UNSTEADY BLADE ROWS IN HIGH-SPEED FLOW

Final Scientific Report

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by

M. Kurosaka
General Electric Company
Corporate Research and Development
Schenectady, NY 12301

and

University of Tennessee Space Institute
Tullahoma, Tennessee 37388

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This report covers analytic investigations over three years sponsored by the Air Force Office of Scientific Research directed toward defining the instability boundaries of low incidence supersonic compressor flutter. A closed form expression was developed for the unsteady pressure distribution for a flat plate cascade in supersonic flow which is valid for a frequency range of practical interest. The work was extended to symmetric parabolic arc airfoils in cascade. Finite thickness was shown to have a first order effect on the flow field. Coupling the flow analysis with blade vibration modes indicates a bending

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) instability in the frequency range of conventional design in general agreement with experimental data. An expression for the frequency of the unsteady pressure along the stationary rotor casing due to a vibrating blade row was derived. It indicates casing treatment tuned approximately to blade passing frequency and its harmonics sh uld be effective in absorbing this unsteady flow energy.

FOREWORD

This technical report covers an investigation on unsteady supersonic flutter carried out, starting on January 1, 1974 and ending on December 31, 1977. The research was sponsored by the Air Force Office of Scientific Research under Contract F 44620-74-C-0040.

For most of the contract period, the research was carried out at the Contractor's site, General Electric Corporate Research and Development, Schenectady, New York; due to the unexpected departure of the principal investigator, M. Kurosaka, from the General Electric Company to the University of Tennessee Space Institute on September 1, 1977, it was then subcontracted to UTSI, where it was continued and completed.

Aiding the principal investigator, I. H. Edelfelt of the General Electric CRD performed most of the computational task and C. E. Danforth, then chief consulting engineer of aeromechanics of G. E. Aircraft Engine Business Group, Evendale, Ohio, served as a consultant to the program; J. Q. Chu of UTSI, a graduate research assistant, also helped the principal investigator.

The contract was monitored by Lt. Colonel R. C. Smith, Program Manager, Directorate of Aerospace Sciences, Air Force Office of Scientific Research, United States Air Force, Bolling Air Force Base, D. C.

This final report supersedes all the previous interim reports and includes all the technical papers, both published and under preparation, which were written under the research contract.

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OBJECTIVES OF RESEARCH

Overall objective was to develop an analytical tool which enables one to define instability boundaries for supersonic unstalled flutter of aircraft engine fans and compressors and also to provide insights into its prevention.

The region of the supersonic instability often falls on the operating line of a high speed fan; consequently, the existence of such flutter presents itself as a barrier problem for the design of high Mach number fans. In order to overcome this serious obstacle, it is necessary to improve at an accelerated pace our capability to define and, at the same time, to minimize the region of blade instability.

The present program was comprised of three phases. In Phase 1, a simplified model of flat-plate airfoils was adopted and its objective was directed to extending our previous low frequency analysis to a higher frequency of practical interest so as to predict oscillatory forces acting on unsteady supersonic airfoils in a cascade subject to subsonic axial velocity. Phase 2 was conducted so as to remove the restriction of flat-plate airfoils; this extension is necessitated because there are substantial experimental data which indicate the importance of airfoil shape. Phase 3 addressed the effect of flow three-dimensionality, particularly the influence of radial velocity gradient in steady base flow and the means of flutter suppression by the provision of liners on outer casing walls.

SUMMARY OF SIGNIFICANT RESULTS

Since all seven technical papers, which are either already published or under preparation for publication, are attached in this report, we relegate the necessary details to them and here summarize only the important results and conclusions.

In Phase 1, airfoils were assumed to be flat plates. An expression of unsteady pressure distribution valid for the range of frequency parameter of practical interest was derived in closed form which is applicable to any cascade geometry and arbitrary motion of airfoils; this was an extension of our previous low frequency analysis (ASME Transactions, Journal of Engineering for Power, 1974, Vol. 95, January, pp. 13-31). When the unsteady pressure distribution was used as input to flutter prediction, the following two major conclusions were established: the zone of torsional instability tends to shrink as the frequency parameter increases, and at the value of frequency parameter of conventional design, the bending instability is predicted, the latter being in agreement with the experimental data. The details of these were presented at the IUTAM Symposium on Aeroelasticity in Turbomachinery, Paris, October, 1976, and published in Revue Francaise de Mecánique, Numéro Special, 1976, pp. 57-64, appended here as Appendix 1. In addition, our discussion on the issue of 'resonance' in a supersonic cascade has been published in the AIAA Journal, vol. 13, No. 11, November, 1975, pp. 1514-1516 (attached as Appendix 2).

In Phase 2, the restriction of flat plate was removed. The effect of airfoil shape was examined first for an isolated airfoil and the cumulative effects of nonlinearity on the unsteady pressure in the far field was studied. For an oscillating airfoil whose contour is of parabolic arcs, an analytical expression of unsteady flow field was obtained. The results clearly indicate that the effect of airfoil shape deeply affects the unsteady flow in the far field. This was published in the Journal of Fluid Mechanics, vol. 83, part 4, pp. 751-773, 1977, appearing here as Appendix 3. The obvious implication of this result for a single airfoil was indeed borne out in the subsequent extension to the cascaded airfoils where the importance of the airfoils shape was in fact confirmed, the details of which are described in Appendix 4.

In Phase 3 the effect of the flow three-dimensionality on flutter boundary was examined. The objective here was not so much on the precise prediction of the three-dimensional unsteady flow field, which would amount to an extremely complicated task; rather it was deemed more beneficial to focus attention on the effect of the surrounding casing walls and specifically to examine whether the use of sound absorbing material for the wall might significantly relieve the fluctuation of pressure on the airfoils. Since the critical section of the fluttering airfoils in supersonic flutter is located near the very tip of the airfoils, the provision of the acoustically treated wall on the adjacent outer casing would be directly effective in relieving the unsteady fluctuation and eventually reduce the level of unsteady pressure acting on the airfoil surface. Based on this idea, an analysis was carried out for a model problem where an isolated, three-dimensional airfoil is placed in a duct whose upper wall is acoustically treated. The flow in the duct is supersonic and initially assumed to be uniform; the airfoil is oscillating in the transverse direction. The solution indicates, as expected, the unsteady pressure field produced by the motion of the airfoil is significantly affected by the presence of the acoustically treated "soft" wall, as described in Appendix 5. During this phase, an additional analysis was made to examine the effect of the spanwise variation of the incoming steady flow. This was done in order to assess the refraction effect of the acoustic signal due to its interaction with the non-uniform steady velocity field where the fans/compressors bladings are immersed. This presented a challenging task of investigating the nature of the acoustic wave propagating through non-uniform flow, which is mathematically described by governing equations with varying coefficients. From our effort to surmount this difficulty evolved a powerful but simple technique which was found to be highly effective in solving a general class of problems involving unsteady disturbances propagating through non-uniform media; this is reported in Appendix 6. This general technique was then applied to examine specifically the effect of velocity gradient upon the unsteady pressure on supersonic airfoils, where, as shown in Appendix 7, the three-dimensional effect was quantified.

APPENDIX 1

"Some Recent Developments in Unsteady Aerodynamics of a Supersonic Cascade." presented at the Symposium of International Union of Theoretical and Applied Mechanics on Aeroelasticity in Turbomachines; published in Revue Francaise de Mécanique, Numéro Spécial, 1976, pp. 57-64.

SOME RECENT DEVELOPMENTS IN UNSTEADY AERODYNAMICS OF A SUPERSONIC CASCADE

by

M. KUROSAKA and I. H. EDELFELT

1. *INTRODUCTION*

Quite unexpectedly, the advent of high speed turbo-fan aircraft engines unfolded a generically new type of blading instability -- supersonic unstalled flutter. The instability, which until recently had been virtually unheard of, is found to occur when the tip speed exceeds sonic velocity; since the unstable regions spread over the operating line where the incidence is small, the flow over the bladings is not stalled. In order to develop a consistently reliable means of averting this trouble, the unsteady aerodynamic forces acting on the oscillating airfoils in a fan must be accurately predicted. In the present paper, we shall attempt to present recent developments of the unsteady aerodynamics of supersonic cascades carried out for the past several years at the Research and Development Center, General Electric Company; this is essentially an extension of our previous work, [1]. The other related works in this area which have appeared until now are [2] to [10]. While it will be attempted to make the paper as self-contained as possible starting from the description of the relevant background, prevailing emphasis will be focused on the recent and hitherto unpublished analytical results, which we hope to be of both sufficient interest and practical importance.

2. *CHARACTERISTICS AND PHYSICAL MECHANISM OF SUPERSONIC UNSTALLED FLUTTER*

When in flutter, the bladings vibrate sinusoidally at their natural frequencies; both predominantly bending (or flexural) and torsional motion of the airfoil tip section have been observed in unstalled supersonic flutter. The contour of the airfoil is known to introduce a significant, first-order effect on the flutter boundary. As an example, for an airfoil which had experienced unstalled supersonic flutter, apparently minor modifications in the airfoil shape succeeded in removing the instability. The back pressure also affects the unstable region but lowering the back pressure itself does not cure the flutter problem.

As regards the physical mechanism of the unstalled supersonic flutter, the first obvious cause that comes to mind is the self-sustained shock-boundary layer oscillation. The shocks impinging on the boundary layer often cause its separation. For an isolated airfoil in the transonic range where a shock appears over the airfoil surface, such shock-boundary layer interaction has often asserted to be capable of sustaining self-induced oscillation. If this is indeed true, similar phenomenon can certainly be expected even in a cascade. In fact, visualization studies of the fluttering airfoils in a supersonic cascade apparently show such oscillatory movement of the shock-boundary layer. Since the shock-boundary oscillation is the phenomenon most conspicuously visible in such a study, one is irresistably tempted to conclude that this is the driving mechanism of the supersonic unstalled flutter, where the oscillatory movement of large pressure rise associated with the impingement of the shock may be directly responsible. However, before one hastily jumps to the conclusion, we have to address the question of whether the shock-boundary layer oscillation is the real cause of the flutter or the aftereffect of the flutter, which itself is initiated by some other mechanism. Indeed, a careful review of the issue leads to what appears to be an inescapable conclusion -- the self-sustained oscillation between the shock and the boundary layer does not play a major role in the present instability. For example, in [11], Liepmann and Ashkenas found that, although the oscillation of a shock was observed for a transonic airfoil placed in a wind tunnel, the removal of the sources of disturbance, which otherwise always exist in any wind tunnel, virtually eliminated the oscillation of the shock. Other available experimental evidences tend to support the similar conclusion that the self-sustained oscillation between the shock and the boundary layer is a weak fluctuation.

Then, what is the predominent cause of the supersonic unstalled flutter? It is well-known that for an isolated airfoil in supersonic flow where the shocks are attached to both the leading and trailing edges and do not appear on the airfoil surface, the motion of the airfoil could become unstable under certain circumstances. Such an instability, which occurs in the obvious absence of self-sustained shock-boundary layer oscillation, is caused by the phased lag response of the unsteady flow outside of the boundary layer to the motion of the airfoil. At the debut of supersonic unstalled flutter in the late sixties, we identified this inviscid mechanism as the primary cause of the instability and the validity of the diagnosis has been increasingly buttressed by supporting evidences.

3. *DIFFICULTIES ASSOCIATED WITH THE ANALYSIS*

For the reason just stated, in the analytical formulation, we assume the flow to be inviscid. Also the flutter tends to become more severe as the pressure ratio is lowered. Hence we examine the situation of the low pressure ratio where the flow can be approximately taken to be purely supersonic from inlet to discharge. In addition, in this first attempt, the flow is assumed to be two-dimensional. Even after these idealization, the analysis of the problem on hand is a difficult one. One source of the difficulties results from the fact that in a current design of the fan, the axial velocity is subsonic even when the relative velocity is supersonic. This axial subsonic velocity implies that the Mach waves emanating from the leading edges extend upstream of the cascade front line, Figure (1), and there, in what is called preinterference zone, one has to take the mutual interference into account. Were the axial velocity supersonic, the Mach waves would be confined within the blade passage and consequently there would be no region of mutual interference upstream of the cascade front line. Furthermore, if one looks at the wake region off the airfoil, Figure (1), we note that the wake velocity between Q and R does influence the trailing edge section of the adjacent blade between points S and T. Since one does not à priori know the wake velocity, it must be sought as a part of the solution. (Again for supersonic axial flow, the wake velocity does not influence the pressure distribution on the airfoil surface.)

Another source of difficulty is masked in delicate subtlety and more fundamentally hard to overcome -- the breakdown of the acoustic theory in the far field. As is well-known, acoustic theory in a moving medium is based on two major underlying assumptions: a disturbance propagates at a uniform acoustic velocity and is swept downstream at a constant freestream speed. Although this approximation is sufficiently accurate in the vicinity of the body, the acoustic theory for a supersonic flow is manifestly unfit for the description of the far field; it fails, for example, to reproduce the fanning out or coalescence of Mach waves. The reasons for the breakdown have long been understood: as a wavelet spreads out, two nonlinear effects ignored in the acoustic theory -- the nonuniform acoustic and flow velocities which vary with both place and time -- emerge to exert their influence over a large distance. The nonlinear effects are locally small everywhere, including the far field. However, not only is the disturbance at a given point influenced by the localized, slightly perturbed flow properties but it has been undergoing a continual distortion while propagating through a nonuniform flow field. It is this cumulative distortion or "memory" content of the signal, which encroaches upon the result of the acoustic theory and eventually alters it in the far field. The breakdown of the acoustic theory raises a serious concern for a supersonic cascade if one takes the approach along the same line as adopted in the subsonic cascade aerodynamics where the upwash generated from all the airfoils are summed up within the framework of the acoustic theory.

4. *PASSAGE APPROACH*

The complication arising from the breakdown of the acoustic theory in the far field can be circumvented by resorting to a simple stratagem of what we call the passage approach. In the passage approach, we focus our attention to a strip of the flow field, the L-shaped region bounded by y_1, x_1, x_2 and y_2 axes of Figure (2). Then, the infinite cascade arrangement is replaced by a physically equivalent requirement of flow periodicity. In the case of subsonic flow, this approach would not offer any additional advantage. However, in the present supersonic situation where the range of influence of a given point in the flow is limited within its Mach cone, the attractive feature of the passage approach is that it enables one to formulate the problem involving only those points close to the reference airfoil; for example, the Mach cone emanating from a point along the y_1 axis of Figure (2) and located far from the leading edge completely misses the flow passage between the first and second airfoil and consequently in this formulation the pressure distribution upon the airfoil is unaffected by it. Since the formulation involves only the near field, the flow field of interest can be closely approximated by the acoustic theory, provided the thickness of the airfoil is sufficiently thin and can be neglected. (We will later see that for supersonic airfoils of conventional design, although their contour is relatively thin, it is not quite slender enough to be described adequately by the acoustic theory. Nevertheless, even under such a circumstance, the passage approach still extends its convenience in circumventing the trouble of summing up all the upwash.) Restricting our attention for the time being to flat-plate airfoils, let us examine how we can apply the passage approach to determine, within the framework of the acoustic theory, the flow in the preinterference zone.

If we focus our attention on the flow entering the passage, it is of course not equal to the upstream flow and must be determined as a part of the solution. If the initial conditions along the y_1 axis of Figure (2) are assumed to be known as

$$\bar{\Phi}'(x = 0, y) = f(y) \; ; \; \frac{\partial \bar{\Phi}'}{\partial x}(x = 0, y) = g(x) \qquad (1)$$

where $\bar{\phi}'$ is the spatial amplitude of the perturbed velocity potential, then along the boundary conditions, it is quite straightforward to determine the flow downstream of it. The solution of the acoustic equation subject to the initial conditions and the boundary condition for the first airfoils is given by equation (3.4) of [1]. The expression of $\bar{\phi}'$ given therein is applicable to the preinterference zone where the effect of the second airfoil is not felt (the shaded region of Figure (2)) and it obviously involves the initial conditions, f and g, where are unknown. However, the unknown initial conditions must be equal to the velocity potential, $\bar{\phi}'$, at the corresponding point on the y_2 axis, with the exception of the phase lag. Consequently the initial conditions may be replaced by the velocity potential $\bar{\phi}'$ and we obtain the following equation for the velocity potential:

$$\bar{\phi}'(x, y) = \frac{1}{m}\int_0^{x - my} V(\tau)\exp[-\frac{ik}{c}(x - \tau)]J_0[\frac{k}{M_\infty c}\sqrt{(x - \tau)^2 - m^2y^2}]d\tau$$

$$+ \frac{1}{2}\exp(-ik\frac{x}{c})e^{-i\mu}[\bar{\phi}'(s_1, \frac{1}{m}x + y + \ell) + \bar{\phi}'(s_1, \frac{1}{m}x - y + \ell)]$$

$$+ \frac{m}{2}\exp(-ik\frac{x}{c})e^{-i\mu}\{\int_0^{\frac{1}{m}x + y}\{[\bar{\phi}'_x(s_1, \tau + \ell) + \frac{ik}{c}\bar{\phi}'(s_1, \tau + \ell)]J_0(x, y; \tau)$$

$$- \frac{k}{M_\infty c}\bar{\phi}'(s_1, \tau + \ell)J_1(x, y; \tau)\}d\tau$$

$$+ \int_0^{\frac{1}{m}x - y}\{[\bar{\phi}'_x(s_1, \tau + \ell) + \frac{ik}{c}\bar{\phi}'(s_1, \tau + \ell)]J_0(x, -y; \tau)$$

$$- \frac{k}{M_\infty c}\bar{\phi}'(s_1, \tau + \ell)J_1(x, -y; \tau)\}d\tau\} \qquad (2)$$

where $V(x)$ is the normal component of the fluid velocity on the reference airfoil, $m = (M_\infty^2 - 1)^{\frac{1}{2}}$ where M_∞ is the freestream Mach number, c is the chord, k is $\omega c\, M_\infty^2/U_\infty(M_\infty^2 - 1)$ where ω is the frequency of oscillation and U_∞ is the freestream velocity, j_0 and j_1 are given by

$$j_0(x, y, \tau) = J_0[\frac{k}{M_\infty c}z], j_1(x, y, \tau) = \frac{x}{z}J_1[\frac{k}{M_\infty c}z] \; ; \quad z = \sqrt{x^2 - (\tau - y)^2 m^2}$$

where J_0 and J_1 are Bessel functions of zeroth and first order, respectively, μ is the interblade phase angle and s_1 and ℓ are parameters defining cascade geometry as given in Figure (2). (Strictly speaking, the above equation (2) is valid only for points belonging to the doubly hatched region of Figure (3) of [1] but this restriction is not essential.) We note that in (2), the first term on the right hand corresponds to the solution for an isolated airfoil and known. The bracketed expression in the second term involves $\bar{\phi}'$ in difference form, while the third term contains $\bar{\phi}'$ under the integral sign. Thus we call the equation (2) as the integro-difference equation. We also observe that the integro-difference equation is 'two-dimensional' in the sense that there are two independent variables, x and y. We have to solve (2) for the unknown $\bar{\phi}'$ to determine the preinterference zone.

5. *LOW FREQUENCY SOLUTION*

In our earlier investigation, [1], it was found that by confining our attention to the low frequency solution, it was possible to obtain a solution of the integro-difference equation in closed-form. According to the scheme, we expand all the variables in terms of the frequency parameter and retain only terms linear to frequency. Thus, for instance,

$$\bar{\phi}' = \phi^{(0)} + i\beta\,\phi^{(1)} + \ldots \; ; \quad V = V^{(0)} + i\beta\, V^{(1)}$$

where $\beta = \omega c/U_\infty$. Then the integro-difference equation (2) becomes considerably simplified; the equations corresponding to $\phi^{(0)}$ and $\phi^{(1)}$ are given as equation (3.17) and (3.18) of [1], respectively. The solution for the simplified integro-difference equation is given by equation (3.26) of the same paper, where the first term corresponds to an isolated airfoil solution and the other term represents the correction due to cascade arrangement. For the limit of sonic leading edge, the second term is reduced to zero, leaving only the term corresponding to an isolated airfoil solution. This is what should be expected because of the lack of any preinterference in the sonic leading edge limit.

Pursuant to this determination of the preinterference zone, a solution for the flow downstream of it can be obtained, including those regions affected by the wake. The complete expression for the unsteady pressure distribution is given in closed form by the equations (4.6) and (4.7) of Part 2, [1]. (There is a typographical error in the equation (4.6) and the last term in the bracket should read $\sum_{n=1}^{\infty} H(x_1 - 2n\,\ell m)$ instead of $\sum_{n=0}^{\infty} H(x_1 - 2n\,\ell m)$.) The unsteady pressure given there is in general form and applicable to any cascade geometry. For cascade geometries of practical interest, however, it can be reduced to considerably simpler form. For instance, if one chooses the cascade geometry of Figure (3), which is characterized by the requirement that the bow shock emanating from the leading edge of the airfoil misses the preceding airfoil, the unsteady pressure distribution is given in the following form: if we express the pressure distribution as

$$\Delta c_p = i\beta \frac{4\,h_0}{mc} e^{i\omega t} c_b^{(1)} \text{ (bending)} \; ; \quad \Delta c_p = \frac{4\,\alpha_0}{m} e^{i\omega t} [c_t^{(0)} + \frac{i\beta}{c} c_t^{(1)}] \text{ (torsion)}$$

where h_0 and α_0 are the amplitude of bending and torsional motion, respectively, then for $0 < x < c - s_1 + \ell m$,

$$c_b^{(1)} = 1 - e^{-i\mu} \; ; \quad c_t^{(0)} = 1 - e^{-i\mu} ,$$

$$c_t^{(1)} = x \frac{M_\infty^2 - 2}{M_\infty^2 - 1} - x_0 - e^{-i\mu} \frac{s_1 - \ell m}{M_\infty^2 - 1} - e^{-i\mu} [x \frac{M_\infty^2 - 2}{M_\infty^2 - 1} + s_1 \frac{M_\infty^2 - 2}{M_\infty^2 - 1} - 2\,\ell m - x_0] ,$$

and for $c - s_1 + \ell m < x < c$,

$$c_b^{(1)} = 1 \; ; \quad c_t^{(0)} = 1 , \quad c_t^{(1)} = x \frac{M_\infty^2 - 2}{M_\infty^2 - 1} - x_0 - \frac{e^{-i\mu}}{1 - e^{-i\mu}} \frac{s_1 - \ell m}{M_\infty^2 - 1} , \qquad (3)$$

where x_0 is the position of the torsional axis measured from the leading edge. The point $x = c - s_1 + \ell m$ corresponds to a point on the pressure surface where the shock emanating from the trailing edge of the preceding airfoils impinges (point P of Figure (3)). For the other cascade configurations, once the shock pattern is specified, the pressure distribution of [1] can be reduced to similarly simple form. Applying the pressure distributions to stability calculation based on a two-dimensional model, the flutter is shown to occur at the torsional motion and the flutter boundary is strongly affected by cascade parameters (see Figures (4), (5) and (6) of Part 2, [1]); bending motion is always predicted to be stable according to low frequency analysis.

For the purpose of practical applications, however, an extension of the analysis was called for, because the frequency parameters of the aircraft engine fan is of the order of unity and cannot be categorized as belonging to the low frequency case.

6. *EXTENSION TO HIGHER FREQUENCY RANGE*

Hence, subsequent to this low frequency analysis, we embarked on the extension of the analysis to the frequency range of practical interest. It was found that the best way to solve the integro-difference equation, (2), is to write $\bar{\phi}'$ as

$$\bar{\phi}' = \frac{1}{m} \int_0^{x - my} V(\tau) \exp[-\frac{ik}{c}(x - \tau)] J_0[\frac{k}{M_\infty c} \sqrt{(x - \tau)^2 - m^2 y^2}] d\tau + A + Bx + Cx^2 + Dy^2 + Ex^2 + Fxy^2 + \dots \qquad (4)$$

where the first term corresponds to an isolated airfoil solution and the rest of the terms represents the cascade correction. We note that the odd terms of y, like y, xy and x^2y are missing in the cascade correction term and this is due to the asymmetry of the integro-difference equation with respect to y. The coefficients, A, B, ..., can be determined by substituting them into the integro-difference equation. It turns out that due to the very nature of the difference form involved, one cannot determine coefficients in a successive way; they have to be determined simultaneously by solving an infinite number of linear algebraic equations. For all practical purposes it is sufficient, however, to retain the coefficients up to F and they have been solved in explicit form; numerical checks show that such a solution can satisfy the integro-difference equation quite closely for the frequency parameters of practical interest. As an example of such approximate solution, A, of equation (4) can be written out as

$$A = \frac{\frac{1}{m}\,e^{-i\mu}\int_0^{s_1 - \ell m} V(\tau)\,\exp[-\frac{ik}{c}(s_1 - \tau)]\,J_0[\frac{k}{M_\infty c}\sqrt{(s_1 - \tau)^2 - \ell^2 m^2}]\,d\tau + \ldots}{2(1 - e^{-i\mu})^2 + 4s_1\frac{ik}{c}(1 - e^{-i\mu})e^{-i\mu} - \frac{m^2}{M_\infty}(\frac{k}{c})^2 s_1^2\, e^{-i\mu}(1 + e^{-i\mu}) + \ldots} \qquad (5)$$

This new solution reveals nonuniformity associated with the previous scheme of frequency expansion discussed in Section 5. If we once again look at the equation (3), we note that the low frequency expression possesses a factor $1 - e^{-i\mu}$ as its denominator and hence becomes singular at zero interblade phase angle ($\mu = 0$). As pointed out in [1], from this singular behavior the unique-incidence effect can be recovered in the steady limit. However, the existence of the singularity at any frequency implies that even in the unsteady case, the pressure distribution would become infinitely large at this zero interblade phase angle. The reason for this anomaly of the previous, low-frequency solution can readily be observed if we inspect the expression of the denominator of (5). Namely, as long as the interblade phase angle, μ, is sufficiently far from the value of zero, one can legitimately expand the expression in the power series of the frequency parameter, k. In such an expression, the denominator becomes the leading term, $2(1 - e^{-i\mu})^2$, and one can show that the expression in fact agrees with the corresponding one obtained by the low frequency analysis. However, the leading term (and also the next term) of the denominator of (5) vanishes at zero interblade phase angle and consequently the formal expansion in the frequency parameter breaks down at this point; in other words, the regular perturbation scheme in the frequency is not uniformly valid (though in practical flutter calculation, this particular situation of zero interblade phase angle is of little significance). The present improved solution such as (5) removes this nonuniformity, in addition to extending its range of applicability to the practical frequency. The expression of (5) becomes singular only if both μ and k simultaneously become zero and from this, one can again obtain the unique-incidence effect as the steady limit in exactly the same manner as described in Section 4 of [1]. Except for this condition, there is no other singular point in (5) (see [12]).

The wake velocity can be determined likewise by expanding it in a Taylor series of x about the trailing edge and we obtained a closed-form expression for the pressure distribution now valid up to and including the frequency parameter of practical interest.

Figure (4) shows an example of the unsteady pressure distribution for a cascade arrangement corresponding to Figure (3); the airfoils ($M_\infty = 1.3$, solidity = 1.0, stagger angle = 60°) are executing torsional motion at its mid-chord with interblade phase angle of 90° (a) and 180° (b). The solid lines correspond to the solution obtained by the present scheme and the dashed lines to the low-frequency solution, equation (3). It is of interest to observe that, at these interblade phase angles, the low-frequency solution holds up remarkably well in comparison with the more exact present solution. The stability analysis shows that the region of torsional instability tends to shrink as the value of the frequency parameter increases, but even at the frequency parameter of unity where the isolated airfoil becomes completely stable, there still remains a considerable region of instability for cascaded airfoils. As a matter of considerable interest, the flutter analysis for the bending motion shows that although at low frequency the bending motion is always predicted to be stable, at the frequency parameter of practical interest, 'islands' of bending instability emerge. We feel that the results are encouraging, because, as mentioned in Section 2, the bending mode instability as well as the torsional one was observed for the actual fans in supersonic unstalled flutter.

7. *CUMULATIVE, NONLINEAR EFFECT OF AIRFOIL SHAPE*

We here recall a remark made earlier in Section 2 on the first-order effect of airfoil shape upon supersonic unstalled flutter. As mentioned in Section 3, the acoustic theory does not take into account the nonuniform fluid and acoustic velocity produced by the very presence of the airfoil with thickness. Consequently one has to depart from the conventional acoustic theory in order to capture the quintessential feature of the effects.

Prior to embarking on the description of the analysis, we first consider what one can anticipate, from the physical reasoning, as the consequence of nonuniform flow field. Let us first assume that only a single point on the airfoil is sinusoidally oscillating. When one plots at a given point in flow the time-trace of the disturbance emitted, the departure of the nonuniform acoustic and convective velocity from the uniform ones (acoustic theory) would be graphically revealed, mainly, as the phase difference between the actual signal and the one predicted by the acoustic theory. The phase lag depends on the position and the more one moves away from the source, the more the phase lag would increase. Suppose now that the airfoil, as a whole, is oscillating. Then the above phase for an individual disturbance, which differs from one signal to another, and that alone -- to say nothing of the modification in the amplitude of each signal -- could introduce, when signals are vectorially added, a significant correction to both the amplitude and phase of the unsteady flow in the far field. Thus the nonuniform flow field associated with the airfoil shape would introduce a change in the very substance of the fluctuating pressure (in addition to the usual alteration to be made to the direction of the characteristics). The modification induced to the far field signal has the following implication, which appears to warrant sufficient emphasis: contrary to the situation in the near field, the unsteady signal at a large distance -- even to the first order of small perturbation -- can by no means be separated from such effects as the airfoil shape, camber and angle of attack, which cause the properties of propagation nonuniform. If this is the case, no doubt this appears to explain the significance of the airfoil shape in the flutter boundary of cascaded airfoils.

We intend to confirm these physical expectations and for that purpose we first obtained the unsteady flow off the surface of an isolated oscillating airfoil with thickness, [13]. The upper and lower surfaces of the airfoil are assumed to consist of parabolic arcs. (Since the bow shock is assumed to be attached, the supersonic flow above the upper surface of the airfoil is independent of one below and consequently the analysis can treat the effect of camber as well as thickness.) The governing equation includes the second-order terms which amount, in the far field, to first-order unsteady term; this can be accomplished by the use of the strained coordinate technique of Lighthill, Whitham and Lin. We relegate the details of the analysis to [13] and at present it suffices to point out that in the final result, the unsteady velocity potential can be given, to the first-order of small perturbation, by the following integral representation:

$$\bar{\phi}' = \frac{1}{m}\int_0^s V(\tau)e^{ik(\tau - x)}\exp[i\,\frac{\epsilon k}{U_\infty} N \cdot 2my\, \phi'^{(1)}(s)] \times \exp\{i\,\frac{\epsilon k}{U_\infty}[(N - 1)(\phi^{(1)}(s) - \phi^{(1)}(\tau))$$
$$+ N(s - \tau)\,\phi'^{(1)}(s)]\} \times M[\tfrac{1}{2} - \frac{mk}{4i\,\epsilon\,\alpha\,N\,M_\infty^2},\ 1,\ \frac{i\,\epsilon\,k\,\alpha\,N}{m}(s - \tau)(p - \tau)]d\tau \qquad (6)$$

where $\phi^{(1)}$ is the steady velocity potential, the shape of the airfoil (in the mean position of oscillation) is given by

$$\epsilon(\tfrac{1}{2}\,\alpha\,x^2 + \beta x)$$

and where $N = (\gamma + 1)M_\infty^2/2(m)^2$, s and p are characteristic coordinates based on the steady velocity, and M is the confluent hypergeometric function. Contrary to the acoustic solution which breaks down in the far field, the present solution is uniformly valid in the entire flow. When reduced to various limits, it agrees with such known solution as the Whitham's rule at the steady limit, oscillating flat plate and the wedge solution of Carrier and Van Dyke. More importantly, the above solution for a parabolic airfoil reveals many physical features relevant to the propagation of unsteady disturbance through nonuniform flow.

