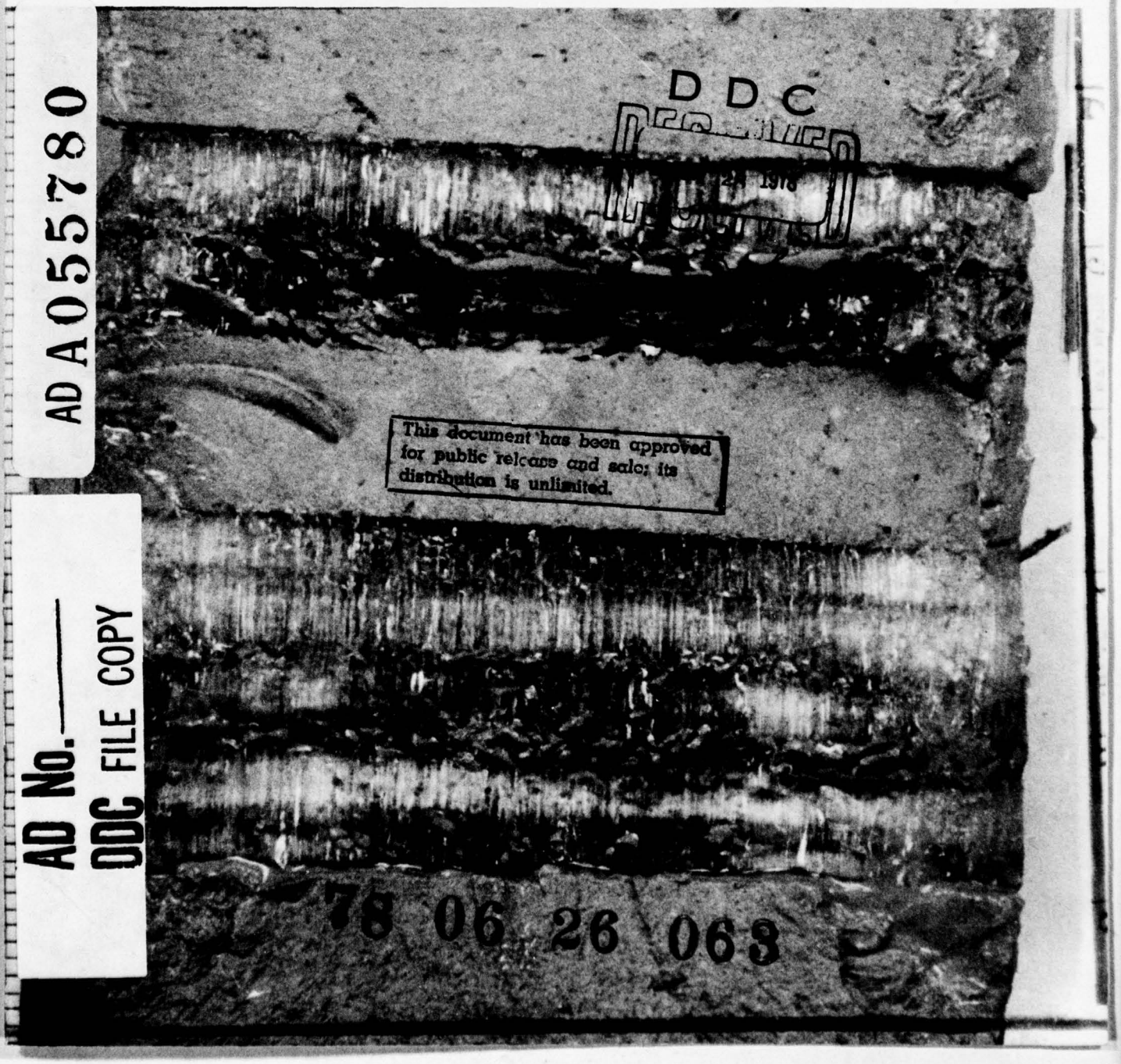
Segregation freezing as the cause of suction force for ice lens formation



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We propose a new freezing mechanism, called segregation freezing, to explain the generation of the suction force that draws pore water up to the freezing surface of a growing ice lens. The segregation freezing temperature is derived by applying thermodynamics to a soil mechanics concept that distinguishes the effective pressure from the neutral pressure. The frost-heaving pressure is formulated in the solution of the differential equations of the simultaneous flow of heat and water, of which the segregation freezing temperature is one of the boundary conditions.

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PREFACE

This report was prepared by Dr. Shunsuke Takagi, Research Physical Scientist, of the Physical Sciences Branch, Research Division, U.S. Army Cold Regions Research and Engineering Laboratory. Funding was provided by DA Project 4A161102AT24, *Research in Snow, Ice and Frozen Ground*, Task A1, *Properties of Cold Regions Materials*, Work Unit 002, *Properties of Frozen Soil* and DA Project 4A161101A91D, *In-House Laboratory Independent Research*, Work Unit 222, *De-Aired Frost Heave Test.*

Richard McGaw and Dr. Samuel Colbeck of CRREL technically reviewed the manuscript of this report.

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ILLUSTRATIONS

TABLES

NOMENCLATURE

c	heat content
D	hydraulic conductivity of unsaturated water
D_0	soil constant introduced in (16)
h	thickness of the ice lens
L	latent heat
p	pressure
q	defined by (39)
r	defined by (39)
S	entropy
T	temperature
t	time
$U_n(\xi)$	defined in (19)
u	= exp (βW), defined at (17)
V	volume
v	flux of water
W	degree of saturation by movable water
w	surcharge on the ice lens, i.e. frost-heaving pressure
x	space coordinate
α	thermal diffusivity
α_{20}	thermal diffusivity of water-saturated soil
β	soil constant introduced in (16)
$\Theta_n(\xi)$	defined in (42)
κ	thermal conductivity
κ_{20}	thermal conductivity of water-saturated soil
λ_0	soil constant introduced in (40)
μ	constant introduced in (7) and formulated in (31)
ν	soil constant introduced in (55)
ξ	variable defined by (20)
ρ	density
χ_s	volume occupied by both soil particles and unmovable water in a unit volume of the soil mass

Subscripts

A	air
I	in-situ freezing
i	ice
S	segregation freezing
s	soil
w	water
1	the first layer in Figure 3, i.e. the ice lens
2	the second layer in Figure 3, i.e. the unfrozen soil

SEGREGATION-FREEZING AS THE CAUSE OF SUCTION FORCE FOR ICE LENS FORMATION

Shunsuke Takagi

INTRODUCTION

The most enigmatic problem in the theory of frost heaving is the generation of the suction force that draws pore water up to the freezing interface of a growing ice lens, increasing its thickness despite the pressure exerted by the overlying burden of the frozen soil and surface load (see cover for photograph of ice lens). Most of the current literature on frost heaving explains the suction by use of the Laplace equation in capillary theory, which gives the pressure difference across a curved meniscus boundary of two different materials. However, capillary theory is not yet proven to be valid on the freezing meniscus of pore water.

In the case of a capillary tube containing air and water separated by a meniscus, the molecules composing the meniscus are stationary, and the Laplace equation gives the pressure difference across the meniscus. The molecules composing a freezing meniscus, however, are constantly renewed; theory does not yet prove whether the Laplace equation is valid or not on such a meniscus. We have found experimental evidence indicating that the Laplace equation is valid on a static ice/water meniscus where molecules are stationary, but not on a freezing meniscus where molecules are renewed.

In their definitive theoretical work in this field, Everett and Haynes (1965) caution that their theory of ice stress derived by applying capillary theory on ice/water menisci may fail when kinematic effects predominate. Koopmans and Miller (1966) measured the capillary potential of the ice/water meniscus and showed that the resulting curve coincided with the soil moisture characteristics, if they substituted the ice/water interfacial tension with the air/water interfacial tension. They took 24 hours to get one point of data. Their experiment shows that capillary theory applies on the static ice/water interface. Penner (1967) and Sutherland and Gaskin (1973) showed that the pressure required to stop ice lens growth was larger than the pressure predicted by capillary theory. Their experiments were kinematic. We may interpret these studies as indicating that the freezing of pore water is not a static effect caused by the capillary pressure but is a kinematic effect caused by the simultaneous flows of heat and water.

We have developed a concept suitable for describing the ice lens formation by using the theory of simultaneous flows of heat and water (Takagi 1959, 1963, 1965, 1970, 1974, 1975, 1977), which is systematized and stated in this report.

SEGREGATION FREEZING

We shall introduce segregation freezing as the agent for creating suction force to draw water to the freezing front and exerting frost-heaving pressure to the overlying burden.

Corte (1962) observed that ice growing upward can carry soil particles floating on the surface (Fig. 1). The explanation of the floating of a soil particle on a heaving ice surface is possible only by assuming that, between the surfaces of the particle and the ice, there exists a thin layer of unfrozen water whose molecules are constantly replenished during the heaving by the

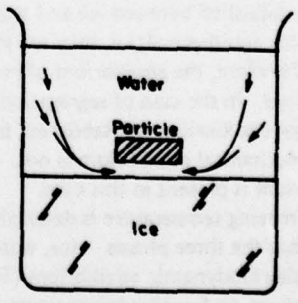

Figure 1. A particle floating on the heaving ice surface.

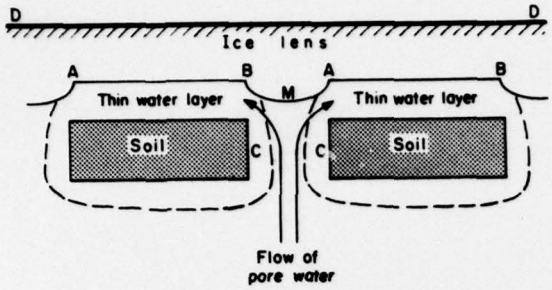

Figure 2. Ice lens forming on the thin water layer.

influx of water from the adjacent reservoir into the freezing front. In other words, we should recognize that, adsorbed or absorbed between the particle and the ice, there exists a heterogeneous layer of water whose "thickness" is maintained at a certain constant value during the freezing process. It should be stated, however, that theoretical physics cannot yet explain the water of this nature and we are quite ignorant of its properties.

In Figures 1 and 2, soil particles are represented by a rectangular shape, because the thickness of the heterogeneous layer is clearly shown in this form. In the case of an actual more complicated shape, the conceptual correction of "thickness" can be made easily.

The freezing of a thin water layer generates suction that draws water, as shown in Figure 1, from the surrounding reservoir. The freezing of water that generates suction will be called *segregation freezing.* In this case the heterogeneous water adsorbed or absorbed between the particle and the ice freezes. In contrast, the freezing of homogeneous free pore water will be called *in-situ freezing.* This freezing mechanism does not generate suction; i.e. the in-situ freezing front advances with the progress of the freezing.

In in-situ freezing, the ice pressure and the water pressure may not necessarily be equal, but mechanical equilibrium is established between ice and water. Ice and water are also equithermal; i.e. they are in thermal equilibrium. Therefore, the equilibrium of in-situ freezing is twofold. In the case of segregation freezing, however, thermal equilibrium is established, but as shown below, mechanical equilibrium is not. Only one type of equilibrium is present in this case.

The in-situ freezing temperature is determined by the condition that the three phases – ice, water, and vapor – are in thermodynamic equilibrium (Takagi 1959). The segregation freezing temperature is determined, as shown in the following, by the two-phase equilibrium.

Let us consider an ice lens resting on soil particles, as shown in Figure 2. If the uppermost part of the thin water layer freezes, water must be sucked in from the neighboring reservoir to recover the original thickness of the thin water layer. Then, if the soil particles stay at the same position during the freezing process, the surface DD rises by the thickness of the frozen portion. This is our explanation of frost heaving. According to our concept, therefore, an ice lens grows on soil particles.

Stress in the thin water layer that sustains the weight of the ice lens plus any surcharge on it acts in the water layer as if the water were solid. However, to calculate the freezing temperature of the thin water layer, i.e. the segregation freezing temperature, the simpler thermodynamic state of the pore water underlying the ice meniscus BMA may be considered instead of the complicated thermodynamic state of the thin water layer under the flat ice surface AB, because we may assume they are equithermal.

The pore water underlying the ice meniscus BMA and the ice lens overlying BMA are, in turn, equithermal, but not in mechanical equilibrium. The weight of the ice lens is not supported by the pore water, but by the thin water layers and the soil particles underlying the layers. The stress of the ice lens, therefore, is independent of the pore water pressure. In soil mechanics terminology, the pore water pressure is *neutral* to mechanical effects, but the stresses of the thin water layers and the soil particles are *effective.* They belong to different categories in terms of mechanical effects (Terzaghi 1942).

For simpler treatment we replace the ice stress with the ice pressure P_i. Then we can describe the thermodynamic equilibrium between the ice lens and the pore water by use of a formula of classical thermodynamics:

$$V_w\, dP_w - S_w\, dT = V_i\, dP_i - S_i\, dT \qquad (1)$$

where V is the specific volume, S the specific entropy, P the pressure, and T the temperature. The suffixes w and i refer to water and ice, respectively. Note that P_i and P_w may not necessarily be equal in this equation. (See Takagi (1965) for a treatment dealing with the tensorial ice stress.)