The numerical results are presented in Figure (5), where the unsteady pressure distribution for a parabolic airfoil (thickness to chord ratio = 2.5%), computed from equation (6), is shown in comparison with the result for a flat-plate airfoil at two different frequencies of oscillation -- $\omega c/U_\infty = 0.1$ in Figure (5a) and $\omega c/U_\infty = 1$ in Figure (5b). There, both the amplitude and phase are plotted as functions of s, or the distance of the root of a straight Mach wave and at three different locations of y. We observed that - though for $\omega c/U_\infty = 0.1$ the effect of the airfoil shape does not become prominent at these locations -- for $\omega c/U_\infty = 1$, except for the close vicinity of the leading edge, it indeed alters the pressure distribution significantly, as anticipated by the physical considerations.

This result for an isolated airfoil appears to explain the importance of airfoil shape in supersonic unstalled flutter and currently efforts are under way to include the above results into cascade analysis. Though evident from the foregoing analysis, it appears worthwhile to emphasize that, in all phases of the present study, efforts are focused on deriving the solution in closed-form or at least retaining its analytical structure in order to facilitate the examination of the roles played by the various parameters and also to minimize the computational time expended for repeated flutter calculations.

ACKNOWLEDGMENT

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M. Kurosaka and I. H. Edelfelt
Corporate Research and Development
General Electric Company
Schenectady, New York, 12301, U. S. A.

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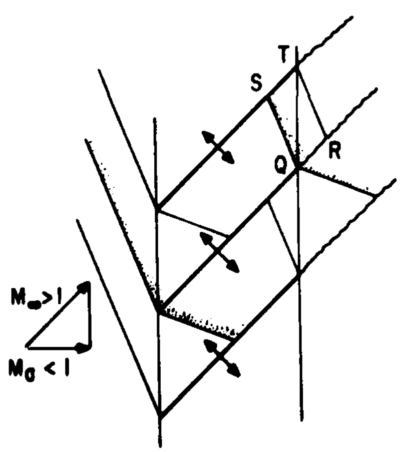

FIGURE 1

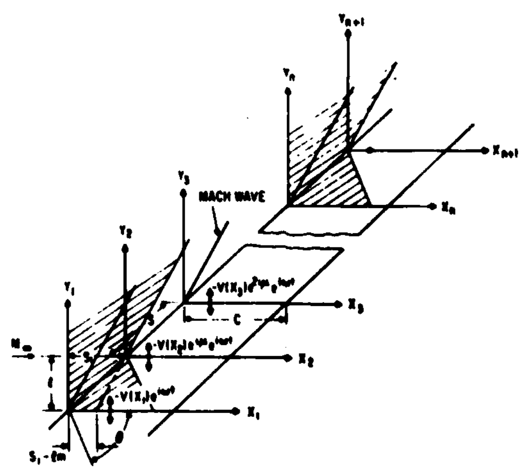

FIGURE 2

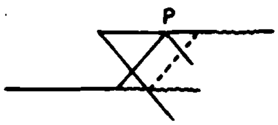

FIGURE 3

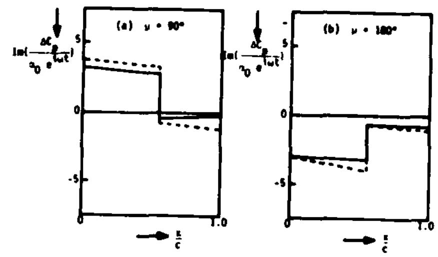

FIGURE 4. PRESSURE DISTRIBUTION

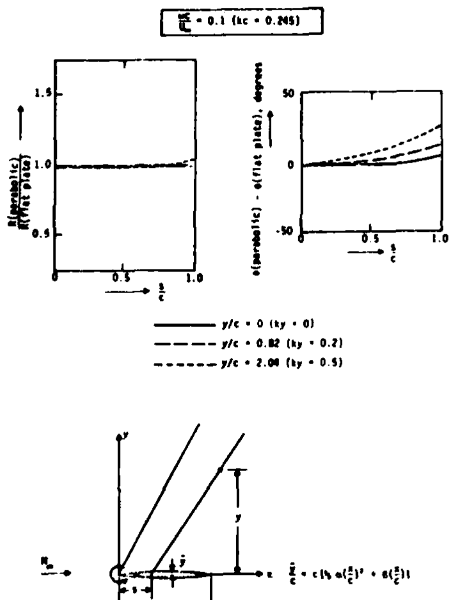

FIGURE 5a. Amplitude and Phase of Unsteady Pressure ($\frac{\omega c}{U_\infty}$ = 0.1)

$$= \frac{p'}{\rho_\infty U_\infty^2 e^{i\omega t}} = R\, e^{i\phi}$$

The ordinate for the figure on the left is the ratio of amplitude R for parabolic-arc airfoil to that of flat plate. The ordinate on the right is the difference in phase φ; M_∞ = 1.3, γ = 1.4, ε = 0.1, α = -1, β = 0.5 (maximum of $\bar{y}$/c = 0.0125), and pivot axis at the leading edge.

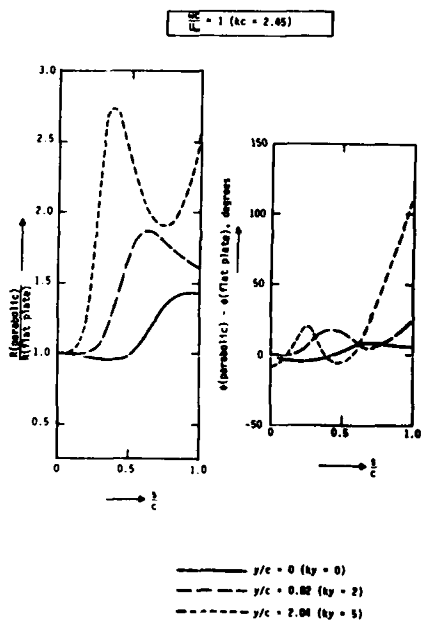

FIGURE 5b. Amplitude and Phase of Unsteady Pressure ($\frac{\omega c}{U_\infty}$ = 1).

APPENDIX 2

"On the Issue of Resonance in an Unsteady Supersonic Cascade." AIAA Journal, Vol. 13, No. 11, November 1975, pp. 1514-1516.

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On the Issue of Resonance in an Unsteady Supersonic Cascade

M. Kurosaka*
General Electric Company, Schenectady, N.Y.

IN a recent paper, Verdon and McCune[1] presented a linearized analysis of an unsteady supersonic cascade with subsonic axial velocity. It is an interesting advancement on Verdon's previous paper.[2] To achieve its objective of computing the pressure distribution, the analysis of Ref. 1 starts out to add the contributions from unsteady disturbances generated at all the oscillating airfoils below the reference airfoil in the cascade. The partial sums of the series were found to oscillate about apparent limiting values in general but, unfortunately, convergence was not proved. Aside from the problem of convergence, Ref. 1 also reports that under certain circumstances the numerical scheme broke down rather inexplicably. Thus it states in the concluding remarks that further work is needed to resolve these questions.

Several years ago, the author of the present Note encountered essentially the same series and found that it diverges at a certain number of discrete points, although these results were never published. The appearance of Ref. 1, therefore, seems a fitting opportunity to point out the divergence of the series, to offer it as a possible explanation for the aforementioned breakdown of the numerical scheme and to discuss its physical implications in regard to resonance and other salient points.

Consider, for example, the following kernel function $K(x)$, which appears in the integral equation (23) of Ref. 1:

$$K(x) = -(1/\mu)S(x)$$

where $S(x)$ is given by

$$S(x) = -\sum_{n=-\infty}^{-1} k\mu^2 n y_A e^{in\Omega} \times \frac{J_1\{k[(x-nx_A)^2-(\mu n y_A)^2]^{1/2}\}}{[(x-nx_A)^2-(\mu n y_A)^2]^{1/2}} \quad (1)$$

where x_A and y_A are the spatial distances between the adjacent airfoils in the cascade defined in Ref. 1, $\mu=(M^2-1)^{1/2}$ where M is the Mach number, $x_A-\mu y_A \geq 0$ (subsonic axial velocity), k is a frequency parameter, and Ω is related to the interblade phase lag σ. The nth term in the sum represents the influence of disturbances generated at the nth ($n<0$) below the first airfoil. We replace n by $-n$, rewrite J_1 in terms of the derivative of J_0 and obtain

$$S(x) = \sum_{n=1}^{\infty} e^{-in\Omega}\left[\frac{\partial J_0(W)}{\partial y}\right]_{y=0} \quad (2)$$

where

$$W(x,y) = k[(x+nx_A)^2-\mu^2(y+ny_A)^2]^{1/2} \quad (3)$$

We will show that $S(x)$ is divergent for certain combinations of parameters. For this, rewrite $W(x,y)$ as

$$W(x,y) = [z^2(n)-2z(n)Z(x,y)\cos\phi(x,y)+Z^2(x,y)]^{1/2} \quad (4)$$

where

$$z(n) = nk(x_A^2-\mu^2 y_A^2)^{1/2}$$
$$Z(x,y) = k(x^2-\mu^2 y^2)^{1/2} \quad (5a)$$

$$\phi(x,y) = -i[\tanh^{-1}(\mu y/x) - \tanh^{-1}(\mu y_A/x_A)] - \pi \quad (5b)$$

This transformation, Eq. (5) may be conveniently achieved by introducing a set of auxiliary variables defined by $x=\rho\cosh\nu$, $\mu y=\rho\sinh\nu$, $x_A=\rho_A\cosh\nu_A$, and $\mu y_A=\rho_A\sinh\nu_A$. Now from Neumann's addition formula,[3] J_0 can be expressed as

$$J_0(W) = \sum_{m=0}^{\infty} \epsilon_m J_m(Z)J_m(z)\cos m\phi$$

where

$$\epsilon_m = 1 \text{ for } m=0 \quad (6a)$$

$$\epsilon_m = 2 \text{ for } m=1,2,\ldots \quad (6b)$$

for any complex values of Z, z, and ϕ. Substituting Eq. (6) into Eq. (2) and assuming the validity of the interchange of the order of differentiation and summation, we obtain

$$S(x) = \sum_{m=0}^{\infty} \epsilon_m \times \left[\frac{\partial[J_m[Z(x,y)]\cos m\phi(x,y)]}{\partial y}\right]_{y=0} \cdot a_m \quad (7)$$

where a_m is given by

$$a_m = \sum_{n=1}^{\infty} e^{-in\Omega} J_m[nk(x_A^2-\mu^2 y_A^2)^{1/2}] \quad (8)$$

In deriving Eq. (7), we have made explicit use of the fact that $z(n)=nk(x_A^2-\mu^2y_A^2)^{1/2}$ which appears in Eq. (8), is independent of x and y, while Z and ϕ are independent of n. The series of a_m is called a Schlomlich series. Here we want to single out a_0 and examine its real part, which is given by

$$\text{Re}[a_0] = \sum_{n=1}^{\infty} \cos(n\Omega) J_0[nk(x_A^2-\mu^2 y_A^2)^{1/2}] \quad (9)$$

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*Fluid Mechanics Engineer, Research and Development Center. Member AIAA.

Now from Ref. 4, for any t and x between 0 and π, the series

$$\sum_{n=1}^{\infty} \cos (nt) J_0(nx)$$

is divergent at $t=x$ and therefore Eq. (9) diverges at

$$\Omega = k(x_A^2 - \mu^2 y_A^2)^{1/2}$$

It is easy to observe that Eq. (9) has another singularity at

$$\Omega = -k(x_A^2 - \mu^2 y_A^2)^{1/2}$$

or more in general has an infinite number of singularities at

$$\Omega = \pm k(x_A^2 - \mu^2 y_A^2)^{1/2} + 2\pi n \quad n=0, \pm 1, \pm 2, \ldots$$

If we express Ω by the interblade phase angle σ, this becomes

$$\sigma + kMx_A = \pm k(x_A^2 - \mu^2 y_A^2)^{1/2} + 2\pi n \quad n=0, \pm 1, \pm 2, \ldots \tag{10}$$

At these points the series $S(x)$ diverges. We take special note that the divergent condition for $S(x)$ is independent of the value of x.

One can check the divergent condition, Eq. (10), more directly by the numerical evaluation of the original series Eq. (1) in the vicinity of divergence. Let the departure from the divergence be Δ, i.e.,

$$\sigma + kMx_A \pm k(x_A^2 - \mu^2 y_A^2)^{1/2} - 2\pi n = \Delta$$

where $\Delta = 0$ corresponds to the divergent condition, Eq. (10). The partial sum of Eq. (1) can then be written, after replacing n by $-\bar{n}$, as

$$S_N(x) = k\mu^2 \sum_{\bar{n}=1}^{N} \exp\{-i\bar{n}[\pm k(x_A^2 - \mu^2 y_A^2)^{1/2} + \Delta]\} \bar{n} y_A \frac{J_1\{k[(x+\bar{n}x_A)^2 - (\mu \bar{n} y_A)^2]^{1/2}\}}{[(x+\bar{n}x_A)^2 - (\mu \bar{n} y_A)^2]^{1/2}} \tag{11}$$

A number of numerical checks have been performed and they in fact confirm the divergence of the series at $\Delta = 0$. For example, for the cascade A of Ref. 1, the real and imaginary part of $S_N(x)$ (divided by $k\mu^2$) at $x=0.5$ are shown in Fig. 1 as functions of Δ for the various values of N [where the positive sign in the argument of the exponential function of Eq. (11) is chosen]. As can be observed immediately, the series tends to diverge at $\Delta = 0$. In addition, we note the following important points;

a) The divergence does not appear when N is 5 or 20. It starts to emerge at $N=200$ and becomes prominent at $N=2000$. In other words, the cause of the divergence is not the effect of the nearby airfoils but the cumulative influence of those airfoils located far from the reference airfoil.

b) As $\bar{n}$ increases, the effect of x on the $\bar{n}$th term in the series, Eq. (11), becomes insignificant.† When we combine this with a), it becomes evident why the divergent condition of Eq. (10) for $S(x)$ is independent of x.

c) For a given value of Δ away from the divergence condition, the partial sum $S_N(x)$ appears to oscillate as N increases. This behavior is in agreement with what was reported in Ref. 1.

Other series in Ref. 1 can also be shown to diverge. For instance, Eq. (20) of Ref. 1 contains a series, which, in the region downstream of the Mach wave emanating from the leading edge of the first airfoil, becomes

$$T(x,\eta,y) = \sum_{n=1}^{\infty} e^{-in\Omega} J_0\{k[(x-\eta+nx_A)^2 - \mu^2(y+ny_A)^2]^{1/2}\} \tag{12}$$

By a method exactly identical to that used for $S(x)$, we can readily show that $T(x,\eta,y)$ again becomes divergent at the condition, Eq. (10).

For the cascade A of Ref. 1, the linear relationship between σ and k as indicated in the divergent condition, Eq. (10) is plotted in Fig. 2 for several values of n. As mentioned, breakdown of the numerical scheme was reported in Ref. 1 for certain combinations of parameters; in Fig. 7 of Ref. 1, for a given value of k, σ was continuously increased starting from 0 and the failure occurred at a certain value of σ. When we add such combinations of σ and k closest to the breakdown situation to the present Fig. 2, where they are designated by circled points, breakdown is observed to occur near the divergent condition.

The present divergent criteria, Eq. (10), are identical with those obtained by Samoilovich.[5] He obtained it under the restricted condition corresponding to $x=\eta=y=0$ in Eq. (12). In such a case, the argument of J_0 becomes proportional to n and the use of the Poisson summation formula readily enables the series to be transformed into another series representation, which possesses the singular points at Eq. (10). In the present derivation, the divergent condition has been obtained for the more general case of any finite values of x, η, and y.

It is a matter of considerable interest to observe that Eq. (10) is formally the same as the resonance condition in a subsonic cascade,[6] and therefore it may be given the physical meaning similar to that discussed in Ref. 7. Thus, for the present divergence of the series associated with the supersonic cascade, Samoilovich also gave the physical interpretation of "resonance." Contrary to the subsonic cascade, it is, however, highly unlikely that resonance at these conditions could indeed occur for a supersonic cascade. The reason is as follows. We have found that the divergence is the direct consequence of the cumulative contribution of those airfoils located far from the reference airfoil; in Fig. 1, the peak does not appear when the number of the preceding airfoils N is 5 or 20, and it starts to emerge only at $N=200$. Needless to say, such an effect is computed within the framework of linearized formulation. It is well known, however, that the linearized treatment of a supersonic flow breaks down in the field far from an airfoil. According to the linear theory, disturbances created by the airfoil would propagate unattenuated, even to infinity, for both the low- and high-frequency limits (for high-frequency behavior, see Ref. 8.) Although in the near field the linearized theory is a good approximation, the contribution of the nonlinear terms become no longer negligible in the far field and there it encroaches on and modifies the effect predicted by the linear theory. Physically, this follows from the fact that, by the time disturbance reaches the far field, two nonlinear effects ignored in the acoustic theory—the convection of disturbance by the *local* and *instantaneous* fluid velocity and its propagation at the nonuniform speed of sound—have cumulatively taken their toll and distorted the shape of the wavelet given by the linearized theory. Thus, as Lighthill[9] puts it: "... the failure of linearized theory ... is explained by the fact that ... while yielding adequate results in a limited region, may yield a worse and worse approximation to the solution farther and farther from where the boundary conditions determining the solution were applied." Consequently, only the influence of a limited numbers of airfoils ($N \approx 5$, say, for a typical cascade) in the neighborhood of the reference airfoils can accurately be predicted by the linearized theory. As the distance from the reference airfoils increases, the nonlinear effect would rapidly alter the linearized con-

†This enables one to check the divergent condition directly by using the asymptotic formula of J_1 in Eq. (11).

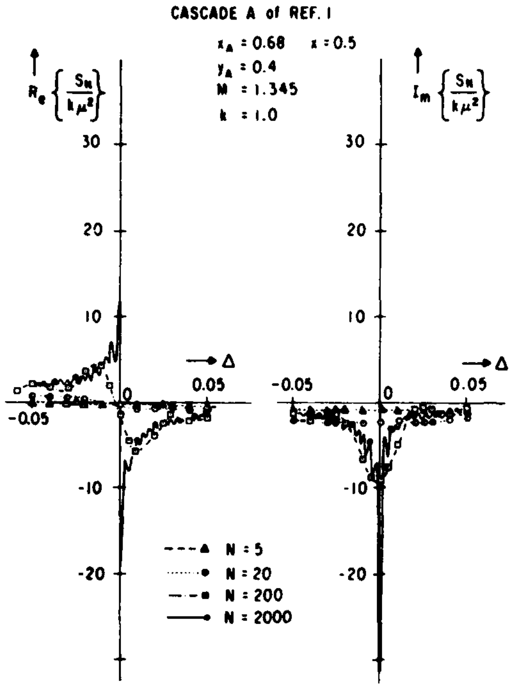
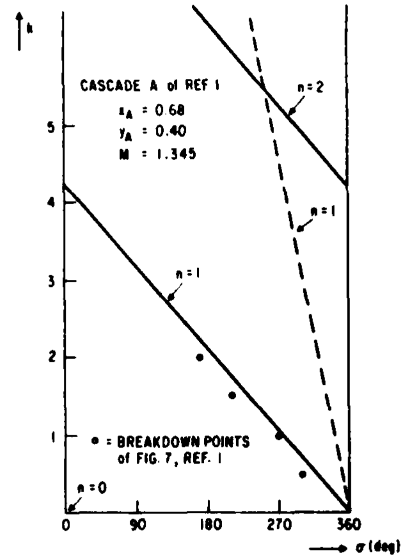


Fig. 1 Divergence phenomena.



$$k = \frac{\sigma - 2\pi n}{\sqrt{x_A^2 - \mu^2 y_A^2} - Mx_A}$$

$$k = \frac{\sigma - 2\pi n}{-\sqrt{x_A^2 - \mu^2 y_A^2} - Mx_A}$$

Fig. 2 σ – k relationship.

tribution whose *pro forma* sum over a large number of airfoils is responsible for "resonance." These considerations therefore cast a serious doubt on the prospect of resonance at the present divergent condition in an actual supersonic cascade, unless evidence to the contrary is found. (When the Mach number is increased in the present subsonic leading edge problem, the solution, including the behavior at 'resonance,' should approach the lowest limit of the supersonic leading edge problem of Ref. 10. Since the latter does not possess any such 'resonance,' this appears to provide additional evidence to the contrary.)

Even when the sum is convergent, the aforementioned failure of the supersonic linearized theory in the far field raises the obvious question on the very scheme of the summation over an infinite array of cascaded airfoils within the framework of linearized analysis. We hasten to add, however, that this does *not* mean that one cannot solve the supersonic cascade problem by linearized analysis. It is indeed feasible to accomplish this by using a different formulation adopted in Ref. 11. The formulation may be called a "passage approach" and attention is focused on the reference passage between two blades. In place of the infinite cascade arrangement, we impose an equivalent periodicity requirement that the flow at any given point in the reference passage be the same as the flow at the corresponding point in the adjacent passage (with the exception of an interblade phase lag). Since the problem is set up within the blade passage and, furthermore, because of the limited domain of dependence of a point in a supersonic flow, it turns out that the formulation involves only the near field where the linearized theory is valid. This approach leads to a closed-form expression for the pressure distribution, correct to the linear order of the frequency. The analysis yields the so-called unique incidence effect as the steady limit, agrees with the sonic limit of the supersonic leading edge analysis, and correctly reflects the effect of the back pressure. The expression does not suffer breakdown at Eq. (10) and this is one more reason to preclude resonance under these conditions‡ in a supersonic cascade.

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‡The expressions of Ref. 11 possess a singularity at σ = 0, which is related to the unique incidence effect.

APPENDIX 3

"Cumulative Nonlinear Distortion of an Acoustic Wave Propagating through an Inhomogeneous Flow," Journal of Fluid Mechanics, Vol. 83, part 4, 1977, pp. 751- 773.

J. Fluid Mech. (1977), *vol.* 83, *part* 4, *pp.* 751–773
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Cumulative nonlinear distortion of an acoustic wave propagating through non-uniform flow

By M. KUROSAKA

Research and Development Center, General Electric Company,
Schenectady, New York 12301†

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In this paper we examine how the unsteady flow field radiated from an oscillating body is altered from the result of acoustic theory as the direct consequence of disturbances propagating through the non-uniform flow produced by the presence of the body. Taking the specific example of an oscillating airfoil placed in supersonic flow and having the contour of a parabolic arc, we derive a closed-form representation for the unsteady flow field in terms of the confluent hypergeometric function. The analytical expression reveals explicitly that, though the body shape has a negligible effect in the near field, it inextricably affects the unsteady flow at a large distance, both in its amplitude and phase, and substantially modifies the results of acoustic theory. In addition, we display the relation of this solution to the 'fundamental solution' and the other salient physical features connected with disturbances propagating through non-uniform flow. The present results recover Whitham's rule in the limit of zero frequency of oscillation and also include, as another special case, the unsteady solution for a wedge obtained by Carrier and Van Dyke.

1. Introduction

As is well known, acoustic theory in a moving medium is based on two major assumptions: that a disturbance propagates at a *uniform* acoustic velocity and is swept downstream at a *constant* free-stream speed. Although this approximation is sufficiently accurate in the vicinity of the body, the acoustic theory for a supersonic flow is manifestly unfit for the description of the far field; it fails, for example, to reproduce the fanning out or coalescence of Mach waves. The reasons for the breakdown have long been understood (e.g. Lighthill 1954): as a wavelet spreads out, two nonlinear effects ignored in the acoustic theory, i.e. the non-uniform acoustic and flow velocities, which vary with both position and time, emerge and exert an influence over a large distance. The nonlinear effects are locally small everywhere, including the far field. However, not only is the disturbance at a given point influenced by the slightly perturbed flow properties at that location but it has been undergoing a continual distortion while propagating through a non-uniform flow field. It is this cumulative distortion or 'memory' content of the signal which encroaches upon the result of acoustic theory and eventually alters it in the far field.

For a steady flow, the task of surmounting the shortcomings of acoustic theory has drawn the attention of Friedrichs (1948), Lighthill (1949) and Whitham (1950, 1952), to mention only a few. These efforts culminated in the following celebrated rule due

† Present address: University of Tennessee Space Institute, Tullahoma, Tennessee 37388.

to Whitham† (1952): to a good approximation, the result of acoustic theory can be amended if one replaces the linearized Mach wave by one revised using linearized velocities but, along this improved Mach wave, retains the values of the fluid properties predicted by acoustic theory. Crudely speaking, then, the only visible consequence of the nonlinearity is the directional change in the Mach waves; the fluid velocities remain essentially unchanged. We reiterate here that the flow is steady in the frame of reference fixed to the body.

In contrast to the above steady flow situation, relatively less attention appears to have been paid to problems where the flow is unsteady, again with respect to the co-ordinate system fixed to the body. To be sure, related studies have been published but they seem mostly to be restricted to a one-dimensional problem and its diverse variants (e.g. Lesser 1970; Romanova 1970; Nayfeh 1975). There have been very few attempts, if any, to obtain, in the spirit of the above steady problems, a complete and uniformly valid solution and then display the global behaviour of the unsteady flow field in either two- or three-dimensional space. Yet there are many important practical problems, like the unsteady aerodynamic interference between a multitude of oscillating bodies in a flow, e.g. flutter of cascaded airfoils, and other similar phenomena, where such an improved prediction of the unsteady flow valid even in the far field is critically needed. Prompted by this, we address here the problem of obtaining a first-order correction to the acoustic field radiated from an oscillating body, accounting for the interaction with the non-uniform flow created by the body itself.

In the case of unsteady flow, the nonlinearity will have additional consequences, as one can anticipate from the following physical reasoning. Let us first assume that only a single point on the body is oscillating sinusoidally. When one plots at a given point in the flow the time trace of the disturbance emitted, the departure of the non-uniform acoustic and convective velocities from the uniform ones (from acoustic theory) will be graphically revealed, mainly, as a phase difference between the actual signal and the one predicted by the acoustic theory. The phase lag depends on the position and, the more one moves away from the source, the more the phase lag will increase. Suppose now that the whole body is oscillating. Then the above phase lag for an individual disturbance, which differs from one signal to another, and that alone (to say nothing of the modification in the amplitude of each signal) could introduce, *when signals are vectorially added*, a pronounced correction to both the *amplitude* and the *phase* of the unsteady flow in the far field. Thus the nonlinearity would cause, in addition to the alteration to be made to the direction of the characteristics, a change in the fluctuating pressure itself. The modification induced in the far-field signal has the following implication, which appears to warrant emphasis: contrary to the situation in the near field, the unsteady signal at a large distance, even to first order in the small perturbation, can by no means be separated from such effects as the body shape, camber and angle of attack, which cause the properties of propagation to be non-uniform. The effect of thickness, for example, would be inextricably embedded in the far-field unsteady signal.

Our present aim is to confirm these expectations and we shall do so by investigating

† In early literature this was referred to as Whitham's hypothesis. Now that it has become well established, it appears more appropriate to call it a rule instead. This rule should not, of course, be confused with another rule, due also to Whitham, relevant to the propagation of a shock through a region of varying cross-sectional area (e.g. Whitham 1974).

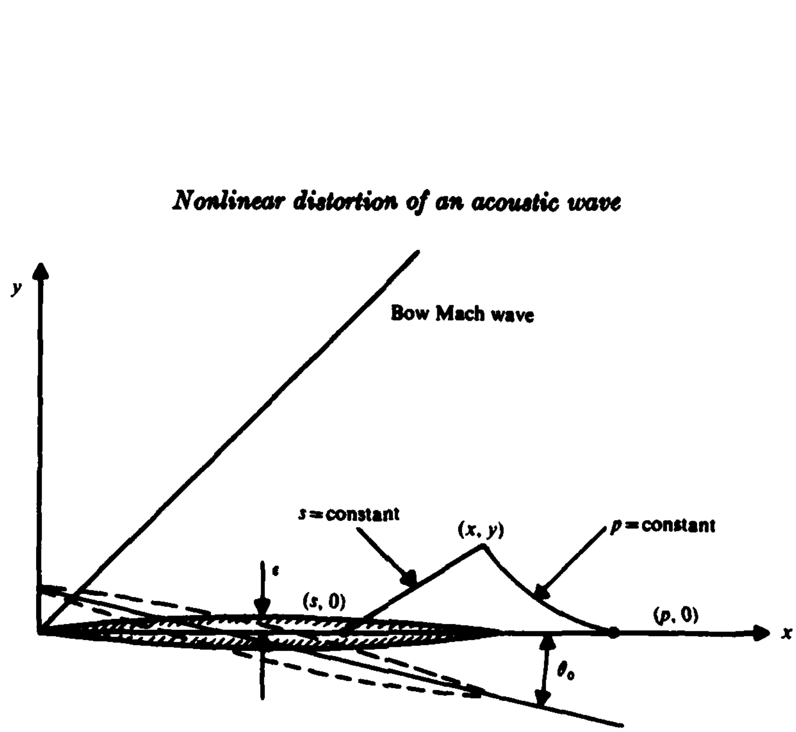


FIGURE 1. Definition sketch.

the effects of a non-planar body, whose presence creates a non-uniform surrounding environment, upon the unsteady flow field. We shall expressly limit our investigations to the case of a two-dimensional slender body whose upper and lower surfaces consist of parabolic convex arcs and which is oscillating sinusoidally in a supersonic flow (figure 1). Though, by confining our attention to this particular shape, we shall inevitably forfeit formal generality, the present approach will give a closed-form solution which is amenable to detailed study; from this we hope to glean the essential features of the flow non-uniformities. With regard to figure 1 again, the thickness of the body is characterized by a parameter ϵ and the amplitude of oscillation by θ_0. We shall examine the cumulative effects of the second-order terms, which ascend in the far field to a first-order unsteady term $O(\theta_0)$. There are three second-order terms, $O(\epsilon\theta_0)$, $O(\theta_0^2)$ and $O(\epsilon^2)$, of which only the first two are relevant for the present unsteady problem. If one assumes $\epsilon \gg \theta_0$, one can discard the term $O(\theta_0^2)$, whose presence would cause undesirable higher harmonics. With this assumption, we are now in a position to focus attention on the remaining, $O(\theta_0\epsilon)$ term, which represents the genuine coupling effect of present interest. It should be remembered, however, that, as pointed out by Hayes (1954) for steady flow, only a few selected second-order terms contribute cumulatively to the first-order effects. Hence we shall pick out, by the use of the strained co-ordinate technique, those terms $O(\theta_0\epsilon)$ whose cumulative effects amount to $O(\theta_0)$ in the far field. Thus our aim is clearly different from Van Dyke's (1953*a*) second-order theory for an oscillating airfoil including the effect of thickness. There, because of his interest in the flow on the airfoil surface, combined with a situation involving only slow oscillations, he used a regular perturbation scheme in θ_0 and ϵ and obtained a solution to $O(\theta_0\epsilon)$; consequently, the hierarchical ascent of terms of second order to first order in the far field was neither expected to take place nor was his concern. On the contrary, our interest centres on just such an evolutionary, ascending process.

In the next section we shall begin with the governing equation and simplify it in §3 by employing the strained co-ordinate technique. In §§4 and 5, we shall describe the procedure for solving this simplified equation. We set out to obtain the corresponding Riemann function appropriate for a parabolic airfoil; the Riemann function can be constructed explicitly and exactly in terms of the confluent hypergeometric function. With the Riemann function thus obtained, the solution, equation (5.4), follows from it without much difficulty. In §6, before embarking on a physical interpretation of the solution, we pause and confirm that the present results can be reduced, through the limiting properties of the confluent hypergeometric function, to some known results. In the limit of zero frequency of oscillation, we shall recover Whitham's rule; for an oscillating wedge with small apex angle, the present result will embrace, as a special case, the solution obtained by Carrier (1949) and Van Dyke (1953*b*). We shall resume the discussion of the curved airfoil in §7, where we observe that Tricomi's (1949) expansion formula for the confluent hypergeometric function is ideally suited to the extraction of a physical interpretation; the gradual ascent of second-order terms to alter the acoustic signal in the far field will become effortlessly visible; and there the effect of body shape will be found to be tenaciously inseparable from the unsteady flow field. This will be followed in §8 by further description of salient physical features related to the disturbances propagating through the non-uniform medium.

2. Problem formulation

The governing equation for the perturbed velocity potential Φ is given to second order, to which order the flow can still be regarded to be irrotational, by

$$\Phi_{yy}-m^2\Phi_{xx}-2\frac{U_\infty}{a_\infty^2}\Phi_{xt}-\frac{1}{a_\infty^2}\Phi_{tt}$$
$$=\frac{M_\infty^2}{U_\infty}\left\{(\gamma-1)\left(\Phi_x+\frac{1}{U_\infty}\Phi_t\right)(\Phi_{xx}+\Phi_{yy})+2\Phi_x\Phi_{xx}+2\Phi_y\Phi_{xy}\right.$$
$$\left.+\frac{2}{U_\infty}(\Phi_x\Phi_{xt}+\Phi_y\Phi_{yt})\right\}, \qquad (2.1)$$

where the perturbed velocity components (u', v') are related to Φ by

$$\Phi_x = u', \quad \Phi_y = v',$$

U_∞ is the free-stream velocity, a_∞ the speed of sound in the free stream, $M_\infty = U_\infty/a_\infty$, $m = (M_\infty^2-1)^{\frac{1}{2}}$ and γ is the adiabatic exponent of the gas. We express, according to Van Dyke (1953*a*), the co-ordinate of the moving upper surface as

$$y = \epsilon f(x) - \theta_0 e^{i\omega t} g(x), \qquad (2.2)$$

where $\epsilon f(x)$ ($\epsilon \ll 1$) designates the shape of the body in its mean position of oscillation and the second term represents its harmonic motion with frequency ω. The two small non-dimensional parameters ϵ and θ_0 characterize the slenderness of the body and the amplitude of motion, respectively. As long as the shock remains attached, we need consider only the flow above the upper surface. The boundary condition on the surface of the airfoil, as given by Van Dyke (1953*a*) to second order, is

$$\Phi_y = (U_\infty+\Phi_x)(\epsilon f' - \theta_0 e^{i\omega t} g') - i\omega\theta_0 e^{i\omega t} g - (\epsilon f - \theta_0 e^{i\omega t} g)\Phi_{yy} \quad \text{at} \quad y = 0. \qquad (2.3)$$

Also, Φ vanishes upstream of the bow shock, whose position moves in time. Since the flow variables are discontinuous at the shock and, strictly speaking, do not possess

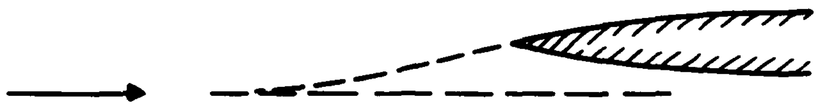


FIGURE 2. Smoothing technique of Van Dyke.

derivatives there, the governing equation is not formally satisfied. Hence in principle jump conditions across the shock, which is moving and whose temporal position is unknown *a priori*, must be imposed to ensure the conservation of mass, momentum and energy there; this would introduce complications. However, this knotty problem can be completely circumvented by the smoothing technique, which was first devised by Courant & Friedrichs (1948, p. 365) for steady flow and later extended to the unsteady case by Van Dyke (1953*a*). We first imagine that an extension has been added to the leading edge of the actual airfoil: a sufficiently smooth and flexible tip of such a shape and moving in such a way as to prevent the formation of the shock in the flow above the upper surface (figure 2). We then regard the desired solution as the limit as the extension shrinks. Once this device has been employed, as here, the need to impose jump conditions at the shock can be eliminated for the solutions up to second order. (Also, whenever necessary, we shall hereafter regard the discontinuity in the flow variables at the shock in the sense of the above limiting process.) The smoothing technique provides, in effect, a formal justification for the following point of view: the global behaviour of the unsteady flow downstream of the bow shock can, to a good approximation, be determined essentially independently of the presence of the shock and various complications arising from its motion (except in the close vicinity of the shock, where such a solution fails); the situation is akin to the familiar steady problem (Whitham 1952).

Following Van Dyke a little further, we separate the perturbed velocity potential into a steady and time-dependent part by writing

$$\Phi = \epsilon\phi(x,y:\epsilon) + \theta_0 \exp\,[i(\omega t - kx)]\,\psi(x,y:\epsilon,\theta_0), \tag{2.4}$$

where $k = M_\infty^2 \omega / m^2 U_\infty$. The first term represents the steady base flow and ψ in the second term corresponds to the unsteady flow; our interest is focused on ψ. We substitute (2.4) into (2.1) and (2.3) and assume $\epsilon \gg \theta_0$, as stated in the introduction. We thus obtain the following two sets of equations: for ϕ

$$\epsilon[-m^2\phi_{xx} + \phi_{yy}] = \epsilon^2 (M_\infty^2/U_\infty)[m^2(N-1)\phi_x^2 + \phi_y^2]_x, \tag{2.5a}$$

with the boundary condition

$$\epsilon\phi_y = \epsilon U_\infty f' + \epsilon^2(\phi_x f' - f\phi_{yy}) \quad \text{at} \quad y = 0, \tag{2.5b}$$

and for ψ

$$\begin{aligned} \theta_0[-m^2\psi_{xx} + \psi_{yy} - (km/M)^2\psi] = {} & 2(\epsilon\theta_0/U_\infty)\{M_\infty^2[m^2(N-1)\phi_x\psi_x + \phi_y\psi_y]_x \\ & - ik[(2N-1)m^2\phi_x\psi_x + Nm^2\psi\phi_{xx} + \phi_x\psi_y] - N(km/M_\infty)^2\phi_x\psi\}, \end{aligned} \tag{2.6a}$$

where $N = \frac{1}{2}(\gamma+1)M_\infty^2/m^2$, with the boundary condition

$$\begin{aligned} \theta_0\psi_y = {} & -\theta_0[U_\infty g' + (ikU_\infty m^2/M_\infty^2)g]e^{ikx} \\ & + \epsilon\theta_0[f'(\psi_x - ik\psi) + (-\phi_x g' + g\phi_{yy})e^{ikx} - f\psi_{yy}] \quad \text{at} \quad y = 0. \end{aligned} \tag{2.6b}$$

In obtaining the above equation, some simplification on the right-hand side has been achieved by using the expressions for the left-hand side and neglecting terms of higher than the second order. It should be noticed that, although (2.5*a*) is nonlinear in ϕ, equation (2.6*a*), the basis of this paper, is a *linear*† function of ψ involving variable coefficients. Both ϕ and ψ vanish upstream of the bow shock.

3. Application of strained co-ordinate technique

The right-hand sides of (2.5*a*) and (2.6*a*) are of higher order than the left-hand sides. Consequently, if one uses a regular perturbation scheme, they successively yield the first- and second-order equations of Van Dyke with the right-hand sides either zero or expressible in terms of the first-order velocities, respectively; the first-order equation for ψ, in particular, is the (reduced) acoustic equation and it obviously precludes the ascent of the terms on the right-hand side to first order. In order to achieve our stated objective of examining such an evolutionary process, we shall employ the strained co-ordinate technique instead: this is the point of departure of the present analysis. Although the original strained co-ordinate technique developed by Lighthill and Whitham involves only a single family of characteristics, the present unsteady problem requires two families of characteristics for adequate description of the flow field. It is therefore convenient to use Lin's (1954) extension of the strained co-ordinate technique (see also Oswatitsch 1962) or the analytic method of characteristics, which enables one to treat the case of two families of characteristics. According to this method, the independent variables (x, y) as well as the dependent variables are to be expanded, with the characteristic parameters s and p regarded as new independent variables:

$$x = x^{(0)}(s,p)+\epsilon x^{(1)}(s,p)+\dots, \tag{3.1a}$$

$$y = y^{(0)}(s,p)+\epsilon y^{(1)}(s,p)+\dots, \tag{3.1b}$$

$$\epsilon\phi = \epsilon\phi^{(1)}(s,p)+\epsilon^2\phi^{(2)}(s,p)+\dots, \tag{3.2a}$$

$$\theta_0\psi = \theta_0\psi^{(1)}(s,p)+\epsilon\theta_0\psi^{(2)}(s,p)+\theta_0^2\psi^{(3)}(s,p)+\dots, \tag{3.2b}$$

where s and p are constant along the corresponding characteristic curves, respectively. With respect to the characteristic curve, we first observe that, comparing (2.5*a*) and (2.6*a*), all the coefficients of the second derivatives in the steady equation are the same as the corresponding ones in the unsteady part. This dictates, then, that the characteristic curves for both the steady and the unsteady equations are identically the same and given by

$$\left.\frac{dy}{dx}\right|_{s=\text{const}} = \frac{1}{m}\left[1-\epsilon\frac{M_\infty^2}{U_\infty}(N-1)\phi_x+\epsilon\frac{M_\infty^2}{U_\infty}\frac{1}{m}\phi_y\right]+\dots, \tag{3.3a}$$

$$\left.\frac{dy}{dx}\right|_{p=\text{const}} = -\frac{1}{m}\left[1-\epsilon\frac{M_\infty^2}{U_\infty}(N-1)\phi_x-\epsilon\frac{M_\infty^2}{U_\infty}\frac{1}{m}\phi_y\right]+\dots. \tag{3.3b}$$

† Besides the usual shock emanating from the leading edge (and the one at the trailing edge, which does not matter for the flow field upstream of it), no additional shock is created owing to the motion of the airfoil; consequently, the entire unsteady flow can be uniformly described by the linearised equation.

Into these we substitute (3.1*a*, *b*) and (3.2*a*) and equate the coefficients of equal powers of ϵ. We then determine successively, using the boundary condition (2.5*b*), the terms in the series expansion; while the zeroth-order terms in (3.1), $x^{(0)}$ and $y^{(0)}$, give the expression for the characteristic parameters corresponding to acoustic theory, the first-order terms $x^{(1)}$ and $y^{(1)}$, give the desired nonlinear correction. We direct attention towards the fact that the procedure is dependent wholly on the steady flow and excludes the unsteady part (3.2*b*). This process of co-ordinate stretching in steady flow being familiar, it suffices here to write down the following results:

$$\phi^{(1)} = -H(s)(U_\infty/m)f(s), \tag{3.4}$$

$$s = x - my - \epsilon myN(M_\infty^2/U_\infty)d\phi^{(1)}(s)/ds, \tag{3.5a}$$

$$p = x + my - (\epsilon/2U_\infty)(N-2)M_\infty^2[\phi^{(1)}(s) - \phi^{(1)}(p)], \tag{3.5b}$$

where $H(s)$ is a unit step function. The above expressions for s and p have been put in the present form by rewriting the results corresponding to (3.1*a*, *b*). Geometrically, s represents, as shown in figure 1, the root of the straight Mach wave passing through a given point (x, y) and along this s remains constant (Van Dyke 1975); likewise, p represents the root of the cross Mach wave, along which p remains constant. (As a matter of fact, the constants of integration in (3.3), their choice being at our disposal, are so adjusted that, at $y = 0, x = s = p$.) Equation (3.4) indicates that the steady, first-order velocity potential is dependent on s only and it obviously embodies Whitham's rule.

Having thus specified s and p, we then substitute the expansion for the unsteady part (3.2*b*) into (2.6). In obtaining the equation for the leading term $\psi^{(1)}$, we proceed with caution and retain the terms associated with k on the right-hand side because, for sufficiently high frequencies, they could become comparable with the terms on the left-hand side; the terms not associated with k can be neglected. One thus obtains

$$\theta_0\left\{-4\psi_{sp}^{(1)} + \frac{4i}{U_\infty}N\epsilon k\phi'^{(1)}\psi_s^{(1)} + \frac{4i}{U_\infty}(N-1)\epsilon k\phi'^{(1)}\psi_p^{(1)} + \left[-\left(\frac{k}{M_\infty}\right)^2 + \frac{2i}{U_\infty}N\epsilon k\phi''^{(1)}\right]\psi^{(1)}\right\} = 0, \tag{3.6a}$$

where $\phi'^{(1)}$ designates the derivative of $\phi^{(1)}$ with respect to s; in differentiating $\phi^{(1)}$, we recall and envisage the smoothing process described in §2 and discard the term associated with the delta function. (When obtaining (3.6*a*), the term $(k/M_\infty)^2$ in the braces initially appears as $(k/M_\infty)^2[1-2(N/U_\infty)\epsilon\phi'^{(1)}]$ but the second term in the square brackets is neglected.) The boundary condition (2.6*b*) becomes

$$\theta_0\{i\epsilon kf'(s)\psi^{(1)} - m\psi_s^{(1)} + m\psi_p^{(1)} + V(s)\,e^{iks}\} = 0 \quad \text{at} \quad s = p,$$

where

$$V(x) = U_\infty g'(x) + (ikU_\infty m^2/M_\infty^2)g(x). \tag{3.6b}$$

Also, the upstream condition becomes

$$\psi^{(1)} = 0 \quad \text{for} \quad s < 0. \tag{3.6c}$$

It is convenient at this point to introduce the function F defined by

$$\psi^{(1)} = \exp[i(\epsilon k/U_\infty)N(p-s)\phi'^{(1)}(s)]F. \tag{3.7}$$

Then (3.6a) becomes, to the order consistent with the present approximation,

$$\theta_0\left\{F_{sp}+\frac{i\epsilon k}{U_\infty}[-(2N-1)\phi'^{(1)}(s)+(p-s)N\phi''^{(1)}(s)]F_p + \left[\left(\frac{k}{2M_\infty}\right)^2+\frac{i\epsilon k}{2U_\infty}N\phi''^{(1)}(s)\right]F\right\}=0, \qquad (3.8a)$$

with the boundary conditions

$$\theta_0\left\{i\epsilon k\left[-\frac{1}{m}f'(s)-\frac{2}{U_\infty}N\phi'^{(1)}(s)\right]F+F_s-F_p-\frac{1}{m}V(s)e^{iks}\right\}=0 \quad \text{at} \quad s=p, \qquad (3.8b)$$

and

$$F=0 \quad \text{for} \quad s<0. \qquad (3.8c)$$

In (3.8a,b), the factor θ_0 is retained as a reminder that the equations are valid to order θ_0, the higher-order terms such as those $O(\epsilon\theta_0)$ in (3.2b) being neglected. Our aim is to obtain the explicit solution for F and we shall do so for an airfoil whose shape consists of parabolic arcs.

4. Construction of the Riemann function

If $\phi'^{(1)}(s)$ were either zero or a constant, (3.8a) would be reduced to the telegraph equation. In the present case of a parabolic-arc airfoil, $f(x)$ in (2.2) is quadratic in x and from (3.4) the derivative $\phi'^{(1)}$ is linear in s. Thus (3.8a) is a second-order linear hyperbolic equation whose coefficients are variable (and linear in s). It is well known that the solution of any second-order linear hyperbolic equation can be expressed in the form of an integral representation, once the corresponding Riemann function has been obtained (e.g. Courant & Hilbert 1962, p. 449). If, in general, u satisfies

$$\mathscr{L}[u]\equiv u_{xy}+au_x+bu_y+cu=0,$$

where a, b and c are given functions of x and y, then u can be represented by an integral along the boundary (where Cauchy data are assumed to be prescribed) whose integrand involves the Riemann function R of the operator $\mathscr{L}$. R does not satisfy the operator equation $\mathscr{L}(R)=0$ but rather satisfies the adjoint operator equation

$$\mathscr{L}^*[R]\equiv R_{xy}-(aR)_x-(bR)_y+cR=0.$$

For our purpose, it is convenient to derive first, instead of R, the Riemann function R^* of the adjoint operator which satisfies the operator equation for $\mathscr{L}$ itself; then we obtain R through the symmetry property of the Riemann functions. For the present equation (3.8a), the Riemann function of the adjoint operator $R^*(\xi,\eta;s,p)$ satisfies the following three conditions (Courant & Hilbert, *ibid.*):

(a) $$\mathscr{L}_{\xi,\eta}[R^*]=R^*_{\xi,\eta}+\frac{i}{U_\infty}\epsilon k[-(2N-1)\phi'^{(1)}(\xi)+(\eta-\xi)N\phi''^{(1)}(\xi)]R^*_\eta + \left[\left(\frac{k}{2M_\infty}\right)^2+\frac{iN}{2U_\infty}\epsilon k\phi''^{(1)}(\xi)\right]R^*=0; \qquad (4.1a)$$

(b) along AC in figure 3,

$$\frac{1}{R^*}\frac{\partial R^*}{\partial\eta}=0 \quad \text{on} \quad \xi=s, \qquad (4.1b)$$

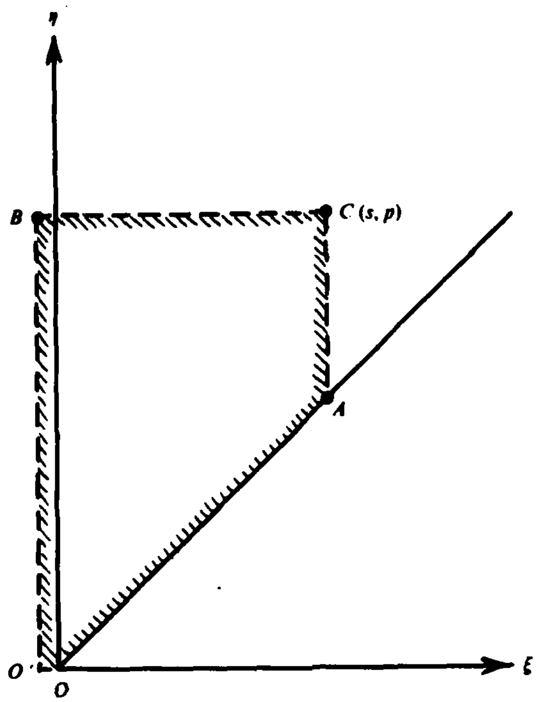


FIGURE 3. Integration contour.

and along BC,

$$\frac{1}{R^*}\frac{\partial R^*}{\partial \xi} = \frac{i}{U_\infty}\epsilon k[(2N-1)\phi'^{(1)}(\xi)-(\eta-\xi)N\phi''^{(1)}(\xi)] \quad \text{on} \quad \eta = p; \tag{4.1c}$$

(c) $R^*(s,p;s,p) = 1.$ (4.1d)

Integrating (4.1c) and determining the constant of integration from (4.1d), we obtain

$$R^*(\xi,p;s,p) = \exp\mu^*, \tag{4.2}$$

where

$$\mu^* = (i/U_\infty)\epsilon k\{(N-1)[\phi^{(1)}(\xi)-\phi^{(1)}(s)] - N[p(\phi'^{(1)}(\xi)-\phi'^{(1)}(s)) - (\xi\phi'^{(1)}(\xi)-s\phi'^{(1)}(s))]\}.$$

If we write

$$R^*(\xi,\eta;s,p) = \exp(\mu^*)M(z), \tag{4.3}$$

where

$$z = -(i/U_\infty)N\epsilon k(\xi-s)(\eta-p)\phi''^{(1)}, \tag{4.4}$$

then for a parabolic-arc airfoil, for which $\phi''^{(1)}$ is a constant, (4.1a) is reduced to the following ordinary differential equation:

$$zM''+(1-z)M'-aM = 0, \tag{4.5}$$

where

$$a = \tfrac{1}{2}+U_\infty k(4i\epsilon NM_\infty^2\phi''^{(1)})^{-1}.$$

This is known as Kummer's equation and its only solution which satisfies (4.1*d*) is the following confluent hypergeometric function (e.g. Slater 1960, p. 2):

$$M \equiv M(a, 1, z), \tag{4.6}$$

defined by

$$M(a,b,z) = 1+\frac{az}{b}+\frac{a_2 z^2}{b_2 2!}+\dots+\frac{a_n z^n}{b_n n!}+\dots,$$

where

$$a_n = a(a+1)(a+2)\dots(a+n-1), \quad \text{for } n = 1, 2, \dots,$$

and

$$a_0 = 1.$$

Hence (4.3) becomes

$$R^*(\xi,\eta;s,p) = \exp(\mu^*)M(a,1,z). \tag{4.7}$$

Along $\xi = s$, $R^*(s,\eta;s,p) = 1$ and this obviously satisfies the remaining requirement (4.1*b*) for the Riemann function.

The Riemann function $R(\xi,\eta;s,p)$ may be immediately derived from R^* through the symmetry property of the Riemann function (Courant & Hilbert 1962, p. 454) by replacing ξ and η with s and p, respectively. Thus we obtain

$$R(\xi,\eta;s,p) = \exp(\mu)M(a,1,z), \tag{4.8}$$

where

$$\mu = (i/U_\infty)\epsilon k\{(N-1)[\phi^{(1)}(s)-\phi^{(1)}(\xi)]-N[\eta(\phi'^{(1)}(s)-\phi'^{(1)}(\xi)) - (s\phi'^{(1)}(s)-\xi\phi'^{(1)}(\xi))]\},$$

$$a = \tfrac{1}{2}+U_\infty k(4i\epsilon N M_\infty^2 \phi''^{(1)})^{-1}$$

and

$$z = -(i/U_\infty)\epsilon k N(s-\xi)(p-\eta)\dot{\phi}''^{(1)}.$$

5. An integral representation of the solution

Once the Riemann function has been thus derived, one is in a position to employ Riemann's formula (Courant & Hilbert, *ibid.*) to obtain the integral representation of F in (3.8*a*), provided that Cauchy data are prescribed on the boundary. Unfortunately, the present boundary condition (3.8*b*), which applies along the segment OA of figure 3 (this corresponds to the x axis of figure 1), is not Cauchy data. Rather, it expresses a linear relationship between the function F and its derivatives; this induces some complication. If one applies Riemann's formula to the contour around the shaded region of figure 3 ($OACBO'$), although the contributions from the line segments AC, CB, BO' and OO' vanish identically, one ends up with an integral along OA; since it turns out that the integral involves the value of F, which is unknown as yet, one has to solve a complicated integral equation to determine it.

The difficulty is by no means unique to the parabolic airfoil, and in fact the same complication arises even in the more simplified situation of a flat-plate airfoil, where $\phi''^{(1)}$ is zero. In such a case, (3.8*a*) is reduced to the telegraph equation and the corresponding Riemann function is a Bessel function (e.g. Courant & Hilbert, *ibid.*):

$$R(\xi,\eta;s,p) = J_0\{(k/M_\infty)[(s-\xi)(p-\eta)]^{\frac{1}{2}}\}. \tag{5.1}$$

To construct the flat-plate solution, Temple & Jahn (1945) used this and applied Riemann's formula for a closed curve; the contour around the shaded region of figure 3 is none other than their path of integration. Their final result for F at a general point in the flow was left in a somewhat awkward form involving, inside the integral along the segment OA, the unknown values of F to be evaluated there, although the undesirable term vanishes from the integral for a point on the surface of airfoil, i.e. on OA. It turns out, however, that one can advance a step further and eliminate the term entirely. More specifically, by substituting the expression derived for F on the surface of the airfoil into the integral representation for an arbitrary point and noting an identity involving a product of Bessel functions, F can be written exactly as the following integral of the Riemann function:

$$F(s,p) = \frac{H(s)}{m}\int_0^s V(\tau)e^{ik\tau}R(\xi = \tau, \eta = \tau; s, p)\,d\tau \qquad (5.2a)$$

$$= \frac{H(s)}{m}\int_0^s V(\tau)e^{ik\tau}J_0\left(\frac{k}{M_\infty}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right)d\tau. \qquad (5.2b)$$

This expression is, of course, the well-known flat-plate solution obtainable by a number of other methods (e.g. Miles 1959, p. 50).

Motivated by (5.2a), in the present case of a parabolic airfoil we try

$$F(s,p) = \frac{H(s)}{m}\int_0^s V(\tau)e^{ik\tau}R(\xi = \tau, \eta = \tau; s, p)\,d\tau, \qquad (5.3)$$

where R is now given by (4.8) and this can be directly verified to satisfy the governing equation (3.8a). Also, substituting this into the boundary condition (3.8b) and recalling that it is valid to $O(\theta_0)$, it can be shown by using some of the results obtained by the present author (1974) that the boundary condition is indeed satisfied to the same order, the details being given in appendix A. From (5.3), F obviously vanishes for $s < 0$. Hence (3.8c) is satisfied and (5.3) is in fact the solution sought. Before we write down the final solution explicitly, we restore, in order to obtain $\psi^{(1)}$ in (3.7), the exponential factor, which may be written to the present order of approximation as

$$\exp\left[i\frac{\epsilon k}{U_\infty}N(p-s)\phi'^{(1)}(s)\right] \simeq \exp\left[i\frac{\epsilon k}{U_\infty}N2my\phi'^{(1)}(s)\right].$$

When we collect all the results obtained so far, we have the following: if the airfoil shape in the mean position is given by

$$\epsilon f(x) = \epsilon(\tfrac{1}{2}\alpha x^2 + \beta x),$$

where $\alpha < 0$ (a convex surface), and the co-ordinate of the moving upper surface is given by

$$y = \epsilon f(x) - \theta_0 e^{i\omega t}g(x),$$

where the amplitude of the motion $g(x)$ is an arbitrary function of x, then the leading term of the unsteady part of the velocity potential, $\psi^{(1)}$ in (3.2b), becomes

$$\psi^{(1)}(s,p) = \frac{H(s)}{m}\int_0^s V(\tau)\exp(ik\tau)\exp\left[i\frac{\epsilon k}{U_\infty}N2my\phi'^{(1)}(s)\right]$$
$$\times\exp\left\{i\frac{\epsilon k}{U_\infty}[(N-1)(\phi^{(1)}(s)-\phi^{(1)}(\tau)) + N(s-\tau)\phi'^{(1)}(s)]\right\}$$
$$\times M\left[\frac{1}{2} - \frac{mk}{4i\epsilon\alpha NM_\infty^2}, 1, \frac{i\epsilon k\alpha N}{m}(s-\tau)(p-\tau)\right]d\tau, \qquad (5.4)$$

where

$$V(x) = U_\infty g'(x) + \frac{ikU_\infty m^2}{M_\infty^2} g(x), \quad k = \frac{\omega}{U_\infty}\left(\frac{M_\infty}{m}\right)^2, \quad N = \frac{\gamma+1}{2}\left(\frac{M_\infty}{m}\right)^2,$$

$$\phi^{(1)}(s) = (-U_\infty/m)(\tfrac{1}{2}\alpha s^2 + \beta s),$$

$$s = \frac{x - my + M_\infty^2 N\epsilon\beta y}{1 - M_\infty^2 N\epsilon\alpha y}, \quad p = x + my - \frac{\epsilon}{2U_\infty}(N-2)M_\infty^2[\phi^{(1)}(s) - \phi^{(1)}(p)],$$

$$|(\epsilon/m)(\alpha s + \beta)| \ll 1.$$

This integral representation is the solution we have been seeking.† (The last inequality is a restriction due to the assumption of a small perturbation.) Before attempting to extract physical meanings, we pause in the next section to observe that the present solution embraces the various known results as special limiting cases.

6. Limiting cases

6.1. *Steady limit*

In the limit $\omega \to 0$ or $k \to 0$, from the limiting form $M(a, b, 0) = 1$ of the confluent hypergeometric function (e.g. Abramowitz & Stegun 1964, p. 108), (5.4) is immediately reduced to

$$\psi^{(1)} = \frac{H(s)}{m}\int_0^s V(\tau)\,d\tau$$

$$= \frac{H(s)}{m} U_\infty g(s), \tag{6.1}$$

from (3.6*b*). This is Whitham's rule for steady flow and becomes identical to (3.4) if we replace f by $-g$. We wish to emphasize that g is an arbitrary function and that we have recovered the above as the limit for zero frequency of oscillation.

6.2. *Oscillating flat-plate airfoil*

In the limit $\epsilon \to 0$, when we note that (Abramowitz & Stegun 1964, p. 506)

$$\lim_{a\to\infty} M(a, 1, -z/a) = J_0(2z^{\frac{1}{2}}),$$

(5.4) becomes at once

$$\psi^{(1)} = \frac{H(x-my)}{m}\int_0^{x-my} V(\tau)\,e^{ik\tau} J_0\left(\frac{k}{M_\infty}[(x-\tau)^2 - m^2y^2]^{\frac{1}{2}}\right)d\tau, \tag{6.2}$$

which is precisely the well-known flat-plate solution; the physical meaning of this integral representation was given by the present author (1974).

† In this connexion, it is of interest to note that Goldstein & Rice (1973) found a solution for sound propagating through a uniform shear flow in terms of the parabolic cylinder function, which is intimately connected with the confluent hypergeometric function.