The meaning of the variations denoted by the total differentials in this equation must be clarified. We choose the datum state (i.e. the starting point of the variation) to be the state of in-situ freezing. The temperature at the datum state, therefore, is the in-situ freezing temperature T_f. We raise the pressure of the ice at the datum state by

$$dP_i = w + \rho_f h \qquad (2)$$

where w is the surcharge overlying the ice lens, h the thickness of the ice lens, and ρ_i the density of ice. We do not change the pressure of the pore water:

$$dP_w = 0. \tag{3}$$

We assume that the soil column underlying the soil particles is incompressible, so that no disturbance can intrude into the system during the proposed pressure increase. During the process, we maintain the thermodynamic equilibrium between the pore water and the ice lens by keeping (1) valid, but leave the temperature free to change. Note that $w + \rho_i h$ is the surcharge on the ice at the freezing front, which may be interpreted, if frost heave actually takes place, to be the frost-heaving pressure.

Thus we can reach the final stage of the formulation. We find T_s the segregation-freezing temperature

$$T_s = T_l[1 - (w + \rho_i h)/(\rho_i L)] \tag{4}$$

by letting $dT = T_s - T_l$ and $S_i = L/T_l$ in (1), where T_l is the in-situ freezing temperature and L the latent heat. Therefore, T_s is always less than T_l, the difference being determined by the ice pressure increment, i.e. the frost-heaving pressure.

If we consider that the stress in the ice lens is determined by the configuration of the ice surface, the stress is not necessarily uniform in the ice lens. The nonuniform stress caused by the capillary force is considered by Everett and Haynes (1965). The difference of the capillary forces between the top and bottom menisci is considered by Loch and Miller (1975) to explain the cause of the flow of ice molecules in the growing ice lens. However, the capillary force does not seem to be directly related to the crystal growth.

In supercooled water, ice crystals grow with sharp edges (Hobbs 1974, Glen 1974) and frequently form dendrites. They grow against the chemical potential gradient in the solid; their growth rate is determined by the heat transfer and the availability of the growth material in the liquid. When the water temperature is very close to the ice temperature, however, ice grows into the water forming a smooth ice surface (Glen 1974). The growth rate in this case is still determined, we believe, by the heat transfer and the availability of the growth material in the liquid, although we could not find any reference that clearly states this. The ice stress caused by the ice/water menisci does not seem to be a cause of crystal growth.

We showed (Takagi 1965) that the ice stress given on the right-hand side of (2) is the normal stress component in the vertical direction. This normal stress may be interpreted to be the overall representative value of the ice stress in the segregation freezing, in the same sense as the overall representative ice stress in the in-situ freezing is interpreted (Takagi 1959) to be atmospheric. Obviously, the stress of the ice forming inside a pore of soil is higher than atmospheric pressure by the amount of the capillary pressure caused by the curved ice surface. The formula of the in-situ freezing temperature, derived by assuming the ice pressure to be equal to atmospheric pressure is, however, confirmed experimentally (Schofield 1935, Williams 1964, Low et al. 1968). Atmospheric ice pressure, therefore, may be the overall representative value of the internal stress of the ice freezing in-situ, and choosing atmospheric pressure is probably a convenient way of avoiding the variability of the internal ice pressure in in-situ freezing. Choosing the ice stress expressed by the right-hand side of (2) in the formulation of the segregation freezing temperature should therefore be interpreted in the same sense as choosing the atmospheric ice pressure in the formulation of the in-situ freezing temperature.

ANALYSIS

We shall use (4) of segregation freezing temperature as one of the boundary conditions of the simultaneous flows of heat and water to analyze the formation of a single ice lens. We shall make the physical system as simple as possible to keep the analysis feasible.

We assume the unfrozen soil underlying the ice lens to be incompressible under the action of the surcharge and, moreover, under the action of the flows of heat and water. At present this assumption is needed because the currently available water flow equations do not include volume change caused by absorption and depletion of water. Also, we do not yet know the constitutive equations of soils to describe the deformation due to surcharge and water content variations. Unification of hydraulics and mechanics still seems to be a remote goal. The assumption of incompressibility obviates these difficulties. Furthermore, this assumption simplifies the analysis, because the segregation freezing front overlying an incompressible unfrozen soil layer stays at the initial level until in-situ freezing replaces the ongoing segregation freezing.

In this system the freezing front starts to descend when in-situ freezing begins. The selection rule, stating which of the two processes should start, emerges at the end of the analysis.

We assume that segregation freezing takes place at the ground surface. Then, we may not consider the complicated flow of unfrozen water in the frozen region. In fact, we are going to analyze the frost needle formation on the ground surface.

According to the present theory, frost needles grow on soil particles. Pore water between soil particles may or may not freeze, because $T_s \leqq T_l$. In this analysis, we disregard the individuality of the frost needles (as seen on the cover) and suppose that the ice lens formation and heat and water flows are uniform in the horizontal direction. In other words, we suppose that the flows are one-dimensional in the vertical direction. The aim of this analysis is not the formulation of actuality but the clarifying of the implication of our assumptions.

We will analyze only for the limit of $t \to 0$. For $t \to 0$, we can linearize the highly nonlinear equations of simultaneous flows of heat and water, and can solve them analytically.

Before entering into the details of the analysis, it is appropriate to give an overview of the analysis.

First, we shall solve the heat conduction in the nascent ice layer by Portnov's (1962) method, of which the essence is given in Appendix A. The boundary temperature conditions are the step-change air temperature T_A at the upper side of the ice lens AA in Figure 3, where $x = -h(t)$ and the segregation freezing temperature T_s at its lower side SS, where $x = 0$. This solution enables us to express the temperature gradient at SS as a function of T_A and T_s.

Second, we shall solve the unsaturated water flow in the unfrozen region; i.e. we shall determine the water content $W(x,t)$. The boundary condition at $x = 0$ is that the water content at $x = 0$ suddenly drops to a certain unknown value $W(0,0)$ at the outset of the ice lens formation. We assign an arbitrary number $W(0,0)$ to the boundary value at $x = 0$ and $t = 0$. The initial condition is that $W(x,0) =$ constant for $0 < x < \infty$. The boundary condition at $x = \infty$ is that $W(\infty,t) =$ constant. These two constants must obviously be equal to each other. The solution of this problem enables us to calculate the flux of water entering the freezing front. All this water becomes ice to form frost needles; thus, we can calculate the ice lens growth rate dh/dt.

Third, we shall solve the equation of the double heat transfer, convected by the water flow and conducted through the soil mass, by using the segregation freezing temperature given by (4) as one of the boundary conditions. We evaluate the thermal conductivity and heat content in the duplicate heat transfer equation by use of the water content distribution $W(x,t)$ found above. The solution of the duplicate heat transfer enables us to calculate the temperature gradient at the segregation freezing front as a function of the surcharge w and the boundary water content value $W(0,0)$.