6.3. *Oscillating wedge*

The third case which invites comparison with the present result is that of an oscillating wedge. In the limit $\alpha \to 0$, with the aid of the limiting formula cited in §6.2 we obtain

$$\psi^{(1)} = \frac{H(s)}{m}\int_0^s V(\tau)\, e^{ik\tau} \exp(-2iN\epsilon\beta ky) \exp\left[-\frac{i}{m}\epsilon\beta k(2N-1)(s-\tau)\right] \times J_0\left(\frac{k}{M_\infty}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right) d\tau, \tag{6.3}$$

where

$$s = x - my + M_\infty^2 N\epsilon\beta y,$$

$$p = x + my - (N-2) M_\infty^2 \epsilon\beta y$$

and $\epsilon\beta$ is the semi-vertex angle of the wedge.

In order to confirm the agreement of this formula with that obtained by previous workers, we first restore the factor e^{-ikx} to (6.3). It is convenient to rotate the co-ordinate system from (x, y) to (x_2, y_2), where x_2 is parallel to the upper surface of the wedge and y_2 normal to it. At the same time, we refer the flow properties to the mean steady flow behind the shock instead of those upstream of the shock and designate them by a subscript 2. Furthermore, we change the integration variable from τ to $\eta \equiv \tau(1 + m_2\epsilon\beta)$. All this transforms the right-hand side of (6.3), upon discarding negligible quantities, into the following expression:

$$\theta_0 e^{-ikx}\psi^{(1)} \sim \theta_0 \frac{1}{m_2} H(x_2 - m_2 y_2) \int_0^{x_2 - m_2 y_2} V(\eta) \exp(-ik_2 x_2) \exp(ik_2\eta) \times J_0\left(\frac{k_2}{M_2}[(x_2-\eta)^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right) d\eta. \tag{6.4}$$

This is identical to the flat-plate solution (6.2) if the latter is expressed in terms of the (x_2, y_2) co-ordinate system and the flow properties downstream of the shock. This result is not unexpected, since it is known that, if one takes the second-order equation for the unsteady component of the velocity potential to $O(\epsilon\theta_0)$ and expresses it in terms of these co-ordinate systems and flow variables, then for a wedge it exactly reduces to the acoustic equation. (The reason why the relationship (6.4) is approximate rather than exact is obviously due to the fact that, in the course of applying the strained co-ordinate technique, some non-essential second-order terms have been discarded.)

Carrier (1949) obtained a solution for a wedge *oscillating at its apex*; the solution was derived in a more generalized way by including the rippling motion of the shock and, in addition to the irrotational component of the flow, rotational flow behind the shock. His solution was later generalized to include the case of a moving vertex by Van Dyke (1953*b*), who also corrected typographical errors in Carrier's paper. The solution was expressed in the form of a series involving Bessel functions. In order to facilitate direct comparison, we recast the present solution (6.4) in the following alternative form:

$$\theta_0 \frac{1}{m_2}\int_0^{x_2 - m_2 y_2} V(\eta) \exp[-ik_2(x_2-\eta)]\, J_0\left[\frac{k_2}{M_2}[(x_2-\eta)^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right] d\eta = -\theta_0 a_2 \sum_{\nu=1}^{\infty} b_\nu e^{-\nu\phi} J_\nu\left[\frac{k_2}{M_2}[x_2^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right] \exp(-ik_2 x_2), \tag{6.5a}$$

where

$$\tanh\theta = m_2 y_2/x_2,$$
$$b_\nu = (iM_2\nu/k_2 m_2)[t^\nu + (-t)^{-\nu}] + b\cos(\epsilon\beta)[t^\nu - (-t)^{-\nu}], \tag{6.5b}$$
$$t = i(M_2 + m_2)$$

and where $V(x_2) = U_2 + i\omega(x_2 - b\cos\epsilon\beta)$, b being the pivotal position of the oscillating wedge measured from the apex. The above identity is given in appendix B. Now Carrier's solution for the irrotational component of the flow becomes, in the present notation,

$$\theta_0 a_2 \sum_{\nu=1}^{\infty} [a_\nu \cosh\nu\theta + b_\nu \sinh\nu\theta]\, J_\nu\left[\frac{k_2}{M_2}[x_2^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right]\exp(-ik_2 x_2). \tag{6.6}$$

(The expression for b_ν given in (6.5*b*) is the corrected one given by Van Dyke 1953*b*.) Carrier showed that as long as the shock is sufficiently weak

$$a_\nu \doteqdot -b_\nu,\dagger$$

and in such a case (6.6) is indeed identical to the right-hand side of (6.5*a*). This agreement naturally endorses the present viewpoint that the global behaviour of the unsteady flow downstream of the weak bow shock can be determined essentially independently of the presence and movement of the shock.

7. Alternative representation of the solution and interpretation

Returning now to the immediate subject of a parabolically curved airfoil, the solution as given in (5.4) is not appropriate for extracting its physical significance. Such an interpretation will, however, be obvious once we recast (5.4) in a more revealing form by making use of the following Tricomi (1949) expansion formula for the confluent hypergeometric function in a series of Bessel functions:

$$M(a,b,x) = \Gamma(b)(\lambda x)^{\frac{1}{2}(1-b)}\exp(\tfrac{1}{2}x)\sum_{n=0}^{\infty} A_n\left(\frac{x}{4\lambda}\right)^{\frac{1}{2}n} J_{n+b-1}[2(\lambda x)^{\frac{1}{2}}] \quad \text{for} \quad \operatorname{Re} b > 0, \tag{7.1}$$

where λ is the Whittaker parameter, given by $\lambda = \frac{1}{2} - a$, and.

$$A_0 = 1, \quad A_1 = 0, \quad A_2 = \tfrac{1}{2},$$
$$(n+2)A_{n+2} = (n+1)A_n - 2\lambda A_{n-1}.$$

When we insert this into (5.4), we obtain

$$\psi^{(1)}(s,p) = \frac{H(s)}{m}\int_0^s V(\tau)\, e^{ik\tau}\exp(-i\epsilon k\sigma)$$
$$\times\left\{J_0\left(\frac{k}{M_\infty}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right) + \sum_{n=2}^{\infty} A_n\left[-\left(\frac{N\epsilon\alpha M_\infty}{m}\right)^2 (s-\tau)(p-\tau)\right]^{\frac{1}{2}n}\right.$$
$$\left.\times J_n\left(\frac{k}{M_\infty}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right)\right\} d\tau, \tag{7.2a}$$

† Van Dyke (1953*b*, also private communication) proved that for a small wedge angle

$$a_1/b_1 = -1 + 2ib\theta_0\epsilon/b_1 + O(\epsilon^2).$$

As for the rotational component of the flow, the first term of its series representation may be shown to be $O(\theta_0\epsilon^{\frac{1}{2}})$.

where

$$A_2 = \tfrac{1}{2}, \quad A_3 = -\tfrac{2}{3}\lambda, \quad A_4 = \tfrac{3}{8}, \tag{7.2b}$$

$$(n+1)A_{n+1} = nA_{n-1} - 2\lambda A_{n-2}, \quad \lambda = mk(4iM_\infty^2 N\epsilon\alpha)^{-1},$$

$$\begin{aligned}\sigma = & -2Nmy\frac{1}{U_\infty}\phi'^{(1)}(s) - (N-1)\frac{1}{U_\infty}[\phi^{(1)}(s) - \phi^{(1)}(\tau)] - N(s-\tau)\frac{1}{U_\infty}\phi'^{(1)}(s) \\ & - \frac{\alpha N}{2m}(s-\tau)(p-\tau) \\ \simeq & -\frac{Nmy}{U_\infty}[\phi'^{(1)}(s) + \phi'^{(1)}(\tau)] - \frac{1}{U_\infty}(2N-1)[\phi^{(1)}(s) - \phi^{(1)}(\tau)].\end{aligned}$$

(All three of the limiting cases of the preceding section are now directly derivable from the present form: for example, when k is set equal to zero, the result (6.1) follows at once.) Equation (7.2a) immediately surrenders itself to the following physical interpretation. Let us first examine the flow field near the leading edge, where both y and s are small. Then (7.2a) becomes, approximately,

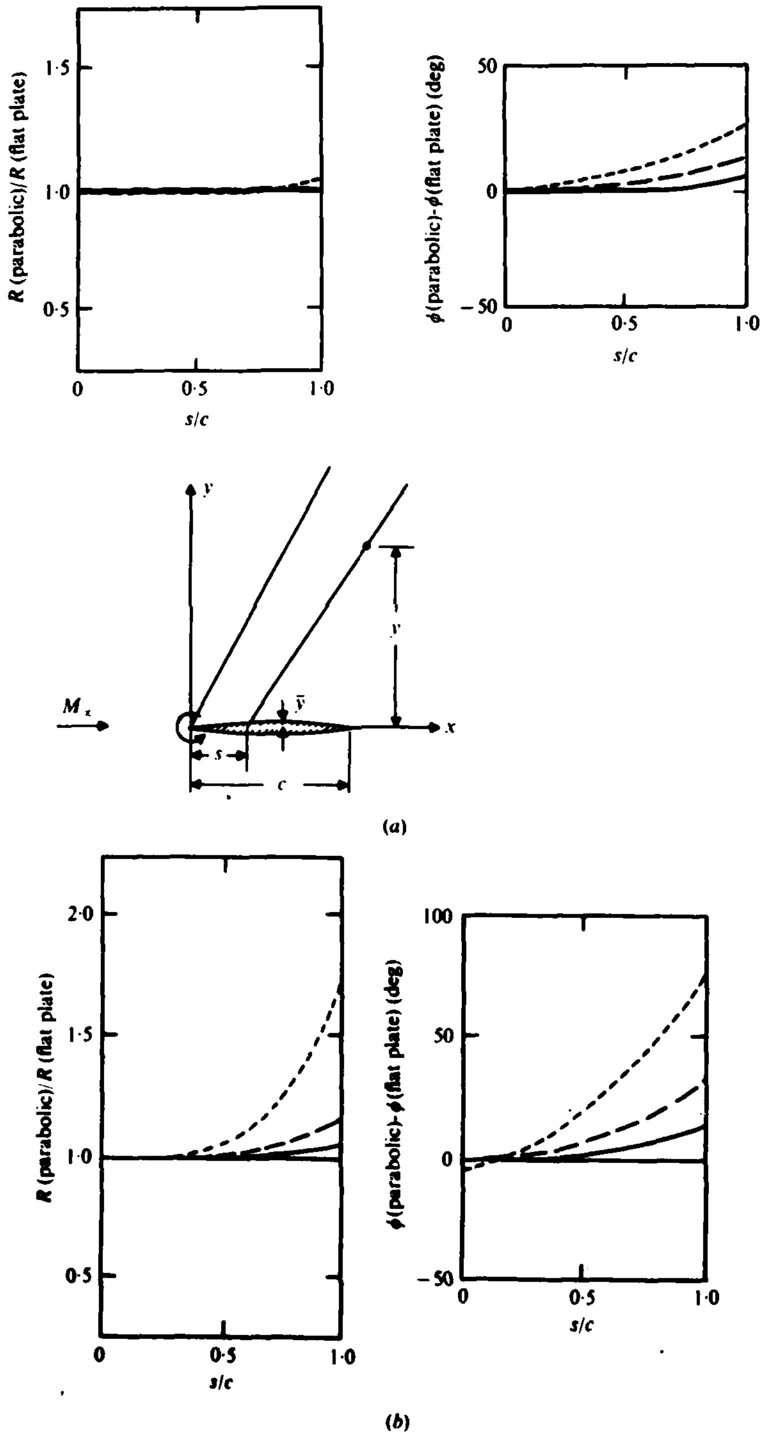
$$\psi^{(1)} = \frac{H(x-my)}{m}\int_0^{x-my} V(\tau)\, e^{ik\tau} J_0\left(\frac{k}{M_\infty}[(x-\tau)^2 - (my)^2]^{\frac{1}{2}}\right) d\tau.$$

This is the flat-plate solution (6.2), and in this region the effect of the body shape is indiscernible as yet; the unsteady flow field is completely separated from the non-uniform, steady flow. Physically the decoupling occurs because the unsteady disturbance, having travelled only a short distance from the leading edge, has suffered little distortion.

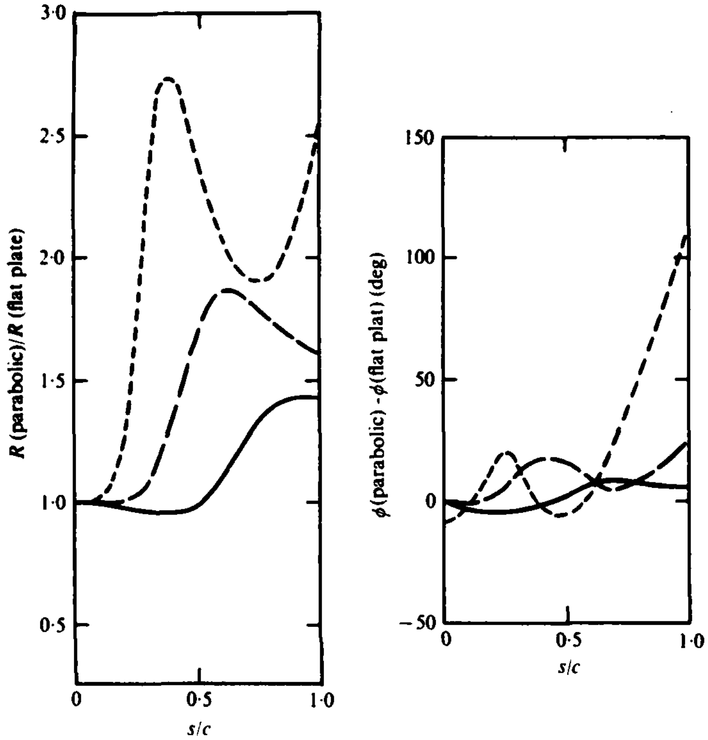
We now move away from the leading edge by increasing the value of y while keeping the value of s constant (along the straight Mach wave) or penetrate downstream by increasing the value of s while keeping y constant. In either case, if we look at the integrand of (7.2a) or the signal emitted at a point τ on the airfoil, the complex exponential term, which can be written as

$$\exp(-i\epsilon k\sigma) = \exp\left\{\frac{i\epsilon k}{U_\infty}[Nmy(\phi'^{(1)}(s) + \phi'^{(1)}(\tau)) + (2N-1)(\phi^{(1)}(s) - \phi^{(1)}(\tau))]\right\},$$

immediately discloses the following key aspect: no matter how slender ($\epsilon \ll 1$) the airfoil may be, this phase shift (induced by the presence of the body) will eventually amount to an increasing delay at a large distance y or s. Moreover, it is also crucial to recognize here that the phase lag of the signal received at a position s depends not only on the local flow at that point, but also, through the very difference in the steady velocity potential, i.e. $\phi^{(1)}(s) - \phi^{(1)}(\tau)$, upon the entire flow field which the signal has traversed; the disturbance 'remembers' its past. Thus we might call this exponential factor the phase memory, a term commonly used in connexion with the propagation of a radio wave through a stratified ionosphere (e.g. Budden 1961). As stated in the introduction, the existence of phase memory, which differs from one signal to another, is by itself quite sufficient to induce, upon superposition, a change in the amplitude of the unsteady flow field. The change is, however, further enhanced because the shape of the airfoil alters even the amplitude of the individual signal in the far field when the contributions from the higher-order terms of (7.2a) in the series of Bessel functions begin to surface. Thus, in the far field the airfoil shape, in its effect of causing non-



FIGURES 4 (a, b). For legend see next page.



(c)

Figure 4. Amplitude and phase of unsteady pressure; $-p'(\rho_\infty U_\infty^2 e^{i\omega t})^{-1} = Re^{i\phi}$. The ordinate for the left figure of each pair is the ratio of the amplitude R for a parabolic-arc airfoil to that for a flat plate. The ordinate on the right is the difference in phase ϕ; $M_\infty = 1{\cdot}3$, $\gamma = 1{\cdot}4$, $\epsilon = 0{\cdot}1$, $\alpha = -1$, $\beta = 0{\cdot}5$ (max $\bar{y}/c = 0{\cdot}0125$) and the pivot axis is at the leading edge. $\bar{y}/c = \epsilon[\frac{1}{2}\alpha(x/c)^2 + \beta(x/c)]$. (a) $\omega c/U_\infty = 0{\cdot}1$ ($kc = 0{\cdot}245$): ——, $y/c = 0$ ($ky = 0$); – –, $y/c = 0{\cdot}82$ ($ky = 0{\cdot}2$); ----, $y/c = 2{\cdot}04$ ($ky = 0{\cdot}5$). (b) $\omega c/U_\infty = 0{\cdot}3$ ($kc = 0{\cdot}735$): ——, $y/c = 0$ ($ky = 0$); – –, $y/c = 0{\cdot}82$ ($ky = 0{\cdot}6$); ----, $y/c = 2{\cdot}04$ ($ky = 1{\cdot}5$). (c) $\omega c/U_\infty = 1$ ($kc = 2{\cdot}45$): ——, $y/c = 0$ ($ky = 0$): – –, $y/c = 0{\cdot}82$ ($ky = 2$); ----, $y/c = 2{\cdot}04$ ($ky = 5$).

uniform surrounding flow, is inextricable from the unsteady flow field and deeply affects both its phase and amplitude, as well as the directional change in the characteristic curves.

This point is illustrated in figure 4, where the unsteady pressure distribution for a parabolic airfoil (max $\bar{y}/c = 0{\cdot}0125$), computed from (7.2),† is compared with the result for a flat-plate airfoil at three different frequencies of oscillation: $\omega c/U_\infty = 0{\cdot}1$ in figure 4(a), $\omega c/U_\infty = 0{\cdot}3$ in figure 4(b) and $\omega c/U_\infty = 1$ in figure 4(c). There, both the amplitude R and phase ϕ are plotted as functions of s, i.e. the distance between the root of a straight Mach wave and the leading edge, and at three different values of y. (If the flow were steady then, regardless of y, the amplitude would remain the same along the characteristics s = constant.) We observe that, though for $\omega c/U_\infty = 0{\cdot}1$ the effect of the airfoil shape does not become prominent at these values of y, it begins

† For numerical computations, (7.2) is also more convenient than (5.4).

to emerge at $\omega c/U_\infty = 0{\cdot}3$; and for $\omega c/U_\infty = 1$, except for the close vicinity of the leading edge, it indeed alters the pressure distribution substantially.

The profound modification of the unsteady linear theory displayed here raises an obviously disquieting thought on the upshot of the acoustic theory when multi-body aerodynamic interference is involved and deepens concern expressed (Kurosaka 1975) with regard to some of the consequences arising from a *pro forma* sum of linearized unsteady upwashes.

8. Further interpretation

Pursuing the physical interpretation further, we seek the connexion between (7.2a) and the 'fundamental' solution. We shall not, however, merely reconstruct (7.2a) by the superposition of the fundamental solution. Rather, we shall reverse the usual process and obtain the fundamental solution from (7.2a): that is to say, we regard (7.2) as the spectrum at frequency ω or the Fourier transform and take its inverse transform so as to derive the transient response to an arbitrary time-dependent motion of the airfoil. The 'fundamental' solution will arise naturally in the course of obtaining the transient response (Miles 1959, p. 53). Let us go back to (2.4) and rewrite the unsteady part in a more general way as

$$\Phi = \epsilon\phi + \theta_0 \Omega(x, y : t).$$

Then the Fourier transform $\tilde{\Omega}(\omega)$ of Ω (its leading part) is equal to $e^{-ikx}\psi^{(1)}$, $\psi^{(1)}$ being given by (7.2a), provided that V is regarded as the Fourier transform $\tilde{V}$ of itself, i.e.

$$\tilde{\Omega}(\omega) = \frac{1}{m}\int_0^s \tilde{V}(\tau)\exp\left\{-i\frac{\omega M_\infty}{a_\infty m^2}[(x-\tau)+\epsilon\sigma(\tau)]\right\}$$
$$\times\left\{J_0\left[\frac{\omega}{a_\infty m^2}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right]\right.$$
$$\left.+\sum_{n=2}^{\infty} A_n(\omega)c_n J_n\left[\frac{\omega}{a_\infty m^2}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right]\right\}d\tau,$$

where $$c_n = \left[-\left(\frac{N\epsilon\alpha M_\infty}{m}\right)^2 (s-\tau)(p-\tau)\right]^{\frac{1}{2}n}.$$

Taking the inverse transform

$$\Omega = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\tilde{\Omega}e^{i\omega t}d\omega,$$

we obtain, by convolution,

$$\Omega(x,y,t) = \frac{1}{m}\int_0^s d\tau\int_{-\infty}^{\infty} V(\tau, t-\xi)F(\xi)d\xi. \tag{8.1a}$$

Here

$$[\partial\Omega/\partial y]_{y=0} = V(x,t)$$

and $$F(\xi) = \frac{1}{\pi}\sum_{n=0}^{\infty} c_n F_n, \tag{8.1b}$$

where, for example,

$$F_0 = r^{-1}H(b-|a|),\quad F_1 = 0,\quad F_2 = -(2r)^{-1}H(b-|a|)\cos 2\theta,$$
$$F_3 = \frac{i}{12N\epsilon\alpha M_\infty a_\infty mb}H(b-|a|)\left\{-\frac{8}{r}\cos 2\theta - \frac{2b^2}{r^3}+\ldots\right\}$$

and where

$$a = \xi - \frac{M_\infty}{a_\infty m^2}(x-\tau+\epsilon\sigma), \quad b = \frac{1}{a_\infty m^2}[(s-\tau)(p-\tau)]^{\frac{1}{2}},$$
$$\cos\theta = a/b, \quad r = (b^2-a^2)^{\frac{1}{2}}.$$

We note that $\Omega(x,y,t)$ contains, through F_n in (8.1*b*), the term $1/r$, which can be written as

$$\frac{1}{r} = a_\infty m \Big/ \left\{\frac{1}{m^2}(s-\tau)(p-\tau) - \left[a_\infty m\xi - \frac{M_\infty}{m}(x-\tau+\epsilon\sigma)\right]^2\right\}^{\frac{1}{2}}. \tag{8.2}$$

The meaning will become immediately recognizable if we note that at $\epsilon = 0$ the denominator of (8.2) may be reduced, after some algebra, to

$$\{(a_\infty\xi)^2 - (x-\tau-U_\infty\xi)^2 - y^2\}^{\frac{1}{2}}.$$

This represents, when set equal to zero, a circular wave front of a disturbance which was emitted at a source point $(\tau, 0)$ and is propagating through uniform flow after a time ξ. Thus the denominator of (8.2), when put equal to zero, i.e.

$$\frac{1}{m^2}(s-\tau)(p-\tau) - \left[a_\infty m\xi - \frac{M_\infty}{m}(x-\tau+\epsilon\sigma)\right]^2 = 0, \tag{8.3}$$

now describes the *distorted* wave front propagating in a non-uniform flow field. In fact, we can directly show that the expression for ξ obtained from (8.3) does satisfy, within the approximation consistent with the present analysis, the appropriate eikonal equation at large distances; (8.2) is indeed the fundamental solution. In general, for a given point (x, y) in flow and for a given source point $(\tau, 0)$, there are two values of ξ satisfying (8.3): one corresponds to the time when the disturbance first arrives at (x, y) and the other to the time when it departs from (x, y). In the particular case when the point (x, y) is located such that either

$$s = \tau \quad \text{or} \quad p = \tau,$$

there is only one such moment for ξ, which implies that the wave front is tangential to either $s = \tau$ or $p = \tau$. $s = \tau$ corresponds to the straight Mach wave, whose root is located at $(\tau, 0)$; $p = \tau$ is the cross Mach wave passing through the same point. Hence, as expected, two families of Mach waves passing the source point form envelopes for the disturbance emitted from the source. In particular, the time required for the signal to arrive at a point on the straight Mach line $s = \tau$ is given by

$$\xi = \frac{M_\infty y}{a_\infty m}\left[1 + \epsilon(\alpha s + \beta)N\frac{(2-M_\infty^2)}{m}\right]. \tag{8.4}$$

It is of interest to note that this can be obtained in the following, more physical way. The wave-front velocity **c** is in general the vectorial sum of the local acoustic speed in the direction of the normal **n** to the front and the convective fluid velocity, i.e. $\mathbf{c} = a\mathbf{n} + \mathbf{u}$. However, along the enveloping Mach waves, which are tangential to the wave front, the acoustic speed does not contribute to the component of the wave-front velocity in the direction parallel to the Mach wave; only the fluid velocity contributes. In particular, along the straight Mach line the component of the fluid velocity or the wave-front velocity remains constant. If we divide the distance from the source $(\tau, 0)$ to the point (x, y) by the component of the flow velocity in the direction of the straight Mach wave, we can directly derive (8.4), as the time elapsed.

9. Concluding remarks

It has been our aim to find a uniformly valid solution for the unsteady flow field and examine it in detail. We have shown, through an explicit solution obtained for the specific case of a parabolic-arc airfoil oscillating in supersonic flow, that the prediction of the unsteady signal in the far field demands the detailed description of the contour of the moving boundary. The non-uniform surrounding flow produced by the very presence of the body, no matter how slender it may be, cumulatively and inextricably affects both the amplitude and the phase of the unsteady disturbance at a large distance from the leading edge.

As a further related effort, it would appear to be worth while to pursue a study for other airfoil shapes so as to enlarge our stock of particular solutions. With regard to the question of similar cumulative, first-order effects of nonlinearity in subsonic flow, we still remain uncertain. It is intriguing, however, to note that in a very recent paper of Goldstein & Atassi (1976), where an exact second-order solution is obtained for an airfoil subject to a convected gust, the incoming gust, in its nonlinear interaction with the steady non-uniform flow field, is found to suffer distortion in wavelength in a manner akin to the present supersonic result though the flow treated there is incompressible.

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Appendix A

In this appendix we shall show that the expression for F given by (5.3) does satisfy the boundary condition (3.8*b*) to order θ_0. We denote the left-hand side of (3.8*b*) by

$$l(F) \equiv \theta_0 \left\{ i\epsilon k \left[-\frac{1}{m} f'(s) - \frac{2}{U_\infty} N \phi'^{(1)}(s) \right] F + F_s - F_p - \frac{1}{m} V(s)\, e^{iks} \right\}, \qquad \text{(A 1)}$$

and we shall prove that along $s = p$ this vanishes, to order θ_0. Substituting (5.3) into the above, one obtains for $s > 0$

$$l(F) = \frac{\theta_0}{m} \int_0^s V(\tau)\, e^{ik\tau} \exp\left(i \frac{\epsilon k}{U_\infty} \nu \right)$$
$$\times \left\{ i\epsilon k \left[-\frac{1}{m} f'(s) - \frac{2N}{U_\infty} \phi'^{(1)} + \frac{1}{U_\infty} (\nu_s - \nu_p) \right] \mathcal{M} + (\mathcal{M}_s - \mathcal{M}_p) \right\} d\tau, \qquad \text{(A 2)}$$

where

$$\nu = (N-1)[\phi^{(1)}(s) - \phi^{(1)}(\tau)] - N(\tau - s)\phi'^{(1)}(s),$$

$$\mathcal{M} = M\left[\frac{1}{2} + \frac{U_\infty k}{4i\epsilon N M_\infty^2 \phi''^{(1)}},\ 1,\ -\frac{i}{U_\infty} \epsilon k N (s-\tau)^2 \phi''^{(1)} \right].$$

We observe that along $s = p$

$$\nu_s = (2N-1)\phi'^{(1)}(s) - N(\tau - s)\phi''^{(1)}, \quad \nu_p = 0, \quad M_s = M_p. \tag{A 3}$$

Furthermore, by Tricomi's expansion formula cited in §7, the confluent hypergeometric function M can be expressed as

$$M = \exp\left[-\frac{1}{2U_\infty} i\epsilon k N(s-\tau)^2 \phi''^{(1)}\right] \times \sum_{n=0}^{\infty} A_n \left[i\epsilon \frac{M_\infty}{U_\infty} N(s-\tau)\phi''^{(1)}\right]^n J_n\left[\frac{k}{M_\infty}(s-\tau)\right], \tag{A 4}$$

where

$$A_0 = 1, \quad A_1 = 0, \quad A_2 = \tfrac{1}{2}$$

and the other, higher-order A_n are the same as those given in (7.2*b*). The leading term of M is given by

$$M \sim \exp\left[-\frac{1}{2U_\infty} i\epsilon k N(s-\tau)^2 \phi''^{(1)}\right] J_0\left[\frac{k}{M_\infty}(s-\tau)\right]. \tag{A 5}$$

From (A 3) and (A 5), (A 2) becomes

$$l(F) = \theta_0 \epsilon k \left\{\frac{1}{m}\int_0^s V^*(\tau) \exp(iK\tau) \exp\left(i\frac{\epsilon k}{U_\infty}\nu\right) \exp\left[-\frac{i\epsilon k}{2U_\infty} N(s-\tau)^2 \phi''^{(1)}\right] \times J_0\left[\frac{k}{M_\infty}(s-\tau)\right] d\tau\right\}, \tag{A 6}$$

where

$$V^*(\tau) = -V(\tau)\, i\, U_\infty^{-1} N(\tau - s)\phi''^{(1)}.$$

Equation (A 6) can contribute to $O(\theta_0)$ only when ϵk is such that, if properly non-dimensionalized, $O(\epsilon k) = 1$ or $k = O(1/\epsilon)$. For such large values of k, we apply the following method of obtaining an asymptotic expansion (Kurosaka 1974): we first express J_0 in terms of an integral involving an exponential and use the stationary-phase method repeatedly. This yields

$$\int_0^s = O\left(\frac{1}{k}\right),$$

and (A 6) becomes

$$l(F) = O(\theta_0 \epsilon),$$

which is of higher order than $O(\theta_0)$; the other terms of (A 4) may similarly be shown to be of higher order. Hence to $O(\theta_0)$, $l(F) = 0$.

Appendix B

In this appendix we shall prove the identity (6.5*a*):

$$\theta_0 \frac{1}{m_2}\int_0^{x_2 - m_2 y_2} V(\eta) \exp(ik_2\eta) \exp(-ik_2 x_2) J_0\left(\frac{k_2}{M_2}[(x_2-\eta)^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right) d\eta$$
$$= -a_2\theta_0 \sum_{\nu=1}^{\infty} b_\nu e^{-\nu\theta} J_\nu\left(\frac{k_2}{M_2}[x_2^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right) \exp(-ik_2 x_2), \tag{B 1}$$

where

$$x_2 \geqslant m_2 y_2, \quad \tanh\theta = \frac{m_2 y_2}{x_2}, \quad b_\nu = \frac{iM_2\nu}{k_2 m_2}[t^\nu + (-t)^\nu] + b\cos(\epsilon\beta)[t^\nu - (-t)^{-\nu}],$$

$$t = i(M_2 + m_2), \quad V(x_2) = U_2 + i\omega(x_2 - b\cos\epsilon\beta). \tag{B 2}$$

First we write

$$V(\eta)\exp(ik_2\eta) = \frac{a_2}{\eta}\left[\left(M_2 - \frac{i\omega b}{a_2}\cos\epsilon\beta\right)[\eta\exp(ik_2\eta)] + \left(i\frac{\omega}{a_2}\right)[\eta^2\exp(ik_2\eta)]\right].$$

As suggested by Carrier (1949), we expand $\eta\exp(ik_2\eta)$ and $\eta^2\exp(ik_2\eta)$ in series of Bessel functions, through the generating function of the Bessel functions, and obtain

$$V(\eta)\exp(ik_2\eta) = -\frac{a_2}{\eta}\sum_{\nu=1}^{\infty} b_\nu \nu m_2 J_\nu\left(\frac{k_2}{M_2}\eta\right).$$

Substitution of this into the left-hand side of (B 1) yields

$$-a_2\theta_0 \sum_{\nu=1}^{\infty} b_\nu \nu \exp(-ik_2x_2)F(x_2), \tag{B 3}$$

where

$$F(x_2) = \int_0^{x_2}\left\{H[x_2 - m_2y_2 - \eta]\,J_0\left[\frac{k_2}{M_2}[(x_2-\eta)^2 - (m_2y_2)^2]^{\frac{1}{2}}\right]\right\}\left\{\frac{1}{\eta}J_\nu\left(\frac{k_2}{M_2}\eta\right)\right\}d\eta. \tag{B 4}$$

If we take the Laplace transform $\mathscr{F}$ of $F(x_2)$, defined by

$$\mathscr{F} = \int_0^\infty \exp(-sx_2)F(x_2)\,dx_2,$$

then, by convolution, we obtain

$$\mathscr{F} = \frac{1}{\nu}(k_2/M_2)^\nu\left[s^2 + \left(\frac{k_2}{M_2}\right)^2\right]^{-\frac{1}{2}}\left\{s + \left[s^2 + \left(\frac{k_2}{M_2}\right)^2\right]^{\frac{1}{2}}\right\}^\nu \exp\left\{-m_2y_2\left[s^2 + \left(\frac{k_2}{M_2}\right)^2\right]^{\frac{1}{2}}\right\}.$$

Inverting this gives (e.g. Erdélyi *et al.* 1954, p. 250)

$$F(x_2) = \frac{1}{\nu}e^{-\nu\theta}J_\nu\left(\frac{k_2}{M_2}[x_2^2 - (m_2y_2)^2]^{\frac{1}{2}}\right) \tag{B 5}$$

for $x_2 \geqslant m_2y_2$. By substituting (B 5) into (B 3), one may establish the required identity (B 1).

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APPENDIX 4

"The Effect of Airfoil Contour upon the Unsteady Aerodynamics of Supersonic Cascades"

The Effect of Airfoil Contour upon the Unsteady Aerodynamics of Supersonic Cascades

*M. Kurosaka**

The University of Tennessee Space Institute
Tullahoma, Tennessee 37388

and

I. H. Edelfelt†

General Electric Research and Development Center, Schenectady, New York

1. *Introduction.*

It is now an indisputable fact that a subtle change in airfoil contour induces significant modification upon the unsteady aerodynamics of supersonic cascades and consequently results in substantial shift in its flutter boundaries. Its phenomological description is briefly given in Ref. 1 along with physical explanation based upon the concept of 'phase memory'; for an isolated airfoil of parabolic contour, a detailed analysis substantiating this concept is presented in Ref. 2., where the profound effect of airfoil contour upon the unsteady far field around an oscillating airfoil is in fact quantified.

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*Associate Professor, Mechanical and Aerospace Engineering.

†Fluid Mechanics Engineer.

The objective of the present paper is to extend the isolated airfoil analysis of Ref. 2 to cascade arrangement; in doing so, we shall rely heavily on the formulation and results of Ref. 2, particularly in the usage of Riemann function. More specifically, in analyzing the so-called pre-interference[3] zone, which is invariably the most difficult portion of the problem on hand, we shall make exclusive use of Riemann's integral representation of a solution for hyperbolic equations[4]; applying it to an appropriate contour near the entrance section of cascades adopting the passage approach[2] and with the aid of cascade periodicity condition, we shall be able to determine the entire flow field in the pre-interference zone.

Based upon these analysis, our results indeed confirm the significant effect of airfoil contour upon unsteady pressure distribution acting on the surface of cascaded airfoils.

2. *Formulation.*

Consider oscillating airfoils in a cascade where the coordinate of moving surface of the n-th airfoil is expressed by

$$y = \epsilon f(x) - \theta_0 e^{i\omega t} e^{in\mu} g(x) , \qquad (1)$$

and the coordinate system (x,y) is shown in Figure 1; in the above $\epsilon f(x)$ ($\epsilon \ll 1$) designates the shape of the airfoil in its mean position of oscillation and the second term represents its harmonic motion with frequency, ω and interblade phase angle, μ; here we shall restrict ourselves to parabolically shaped airfoil where f(x) is a quadratic of x. The two small non-dimensional parameters, ϵ and θ_0 characterize the slenderness of the airfoil and the amplitude of motion, respectively. Far upstream of the cascade, the flow velocity is equal to u_∞ and the Mach number, M_∞. Following Ref. 2, we separate the perturbed velocity potential into a steady and time-dependent part by writing

$$\Phi = \epsilon\phi^{(1)} + \theta_0 \exp\left[i(\omega t - kx)\right] \psi^{(1)}, \qquad (2)$$

where $k = M_\infty{}^2\omega/m^2U_\infty$; the first term represents the steady, base flow and the second term corresponds to the unsteady flow; our present interest is, of course, focused on ψ. It turns out to be convenient to introduce the characteristic coordinate (s,p) defined by

$$\left.\begin{aligned} s &= x - my - \epsilon myN\ (M_\infty{}^2/U_\infty)\ d\phi^{(1)}(s)/ds, \\ p &= x + my - (\epsilon/2U_\infty)\ (N-2)\ M_\infty{}^2\ [\phi^{(1)}(s) - \phi^{(1)}(p)], \end{aligned}\right\} \qquad (3)$$

where $N = \frac{1}{2}(\gamma + 1) M_\infty{}^2/m^2$, and $m = (M_\infty{}^2 - 1)^{\frac{1}{2}}$, γ is the adiabatic exponent of the gas; instead of the Cartesian coordinate system, we shall hereafter use the above characteristic coordinates as the independent variables. The steady part of the velocity potential, $\phi^{(1)}$, is given by

$$\phi^{(1)} = -H\,(s)\,\frac{U_\infty}{m}\,f\,(s)\ , \tag{4}$$

where H is the step function. With regard to the unsteady part $\psi^{(1)}$, upon introduction of the function F defined by,

$$\psi^{(1)} = \exp\,[i(\varepsilon k/U_\infty)\,N\,(p - s)\,\phi'^{(1)}(s)\,F], \tag{5}$$

it is found to satisfy the following:

$$F_{sp} + \frac{i\varepsilon k}{U_\infty}[-\,(2N - 1)\,\phi'^{(1)}(s) + (p - s)\,N\,\phi''^{(1)}(s)]\,F_p$$

$$+\,[(\frac{k}{2M_\infty})^2 + \frac{i\varepsilon k}{2U_\infty}\,N\,\phi''^{(1)}(s)]\,F = 0, \tag{6}$$

with the boundary conditions on the airfoil surface

$$i\varepsilon k\,[-\,\frac{1}{m}\,f'(s) - \frac{2}{U_\infty}\,N\,\phi'^{(1)}(s)]F + F_s - F_p - \frac{1}{m}\,V(s)\,e^{iks} = 0\ , \tag{7}$$

where

$$V(x) = [U_\infty g'(x) + (ik\,U_\infty m^2/M_\infty{}^2)\,g(x)]\,e^{in\mu}. \tag{8}$$

The initial condition is that at any given point, say Q_1, of Figure 1 just upstream of the shock emanating from the leading edge of the reference airfoil, the unsteady velocity potential there must be equal to the one at the corresponding point, Q_2, upstream of the leading edge shock of the following airfoil, with the exception of the phase lag, i.e.,

$$[e^{-ikx}\,\psi^{(1)}]_{Q_1} = [e^{-ikx}\,\psi^{(1)}]_{Q_2}\, e^{-i\mu}. \tag{9}$$

It is convenient at this point to separate F into the following two parts:

$$F = F^{(1)} + F^{(2)} . \tag{10}$$

Of course, both $F^{(1)}$ and $F^{(2)}$ should satisfy the governing equation; for $F^{(1)}$, we assign the full boundary condition of (7) and for $F^{(2)}$, we assign the initial conditions corresponding to (9). Then, the initial condition for $F^{(1)}$ is that it simply vanishes upstream of the leading edge bow shock of the reference airfoil and the boundary condition for $F^{(2)}$ takes the form similar to (7) but with V(s) set equal to zero. Evidently, $F^{(1)}$ then corresponds to the one for an isolated airfoil and it is given in Ref. 2 to be

$$F^{(1)} = \frac{H(s)}{m}\int_0^s V(\tau)\,\exp\,(ik\tau)\,\exp\left\{\left[\frac{i\varepsilon k}{U_\infty}(N-1)\,(\phi^{(1)}(s)-\phi^{(1)}(\tau)) + N\,(s-\tau)\,\phi'^{(1)}(s)\right]\right\}$$

$$\times\, M\,[\frac{1}{2} - \frac{mk}{4i\varepsilon\alpha\, NM_\infty{}^2},\ 1,\ \frac{i\varepsilon k\alpha N}{m}(s-\tau)(p-\tau)]\,d\tau\ , \tag{11}$$

where M is the confluent hypergeometric function.

3. *Solution.*

Our present aim is to determine $F^{(2)}$, which satisfies the governing equation (6) along with the following boundary condition

$$i\epsilon k\ [-\frac{1}{m} f'(s) - \frac{2}{U_\infty} N\ \phi'^{(1)}(s)]\ F^{(2)} + F_s^{(2)} - F_p^{(2)} = 0\ , \qquad (12a)$$

and the initial condition

$$[e^{-ikx}\ F^{(2)}]_{Q_1} = [e^{-ikx}(F^{(1)} + F^{(2)})]_{Q_2}\ e^{-i\mu}\ . \qquad (12b)$$

The governing equation (6) is hyperbolic; it is well known that the solution of any second-order linear hyperbolic equation can be expressed in the form of integral representation, once the corresponding Riemann function R is obtained[4]. More specifically, for any point C of Figure 1, whose characteristic coordinates are given to be (s,p), the Riemann's integral representation can be expressed as

$$F^{(2)}(s,p) = \frac{1}{2}\ [F^{(2)}(A)\ R\ (A) + F^{(2)}\ (B)\ R\ (B)]$$
$$+ \frac{1}{2}\int_{BO}\left\{\left[[-F_\eta^{(2)}R + F^{(2)}R_\eta]\ d\eta + [F_\zeta^{(2)}\ R - F^{(2)}R_\zeta + 2bF^{(2)}R]\ d\zeta\right]\right\}$$
$$+ \frac{1}{2}\int_{OA}\left\{\left[[-F_\eta^{(2)}\ R + F^{(2)}\ R_\eta]\ d\eta\ + [F_\zeta^{(2)}R - F^{(2)}R_\zeta + 2bF^{(2)}R] d\zeta\right]\right\}, \qquad (13a)$$

where, as shown in Figure 1, A and B are intercepts of the characteristics passing the point C and the airfoil and the bow shock, respectively, and O is the origin; the line integrals should be carried out in the sense indicated above, that is, in the first integral from point B to point O and in the second integral, from point O to point A. Furthermore, in (13) b is equal to

$$b = \frac{i\varepsilon k}{U_\infty}\left[-(2N-1)\,\phi'^{(1)}(\xi) + (\eta - \xi)\,N\,\phi^{(1)''}(\xi)\right], \tag{13b}$$

and R, the Riemann function, is given by

$$R(\xi, \eta, s, p) = \exp \nu \cdot M\,(a, 1, z), \tag{13c}$$

where

$$\nu = i\varepsilon k\,\frac{1}{U_\infty}\left\{(N-1)\,(\phi^{(1)}(s) - \phi^{(1)}(\xi)) - N\,[\eta\,(\phi'^{(1)}(s) - \phi'^{(1)}(\ ))\right. \\ \left. -(s\phi^{(1)'}(s) - \xi\phi^{(1)'}(\xi))]\right\}, \tag{13d}$$

and where

$$a = \frac{1}{2} + \frac{U_\infty k}{4i\varepsilon\, NM_\infty^{2}\, \phi''^{(1)}}, \tag{13e}$$

$$z = -\frac{i}{u_\infty}\,\varepsilon kN\,(s - \xi)\,(p - \eta)\,\phi''^{(1)}. \tag{13f}$$

We rewrite the second integral appearing in (13a) with the aid of (12a) and (13b) and, noting that along the path OA, $\zeta = \eta$, (13a) becomes

$$F^{(2)}(s,p) = \frac{1}{2}\,[F^{(2)}(A)\,R(A) + F^{(2)}(B)\,R(B)]$$

$$+ \frac{1}{2} \int_{BO} \left\{ \left[-F_{\eta}^{(2)} R + F^{(2)} R_{\eta} \right] d\eta + \left[F_{\zeta}^{(2)} R - F^{(2)} R_{\zeta} + 2bF^{(2)} R \right] d\zeta \right\}$$

$$+ \frac{1}{2} \int_{0}^{s} F^{(2)} (\xi , \xi) \left[R_{\eta} - R_{\zeta} + \frac{i\epsilon k}{m} R (2N - 1) \quad f'(\xi) \right]_{\zeta = \eta} d\xi . \tag{14}$$

We then replace, from (12b), those values of $F^{(2)}$ to be evaluated along the path OB located just upstream of the first shock by the ones at the corresponding points just upstream of the second shock. By this, all the values of $F^{(2)}$ appearing both sides of the equation become those downstream of the first shock. Consequently $F^{(2)}$ involved there does not contain any discontinuity and regarding the resulting expression as an integral equation, one solves it by the collation method. This determines $F^{(2)}$ and hence we obtain the flow field in the pre-interference zone. Pursuant to this, we obtain the flow field between the blade passage by the enperposition principle.

4. *Discussion on the Results.*

The computed pressure distributions are shown in Figure 2 - 5. In all of the figures, the shape of the airfoil, f(x) of (1) is as follows:

$$f(x) = -1.7633\,(x^2 - xc)\ ,$$

where c is the airfoil chord; the airfoil is executing torsional motion at its mid chord, i.e. g(x) in (1), is given by

$$g(x) = x - c/2.$$

In addition,

$M_\infty = 1.40$

Stagger angle = 65°,

Solidity = 1.589

μ, interblade phase angle = 180°

The trajectory of the shock locus at its steady position, which is needed in solving (14), is computed according to the routine used in Ref. 5.

Figure 2 shows both the real and imaginary part of the pressure coefficients, defined by $C_p = (p - p_\infty)/\frac{1}{2}\,\rho_\infty U_\infty{}^2$, and compare the present results obtained for $\varepsilon = 0$ (the limits of zero thickness airfoil) with the ones previously computed for flat plate airfoils[1]; as expected, both results agree completely. Figures 3 to 5 compare the present results for $\varepsilon = 0.1$ with the ones for flat plate airfoils at three different values of k and, as might be anticipated, the effect of airfoil contour indeed induces significant modification.

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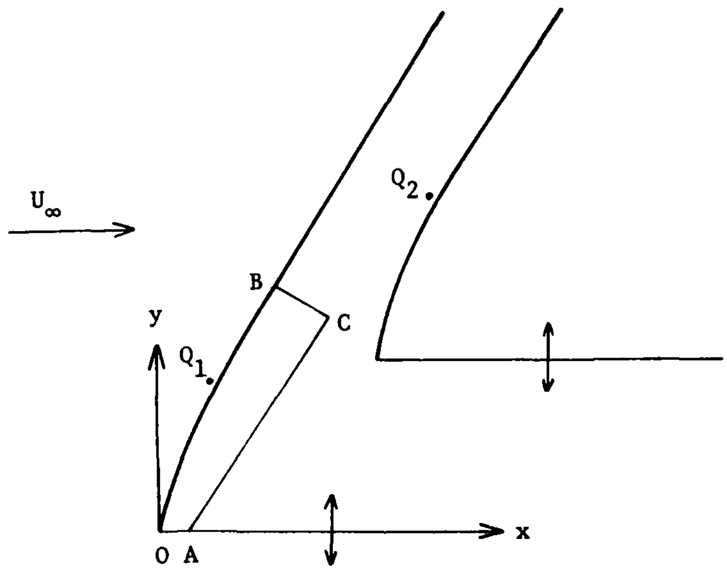

Figure 1.

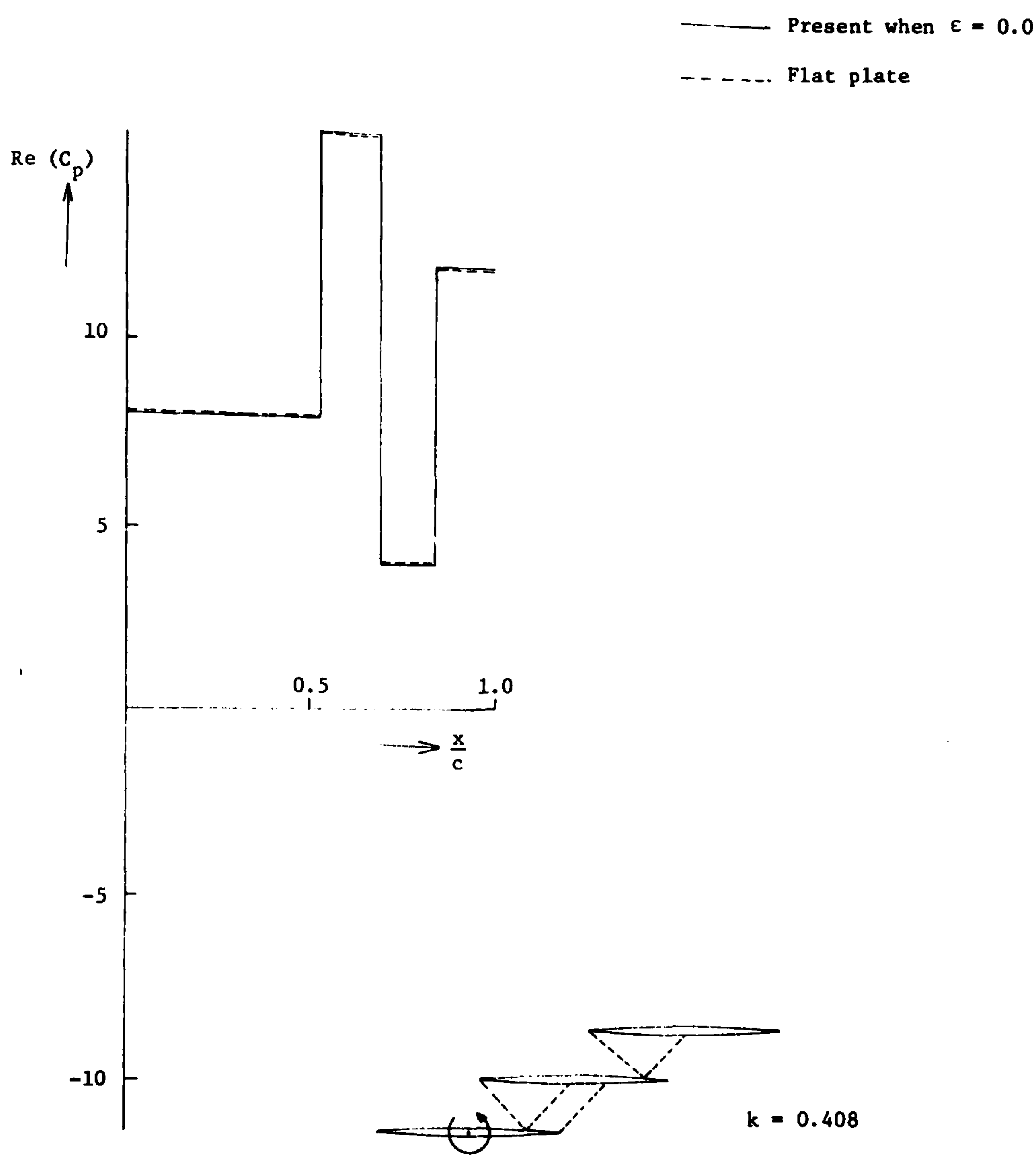

Figure 2(a).

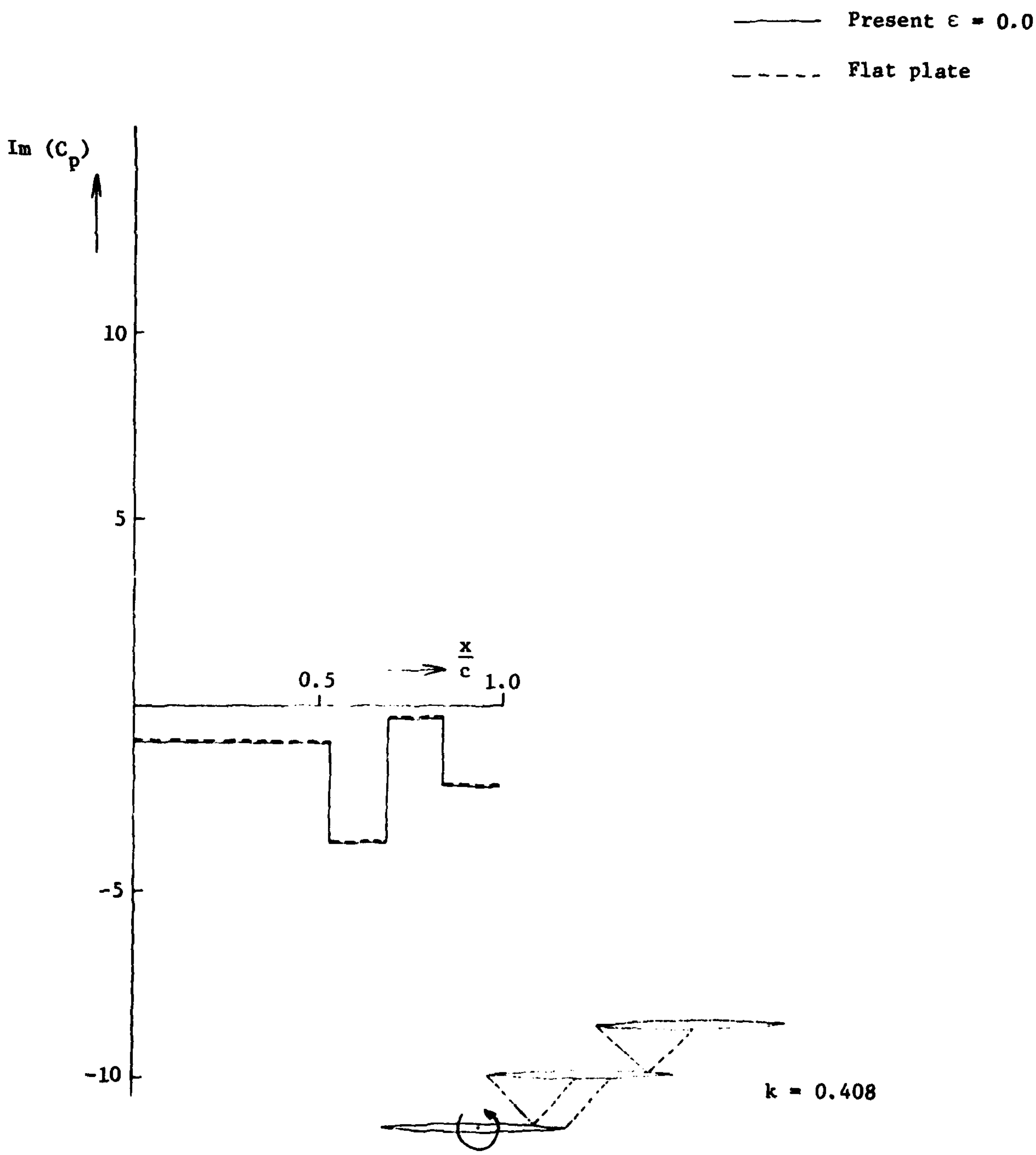

Figure 2(b).

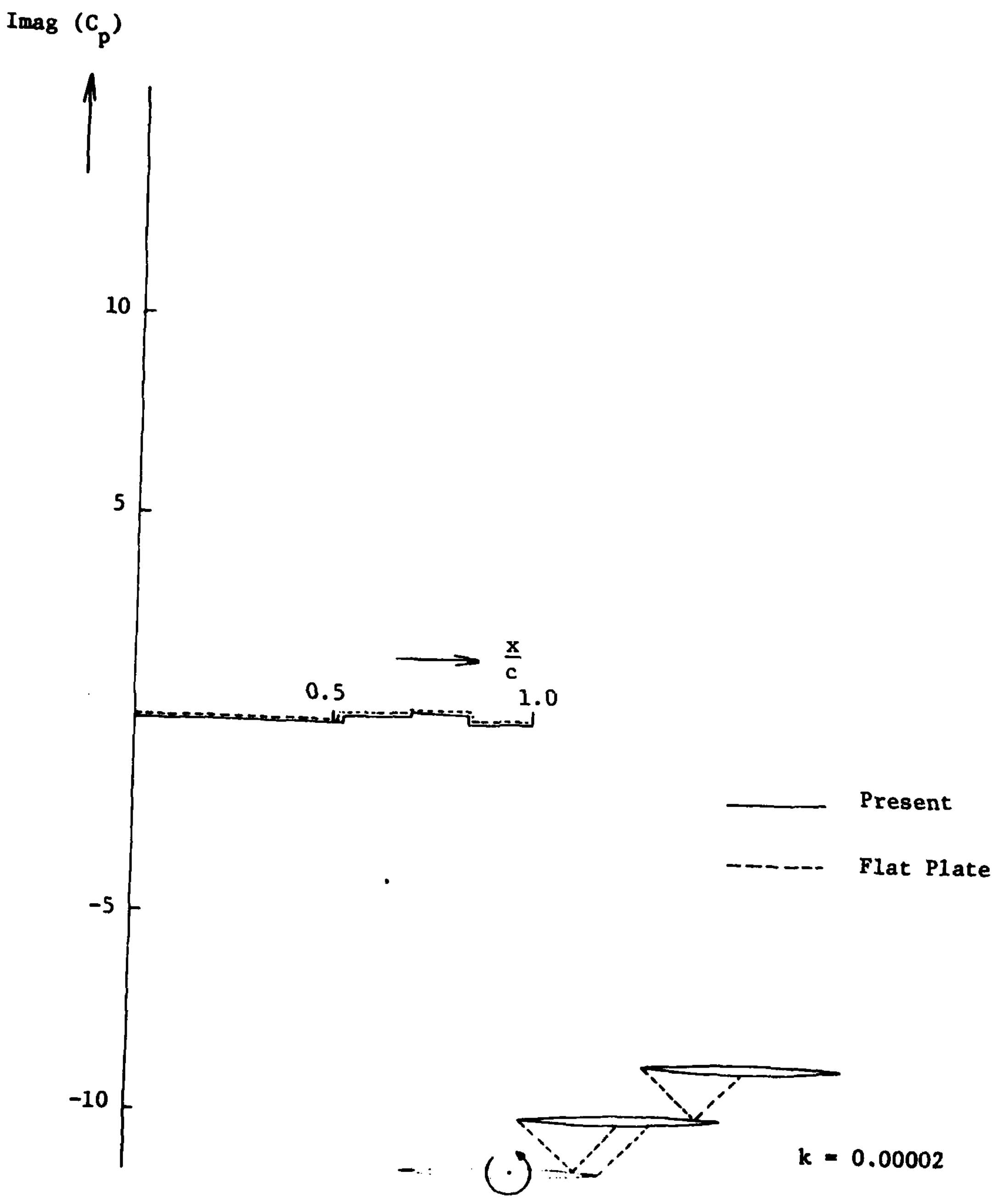

Figure 3.

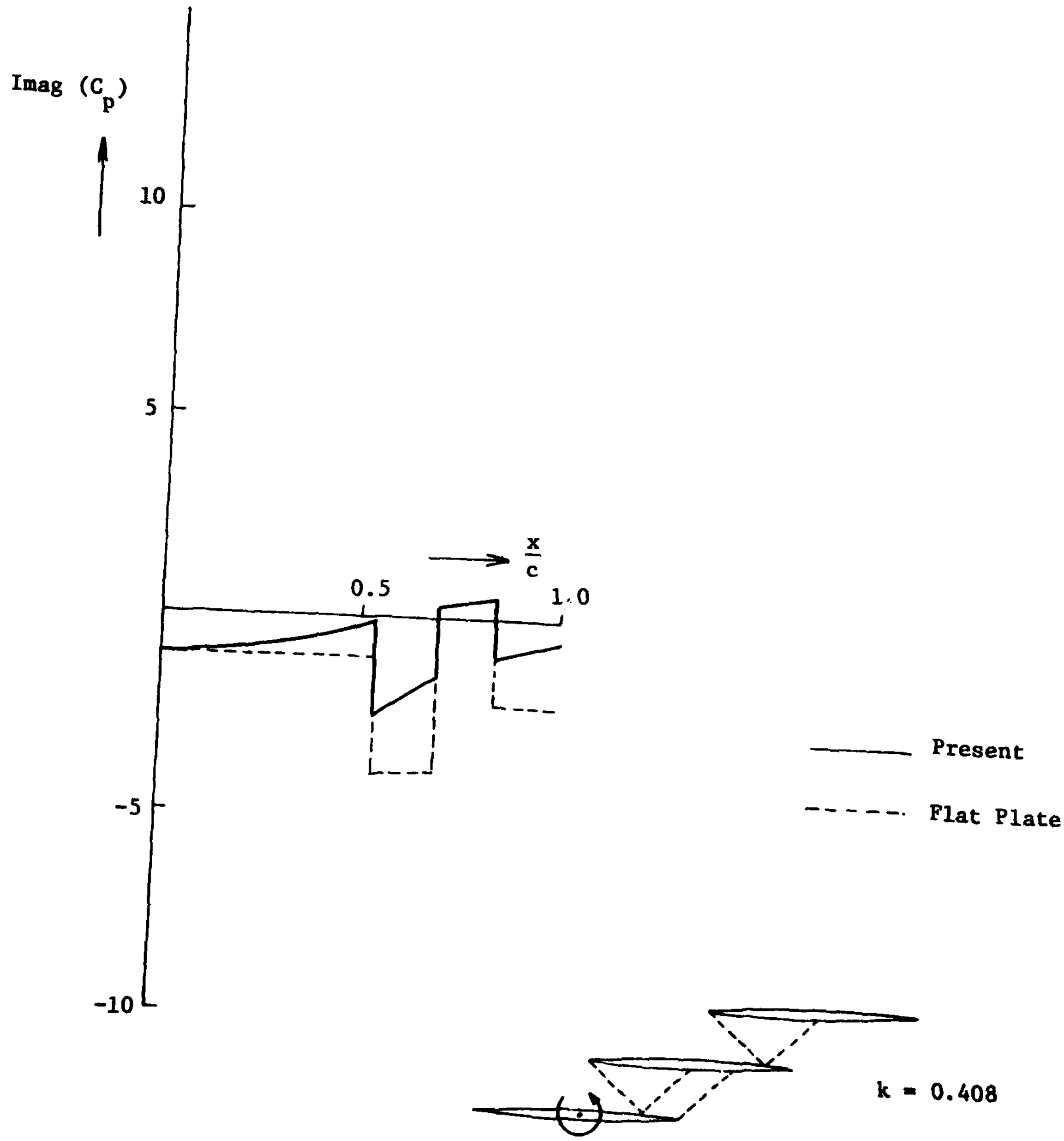

Figure 4.