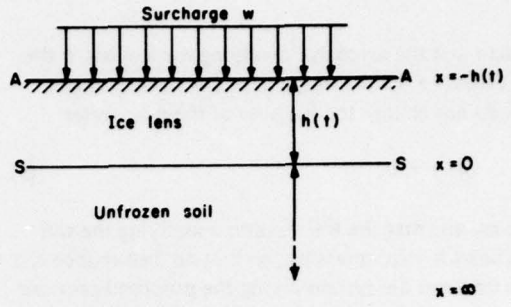

Figure 3. Analysis of ice lens formation.

Finally, we shall use the energy balance equations at the segregation freezing front. We substitute the ice lens growth rate dh/dt and the two temperature gradient equations previously formulated at both sides of the freezing front into the energy balance equation. Then, we can find surcharge w, i.e. the frost-heaving pressure, in terms of the air temperature T_A and the boundary water content value $W(0,0)$.

The selection rule is given by the expression of w. If the frost-heaving pressure w is zero or positive, segregation-freezing begins. If the frost-heaving pressure is negative, in-situ freezing begins. Soil data in this calculation were collected from many sources, and the soils were not incompressible; however, the result is deemed reasonable.

Heat conduction in the nascent ice layer

The ice lens is lifted as a whole at the rate of dh/dt. Applying the theory of heat conduction in a moving medium (Carslaw and Jaeger 1959), we have the differential equation of the temperature T_1 of the growing ice lens:

$$\frac{\partial T_1}{\partial t} - \frac{dh}{dt}\frac{\partial T_1}{\partial x} = \alpha_i \frac{\partial^2 T_1}{\partial x^2} \qquad (5)$$

where α_i is the thermal diffusivity of ice. Let

$$z = x + h(t)$$

and then (5) becomes

$$\frac{\partial T_1}{\partial t} = \alpha_i \frac{\partial^2 T_1}{\partial z^2},$$

where $0 \leqq z \leqq h(t)$. The solution of this problem by using Portnov's (1962) method is shown in Appendix A.

To consider the limit of $t \to 0$, we take only the first (i.e. $n = 0$) terms in (A6) and (A8) in Appendix A, and approximate them by

$$T_1(x,t) = a + b \operatorname{erfc}\left[\frac{x + h(t)}{2\sqrt{\alpha_i t}}\right] \tag{6}$$

and

$$h(t) = 2\mu\sqrt{\alpha_i t}, \tag{7}$$

where a, b, and μ are unknown constants. To determine a and b, we use the conditions:

$$T_1(0,t) = T_s \tag{8}$$

and

$$T_1(-h(t),t) = T_A, \tag{9}$$

where T_A is the air temperature on the ice/air interface AA. Considering that

$$h(0) = 0, \tag{10}$$

we denote the value of T_s at $h = 0$ by

$$T_s(0) = T_1\{1 - w/(\rho_i L)\}. \tag{11}$$

Thus we have

$$a = \frac{T_s(0) - T_A \operatorname{erfc}\mu}{\operatorname{erf}\mu} \tag{12}$$

and

$$b = \frac{T_A - T_s(0)}{\operatorname{erf}\mu}. \tag{13}$$

The constant μ will be given at the end of the next section.

Water flow in the unfrozen soil

We express the one-dimensional flow of unsaturated water with the following equation (Miller and Klute 1967):

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x}\left[D(W)\frac{\partial W}{\partial x}\right] \tag{14}$$

where $W(x,t)$ is the degree of saturation by the movable pore water such that $W = 1$ when the pore is saturated with the movable water and $W = 0$ when no movable water exists in the pore, and $D(W)$ is the hydraulic conductivity of unsaturated water flow as a function of W. We believe the degree of saturation is more convenient for mathematical analysis than the water content conventionally used in soil engineering. The cross-effect of the temperature gradient on the water flow may not be included in (14), because our interest is in the flow of liquid water, not in the flow of water vapor (Philip and DeVries, 1957). A question, "What will happen if air is unavailable to the frost-heaving system?" is discussed in Appendix B.

The flux $v(x,t)$ of liquid water is given as

$$v(x,t) = (1 - \chi_s)\, D(W)\, (\partial W/\partial x) \tag{15}$$

where χ_s is the volume occupied by both soil particles and unmovable water in a unit volume of the soil mass. The volume of movable water in the unit volume is $(1 - \chi_s)W$. We have formulated $v(x,t)$ in (15) to be positive in the upward direction.

We assume the diffusion coefficient $D(W)$ in (14), following Gardner (1958a, 1959), to be

$$D(W) = D_0 \exp(\beta W) \tag{16}$$

where D_0 and β are constants. However, this form of $D(W)$ is inadequate, because it does not become constant in the neighborhood of $W = 1$ as required by Darcy flow (i.e. the water flow saturating pores). A few years after the analysis presented here was finished, we found another formula (Gardner 1958b) that satisfies this requirement. This analysis was not revised, however, because we believe that this defect in our data of water content is tolerable, as will be shown in the numerical analysis section. A more realistic analysis should be attempted when the rigid soil frost-heaving test will become available.

Use of u defined by

$$u = \exp(\beta W) \tag{17}$$

simplifies eq 14, where $D(W)$ is given by (16), to

$$\frac{1}{D_0}\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2}. \tag{18}$$

In view of Portnov's (1962) formulation (App. A), we may assume $u(x,t)$ to be in the following form:

$$u(x,t) = \sum_{n=0}^{\infty} (D_0 t)^{n/2}\, U_n(\xi) \tag{19}$$

where ξ is defined by

$$\xi = x/\sqrt{2D_0 t} \tag{20}$$

and $U_n(\xi)$ is a function of ξ only. To discuss the limit $t \to 0$, we need only the first term:

$$u(x,t) = U_0(\xi). \tag{21}$$

Then (18) becomes

$$\xi \frac{dU_0}{d\xi} + U_0 \frac{d^2U_0}{d\xi^2} = 0. \tag{22}$$

Use of a single independent variable ξ demands that the boundary condition at $x = \infty$ at time $t > 0$ and the initial condition in the region $0 < x < \infty$ reduce to a single condition:

$$U_0(\infty) = \exp(\beta W_\infty) \tag{23}$$

where W_∞ is a constant such that

$$W(0,0) < W_\infty \leqq 1. \tag{24}$$

We determine the boundary value $U_0(0)$ as follows. Use of (19) in (17) shows us that $W(x,t)$ can be expressed with a series similar to (19). For the limit of $t \to 0$, taking only the first term, $W(x,t)$ may be approximated by

$$W(x,t) \doteqdot \psi_0(\xi), \tag{25}$$

where $\psi_0(\xi)$ is related to $U_0(\xi)$ by

$$U_0(\xi) = \exp(\beta\psi_0(\xi)). \tag{26}$$

Because $\psi_0(\xi)$ is continuous, it must satisfy

$$W(0,0) = \psi_0(0). \tag{27}$$

Thus we get

$$U_0(0) = \exp(\beta W(0,0)). \tag{28}$$

To solve (22) with the boundary conditions (23) and (28), we used Scott and Hank's (1962) method. Assuming an arbitrary value of $U_0'(0)$ and given $U_0(0)$ and $U_0'(0)$, one can compute the higher derivatives $U_0^{(n)}(0)$ ($n \geqq 2$) by use of (22). Thus, one can formulate a Taylor series in the neighborhood of $\xi = 0$. If the convergence deteriorates as ξ increases, one can analytically continue the Taylor series to a new series that better converges in a new range of larger values of ξ. Repeating the analytical continuation as many times as necessary, one can find the value $U_0(\infty)$, which, however, in general is different from the given value of $U_0(\infty)$. One renews $U_0'(0)$ and repeats the same procedure until $U_0(\infty)$ agrees with the given value. Thus, the Scott and Hanks method enables us to express $U_0'(0)$ numerically in terms of $U_0(0)$. In this way we can find the numerical solution of $U_0(\xi)$ involving $W(0,0)$ as a parameter.