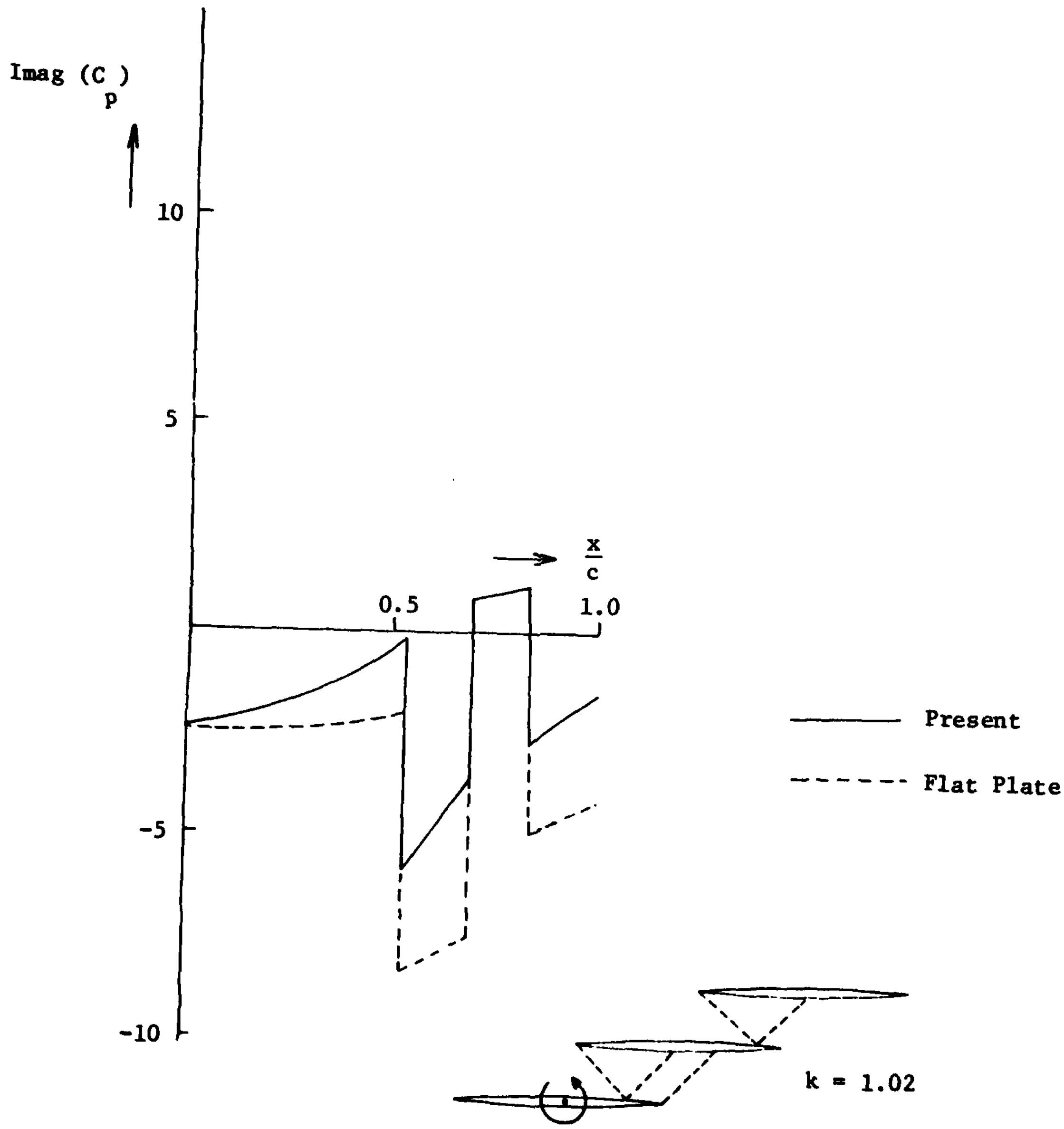

Figure 5.

APPENDIX 5

"Suppressive Effect of Liners on Supersonic Compressor Flutter"

SUPPRESSIVE EFFECT OF LINERS ON SUPERSONIC COMPRESSOR FLUTTER

*M. Kurosaka**

The University of Tennessee Space Institute
Tullahoma, Tennessee

and

I. H. Edelfelt†

General Electric Research and Development
Center, Schenectady, New York

I. Introduction.

One of the treacherous flow-induced vibration problems in high-speed aircraft engines is the so-called supersonic flutter, which, if allowed to persist, may be capable of inflicting excessive damage to their structural integrity. For more detailed information on this, we relegate to elsewhere[1]. Here we only remark that flutter takes place when the relative velocity at the tip exceeds sonic speed and at the time it occurs near the operating line, the flow remains unstalled. Because of its critical importance, there have in recent years an accelerated activity in our effort to understand and quantify the phenomena[2]. However, due to the very complexity of the problem on hand, past emphasis, in both theoretical and experimental research, has been laid upon two-dimensional, cascade problem. Highly

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*Associate Professor, Mechanical and Aerospace Engineering. Associate Fellow AIAA. Also consultant, General Electric Company.

†Fluid Mechanics Engineer.

useful these approaches have been in contributing to our understanding of the phenomena, there always been a nagging question upon the validity of the two-dimensional approximation to what is in reality three-dimensional flow through turbomachines. The question is certainly a legitimate one, since even in the steady situation where bladings do not vibrate, it has long been recognized that the three-dimensionality of flow induces substantial modification to cascade representation[3,4,5]. If once the compressor bladings start to oscillate, two extra factors introduce additional deviation, both attributing to influence the propagation of disturbances emanating from the airfoils oscillating with their natural frequency.* Firstly, the encasing walls serve to reflect the incident walls and thus affect the unsteady flow field; secondly, strong radial gradient in steady, base flow causes refraction of transmitting waves. The full detailed analysis of three-dimensional flow through turbomachinery being dishearteningly difficult even for steady flow, the satisfactory treatment incorporating the above features --- in addition to retaining other necessary constituents such as contour of individual airfoil and geometry associated with the arrangement of bladings --- appears, in spite of some recent attempt[6], to be formidable. Consequently, instead of attacking the subject as such, we turn our present attention toward

*Parentically, if the natural frequency of the airfoil is ω_n and the fundamental blade passing frequency is ω_b, it can be easily shown that the frequency perceived by the stationary observer is equal to

$$\omega_n + \omega_b (n - \mu/2\pi),$$

where n is an integer and μ is the interblade phase lag.

the following problem, which seems to be more crucial in practical applications: suppose one replaces outer casing of compressors with material which is capable of absorbing unsteady fluctuations, what would be its effect on compressor flutter boundary? The installation of acoustic lining on casing walls of aircraft engines is, of course, widely used practice to alleviate the aeroacoustic noise but its potential benefit upon the flutter suppression does not appear to be exploited. However, its obvious advantage appears to be worthy of exploration, since owing to the very capacity of sound absorbing material to relieve the flow fluctuation, the unsteady pressure acting upon the bladings would directly be reduced. Moreover, in supersonic flutter the tip portion of the bladings is the most critical region immersed in high supersonic flow; therefore, the provision of acoustic liners on the surface of outer casing appears -- due to its closest vicinity to bladings -- to produce an immediate beneficial effect upon the tip. Furthermore, since the installation of lining material can be carried out entirely independent of the aerodynamic performance of bladings, it does not interfere with or compromise the other various design consideration of compressors.

To appraise this concept, in the present paper we formulate and analyze a simplified model problem where an isolated oscillating airfoil is placed in a supersonic duct whose upper wall is lined with sound absorbent material. By evaluating the influence of wall liners upon unsteady pressure received at points off the surface of the isolated airfoil, we assess its effect in turbomachines, whose individual blading is, of course, subject to far-field upwash generated by the other members of airfoils.

II. Model Problem Formulation.

Consider an isolated airfoil placed in a two-dimensional duct (Figure 1) where the base, steady flow moves in the x direction with supersonic velocity, U; U is assumed to be uniform across the duct and there is no variation in other, steady-state flow variables. The airfoil executes small-amplitude harmonic motion in the transverse z direction; one surface wall of the duct located at y = h is provided with sound absorbent material having specified impedance while the other wall of the duct is untreated. If one represents the unsteady quantities by primes, the linearized governing equations become

$$\left.\begin{aligned}
&\frac{1}{c^2}\left(\frac{\partial p'}{\partial t} + U\frac{\partial p'}{\partial x}\right) + \rho\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) = 0,\\
&\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial x},\\
&\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial y},\\
&\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial z},
\end{aligned}\right\} \qquad (1)$$

where c is the unperturbed speed of sound, ρ the ambient density; u', v', w' are perturbed velocities in the x, y and z directions, respectively; p' is the perturbed pressure and t is the time. From all of these above perturbed quantities, we separate the time-dependent factor by writing $p' = \overline{p'}e^{i\omega t}$, $u' = \overline{u'}e^{i\omega t}$, etc. and (1) becomes

$$\left.\begin{aligned}
&\frac{1}{c^2}\, i\omega\, \bar{p}' + \frac{1}{c^2}\, U \frac{\partial \bar{p}'}{\partial x} + \rho \left(\frac{\partial \bar{u}'}{\partial x} + \frac{\partial \bar{v}'}{\partial y} + \frac{\partial \bar{w}'}{\partial z}\right) = 0, \\
&i\omega\, \bar{u}' + U \frac{\partial \bar{u}'}{\partial x} = -\frac{1}{\rho} \frac{\partial \bar{p}'}{\partial x}, \\
&i\omega\, \bar{v}' + U \frac{\partial \bar{v}'}{\partial x} = -\frac{1}{\rho} \frac{\partial \bar{p}'}{\partial y}, \\
&i\omega\, \bar{w}' + U \frac{\partial \bar{w}'}{\partial x} = -\frac{1}{\rho} \frac{\partial \bar{p}'}{\partial z}.
\end{aligned}\right\} \tag{2}$$

The boundary condition on the airfoil is that the z-component of fluid velocity is prescribed and equal to, say, W (x,y)

$$\bar{w}'\ (x,y,\ z = 0) = W\ (x,y)\ . \tag{3}$$

The condition on the hard wall is

$$\bar{v}'\ (x,\ y = 0,\ z) = 0\ , \tag{4}$$

and on the soft-wall

$$\left.\frac{\bar{p}'}{\bar{v}'}\right|_{y = h} = -\frac{\rho c}{A} \tag{5}$$

where A is the specific acoustic admittance and is assumed to be constant. The initial conditions are such that upstream of the leading edge Mach cone, the flow remains unperturbed or equivalently

$$\bar{p}' = \bar{u}' = \bar{v}' = \bar{w}' = 0. \tag{6}$$

Because of the supersonic nature of the flow, we focus our attention to the flow for $z > 0$ (above airfoil), which is uncoupled from the flow for $z < 0$ (below airfoil).

III. Analytical Solution.

In order to derive an analytical solution, we take the Laplace transform of all the perturbed quantities in the x direction, defined, for example, by

$$\tilde{p} = \int_0^\infty e^{-sx}\, \bar{p}'\ (x,y,z)\ dx,$$

and obtain the following:

$$\frac{1}{c^2}\, i\omega\, \tilde{p} + \frac{1}{c^2}\, Us\, \tilde{p} + \rho\, (s\, \tilde{u} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z}) = 0\ , \tag{7a}$$

$$i\omega\, u + Us\, \tilde{u} = -\frac{1}{\rho}\, s\, \tilde{p}\ , \tag{7b}$$

$$i\omega\, \tilde{v} + Us\, \tilde{v} = -\frac{1}{\rho}\, \frac{\partial \tilde{p}}{\partial y}\ , \tag{7c}$$

$$i\omega\, w + Us\, \tilde{w} = -\frac{1}{\rho}\, \frac{\partial \tilde{p}}{\partial z}\ , \tag{7d}$$

with the following boundary conditions:

$$\tilde{w}\Big|_{z\,=\,0} = \tilde{W}\ , \tag{8a}$$

$$\tilde{u}\Big|_{y\,=\,0} = 0\ , \tag{8b}$$

$$\tilde{p}\Big|_{y\,=\,h} = -\frac{\rho c}{A}\ \tilde{v}\Big|_{y\,=\,h}\ . \tag{8c}$$

Elimination of $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ from the above equations at once yields the following expression for $\tilde{p}$:

$$\frac{\partial^2 \tilde{p}}{\partial y^2} + \frac{\partial^2 \tilde{p}}{\partial z^2} - [m^2 s^2 + \frac{2i\omega Us}{c^2} - \frac{\omega^2}{c^2}]\ \tilde{p} = 0 \ , \tag{9a}$$

with the boundary conditions,

$$\left.\frac{\partial \tilde{p}}{\partial z}\right|_{z = 0} = -\rho\ (i\omega + Us)\ \tilde{w}\ (y), \tag{9b}$$

$$\left.\tilde{p}\right|_{y = 0} = 0 \ , \tag{9c}$$

$$\left.\tilde{p}\right|_{y = h} = \frac{c}{A(i\omega + Us)} \left.\frac{\partial \tilde{p}}{\partial y}\right|_{y = h} . \tag{9d}$$

The solution for $\tilde{p}$ is immediately obtained to be

$$\tilde{p} = \sum_{n = 0}^{\infty} \rho\ (i\omega + Us)\ \frac{1}{\alpha_n}\ e^{-\alpha_n z} \cdot \tilde{\delta}_n \cos \beta_n y, \tag{10a}$$

where β_n is given as solutions of

$$\beta_n h \cdot \tan \beta_n h = - \frac{A(i\omega + Us)h}{c} , \tag{10b}$$

and α_n is related to β_n by

$$\alpha_n{}^2 - \beta_n{}^2 - m^2\ [s^2 + \frac{2i\omega Us}{c^2} - \frac{\omega^2}{c^2}] = 0, \tag{10c}$$

where $\tilde{b}_n$ are the coefficients associated with the expansion of $\tilde{W}(y)$ in the series of $\cos \beta ny$, i.e.,

$$\tilde{W}(y) = \sum_{n=0}^{\infty} \tilde{b}_n \cos \beta_n y \qquad (10d)$$

or

$$\tilde{b}_n = [\int_0^h \tilde{W}(y) \cos (\beta_n y)\, dy)/[\int_0^h (\cos \beta_n y)^2\, dy] \;. \qquad (10e)$$

If we denote the inverse Laplace transform by $\bar{L}^{1}$, the inverted form of (10a) may be expressed as

$$\overline{p'} = \sum_{n=0}^{\infty} \rho\, [i\omega p_1 + Up_2] \;, \qquad (11a)$$

where

$$p_1 = \bar{L}^{1} [(\frac{1}{\alpha_n} e^{-\alpha_n z}) (\tilde{b}_n \cos \beta_n y)] \;, \qquad (11b)$$

$$p_2 = \bar{L}^{1} [(s \frac{1}{\alpha_n} e^{-\alpha_n z}) (\tilde{b}_n \cos \beta_n y)]. \qquad (11c)$$

In order to evaluate (11b) through convolution, we have to invert the following two transforms, respectively:

$$\frac{1}{\alpha_n} e^{-\alpha_n z} \quad \text{and } \tilde{b}_n \cos \beta_n y \;.$$

(The other transform $s/\alpha_n \, e^{-\alpha_n z}$ appearing in (11c) will be promptly obtained, once $1/\alpha_n \, e^{-\alpha_n z}$ is inverted, as described shortly.) The inversion of the above obviously demands the explicit expression of β_n and in order to obtain it, we assume that the specific acoustic admittance, A, is small and then (10b) yields approximately

$$\beta_0 = [- \frac{A(i\omega + Us)}{ch}]^{\frac{1}{2}} , \tag{12a}$$

$$\beta_n = \frac{2n\pi}{h} [1 - \frac{A(i\omega + Us)h}{(2n\pi)^2 c}], \; n = 1, 2 \text{ ---} \tag{12b}$$

A. *Inversion of* $\frac{1}{\alpha_n} e^{-\alpha_n z}$.

Using these expressions and with the aid of the following well-known inversion formulae[7],

$$L^{-1} \left\{ (s + \alpha)^{-\frac{1}{2}} (s + \beta)^{-\frac{1}{2}} \exp [-b \, (s + \alpha)^{\frac{1}{2}} (s + \beta)^{\frac{1}{2}}] \right\}$$

$$= H \, (x - b) \, \exp [- \frac{1}{2} (\alpha + \beta)x] \, J_0 \, [\frac{1}{2} (\alpha - \beta) \, (x^2 - b^2)^{\frac{1}{2}}] \, ,$$

where J_0 is the Bessel function, $1/\alpha_n \, e^{-\alpha_n z}$ is readily inverted to be

$$L^{-1} \, [\frac{1}{\alpha_0} e^{-\alpha_n z}]$$

$$= \frac{1}{m} H \, (x - mz) \, \exp \left\{ [- \frac{i \omega U}{c^2 m^2} + \frac{AU}{2hcm^2}] \, x \right\}$$

$$\times J_0 \left\{ [\frac{\omega^2}{c^2 m^4} + \frac{i\omega A}{hcm^4}]^{\frac{1}{2}} \, (x^2 - m^2 z^2)^{\frac{1}{2}} \right\} , \tag{13a}$$

and

$$\bar{L}^{1}\left[\frac{1}{\alpha_n}\bar{e}^{\alpha_n z}\right]$$

$$= \frac{1}{m} H\ (x - mz)\ \exp\left\{\left[-\frac{i\omega U}{c^2 m^2} + \frac{AU}{hcm^2}\right] x\right\}$$

$$\times J_0\left\{\left[\frac{\omega^2}{c^2 m^4} + \frac{2i\omega A}{hcm^4} + \left(\frac{2n\pi}{hm}\right)^2\right]^{\frac{1}{2}} (x^2 - m^2 z^2)^{\frac{1}{2}}\right\}, \qquad (13b)$$

where n = 1, 2, --- .

If we denote the right hand side of (13) as (x), i.e.,

$$L^{-1}\left[\frac{1}{\alpha_n}\bar{e}^{\alpha_n z}\right] = \theta\ (x)\ , \quad n = 0,\ 1,\ ---$$

then the inversion of $s/\alpha_n\ \bar{e}^{\alpha_n z}$ appeared in (11c) is immediately given to be[8]

$$L^{-1}\left[\frac{s}{\alpha_n}\bar{e}^{\alpha_n z}\right] = \theta'(x)\ . \qquad (13c)$$

B. *Inversion of $\tilde{b}_n \cos \beta_n y$.*

In order to invert $\tilde{b}_n \cos\beta_n y$ of (11b), we recognize that if the normal fluid velocity W of (3) is independent of y, and is a function of x only, for instance, equal to q (x), i.e.,

$$W(x,y,\ z = 0) = q\ (x),$$

then

$$\tilde{W}(y) = \tilde{q},$$

where $\tilde{q}$ is a constant, (10e) yields, to the present order of approximation,

$$\tilde{b}_0 = \tilde{q} \ , \tag{14a}$$

$$\tilde{b}_n = -\frac{Ah}{2(n\pi)^2 c}[i\omega\,\tilde{q} + Us\,\tilde{q}] \ , \tag{14b}$$

where n = 1, 2 --- . Making use of these and noting

$$L^{-1}(s\,\tilde{q}) = q'(x) + q(0)\ \delta(x),$$

where $\delta(x)$ is the delta function, we obtain the inverse of $\tilde{b}_n \cos\beta_n y$ as

$$L^{-1}[\tilde{b}_0 \cos\ \beta_0 y] = q(x) + \frac{y^2}{2}\frac{UA}{ch}[q'(x) + \frac{i\omega}{U} q\ (x)], \tag{15a}$$

and

$$L^{-1}[\tilde{b}_n \cos\beta_n y] = \frac{-hA}{2(n\pi)^2 c}\ \cos\ (\frac{2n\pi y}{h})$$

$$\times\ [i\omega q(x) + U\ (q'(x) + q(0)\,\delta\,(x))], \tag{15b}$$

where n = 1, 2, ---; the derivation of (15b) is straightforward, while that of (15a) requires some consideration, and is therefore given in Appendix.

C. Final Solution.

With the aid of (13) and (15), (11a) becomes as follows:

$$\frac{p'}{\rho} = \frac{1}{m} f_0 (mz)\ B\ (x - mz) + \frac{1}{m} \int_0^{x - mz} g_0 (x - t)\ B(t)\ dt$$

$$- \sum_{n=1}^{\infty} \varepsilon \cos \left(\frac{2n\pi y}{h}\right) \Bigg\{ [i\omega\ q(x - mz) + U\ q'\ (x - mz)]\ f_n\ (mz)$$

$$+ U\ q(0)\ H\ (x - mz)\ g_n\ (x)$$

$$+ \int_0^{x - mz} [i\omega\ q(t) + U\ q'(t)]\ g_n(x - t)\ dt \Bigg\}, \tag{16}$$

where $f_0(x) = U \exp (\gamma_0 x)\ J_0\ [\delta_0\ (x^2 - m^2 z^2)^{\frac{1}{2}}$,

$$g_0(x) = \Bigg\{ \left(- \frac{i\omega}{m^2} + \frac{AU^2}{2hcm^2}\right) J_0\ [\delta_0(x^2 - m^2 z^2)^{\frac{1}{2}}$$

$$- U\ \delta_0\ x\ (x^2 - m^2 z^2)^{-\frac{1}{2}} J_1\ [\delta_0(x^2 - m^2 z^2)^{\frac{1}{2}}\ ] \Bigg\} \exp\ (\gamma_0 x),$$

and where

$$\gamma_0 = \frac{-\ i\omega\ U}{c^2 m^2} + \frac{AU}{2hcm^2}\ , \quad \delta_0 = \left[\frac{\omega^2}{c^2 m^4} + \frac{i\omega A}{hcm^4}\right]^{\frac{1}{2}},$$

$$B(x) = q\ (x) + \frac{y^2 UA}{2\ ch}\ [q'(x) + \frac{i\omega}{U}\ q(x)],$$

$$\varepsilon = \frac{Ah}{2(n\pi)^2 cm},$$

$$f_n(x) = U \exp\ (\gamma x)\ J_0\ [\delta_n (x^2 - m^2 z^2)^{\frac{1}{2}}]$$

$$g_n(x) = \left\{ \left(-\frac{i\omega}{m^2} + \frac{AU^2}{hcm^2} \right) J_0\ [\delta_n\ (x^2 - m^2 z^2)^{\frac{1}{2}}] - U\delta_n\ x\ (x^2 - m^2 z^2)^{-\frac{1}{2}}\ J_1\ [\delta_n (x^2 - m^2 z^2)^{\frac{1}{2}}] \right\} \exp\ (\gamma x),$$

and where

$$\gamma = -\frac{i\omega U}{c^2 m^2} + \frac{AU}{hcm^2}, \quad \delta_n = \left[\frac{\omega^2}{c^2 m^4} + \frac{2i\omega A}{hcm^4} + \left(\frac{2n\pi}{hm}\right)^2\right]^{\frac{1}{2}}.$$

In the limist of zero acoustic admittance, one can immediately show that the above expression will be reduced to a solution for two-dimensional, oscillating airfoil immersed in supersonic flow[9].

IV. Discussion.

Figure 2 show the comparison of the unsteady pressure distribution between treated and untreated walls; the ordinate is the absolute magnitude of unsteady pressure and the abscissa is the distance measured from the bow shock, $x - mz$, plotted at varous values of z. The additional parameters are as follows:

airfoil chord = 1.0,

M_∞ (Mach number) = 1.3,

$h = 1.5$,

$k = \omega/U \cdot M_\infty = 0.5$,

$A = -1.140 + 0.547i$ (for treated wall)

$= 0$ (for untreated wall).

The airfoil is executing bending motion and its instantaneous position is given by $z = e^{i\omega t}$; the unsteady pressure in the figure is evaluated on the top surface, i.e., $y = h$.

It is clearly evident that the provision of acoustic liners significantly attenuates the unsteady pressure fluctuation and tends to suppress the occurrence of flutter.

Acknowledgement.

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APPENDIX

Here we shall derive the inversion of (15a): we note that $\cos \beta_0 y$ becomes, from (12a)

$$\cos \beta_0 y = \cos \left\{ y[- \frac{A(i\omega + Us)}{ch}]^{\frac{1}{2}} \right\}.$$

Taking the inverse transform, we obtain

$$L^{-1} [\cos \beta_0 y] = \exp (-i \frac{\omega x}{U}) L^{-1} \left\{ \cos [y(- \frac{UA}{ch})^{\frac{1}{2}} \sqrt{s}] \right\}$$

$$= \exp (-i \frac{\omega x}{U}) L^{-1} [1 + \frac{y^2 UA}{2 ch} s + 0 (A^2) + -]$$

$$= \exp (-i \frac{\omega x}{U}) [\delta(x) + \frac{y^2 UA}{2ch} \delta'(x) + 0 (A^2) + --].$$

From (14a) and with the aid of convolution, one obtains (15a).

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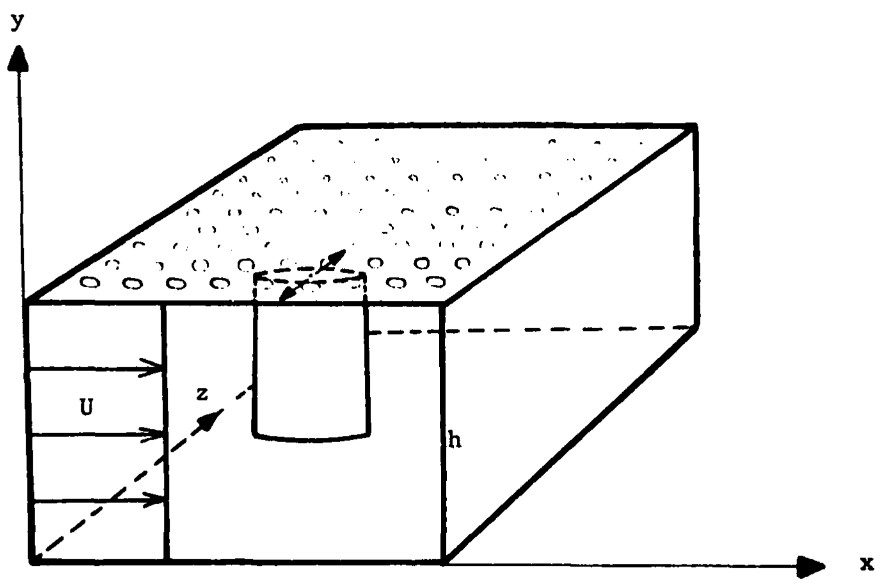

Figure 1. Definition Sketch.

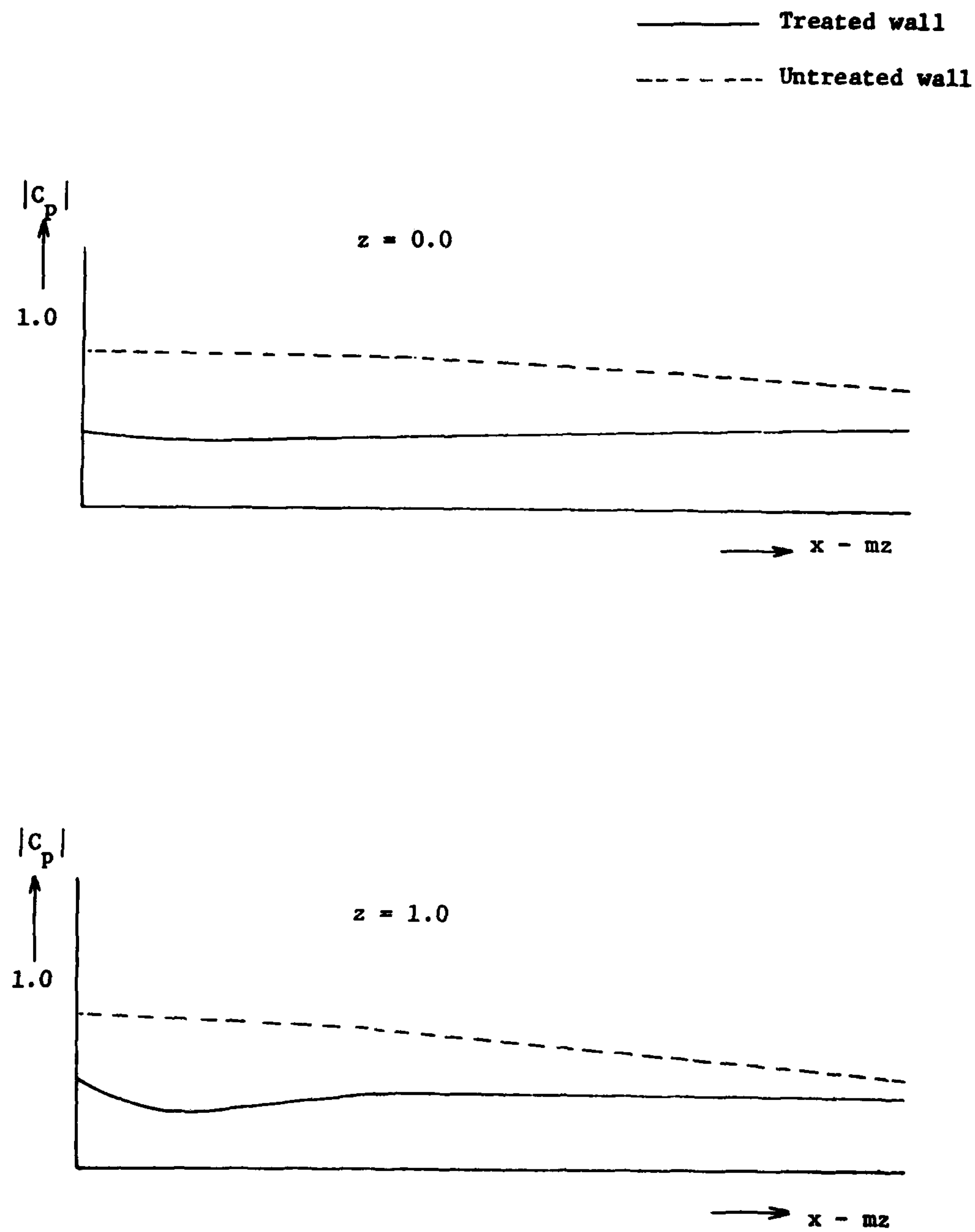

Figure 2a. Attenuation of Unsteady Pressure (Bending Motion).