We may approximate $v(x,t)$ by substituting $\psi_0(\xi)$ in (25) for W in (15); thus, we find

$$v(x,t) \doteqdot \frac{\sqrt{D_0}}{\sqrt{2\beta}} (1 - \chi_s) \frac{dU_0}{d\xi} \frac{1}{\sqrt{t}}. \tag{29}$$

by use of (26). The balance of mass at the freezing front $x = 0$ is given by

$$\rho_i \frac{dh}{dt} = \rho_w v(0,t). \tag{30}$$

Use of (29) in (30) yields the differential equation of $h(t)$, which on integration with the initial condition $h(0) = 0$ yields μ introduced in (7):

$$2\mu\sqrt{\alpha_i} = (1 - \chi_s) \frac{\rho_w}{\rho_i} \frac{\sqrt{2D_0}}{\beta} U_0'(0). \tag{31}$$

Heat transfer in the unfrozen soil

We shall formulate the equation of the double heat transfer, convected by the water flow and conducted through the soil mass. The heat content c_2 per unit volume of unfrozen soil mass is

$$c_2 = c_w W (1 - \chi_s) + c_s \chi_s \tag{32}$$

where c_s is the heat content per unit volume of soil particles including unmovable water, and c_w the heat content per unit volume of water. Let $T_2(x,t)$ be the temperature of the unfrozen soil mass. The convective heat flow formulated by

$$Q = c_w T_2 v \tag{33}$$

is positive upward because the flux $v(x,t)$, defined by (15), is positive upward. The conductive heat flow R is given by

$$R = -\kappa_2 \partial T_2/\partial x \tag{34}$$

where κ_2 is the thermal conductivity of the soil mass. Because x is positive downward, R is positive downward. The duplicate heat transfer is formulated by

$$\partial\{c_2 T_2)/\partial t = \partial Q/\partial x - \partial R/\partial x. \quad (35)$$

To simplify (35), we derive the relation

$$\partial c_2/\partial t = c_w \partial v/\partial x \quad (36)$$

by differentiating (32) with regard to t and using (14) and (15) on the assumption that $c_s \chi_s$ = constant, i.e. that soil particles including unmovable water do not move. Substituting (33) and (34) into (35) and using (36), we find

$$c_2 \frac{\partial T_2}{\partial t} - c_w v \frac{\partial T_2}{\partial x} = \frac{\partial}{\partial x}\left(\kappa_2 \frac{\partial T_2}{\partial x}\right). \quad (37)$$

It may be verified that the equation of the triple heat transfer (Philip and DeVries 1957, and DeVries 1958), containing convection by water vapor, convection by liquid water, and conduction through the soil mass, reduces to (37) when the former is simplified on the assumption that the flow of water vapor is negligible.

Equation (37) of the duplicate heat transfer includes the effect of the water flow v on the temperature gradient, but (14) of the unsaturated water flow does not include the reciprocal relationship, i.e. the effect of the temperature gradient on the water flow. The validity of Onsager's reciprocal relationship (DeGroot and Mazur 1962) is not claimed in this paper; nor does this relationship hold in the theory by Philip and DeVries (1957) and DeVries (1958). The neglect of the relationship seems to be natural, although not yet proven, in the simultaneous flows of heat and water through soil, because heat can penetrate soil particles but water cannot — a condition that is not considered in theoretical physics for proving Onsager's relationship.

We cast (32) into a form convenient to the heat flow analysis

$$c_2 = c_w q (1 - r + rW) \quad (38)$$

by introducing two constants q and r through the following two equations.

$$\begin{aligned} qr &= 1 - \chi_s \\ q(1-r) &= c_s \chi_s / c_w. \end{aligned} \quad (39)$$

The constant qc_w expresses the heat contained in the water-saturated soil mass. To prove this, note that the sum of the two equations in (39) yields the relation

$$qc_w = c_w (1 - \chi_s) + c_s \chi_s$$

whose right-hand side is the one found by letting $W = 1$ in the right-hand side of (32). The constant r is in the range $0 < r < 1$, because dividing the first equation of (39) with the one at the top of this column shows that $r = c_w(1 - \chi_s)/[c_w(1 - \chi_s) + c_s \chi_s]$.

We used Kersten's (1949) equation

$$\kappa_2 = \kappa_{20} (1 + \lambda_0 \log W) \quad (40)$$

to express the thermal conductivity κ_2 of unsaturated soil. The constant κ_{20} is the thermal conductivity for the saturated condition ($W = 1$), given by

$$\kappa_{20} = c_w \rho_w q \alpha_{20} \quad (41)$$

where α_{20} is the thermal diffusivity of the saturated soil. The constant λ_0 is a soil constant.

In view of Portnov's formulation (App. A), we may assume $T(x,t)$ to be in the following form:

$$T_2(x,t) = \sum_{n=0}^{\infty} (D_0 t)^{n/2} \Theta_n(\xi) \quad (42)$$

where we have introduced a function Θ_n of ξ only. Taking the lowest term of t, we approximate $T_2(x,t)$ with

$$T_2(x,t) \doteq \Theta_0(\xi). \quad (43)$$

Substituting T_2 from (43), v from (29), c_2 from (38), and κ_2 from (40), (37) transforms to

$$\frac{\alpha_{20}}{D_0} \frac{d^2\Theta_0}{d\xi^2} + \left[F(\xi) + \frac{\alpha_{20}}{D_0} \frac{\lambda \frac{d}{d\xi} \log W_0(\xi)}{1 + \lambda \log W_0(\xi)}\right] \times$$

$$\times \frac{d\Theta_0}{d\xi} = 0 \quad (44)$$

where we have defined

$$F(\xi) = \frac{\xi(1 - r + r\, W_0(\xi)) + \frac{1 - \chi_s}{\beta q} \frac{dU_0}{d\xi}}{1 + \lambda \log W_0(\xi)}. \quad (45)$$