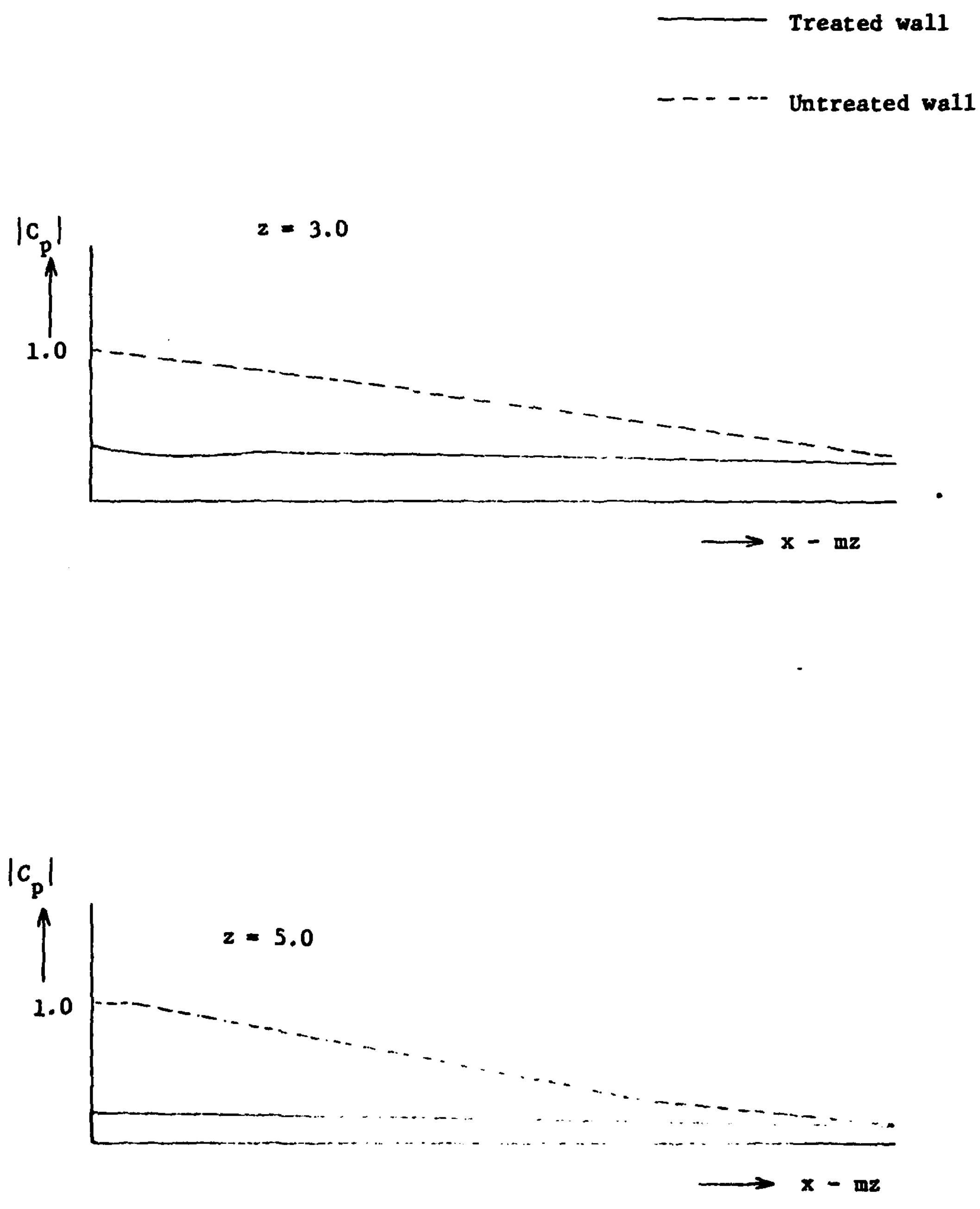

Figure 2b (continued).

APPENDIX 6

"Towards Simpler Representation of Wave Propagation Through Non-Uniform Media"

Towards Simpler Representation of Wave Propagation
Through Non-Uniform Media

M. Kurosaka

The University of Tennessee Space Institute
Tullahoma, Tennessee 37388

1. Introduction.

Pressing problems beset with unsteady flow phenomena in practical applications appear to involve almost invariably the propagation of disturbance through non-uniform media -- such as flow with velocity and temperature gradients. Notable among them are: the flutter problems in turbomachinery bladings where the imparted swirling motion induces substantial radial gradient in steady base flow; the aeroacoustic noises propagating through aircraft engine ducts whose contoured form and the boundary layer growth over its inner surface result in pronounced variation of velocity; and high-intensity sound generated in heat exchangers, gas-cooled reactors and the like whose complicated internals inescapably produce considerable spatial distribution, both in velocity and temperature.

Having been spurred by these compelling incentives, the last decade has witnessed intensified activity in the study of acoustic propagation through non-uniform flow field ; for extensive bibliography in this area, the reader is referred to the recent survey articles by Nayfeh, Kaiser and Telionis (1975) and Goldstein (1976). However, but for few exceptions (e.g. Pridmore-Brown (1958), Shankar (1971), Goldstein and Rice (1973)), the bulk of the work has leaned heavily on entirely computational method. Though, needless to say, extremely useful as these numerical methods are, they may still tend to fall short of achieving the desired goal of revealing the explicit functional interrelationship between the assorted parameters and displaying the underlying physical features. Even in the situations which are amenable to analytical solutions, they appear to suffer from various rigid constraint imposed by the particular choice of velocity profile or the geometry studied; for example, the analytical expressions accrued for unbounded media may not readily be applied to eigenvalue problems associated with ducted flow and often they are altogether unwieldy for physical interpretations.

It therefore appears that what is genuinely needed is a simple, albeit approximate, technique which enables one to cope effectively with the general wave propagating problems. In this paper, we shall describe such a technique which yields an analytical solution of disturbances propagating through arbitrary velocity profile in simple, closed form; and we shall present it from a unified viewpoint where, once the free-space solution is derived, a slight modification of the results will promptly furnish both a solution and physical interpretation even for wave guide problems. The essential idea that we shall promote is not unknown, and it is in fact a variant of 'slowly varying' method, but somehow it does not appear to have been pursued in the way it will be exploited here; with the assumption of small velocity (or temperature) gradient, the technique to be employed here is strikingly simple yet appears to be capable of exposing physical meaning in unmistakably transparent manner.

In the problem of sound propagation through non-uniform media, the original governing equations are invariably reduced into a second-order ordinary differential equation with varying coefficients, the latter reflecting the non-uniformity of velocity (and temperature) profile; the equation is sometimes called Pridmore-Brown's equation.

According to the method suggested perhaps first by Jeffreys and Jeffreys (1956, p. 522), we recast the second-order differential equation, via exponential transformation, into a first-order equation. The equation is of Ricatti's type, nonlinear and inhomogeneous; the transformation is, of course, the one sometimes used as the first step in deriving the WKB formula. In the standard WKB method, one then proceeds to solve the nonlinear first-order equation by iterative process. Here, instead, we seize upon the explicit advantage that under the assumption of small velocity gradient, the first order equation turns into the linear one with constant coefficients; a closed form solution follows immediately from this, providing a free-space solution for sound propagation through non-uniform media. In compact form, our solution (2.11) gives the expression for wave transmission through any arbitrary shear profile; as a special case, we shall prove that the solution, when applied to linearly sheared flow, recovers the result corresponding to the exact solution of Goldstein and Rice (ibid.), which they obtained in terms of Weber's parabolic cylinder function. The present solution also includes, for another specific shear profile, a solution of Miles (1957) and Ribner (1957) for sound transmission through sudden velocity discontinuity. These shall be discussed in the following section, 2.

Most significantly, from the wave-like behaviour of the solution explicitly embodied in the exponential transformation and furthermore, owing to the simplification arising from the assumption of slightly sheared flow, the present solution will conspicuously reveal the physical features of the wave propagation in non-uniform media, which will be described in section 3. For example, we shall readily recognize that for disturbances propagating slightly but arbitrarily sheared profile, the wave front advancing past a given point can simply be constructed by replacing the role of convective velocity in uniform flow by the mean velocity cumulatively averaged up to that point (as might be anticipated from 'slowly varying' concept); this appears to provide the approximate but simpler means than the ones where the wavefront has to be successively constructed by chasing an instantaneous front surface after surface. After these descriptions of physical implication, we shall compare the present solution with the ones derived from WKB formula; to obtain this, we transform the Pridmore-Brown equation into one-dimensional Schrödiger's equation and apply the standard WKB method; we shall observe that the present solution, when rendered into alternative form of infinite series representation, is tantamount to providing all the higher-order terms lacking in the solution based upon the standard WKB method. In the final section 4, we shall show how these solutions obtained for unbounded media can be utilized, with slight modification, to waveguide problems, leading to closed-form solution (4.13). In addition, we shall observe there that, with the aid of free-space solution, a certain interesting phenomena associated with some class of velocity distribution will become physically interpretable.

Our undeviating aim in this paper is to obtain a solution for waves propagating through non-uniform media in a form as uncomplicated as possible and in order to strive for this, it will inevitably be subject to certain formal limitations; nevertheless we shall find that the present approach appears to be highly effective, both in providing a compact expression, from which one can easily detect various trend, and in exposing physical features.

2. *General Solution For Unbounded Medium.*

Consider acoustic waves propagating in a two-dimensional and flowing medium; the base, steady flow is unidirectional, with velocity U in the positive x direction and it varies in the transverse y direction as shown in Figure 1, e.g. U = U(y). For the sake of simplicity, the ambient density and temperature are assumed to be constant, although, if necessary, the present technique can easily be employed for stratified medium as well; the flow will be treated as inviscid. In this section, we consider the medium to be unbounded and in order to fix our ideas, we shall be interested in acoustic waves travelling from $y = -\infty$ and propagating through sheared flow in the positive y direction; at $y = -\infty$, U is assumed to be uniform. The governing equations are given in the linearized form as

$$\left.\begin{aligned}
&\frac{1}{c^2}\left(\frac{\partial p'}{\partial t} + U(y)\frac{\partial p'}{\partial x}\right) + \rho\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \,, \\
&\frac{\partial u'}{\partial t} + U(y)\frac{\partial u'}{\partial x} + v'\frac{dU(y)}{dy} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} \,, \\
&\frac{\partial v'}{\partial t} + U(y)\frac{\partial v'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial y} \,,
\end{aligned}\right\} \tag{2.1}$$

where ρ and c are unperturbed density and the speed of sound, respectively, which are constant; u' and v' are the fluctuating part of the velocity component in the x and y direction, respectively; p' is the perturbed pressure and t is time. Eliminating u' and v' from (2.1), one obtains

$$\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right)\left[(1 - M^2(y))\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} - 2\frac{U(y)}{c^2}\frac{\partial^2 p'}{\partial x \partial t} - \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2}\right]$$

$$- 2\frac{dU(y)}{dy}\frac{\partial^2 p'}{\partial x \partial y} = 0 , \qquad (2.2)$$

where $M(y) = U(y)/c$ is the usual Mach number. If one considers a sinusoidal wave with frequency ω and travelling with wave constant α in the x direction, the fluctuating pressure takes the following form:

$$p' = p(y)\, e^{i(\alpha x - \omega t)} . \qquad (2.3)$$

Parenthetically, this embodies a kinematical condition that the wave number in the x direction, $\alpha/2\pi$, remains always constant even though the waves suffer refraction due to shear flow in the y direction. Into (2.2), we substitute (2.3) and obtain

$$p'' - \frac{2\alpha M'(y)}{\alpha M(y) - k}\, p' + [(\alpha M(y) - k)^2 - \alpha^2]\, p = 0, \qquad (2.4)$$

where $k = \omega/c$ and the primes denote the differentiation with respect to y; the above ordinary equation involving variable coefficients is sometimes called Pridmore-Brown's equation (Pridmore-Brown, 1958).

For the extremely simple situation of $M(y) = \text{constant} = M_0$, say, the solution is, of course, given by

$$p \sim \exp\left[\pm\, i\, y \sqrt{(\alpha M_0 - k)^2 - \alpha^2}\right] . \qquad (2.5)$$

Now consider slightly sheared flow with arbitrary profile

$$M(y) = M_0 + \varepsilon g(y), \tag{2.6}$$

where ε is a small parameter and the shear distribution $g(y)$ is any given function of y; we take M_0 to be the Mach number at $y = -\infty$ (hence $g(-\infty) = 0$). Motivated by the form of (2.5) for uniform flow, we write in the present case

$$p = \exp\left[\pm i y \sqrt{(\alpha M_0 - k)^2 - \alpha^2} + \varepsilon \int_{-\infty}^{y} F(\xi)\, d\xi\right]. \tag{2.7}$$

Substitution of (2.7) into (2.4) at once yields the following equation, to the order of ε, for F:

$$\frac{dF}{dy} \pm 2 i \gamma F + \left[2 (\alpha M_0 - k) \alpha g(y) \mp \frac{2 i \gamma g'(y)}{\alpha M_0 - k}\right] = 0, \tag{2.8a}$$

where

$$\gamma = \sqrt{(\alpha M_0 - k)^2 - \alpha^2}. \tag{2.8b}$$

It is crucial to recognize here that the above first order equation is linear and the coefficients are constant; were the perturbation scheme not explicitly introduced in (2.7), the exponential transformation would have led to the nonlinear Ricatti's equation. In the above (2.8a), we also call our attention to the fact that shear flow profile g (y), which is a function of y, now appears only within the bracket or as non-homogeneous term. With regard to the solution of (2.8a), the homogeneous solutions is obviously equal to $\exp(\mp 2 i \gamma y)$. But since this does not contain any effect of shear, the solution is physically meaningless and, therefore, is discarded; only the particular solution of (2.8a) is relevant. In order to obtain the latter, we take the Fourier transform of (2.8a) and obtain

$$- i\sigma \tilde{F} \pm 2i\gamma\tilde{F} = -2(\alpha M_0 - k)\alpha\tilde{g} \pm \frac{2\alpha\sigma\gamma\,\tilde{g}}{\alpha M_0 - k}, \tag{2.9}$$

where $\tilde{F}$ and $\tilde{g}$ are Fourier transform of F and g, respectively, defined by

$$\tilde{F} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\, e^{i\sigma y}\, dy, \quad \tilde{g} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g\, e^{i\sigma y}\, dy.$$

Solving for $\tilde{F}$ and inverting, we obtain

$$F = \pm i \frac{2\alpha\gamma}{\alpha M_0 - k}\, g(y) + \left[\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2\alpha(\alpha M_0 - k)\right] \int_{-\infty}^{y} g(\xi)\, e^{\mp 2i\gamma(y-\xi)}\, d\xi \,. \tag{2.10}$$

We substitute this into (2.7) and, restoring the dependence with respect to x and t, obtain immediately

$$p' = e^{i(\alpha x - \omega t)}\, e^{\pm i\gamma y} \times \exp\left\{ \pm i\varepsilon \frac{2\alpha\gamma}{\alpha M_0 - k} \int_{-\infty}^{y} g(\xi)\, d\xi + \varepsilon\left[\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2\alpha(\alpha M_0 - k)\right] \int_{-\infty}^{y} d\eta \int_{-\infty}^{\eta} g(\xi)\, e^{\mp 2i\gamma(\eta - \xi)}\, d\xi \right\}. \tag{2.11a}$$

Thus we have already attained the desired goal of obtaining an expression for waves propagating through arbitrarily sheared flow in the positive y direction. (For waves propagation in the negative y direction, one only has to change the lower limits of the integrals from $-\infty$ to $+\infty$.) If the flow is uniform up to certain height, say, $y = y_0$, i.e.,

$$g(y) = 0 \quad \text{for } y < y_0 ,$$

(2.11a) becomes

$$p' = e^{i(\alpha x - \omega t)}\, e^{\pm i\gamma y} \times \exp\left\{ \pm i \frac{2\varepsilon\alpha\gamma}{\alpha M_0 - k} \int_{y_0}^{y} g(\xi)\, d\xi + \varepsilon\left[\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2\alpha(\alpha M_0 - k)\right] \int_{y_0}^{y} d\eta \int_{y_0}^{\eta} g(\xi)\, e^{\mp 2i\gamma(\eta - \xi)}\, d\xi \right\}. \tag{2.11b}$$

Although the case of $\gamma = 0$ or zero incidence angle is physically trivial for the present free-space transmission problem, we record here, for subsequent reference to be used in ducted wave analysis, that at $\gamma = 0$ (2.11b) becomes

$$p' = e^{i(\alpha x - \omega t)} \exp\left[- 2\varepsilon\alpha(\alpha M_0 - k) \int_{y_0}^{y} d\eta \int_{y_0}^{\eta} g(\xi)\, d\xi \right]. \tag{2.11c}$$

For given shear flow, the propagating wave at any point can be determined from either of the above, once the integration involved is carried out. A note of caution is, however, in order here. When the integration over g is carried out for specified shear profile, sometimes the terms in the form of exp ($\pm 2i\gamma y$) appear. Since they are of the same form as the aforementioned homogeneous solution of (2.8a), these terms are parasitic in their nature and therefore they should always be discarded for the reason already stated.

In what is to follow, we shall work out examples for three shear distribution: (a) shear profile with exponentially decaying transition (b) linear shear and (c) sudden velocity discontinuity (see Figure 2). In (b) and (c), we shall observe that the present results agree satisfactorily with the ones based on the available solution.

Example 1. <u>Shear profile with exponential transition</u>:

$$g(y) = 1 - e^{-ay} \quad \text{for } y \geq 0$$

$$\text{and } g(x) = 0, \qquad \text{for } y \leq 0 .$$

Upon substitution of g(y) into (2.11b) where we take $y_0 = 0$ and discarding the terms of exp ($\pm 2i\gamma y$) after carrying out the integral, one obtains, when $\gamma \neq 0$

$$p' = A\, e^{i(\alpha x - \omega t)}\, e^{\pm i\gamma y}$$

$$\times \exp \left\{ \pm\, i\,\varepsilon \frac{2\alpha\gamma}{\alpha M_0 - k} \left(y + \frac{1}{a} e^{-ay}\right) \right.$$

$$+ \varepsilon \left[\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2\alpha(\alpha M_0 - k)\right]$$

$$\times \left[\frac{1}{\pm 2 i \gamma} y - \frac{1}{a \mp 2 i\gamma} \frac{1}{a} e^{-ay}] \right\},$$

where

$$A = \exp \left\{ \mp i \varepsilon \frac{2\alpha\gamma}{a(\alpha M_0 - k)} + [\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2 \alpha (\alpha M_0 - k)] \right. \quad (2.12)$$

$$\left. \times [\frac{1}{(a \mp 2 i\gamma) a} - \frac{1}{(\pm 2 i\gamma)^2} \mp \frac{1}{(a \mp 2 i \gamma) 2 i\gamma}] \right\} .$$

This represents the transmitted wave in the region above $y = 0$ when the incident wave for $y < 0$ is prescribed to be $p' = e^{i(\alpha x - \omega t)} e^{\pm i\gamma y}$. We particularly emphasize here that the above expression provides even the correct amount of the amplitude of the transmitted wave corresponding to the amplitude of incident waves specified to be unity. (If other than unity, the amplitude of the transmitted wave should, of course, be adjusted proportionately). This is the reason why the constant A is deliberately retained in the above and there is no need to multiply it by some other additional factor. One can directly confirm this point by applying, at the interface of $y = 0$, the continuity of pressure and the particle displacement condition between the expression of (2.12), valid for $y \geq 0$, and the expression for $y \leq 0$; the latter is given by the following combination of plane incident and reflecting waves

$$p' = e^{i(\alpha x - \omega t)} e^{\pm i\gamma y} + R e^{i(\alpha x - \omega t)} e^{\mp i\gamma y}, \text{ for } y \leq 0$$

where R is the amplitude of the reflected wave to be determined in the process.

Example 2. Linear Shear : $g(y) = y$ for $y \geq 0$ and $g(y) = 0$ for $y \leq 0$.

In this case, where at $y = 0$ the base flow changes from uniform distribution to linearly varying one, equation (2.11b) becomes as follows:

$$p' = e^{i(\alpha x - \omega t)} e^{\pm i\gamma y}$$

$$\times \exp \left\{ \pm i \frac{\varepsilon\alpha(\alpha M_0 - k)}{2\gamma} y^2 + \varepsilon \left[- \frac{\alpha(\alpha M_0 - k)}{2\gamma 2} + \frac{\alpha}{\alpha M_0 - k} \right] y \right\} . \qquad (2.13)$$

In obtaining this, the term in the form of $\exp(\pm 2i\gamma y)$, which has arisen in the course of integration, is again discarded. As before, (2.13) represents the wave transmitted in the region above $y = 0$ when the incident wave impinging on $y = 0$ is specified to be $p' = e^{i(\alpha x - \omega t)} e^{\pm i\gamma y}$.

For linearly sheared profile, Goldstein and Rice (1973) obtained the exact solution of propagating waves in terms of the parabolic cylinder function. In the present notation, their solution can be expressed as

$$p' = e^{i(\alpha x - \omega t)} \frac{1}{\frac{1}{2} + b} \left[U'(b, \pm \xi) \mp b \xi U(b, \pm \xi) \right], \qquad (2.14)$$

where U is the parabolic cylinder function (e.g. Abramowitz and Stegun (1970), p. 686) defined by the following integral representation:

$$U(b,\xi) = \frac{1}{\Gamma(\frac{1}{2} + b)} e^{-\frac{1}{4}\xi^2} \int_0^\infty \exp\left(-\xi t - \frac{1}{2} t^2\right) t^{b - \frac{1}{2}} dt, \qquad (2.15a)$$

where

$$b = \frac{\alpha}{2i\,\frac{dM}{dy}} \quad \text{and} \quad \xi = \sqrt{\frac{2i}{\alpha\,\frac{dM}{dy}}}\,(k - \alpha M(y)) \; . \tag{2.15b}$$

When the gradient of shear, $\frac{dM}{dy}$, is small, both b and ξ defined above tend to become large. In such a case, one may prove, as shown in Appendix A, that, with the aid of the Darwin's formula (1949) for the parabolic cylinder function, (2.14) is reduced to the present result, equation (2.13). (In lieu of Darwin's formula, it might appear possible to start with the integral representation of (2.15a) and apply the Laplace's method for large values of b and ξ. It turns out that this approach calls for the usage of the generalized Laplace's method and so-called Faxén integral, both of which are discussed in Olver (1974, p. 331-332). The method is, however, apparently somewhat too crude to serve the present need, for, although this accurately reproduces y^2 term in (2.13), it fails to yield the correct coefficients of y.)

Example 3. <u>Sudden Velocity Discontinuity</u>:

$$g\,(y) = \Delta M \text{ for } y > 0$$

$$\text{and} \quad g\,(y) = 0 \quad \text{for } y < 0.$$

When the steady velocity increases suddenly by ΔM at $y = 0$, (2.11b) yields

$$p' = A\,e^{i(\alpha x - \omega t)}\,e^{\pm i\gamma y}\,\exp\left[\pm i\epsilon\alpha\,(\alpha M_0 - k)\,\frac{\Delta M_0 y}{\gamma}\right] , \tag{2.16a}$$

where

$$A = \exp\left\{\varepsilon\Delta M \frac{\alpha[2\gamma^2 - (\alpha M_0 - k)^2]}{2\gamma^2 (\alpha M_0 - k)}\right\} . \tag{2.16b}$$

Once again, this is the wave transmitted in the region above the interface $y = 0$, upon which the incident wave impinges.

The exact solution for sound transmission through a flow velocity discontinuity was obtained by Miles (1957) and, independently, by Ribner (1957). According to them, the transmitted wave is given by

$$p' = T\, e^{i(\alpha x - \omega t)}\, e^{i\alpha y \tan \phi_2} , \tag{2.17a}$$

where

$$T = \frac{2 \sin (2\,\phi_1)}{\sin (2\,\phi_1) + \sin (2\,\phi_2)} . \tag{2.17b}$$

In the above, ϕ_1, and ϕ_2 are incident and refracted wave angle, respectively, measured from the positive x direction. ϕ_1 is given by

$$\frac{\gamma}{\alpha} = \tan \phi_1 ; \tag{2.17c}$$

and ϕ_2 is related to ϕ_1 by the following law of refraction analogous to the Snell's law:

$$\frac{1}{\cos \phi_1} = \frac{1}{\cos \phi_2} + \varepsilon\,\Delta M. \tag{2.17d}$$

When the amount of velocity jump is small, one can readily prove that (2.17) becomes identical to (2.16), the details of which are shown in Appendix B. Additional numerical comparison between the Miles-Ribner's exact solution and the present approximate solution is shown in Table 1, where the transmission coefficient, A of (2.16b) for the present solution, are compared with T of (2.17b) for various incident angles, ϕ_1; they are given both for positive and negative velocity discontinuity. It is clear that the present formula provides sufficiently accurate values for a wide range of incident angles, except for the vicinity of either critical incidence (corresponding to total reflection for positive velocity jump) or the zero incidence in the case of negative velocity jump. (The local discrepancy near these two incidence angles is not quite unexpected, since both correspond to the turning point of (2.4), about which we shall discuss more in section 3; for $\Delta M > 0$, the turning point occurs in the region where $y > 0$ and for $\Delta M < 0$, in $y < 0$.)

3. *Alternative Representation and Physical Interpretation.*

We now return to the general solution, (2.11), representing waves propagating through arbitrary shear profile; though convenient for obtaining compact expression of such waves, it is not appropriately suitable for exposing physical features. Therefore, in order to pave the way for extracting physical interpretation, we recast it in alternative form. Taking the case of shear distribution where the flow remains uniform up to $y = 0$ and whose solution is given by (2.11b), we apply the integration by parts to its double integral appearing in the argument of exponential and obtain

$$p' = e^{i(\alpha x - \omega t)} e^{\pm i\gamma y}$$

$$\times \exp \left\{ \pm i\varepsilon \frac{\alpha(\alpha M_0 - k)}{\gamma} \int_0^y g(\eta)\, d\eta \right.$$

$$+ \varepsilon \left[\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2\alpha(\alpha M_0 - k) \right]$$

$$\times \left(\left[\frac{1}{(2\gamma)^2} g(y) - \frac{1}{(2\gamma)^4} g''(y) + \frac{1}{(2\gamma)^6} g^{(4)}(y) \text{---} \right] \right.$$

$$\left. \left. \pm\ i \left[\frac{1}{(2\gamma)^3} g'(y) - \frac{1}{(2\gamma)^5} g'''(y) + \frac{1}{(2\gamma)^7} g^{(5)}(y) \text{---} \right] \right) \right\}, \tag{3.1}$$

where, as before, the terms in the form of $\exp(\pm 2 i \gamma y)$ originated from the lower limits of the integral are discarded. The above expression, (3.1), is a series expansion of (2.11b) in terms of γ. Though, in general, it is less expedient for computational purpose, on the other hand it readily surrenders to the physical interpretation, whose term-by-term description will be given below; pursuant to this, we shall compare (3.1) with the other series solution, which we derive by applying the standard WKB method.

3.1 *Term Associated with* $\int_0^y g(\xi)\, d\xi$.

Before we consider the first term in the exponential

$$\pm i\,\varepsilon \frac{\alpha(\alpha M_0 - k)}{\gamma} \int_0^y g(\eta)\, d\eta \,, \tag{3.2}$$

we note that for uniform flow ($\varepsilon = 0$), the surfaces of crests and troughs or of constant phase are given by

$$\alpha x \pm \gamma y = \text{constant}, \tag{3.3a}$$

and the surfaces move as time changes. For the present case of $\varepsilon \neq 0$, the term (3.2), which is the leading correction term for large values of γ or at higher frequency, modifies the above into the following:

$$\alpha x \pm' \gamma y \pm \varepsilon \frac{\alpha(\alpha M_0 - k)}{\gamma} \int_0^y g(\eta)\, d\eta = \text{constant} \,. \tag{3.3b}$$

The integral form indicates that phase change is dependent on the entire flow field over which the wave has traversed; hence, the term embodies cumulative, memory content and might be called "phase memory". (Though more popularly found in electromagnetic propagation, this phase memory recently surfaced in the context of unsteady flow field radiated by an oscillating body (Kurosaka, 1977).) In order to appreciate the meaning contained in (3.3b) more fully, we replace the uniform velocity M_0 appearing in the following definition of γ,

$$\gamma = \sqrt{(\alpha M_0 - k)^2 - \alpha^2} \quad ,$$

by the mean velocity averaged up to the vertical position, y, i.e.,

$$\overline{M} = \frac{1}{y} \int_0^y M(\eta)\, d\eta = M_0 + \frac{\varepsilon}{y} \int_0^y g(\eta)\, d\eta \, .$$

Then, if we define $\overline{\gamma}$ to be $[(\alpha\overline{M} - k)^2 - \alpha^2]^{1/2}$, this becomes, to the present order, as

$$\overline{\gamma} = \gamma + \frac{\varepsilon\alpha(\alpha M_0 - k)}{\gamma} \frac{1}{y} \int_0^y g(\eta)\, d\eta \quad ,$$

and (3.3b) becomes

$$\alpha x \pm \overline{\gamma} y = \text{constant} \, . \tag{3.4}$$

Comparison of the above with the expression for uniform flow, (3.3a), immediately reveals the following: the surfaces of constant phase for waves propagating through shear flow passing a vertical point, y, may be given (as a good approximation for shortwave length in the y direction or at high frequency) by replacing the role of uniform velocity by the mean one cumulatively averaged up to that point, y. From the intimate relationship between the behaviour of the waves at high frequency and the wave front (e.g. Whitham, 1974, p. 236), one is naturally led to expect that the same replacement might take place for wave fronts. In order to obtain the direct confirmation of this, we regard the solution, correct to the present order and expressed in terms of the above γ, as the Fourier transform, $\tilde{F}$, with respect to α and ω: that is

$$e^{\pm \bar{\gamma} y} = \tilde{F} , \tag{3.5a}$$

where

$$\bar{\gamma} = [(\alpha\bar{M} - k)^2 - \alpha^2]^{1/2} . \tag{3.5b}$$

The expression for the wavefront will accrue naturally when one takes the inverse transform so as to derive the transient response, (Miles, 1959, p. 53; Kurosaka, 1977). Taking the inverse transform of (3.5a) with regard to α and ω ,

$$F = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i(\alpha x - \omega t)} e^{\pm i\bar{\gamma} y} d\omega, \tag{3.6a}$$

we obtain

$$F = \mp \frac{\partial}{\partial y} \left\{ 2 \, [c^2(t^2 - \frac{1}{c^2} y^2) - (x - \bar{U}t)^2]^{-1/2} \times H \, [c \, (t^2 - \frac{1}{c^2} y^2)^{1/2} - (x - \bar{U}t)] \right\} , \tag{3.6b}$$

where $\bar{U}$ is the averaged shear velocity defined by

$$\bar{U} = \frac{1}{y} \int_0^y U(y) \, dy , \tag{3.7}$$

and H is the step function; the derivation of this is relegated to Appendix C. From (3.6b), it is clear that the wave front is given by

$$c^2t^2 = y^2 + (x - \bar{U}t)^2. \tag{3.8}$$

Comparison with the one for uniform flow, $c^2t^2 = y^2 + (x - U_\infty t)^2$, readily shows that the wave front propagating through non-uniform flow can easily be obtained by replacing the role of uniform velocity by the cumulatively averaged one, as might be anticipated from 'slowly-varying' notion. We emphasize the point that the above approximate but analytical expression offers a simpler way to construct wavefronts than the general method where it has to be successively constructed from the initial surface by the vectorial addition of acoustic velocity, which is in the direction of normal to the surface, and fluid velocity.