The boundary condition at $x = 0$ is

$$\Theta_0(0) = T_s(0). \quad (46)$$

The boundary condition at $x = \infty$ and the initial condition for $0 < x < \infty$ reduces to a single condition

$$\Theta_0(\infty) = T_\infty(0). \tag{47}$$

Equation 44 integrates to

$$\Theta_0(\xi) = \Theta_0'(0)(1 + \lambda \log W_0(0)) \int_0^\xi G(\eta)\, d\eta + \\ + T_s(0) \tag{48}$$

where $\Theta_0'(0)$ is the yet unknown value of $d\Theta_0(\xi)/d\xi$ at $\xi = 0$, and $G(\eta)$ is defined by

$$G(\eta) = \frac{1}{1 + \lambda \log W_0(\eta)} \exp\left[-\frac{D_0}{\alpha_{20}} \int_0^\eta F(\xi)\, d\xi\right]. \tag{49}$$

The boundary condition (46) is satisfied by (48). The boundary condition (47) is satisfied if $\Theta_0'(0)$ is chosen to be

$$\Theta_0'(0) = \frac{T_\infty - T_s(0)}{(1 + \lambda \log W_0(0)) \int_0^\infty G(\eta)\, d\eta}. \tag{50}$$

Energy balance at the segregation-freezing front

The balance of energy at the segregation freezing front is described by

$$\kappa_i \left(\frac{\partial T_1}{\partial x}\right)_0 - (\kappa_2)_0 \left(\frac{\partial T_2}{\partial x}\right)_0 = \rho_i \frac{dh}{dt} L \tag{51}$$

where $(\kappa_2)_0$ is the value of κ_2 found from (40) by letting $W = W(0,0)$. Notations $(\partial T_1/\partial x)_0$ and $(\partial T_2/\partial x)_0$ are values of $\partial T_1/\partial x$ and $\partial T_2/\partial x$ at $x = 0$. For the limit of $t \to 0$ we find

$$\left(\frac{\partial T_1}{\partial x}\right) = \frac{T_s(0) - T_A}{\operatorname{erf}\mu} \frac{1}{\sqrt{\pi \alpha_i t}} e^{-\mu^2} \tag{52}$$

from (6) by use of (13), and

$$\left(\frac{\partial T_2}{\partial x}\right)_0 \doteq \frac{1}{\sqrt{2D_0 t}} \Theta_0'(0) \tag{53}$$

from (43) by use of (50). Thus, we can express w in terms of parameters $W(0,0)$, T_∞, and T_A as

$$\frac{T_1}{\rho_i L} w = \tag{54}$$

$$\frac{\dfrac{\kappa_i}{\sqrt{\pi\alpha_i}} \dfrac{e^{-\mu^2}}{\operatorname{erf}\mu} (T_1 - T_A) - \dfrac{c_w \rho_w}{\sqrt{2D_0}} \dfrac{q\alpha_{20}}{\displaystyle\int_0^\infty G(\eta)\, d\eta} (T_\infty - T_1) - \rho_i \mu \sqrt{\alpha_i L}}{\dfrac{\kappa_i}{\sqrt{\pi\alpha_i}} \dfrac{e^{-\mu^2}}{\operatorname{erf}\mu} + \dfrac{c_w \rho_w q\alpha_{20}}{\sqrt{2D_0} \displaystyle\int_0^\infty G(\eta)\, d\eta}}$$

by substituting (52) and (53) for the temperature gradients in (51), (7) for $h(t)$, and (11) for $T_s(0)$ in (52). In this equation μ is a function of $W(0,0)$ that can be found by expressing $U_0'(0)$ in (31) as a function of $W(0,0)$, which we have derived by use of Scott and Hanks' method; T_1 is also a function of $W(0,0)$ as given in (55) below.

Equation (54) gives the *selection rule.* If the right-hand side of (54) is zero or positive, segregation freezing starts. If it is negative, segregation freezing cannot start but, instead, in-situ freezing begins.

Numerical computation

The only experiment on segregation freezing including the measurement of unsaturated water flow is, to our knowledge, Hoekstra's (1966, 1967). The soil he used was Fairbanks silt, which does not satisfy the assumption of rigid pores; therefore (4) of T_s may not be exact for this soil. The water flow was unsaturated, the initial W being equal to 0.82; therefore (16) may be used for $D(W)$. Although his soil column was of finite length, we may use his data in our analysis, because we consider only the limit of $t \to 0$. The porosity was 0.36 and therefore $\chi_s = 0.64$. We determined the specific density ρ_s of the soil by equating $\rho_s \chi_s$ to the dry density, which was 1670 kg/m^3.

Low et al. (1968) observed almost complete linearity between T_1 and W for a Wyoming Na-bentonite nearly saturated with water. Assuming that this relationship is valid even for other soils, we formulated

$$T_1 = T_0 - \nu (1 - W) \tag{55}$$

where $T_0 = 273.15$ K and ν is a soil constant. We chose $\nu = 0.140$, referring to Keune and Hoekstra (1967).

Kersten's (1949) data of the thermal conductivity of Fairbanks silt gave $\lambda_0 = 0.892$ and $\kappa_{20} = 1.6039$ W/m K.

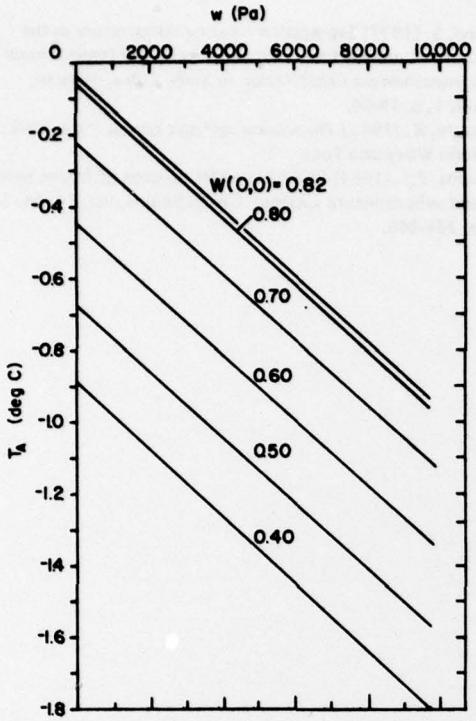

Figure 4. Frost-heaving pressure w as function of T_A and W(0,0).

Table I. Temperatures and degree of saturation for needle ice formation.

$W(0,0)$	T_A(°C)	$T_l = T_s$(°C)
0.4	–0.193	–0.084
0.5	–0.156	–0.07
0.6	–0.115	–0.056
0.7	–0.0732	–0.042
0.8	–0.0347	–0.028
0.82	–0.0279	–0.025

The specific heat of the dry soil given by him was 795 J/kg K; therefore, c_s is given by $c_s = 795\,\rho_s$. Using (39) we find $q = 0.6853$ and $r = 0.5370$.

Values of D_0 and β determined by Hoekstra's (1966, 1967) data were $D_0 = 2.92\times10^{-9}$ m^2/s and $\beta = 2.88$. The method of determination of these values is not mentioned here, because special knowledge of unsaturated water flow was used for their determination as mentioned by us (1970). The values of D_0 and β were reasonable as compared with other soils.

Numerical computation was performed keeping $T_\infty = 5$°C constant and varying T_A and $W(0,0)$. The relation between w and T_A is shown in Figure 4 with $W(0,0)$ as a parameter. The ice lens that forms when $w = 0$, i.e. under atmospheric pressure, is usually called needle ice. Under this condition, T_s is equal to T_l, which may be computed from (55) by using $W(0,0)$ for W. The values thus found are shown in Table I and Figure 4.

Although we have used the simplifying assumption of rigid pores and collected the input data from a variety of sources, the results shown in Figure 4 and Table I are reasonable when compared with observations in the laboratory and in nature.

LITERATURE CITED

Bell, E.T. (1934) Exponential polynomials. *Annals of Mathematics*, vol. 35, no. 2, p. 258-277.

Carslaw, H.S. and J.C. Jaeger (1959) *Conduction of heat in solids.* 2nd ed., Oxford: Clarendon Press.

Corte, A.E. (1962) Vertical migration of particles in front of a moving freezing plane. *Journal of Geological Research*, vol. 67, no. 3, p. 1085-1090.

DeGroot, S.R. and P. Mazur (1962) *Non-equilibrium thermodynamics.* New York: Interscience Publishers.

DeVries, D.A. (1958) Simultaneous transfer of heat and moisture in porous media. *Transactions of American Geophysical Union*, vol. 39, no. 5, p. 909-916.

Everett, D.H. and J.M. Haynes (1965) Capillary properties of some model pore systems with special reference to frost damage. RILEM Bulletin (Réunion internationale des laboratoires d'essais et de recherches sur les matèriaux et les constructions), New Series, no. 27, p. 31-38.

Gardner, W.R. (1958a) Mathematics of isothermal water conduction in unsaturated soil. Highway Research Board Special Report 40, p. 79-87.

Gardner, W.R. (1958b) Some steady-state solutions of the unsaturated moisture flow equation with application to evaporation from a water table. *Soil Science*, vol. 85, p. 228-232.

Gardner, W.R. (1959) Solutions of the flow equation for the drying of soils and other porous media. *Soil Science Society of America Proceedings*, vol. 23, p. 183-187.

Gautschi, W. (1964) Error functions and Fresnel integrals. In *Handbook of Mathematical Functions*, (M. Abramowitz and I. Stegun, Eds.) National Bureau of Standards Applied Mathematics Series, 55. Washington, D.C., United States Department of Commerce, p. 295-329.

Glen, J.W. (1974) The physics of ice. CRREL Cold Regions Science and Engineering Monograph II-C2a. AD 778009.

Hobbs, P.V. (1974) *Ice physics.* London: Oxford University Press.

Hoekstra, P. (1966) Moisture movement in soils under temperature gradients with the cold side temperature below freezing. *Water Resources Research*, vol. 2, no. 2, p. 241-250.

Hoekstra, P. (1967) Moisture movement to a freezing front. In *Geochemistry, Precipitation, Evaporation, Soil-Moisture, Hydrometry.* International Association of Scientific Hydrology (Bern), pub. 78, p. 411-417.

Jackson, R. (1964) The solution of problems involving the melting and freezing of finite slabs by a method due to

Portnov. *Proceedings of the Edinburg Mathematical Society*, vol. 14, ser. II, part 2, p. 109-128.

Kersten, M.S. (1949) Laboratory research for determination of the thermal properties of soils. Final Report to the Corps of Engineers, U.S. Army District, St. Paul, Minnesota.

Keune, R. and P. Hoekstra (1967) Calculating the amount of unfrozen water in frozen ground from moisture characteristic curves. CRREL Special Report 114. AD 659781.

Koopmans, R.W.R. and R.D. Miller (1966) Soil freezing and soil water characteristic curves. *Soil Science Society of America Proceedings*, vol. 30, p. 680-685.

Loch, J.P.G. and R.D. Miller (1975) Tests of the concept of secondary frost heaving. *Soil Science Society of America Proceedings*, vol. 39, p. 1036-1041.

Low, P.F., D.M. Anderson and P. Hoekstra (1968) Some thermodynamic relationships for soils at or below the freezing point. 1: Freezing point depression and heat capacity. *Water Resources Research*, vol. 4, no. 2, p. 379-394.

Miller, E.E. and A. Klule, (1967) The dynamics of soil water. 1: Mechanical forces. (In *Irrigation of Agricultural Lands*, ch. 13.) American Society of Agronomy, Madison, Wisconsin.

Penner, E. (1967) Heaving pressure in soils during unidirectional freezing. *Canadian Geotechnical Journal*, vol. 4, p. 398-408.

Philip, J.P. and D.A. DeVries (1957) Moisture movement in porous materials under temperature gradients. *Transactions of American Geophysical Union*, vol. 38, p. 222-232.

Portnov, I.G. (1962) Exact solution of freezing problem with arbitrary temperature variation on fixed boundary. *Soviet Physics-Doklady*, vol. 7, no. 3, p. 186-189.

Riordan, J. (1946) Derivative of composite functions. *Bulletin of the American Mathematical Society*, vol. 52, p. 664-667.

Riordan, J. (1949) Inversion formulas in normal variable mapping. *Annalen of Mathematical Statics*, vol. 29, p. 419.

Schofield, R.K. (1935) The pF of the water in soil. *Transactions of the 3rd International Congress of Soil Science*, vol. 2, p. 37-48.

Scott, E.J. and R.J. Hanks (1962) Solution of the one-dimensional diffusion equation for exponential and linear diffusivity functions by power series applied to moisture flow in soils. *Soil Science*, vol. 94, no. 5, p. 314-322.

Sutherland, H.B. and P.N. Gaskin (1973) Pore water and heaving pressures developed in partially frozen soils. Permafrost, 2nd International Conference, North American Contribution, National Academy of Sciences, Washington, D.C., p. 409-419.

Takagi, S. (1959) Theory of freezing-point depression of soil water, and a note on the extra-thermodynamics of soil moisture. *Soil Science*, vol. 88, no. 1, p. 25-31.

Takagi, S. (1963) Fundamentals of the theory of frost-heaving. Permafrost, 1st International Conference, National Academy of Sciences-National Research Council, Washington, D.C., p. 203.

Takagi, S. (1965) Principles of frost heaving. CRREL Research Report 140. AD 626174.

Takagi, S. (1970) An analysis of ice lens formation. *Water Resources Research*, vol. 6, no. 3, p. 736-749.

Takagi, S. (1974) Heaving of Fairbanks silt during freezing with and without air available. CRREL Technical Note (unpublished).

Takagi, S. (1975) Fundamentals of ice lens formation. 15th National Heat Transfer Conference, San Francisco, California, Paper no. 28.

Takagi, S. (1977) Segregation-freezing temperature as the cause of suction force. *Proceedings of the International Symposium on Frost Action in Soils*, Lulea, Sweden, vol. 1, p. 59-66.

Terzaghi, K. (1942) *Theoretical soil mechanics.* New York: John Wiley and Sons.

Williams, P.J. (1964) Unfrozen water content of frozen soils and soils moisture suction. *Geotechnique*, vol. 14, no. 3, p. 231-246.

APPENDIX A

Essence of Portnov's method

Portnov (1962) presented an interesting idea for handling the moving boundary heat conduction problems. Jackson (1964) extended it and showed several examples. However, their presentations are still complicated. In the following a simple formulation of Portnov's idea is presented.

It is well-known that a solution of the heat conduction equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

in the infinite region, $-\infty < x < \infty$, is

$$T(x,t) = \frac{1}{2\sqrt{\pi \alpha t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4\alpha t}\right) \phi(\xi)\, d\xi \qquad \text{(A1)}$$

where $\phi(x)$ represents the initial temperature distribution. Use of (A1) enables one to find the solution in the moving boundary region $0 \leq x \leq h(t)$.

Let $\phi(x) = \phi_1(x)$ in $0 < x < \infty$ and $\phi(x) = \phi_2(x)$ in $-\infty < x < 0$. Assuming that $\phi_1(x)$ and $\phi_2(x)$ are analytic in their respective regions, let

$$\phi_1(x) = \sum_{n=0}^{\infty} \phi_1^{(n)} x^n$$

$$\phi_2(x) = \sum_{n=0}^{\infty} \phi_2^{(n)} x^n.$$

Then, one can integrate (A1) to

$$T(x,t) = \frac{1}{2} \sum_{n=0}^{\infty} n! \, (2\sqrt{\alpha t})^n \left\{ \phi_1^{(n)} I^{(n)}\mathrm{erfc}\left(-\frac{x}{2\sqrt{\alpha t}}\right) + \right.$$

$$\left. + \phi_2^{(n)} (-1)^n I^{(n)}\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \right| \qquad \text{(A2)}$$

where $I^{(n)}\mathrm{erfc}x$ is the n-time repeated integral of $\mathrm{erfc}x$, which may be expressed in the form of a single integral:

$$I^{(n)}\mathrm{erfc}x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \frac{(u-x)^n}{n!} e^{-u^2} du. \qquad \text{(A3)}$$

To find (A2), let $\xi - x = 2u\sqrt{\alpha t}$ in the region $0 < \xi < \infty$ and $x - \xi = 2u\sqrt{\alpha t}$ in the region $-\infty < \xi < 0$. Then use of (A3) easily yields (A2). The customary notation $i^{-n}\mathrm{erfc}x$ is rejected here in favor of $I^{(n)}\mathrm{erfc}x$, because i^{-n} in the functional notation can be confused with $(\sqrt{-1})^{-n}$.

The formula changing the negative argument of $I^{(n)}\mathrm{erfc}(-x)$ to the positive argument of $I^{(n)}\mathrm{erfc}x$ (c.f. formula 7.2.11 of Gautschi 1964) simplifies to

$$(-1)^n I^{(n)}\mathrm{erfc}x + I^{(n)}\mathrm{erfc}(-x) = \frac{1}{2^{n-1} n!} E_n(x), \qquad \text{(A4)}$$

when the polynomial $E_n(x)$, defined by

$$E_n(x) = e^{-x^2} \frac{d^n e^{x^2}}{dx^2} \qquad \text{(A5)}$$

$$= i^{-n} H_n(ix)$$

is introduced, where $H_n(x)$ is the Hermite polynomial and $i = \sqrt{-1}$. Using (A4), one can transform (A2) to contain positive arguments only:

$$T(x,t) = \sum_{n=0}^{\infty} (2\sqrt{\alpha t})^n \left\{ \frac{1}{2^n} \phi_1^{(n)} E_n\left(\frac{x}{2\sqrt{\alpha t}}\right) + \right.$$

$$\left. + \frac{(-1)^n n!}{2} \left(\phi_2^{(n)} - \phi_1^{(n)}\right) I^{(n)}\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \right|. \qquad \text{(A6)}$$

Equation (A6) shows that the temperature functions for $x = 0$ and $x = h(t)$ are

$$T(0,t) = \sum_{n=0}^{\infty} a_n t^{n/2}$$

$$T(h(t),t) = \sum_{n=0}^{\infty} b_n t^{n/2}. \qquad \text{(A7)}$$

Therefore, (A6) can express the temperature in the growing ice lens, if $h(t)$ is a power series of $\sqrt{t}$:

$$h(t) = \sum_{n=1}^{\infty} h_n t^{n/2}. \qquad \text{(A8)}$$

The series must begin with $\sqrt{t}$ because $t^{-1/2}$ is in the arguments of functions $E_n(x/(2\sqrt{at}))$ and $I^{(n)}\text{erfc}(x/(2\sqrt{at}))$. Bell's formula (Bell 1934, Riordan 1946, 1949) which gives the nth derivative of a function $f(g(x))$, may be used to express a_n and b_n in (A7) in terms of the derivatives of $E_n(h(t)/(2\sqrt{at}))$ and $I^{(n)}\text{erfc}(h(t)/(2\sqrt{\alpha t}))$ and arbitrary constants $\phi_1^{(n)}$ and $\phi_2^{(n)}$.

APPENDIX B

Frost-heaving without air available

We can theoretically prove that segregation freezing cannot start in a rigid soil whose pores are saturated with de-aired water.

To prove this, we may assume that the flow is one-dimensional. The equation of continuity, div $V = 0$, reduces in one-dimensional flow to $\partial v/\partial x = 0$, where V is the velocity vector, v the vertical component, and x the vertical coordinate. Therefore, v is a function of t only. Portnov's formulation shows that ice thickness h for an initial small period is proportional to $\sqrt{t}$. Substitution of this result into (30) of the balance of mass at the freezing front indicates that v for the initial small period must be proportional to $t^{-1/2}$. In other words, initial velocity is infinite throughout the entire domain, $0 \geqq x > \infty$. Under this initial condition, the problem of water flow cannot be solved and, therefore, segregation freezing cannot start.

We showed experimentally (Takagi 1974) that this theoretical conclusion with regard to rigid soils does not necessarily hold true with regard to deformable soils. However, the mechanics of water flow in deformable soils is not readily understandable.

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Literature cited: p. 9.

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