3.2 Term Associated with g(y).

With respect to the term associated with g(y) in the argument of the exponential in (3.1),

$$\varepsilon \left[\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2\alpha\ (\alpha M_0 - k)\right] \frac{1}{(2\gamma)^2}\, g(y), \tag{3.9}$$

we observe that this is wholly real and consequently represents the primary change of wave amplitude due to the presence of shear. It may be of interest to note its following implication: when we express the amplitude corresponding to the above by A, i.e.,

$$A = \exp \left\{\varepsilon \left[\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2\alpha\ (\alpha M_0 - k)\right] \frac{1}{(2\gamma)^2}\, g(y)]\right\},$$

then

$$\frac{dA}{dy} \sim A \cdot g'(y) \ . \tag{3.10}$$

Therefore, to the first order of short wavelength in y direction or at high frequency, the rate of the change of the amplitude is proportional to the product of velocity gradient at that position and the amplitude itself. (It satisfies the equation analogous to the one for radioactive decay.)

3.3 *Terms Associated with derivatives of g(y)*

The remainder of the terms in the argument of exponential in (3.1),

$$\varepsilon \left[\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2\alpha\ (\alpha M_0 - k)\right]$$

$$\times \left(\left[-\frac{1}{(2\gamma)^4} g''(y) + \frac{1}{(2\gamma)^6} g^{(4)}(y) \text{---}\right]\right.$$

$$\left.\pm i \left[\frac{1}{(2\gamma)^3} g'(y) - \frac{1}{(2\gamma)^5} g'''(y) + \frac{1}{(2\gamma)^7} g^{(5)} y \text{---}\right]\right), \tag{3.11}$$

obviously provides the higher-order correction to both amplitude and phase. If we consider a shear profile where at some point $y = y_1$, all of its derivatives vanish (see Figure 1 again), i.e., $g'(y_1) = g''(y_1) = \text{---} = 0$, the wave transmitting through y does not perceive any local change in steady velocity; and therefore, both the amplitude and phase should remain unaltered while traversing past y_1. This fact is indeed accounted for in the present solution, since in such an instance, (3.11) vanishes completely while (3.2) and (3.9) retain the same value just before and just after leaving y_1.

3.4 *Comparison with the Series Solution Obtainable by the Standard WKB Method.*

We now compare the alternative series representation, (3.1), with the other series solution derivable from the standard WKB formula. For this, we transform the Pridmore-Brown's equation (2.4), into the Schrödinger's equation for one-dimensional quantum-mechanical motion where the standard form of WKB solution is readily available. First, we remove, in (2.4), the term in first derivative by the following standard transformation:

$$p = zY, \tag{3.12a}$$

where

$$z = \alpha M(y) - k \quad . \tag{3.12b}$$

This, together with (2.6), changes (2.4) into

$$Y'' + [\gamma^2 - \varepsilon U(y)]\, Y = 0 \ , \tag{3.13a}$$

where

$$U(y) = -2\,\alpha(\alpha M_0 - k)\, g\,(y) - \frac{\alpha\, g''(y)}{\alpha M_0 - k + \alpha\varepsilon g(y)}$$

$$+ \varepsilon[-\alpha^2\, g\,(y) + \frac{2\alpha^2\,(g'(y))^2}{(\alpha M_0 - k + \alpha\,\varepsilon\, g(y))^2}\,] \ . \tag{3.13b}$$

This is the Schrödinger's equation where U (y) corresponds to potential and so far no approximation has been made. The standard WKB formula (e.g. Morse and Feshbach, 1953, p. 1092 et seq.) is applied here and the approximate solution of (3.13a) is immediately found to be

$$Y \sim \exp\left[\pm i\int_0^y q\,dy - \frac{1}{2}\log q\right] , \tag{3.14a}$$

where $q = \sqrt{\gamma^2 - \varepsilon U(y)}$. If we now assume $\varepsilon \ll 1$, (3.14a) becomes, to the order of ε, as

$$Y \sim \exp\left\{ \pm i\gamma y \pm i\,\frac{\alpha\varepsilon(\alpha M_0 - k)}{\gamma}\int_0^y g(\eta)\,d\eta \right.$$

$$- \varepsilon \cdot 2\alpha\,(\alpha M_0 - k)\,\frac{1}{(2\gamma)^2}\,g\,(y)$$

$$\pm \varepsilon\,\frac{4\alpha\,\gamma^2}{\alpha M_0 - k}\;\frac{i}{(2\gamma)^3}\;g'\,(y)$$

$$\left. - \varepsilon\,\frac{4\alpha\gamma^2}{\alpha M_0 - k}\;\frac{1}{(2\gamma)^4}\,g''\,(y) \right\} \tag{3.15}$$

To this, we restore the factor z of (3.12b), which we recognize to be equivalent, to the order of the present approximation, to

$$z \sim (\alpha M_0 - k)\,\exp\left[\frac{\alpha\varepsilon g(y)}{\alpha M_0 - k}\right] .$$

Multiplying the result by $e^{i(\alpha x - \omega t)}$, we obtain p' as

$$p' = e^{i(\alpha x - \omega t)} e^{\pm i\gamma y}$$

$$\times \exp \left\{ \pm \frac{i\varepsilon\alpha(\alpha M_0 - k)}{\gamma} \int_0^y g(\eta)\, d\eta \right.$$

$$+ \varepsilon \left[\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2\alpha (\alpha M_0 - k)\right] \frac{1}{(2\gamma)^2} g(y)$$

$$\left. + \varepsilon \frac{4\alpha\gamma^2}{\alpha M_0 - k} \left[\pm \frac{i}{(2\gamma)^3} g'(y) - \frac{1}{(2\gamma)^4} g''(y)\right] \right\}. \tag{3.16}$$

We take a special note that the argument of the above exponential function consists of finite terms, in contrast to the infinite series appeared in (3.1). Further scrutiny between the above WKB solution, (3.16), and our series representation, (3.1), reveals that the terms associated with $\int_0^y g(\eta)\, d\eta$ and g (y) agree; the other terms associated with g'(y) and g''(y) are not quite the same; g'''(y) and other higher derivatives are completely missing in the WKB solution. One can show that whereas the present series solution (3.1), or more precisely its y-dependent portion corresponding to p(y) of (2.3), satisfies the original differential equation (2.4) to the order of ε, the WKB solution (3.16) fails, not surprisingly, to do so even to 0 (ε). Indeed, it is straightforward to prove that if the iteration procedure described by Morse and Feshbach (ibid.), which leads to the standard form of WKB solution of (3.14a), had been repeatedly continued, the resulting higher order terms would, to the order of ε, contribute to recover the entire terms missing in (3.16). Consequently, the present infinite series representation, (3.1), does offer the terms of higher order which would be lacking in the WKB solution; the primary expression obtained in the preceding section 2, (2.11), amounts to the summation of these terms into a concise form convenient for arbitrary shear profile. Observe also that without the assumption of $\varepsilon \ll 1$, the solution (3.14a) directly derived from the standard WKB formula would not be amenable to the physical interpretation described previously.

Before concluding this section, we wish to dissipate a potential source of misconception about summation formula, (2.11): on the face of it,this solution might appear to be valid even in the neighborhood of $\gamma = 0$, which is a singular point in the series representation of (3.1) or WKB solution (3.16). (The turning point of (3.14b) is $q = \sqrt{\gamma^2 - \varepsilon U(y)} = 0$ but the expansion for $\varepsilon \ll 1$ has transferred the singular point to $\gamma = 0$, instead). However, except for the case where γ happens to be exactly equal to zero, this is not so; for once the integration for (2.11) is carried out, the terms like $1/\gamma$ reappears. For example, one can immediately recognize this from (2.12), which -- although of compact form than the one that would be obtained from the infinite series representation corresponding to (3.1) -- is still singular; in example 3 of section 2, we also have observed the similar behaviour. To avoid undue complication, we defer any attempt to improve this limitation obviously associated with the regular perturbation scheme to elsewhere, but in the next section we shall observe that even in the present form, the expression for $\gamma = 0$, (2.11c), will be found to be valid and of physical significance in ducted acoustic waves.

4. *Application to Ducted Shear Flow Problem.*

Let us turn our attention toward the problem of acoustic wave propagation through shear flow in ducts; here we shall find that the results of the previous sections obtained for unbounded media can be applied to channel flow with minor modification -- in particular, without any change in the case of the fundamental mode.

The coordinate system is shown in Figure 3 where the duct walls are placed at $y = \pm\ \ell$. Assuming that the walls are rigid or hard, the boundary conditions on the walls are given by

$$\frac{\partial p'}{\partial y} = 0 \qquad \text{at } y = \pm\ \ell \ . \tag{4.1}$$

In order to ascribe definite meaning, the steady, base velocity, M_0, of (2.6), is now taken, without loss of generality, to be the one averaged across the duct, i.e.,

$$M_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} M(y)\, dy, \tag{4.2}$$

and accordingly

$$\int_{-\ell}^{\ell} g(y)\, dy = 0 \ . \tag{4.3}$$

If the flow inside the duct is uniform, it is of course elementary to show that the wave constant in the y direction, γ of (2.8b), takes the following eigenvalues determined from the wall boundary condition:

$$\gamma_n = \frac{n\pi}{\ell} \text{ where } n = 0, \pm 1, \pm 2, --- \tag{4.4}$$

and correspondingly inverting (2.8b), α is given by

$$\alpha_n = \frac{M_0 k \pm \sqrt{k^2 - (1 - M_0^2)\left(\frac{n\pi}{\ell}\right)^2}}{M_0^2 - 1} \text{ for } n = 0, \pm 1, \pm 2, ---, \tag{4.5}$$

where we assign suffix n to α. In the above, the positive sign corresponds to the waves propagating in the upstream direction and the negative sign to the one in the downstream direction.

In the particular case of the fundamental mode of plane wave where $n = 0$, γ for uniform flow becomes zero from (4.4). If we return to the present subject of sheared flow, then for $\gamma = 0$ the solution has previously been obtained as (2.11c) and, when one takes the lower limit of the integrals to be equal to $-\ell$, we obtain the following:

$$p' = e^{i(\alpha x - \omega t)} \exp[-2\varepsilon\alpha(\alpha M_0 - k) \int_{-\ell}^{y} d\eta \int_{-\ell}^{\eta} g(\xi)\, d\xi] . \tag{4.6}$$

Upon differentiation with respect to y, it is immediately apparent from (4.3) that this does satisfy the boundary condition, (4.1); consequently this is in fact the lowest mode for waves propagating in duct; the underlying physical reason why in this case the free-space solution also becomes a ducted wave will be given subsequently. For now, from the expression of α corresponding to $n = 0$ in (4.5), one obtains

$$p_0' = e^{-i\omega t} \exp\left(\pm i k \frac{x}{M_0 \mp 1}\right) \exp\left[\mp \varepsilon \frac{2k^2\ell^2}{(1 \mp M_0)^2} \int_{-1}^{y/\ell} d \int_{-1}^{\eta} g(\xi)\, d\left(\frac{\xi}{\ell}\right)\right], \tag{4.7}$$

where suffix 0 attached to p' denotes the fundamental mode and upper signs inside the exponential correspond to the upstream propagation and the lower signs to the downstream propagation. During the course of the present work, it was communicated by Dr. Savkar that the essentially same expression was previously obtained by him in hitherto unpublished memo (1972), which is now being prepared for publication, by using an entirely different method where a technique akin to PLK method was employed along with certain orthogonality condition for various duct modes; on the other hand, as we have just seen, the present derivation is directly obtained from the free-space solution and not dependent upon the latter requirement.

Of interest here is the comparison of the present expression, (4.7), with Shankar's solution (1971) for initial value problem. He posed and solved a problem where a harmonic plane wave disturbance is suddenly switched on at x = 0 in an initially quiescent medium with slightly sheared flow. For the initial condition

$$p'(0,y,t) = A\, e^{-i\omega t} H(t)$$

where A is constant (plane wave) and H is the step function, his solution for large time is given, according to the present definition of M_0, to be

$$\frac{p'}{A}(x,y,t) = \exp\left\{ -i\omega \left(t - \frac{x}{c(1+M_0)}\right)\right\}$$

$$+ \varepsilon \sum_{n=1}^{\infty} \frac{4}{\pi^2 (1+M_0)^2} \frac{a_n}{n^2} b\, k^2 \cos\left(\frac{n\pi y^*}{b}\right) e^{-i\omega t}$$

$$\times \left[\exp \left\{ \frac{x}{1 - M_0^2} \left(- i\omega \frac{M_0}{c} + i \left[\frac{\omega^2}{c^2} - \frac{n^2\pi^2}{b^2} (1 - M_0^2)\right]^{\frac{1}{2}}\right) \right\} - \exp \left\{ i \frac{\omega x}{c(1 + M_0)} \right\} \right] + 0 (\varepsilon^2) , \qquad (4.8)$$

where $b = 2\ell$, $y^* = y + \ell$ and a_n is the Fourier coefficient of shear profile $g(y) = M^{(1)} (y^*)$ defined by

$$a_n = \int_0^b M^{(1)} (\xi) \cos \left(\frac{n\pi\xi}{b}\right) d\xi .$$

We combine the first term of (4.8) and the terms of the same exponential form appearing inside of the series, i.e.,

$$\exp \left\{ - i\omega \left(t - \frac{x}{c(1 + M_0)}\right) \right\}$$

$$- \varepsilon \sum_{n=1}^{\infty} \frac{4}{\pi^2(1 + M_0)^2} \frac{a_n}{n^2} b k^2 \cos \left(\frac{n\pi y^*}{b}\right) e^{-i\omega t} \exp \left[\frac{i\omega x}{c(1 + M_0)} \right] . \qquad (4.9)$$

Upon differentiating twice with respect to y, it is straightforward to show that, from the Fourier series formula, the sum is indeed equal to the present result of (4.7) for downstream propagation, correct to the order of ε. (The other terms in the series of (4.8) obviously correspond to higher modes associated with the particular choice of the initial condition which is assumed to be a plane wave in Shankar's paper.)

The present expression for the fundamental modes, (4.7), reveals explicitly various features of the shear flow effect. For example, for a parabolic shear flow satisfying the requirement of (4.3), i.e. $g(y) = 1/3 - (y/\ell)^2$

$$p'_0 \sim e^{-i\omega t} \exp\left(\pm i\, k \frac{x}{M_0 \mp 1}\right) \exp\left\{ \pm \varepsilon \frac{k^2 \ell^2}{6(1 \mp M_0)^2} \left(\frac{y}{\ell}\right)^2 \left[\left(\frac{y}{\ell}\right)^2 - 2\right] \right\} \tag{4.10}$$

where the upper and lower signs correspond to upstream and downstream propagation, respectively. Not only the role of amount of shear (ε), frequency (k) and mean Mach number is transparently obvious, but also, when we note that $[(y/\ell)^2 - 2]$ always remains negative across ducts, it is plainly visible that shear layers refract the fundamental mode wave toward the wall for downstream propagation and away from the wall for upstream propagation. The effect has been known from the various results based upon numerically computational scheme (see Nayfeh et al. for full references.); the magnified effect of refraction at higher frequency and greater shear has likewise been recognized. We feel, however, that the present analytical expression displays these effects, perhaps more conspicuously and compactly. The numerical comparison of (4.7) with the results obtained by wholly computational scheme (e.g. Mungur and Gladwell (1969)) for prescribed shear profile such as linear distribution shows the satisfactory agreement, even when the values of $\varepsilon k^2 \ell^2$ appearing inside the exponential of (4.7) become moderate;

therefore in this paper, in the interest of retaining simplicity, we shall not attempt to improve the, formal limitation at large values of $\epsilon k^2 \ell^2$ through the obvious application of singular perturbation procedure.

For higher modes in ducted flow, the free-space solution obtained previously, requires some modification. Since, for uniform flow, the wave constants in the x direction, α_n of (4.5), are dependent upon the uniform velocity, M_0, we expect that for sheared flow, it is dependent on shear; accordingly, we express

$$\alpha_n^* = \alpha_n + \epsilon \chi_n \qquad (4.11)$$

where χ_n is the correction factor due to shear. In general, χ_n has to be evaluated from the boundary condition. However, as in the previous case of fundamental mode of n = 0, which in the present notation corresponds to $\chi_0 = 0$, one can show that even for higher mode χ_n vanishes for a certain shear profile; that is, for odd component of velocity distribution in the y direction, $\chi_n = 0$ or the wave number in the x direction remains unaffected by the presence of shear. This fact appears to be first recognized by Savkar in the aforementioned unpublished memo based upon the duct-mode analysis; in what is to follow, we shall instead show this from physical consideration based upon free-space solution.

The sound field in ducts may in general by regarded in the following, two different ways: in the first interpretation, it can be visualized as the superposition of two sets of free-space waves, one propagating in the positive y direction and the other in the negative y direction; alternatively, it can be viewed that these waves are continually reflected back and forth between two rigid walls. Let us at first adopt the first viewpoint and consider these two waves generated by the initial pressure specified at $x = 0$; for the n-th mode in uniform flow, the wavelength in the y direction is equal to $2\ell/n$, (shown in Figure 4(a) for $n = 1$) with the pressure oscillating at frequency, ω. The wave constant, α, in the x direction is proportional to the number of crests (or troughs) in unit length in that direction. Let us pose the following question: what is the change in the number of crests due to the non-uniform velocity distribution? In order to answer this, we extend the shear profile, originally defined inside the duct, to its outside in periodic, repeated manner in order to cover the entire unbounded space, as shown in Figure 4(b). (For odd component of velocity distribution, the profile in the neighborhood of $y = \pm \ell, \pm 2\ell$, --- may, if necessary, be envisioned to be slightly modified so that the velocity distribution becomes continuously smooth; this is to avoid the unnecessary complication arising from the refraction which would otherwise occur at the interface between velocity discontinuity.) We note that, for any modes, the lines of constant phase in the free space may be given by the curves along which the imaginary part of (3.1) remains constant, i.e.,

$$\alpha x \pm \gamma y \pm \varepsilon \frac{\alpha(\alpha M_0 - k)}{\gamma} \int_{-\infty}^{y} g(\eta)\, d\eta$$

$$\pm \varepsilon\left[\frac{4\alpha\gamma^2}{\alpha M_0 - k} - 2\alpha(\alpha M_0 - k)\right]\left[\frac{1}{(2\gamma)^3} g'(y) - \frac{1}{(2\gamma)^5} g'''(y) + \frac{1}{(2\gamma)^7} \times g^{(5)}(y) \text{ ---}\right] = \text{constant.} \tag{4.12}$$

We observe that because of the periodic extension of the shear profile throughout the entire space and with the aid of (4.3), the integral involved may be reduced to the following:

$$\int_{-\infty}^{y} g(\eta)\, d\eta = \int_{-\ell}^{y} g(\eta)\, d\eta\,.$$

We at once recognize that at the points corresponding to walls, $y = \pm\, \ell$, the integral vanishes. Furthermore, for odd component of velocity distribution, the odd derivatives take the same value at these points: $g'(\ell) = g'(-\ell)$, $g'''(\ell) = g'''(-\ell)$ and so on. Consequently, these promptly lead to the following: in (4.12), all the terms associated with ε or shear take the same value at the points corresponding to walls. Therefore lines of constant phase appear, as shown in Figure (4c), where, for the sake of illustration, the lines are adjusted so that the lines of constant phase for uniform and for non-uniform flow cross each other initially at $y = -\,\ell$. When we now switch to the second viewpoint of regarding the duct wave system as being reflected repeatedly back and forth between the walls, one can immediately recognize that for odd component of velocity distribution, the number of wave crests remains unaffected by the presence of shear; and therefore, in such a case χ_n of (4.11) vanishes. (It is apparent that this does not hold for even component of velocity.) A similar argument can also be used to provide the physical explanation of $\chi_0 = 0$ for $n = 0$ or the lowest mode, for in the corresponding free-space solution of (2.11c), the shear effect appears only as the real part of the argument of the exponential function, leaving the phase surface unaltered.

For general, arbitrary shear profile, the solution for higher mode may be obtained by enforcing the boundary condition (4.1), upon the linear combination of the solutions, which can be obtained, in a manner similar to the one used for free space; in the course of this we naturally have to keep in mind the correction of α_n as expressed in (4.11). The final solution for the n-th mode is as follows:

$$p_n' = \frac{1}{2}(\exp z_n + \exp \bar{z}_n)\, e^{i(\alpha_n + \chi_n) x - i\omega t}, \quad n = 1, 2 --- \qquad (4.13a)$$

where $\bar{z}_n$ is the complex conjugate of z_n defined by

$$z_n = i\gamma_n y + i\,\varepsilon \frac{2\alpha_n \gamma_n}{\alpha_n M_0 - k} \int_{-\ell}^{y} g(\eta)\, d\eta$$

$$+ \varepsilon \left[\frac{4\alpha_n \gamma_n^2}{\alpha_n M_0 - k} - 2\alpha_n (\alpha_n M_0 - k)\right]$$

$$\times \left[\int_{-\ell}^{y} d\eta \int_{-\ell}^{\eta} g(\xi)\, e^{-2i\gamma_n(\eta - \xi)} d\xi\right.$$

$$\left. + \frac{i}{(2\gamma_n)^3} g'(\ell) - \frac{i}{(2\gamma_n)^5} g'''(\ell) + --\right]$$

$$+ \frac{i\varepsilon}{\gamma_n} \chi_n (y - \ell) [M_0(\alpha_n M_0 - k) - \alpha_n], \qquad (4.13b)$$

and where

$$\chi_n = \frac{1}{\ell[M_0(\alpha_n M_0 - k) - \alpha_n]} \left[\frac{\alpha_n \gamma_n^2}{\alpha_n M_0 - k} - \frac{1}{2}\alpha_n(\alpha_n M_0 - k)\right]$$

$$\times \left\{ \frac{1}{(2\gamma_n)^2}[g'(\ell) - g'(-\ell)] - \frac{1}{(2\gamma_n)^4}[g'''(\ell) - g'''(-\ell)] + \text{---} \right\}. \tag{4.13c}$$

It is quite apparent that for odd velocity distribution, $\chi_n = 0$, as we expected before. Needless to say, in the above, as in the free space solution, the parasitic terms of $e^{\pm 2i\gamma_n y}$, which may arise in the course of carrying out the integration for specific velocity distribution, have again to be discarded.

Although the above results are obtained for hard-walled ducts, one can obtain, in similar manner, the expressions for the ones with soft or treated wall as well.

5. *Concluding Remark*.

In summary, it has been our intent to demonstrate that the present method provides a simple and effective means to obtain a solution for waves propagating through aribtrarily sheared flow. Our main results, accrued in closed form for both unbounded media (2.11) and ducted flow (4.13), involve only quadratures of any given shear profile; as special cases, they are found to embrace other known solutions and besides, they can be rendered into forms particularly convenient for the extraction of various physical features. The method appears to be easily applicable to other problems involving disturbance propagation through non-uniform media. (For example, the method is now being found to be particularly instrumental in assessing the effect of spanwise flow variation upon flutter in turbomachinery bladings.)

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APPENDIX A

We shall show that, under the assumption of small velocity gradient, (2.14) expressed in terms of the parabolic cylinder function becomes equal to (2.13). For this we make use of the following formula of the parabolic cylinder function, due to Darwin (1949):

$$U(b, \pm \xi) \sim \bar{Y}^{-\frac{1}{2}} \exp\left\{ \pm i \left(\phi + \frac{g}{\bar{Y}^3} + \frac{\ell}{\bar{Y}^9} + \frac{n}{\bar{Y}^{15}}\right) + \left(\frac{h}{\bar{Y}^6} + \frac{m}{\bar{Y}^{12}} + \text{---}\right)\right\} \qquad \text{(A-1a)}$$

where
$$\bar{Y} = (x^2 - 4a)^{\frac{1}{2}}, \qquad \text{(A-1b)}$$

$$\phi = \frac{1}{4} x \bar{Y} - a \log \frac{x + \bar{Y}}{2\sqrt{a}}, \qquad \text{(A-1c)}$$

$$a = -ib, \qquad \text{(A-1d)}$$

$$x = i^{\frac{1}{2}} \xi, \qquad \text{(A-1e)}$$

and the expression for g, ℓ,. . . etc. are given in Darwin's paper. Since in the present case of Example 2 of Section 2, g(y) of (2.6) is equal to y, it follows

$$\frac{dM}{dy} = \varepsilon,$$

and from (2.15b), (A-1d) and (A-1e) become

$$a = -\frac{\alpha}{2\varepsilon}, \qquad \text{(A-2a)}$$

$$x = i \left(\frac{2}{\alpha\varepsilon}\right)^{\frac{1}{2}} (k - \alpha M_0 - \alpha\varepsilon y) \ . \qquad \text{(A-2b)}$$

Into (A-1c), we substitute the above and, after expanding and discarding a certain constant which is disposable, we obtain

$$\phi = \gamma y - \frac{1}{2} \ \alpha\varepsilon y^2 \ \frac{k - \alpha M_0}{\gamma} + 0 \ (\varepsilon^2) \ , \qquad \text{(A-3)}$$

where γ is defined in (2.8b). Likewise

$$\bar{Y} = i \ \gamma\left(\frac{2}{\alpha\varepsilon}\right)^{\frac{1}{2}} [1 - \frac{\alpha\varepsilon y(k - \ M_0)}{\gamma^2} - \frac{1}{2} \alpha^4 \varepsilon^2 y^2 \frac{1}{\gamma^4} + ---] \ . \qquad \text{(A-4)}$$

By substituting (A-3) and (A-4) into (A-1a), it is found that the terms associated with g, , . . . etc. become negligible and we obtain

$$U(b, \pm \xi) \sim \exp \ (\pm \ i\gamma y) \ \exp \ [\mp \frac{1}{2} \ i\alpha\varepsilon y^2 \ \frac{k - \alpha M_0}{\gamma} - \frac{1}{2} \ \alpha\varepsilon y \ \frac{\alpha M_0 - k}{\gamma}]. \qquad \text{(A-5)}$$

Upon substituting the above into (2.14) and noting that the factor $b\xi$ associated with the second term within the bracket can be expressed as

$$b\xi = \exp \ [\log \ (b\xi)]$$

$$= \frac{\alpha}{2i\varepsilon} \left(\frac{2i}{\alpha\varepsilon}\right)^{\frac{1}{2}} (k - \alpha M_0) \ \exp \left\{ \log \ (1 - \frac{\alpha\varepsilon y}{k - \alpha M_0}) \right\}$$

$$= \frac{\alpha}{2i\varepsilon} \left(\frac{2i}{\alpha\varepsilon}\right)^{\frac{1}{2}} (k - \alpha M_0) \left\{ \exp \ [- \frac{\alpha\varepsilon y}{k - \alpha M_0} + 0 \ (\varepsilon^2)] \right\} ,$$

it follows that, due to this factor, the second term dominates for small ε; from this, (2.14) takes the same form as (2.13).

APPENDIX B

Here, we shall find that for small velocity discontinuity, (2.17) becomes (2.16). From (2.17d), we obtain

$$\phi_2 = \phi_1 - \varepsilon\Delta M \cdot \frac{\cos^2\phi_1}{\sin\phi_1} . \tag{B-1}$$

From (2.17c) and the definition of γ given in (2.8b), $\alpha \tan \phi_2$ appearing in (2.17a) becomes

$$\alpha \tan\phi_2 = \gamma \pm \varepsilon\alpha(\alpha M_0 - k)\frac{\Delta M}{\gamma} . \tag{B-2}$$

With the aid of (B-1), T of (2.17b) becomes

$$T = 1 + \varepsilon\Delta M \frac{\alpha[2\gamma^2 - (\alpha M_0 - k)^2]}{2\gamma^2(\alpha M_0 - k)} , \tag{B-3}$$

which, to the order of ε , is equal to (2.16b). Thus, (2.17) becomes (2.16), to the present order of approximation.

APPENDIX C

We prove that (3.6a) is identical to (3.6b). For this, we rewrite (3.6a) as

$$F = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} \; I(\alpha) \; d\alpha \; , \qquad \text{(C-1a)}$$

where

$$I\ (\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \; e^{\pm i\overline{\gamma} y} \; d\omega \; . \qquad \text{(C-1b)}$$

From the definition of $\overline{\gamma}$ given by (3.5b), (C-1b) becomes

$$I(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \exp\left\{ \pm i \frac{y}{c} \left[(\omega - \alpha c(\overline{M} - 1))(\omega - \alpha c(\overline{M} + 1)) \right]^{\frac{1}{2}} \right\} d\omega. \qquad \text{(C-2)}$$

Now from Campbell and Foster (1961, p. 109, No. 860.0), one notes

$$\int_{-\infty}^{\infty} [(\omega - c(\overline{M} + 1))(\omega - c(\overline{M} - 1))]^{-\frac{1}{2}} \; \overline{e}^{\; i\omega t}$$

$$\times \exp\left\{ \pm i \frac{y}{c} \left[(\omega - \; c(\overline{M} - 1))(\omega - \alpha c(\overline{M} + 1)) \right]^{\frac{1}{2}} \right\} d\omega$$

$$= - 2\pi i \;\; H(t \mp \frac{1}{c} y) \exp\left(-i \; c\alpha \; \overline{M} t\right) J_0 \left[c\alpha \; (t^2 - \frac{1}{c^2} y^2)^{\frac{1}{2}} \right] , \qquad \text{(C-3)}$$

where H is the step function and J_0 is the Bessel function. Differentiating both sides with respect to y, we obtain

$$I(\alpha) = \mp \sqrt{2\pi} \cdot c \exp(-ic\alpha \bar{M} t)$$

$$\times \frac{\partial}{\partial y} \left\{ H(t \mp \frac{1}{c} y) \, J_0 \left[c\alpha \left(t^2 - \frac{1}{c^2} y^2\right)^{\frac{1}{2}} \right] \right\} . \qquad \text{(C-4)}$$

Into (C-1a), we substitute (C-4); with the aid of the following identity

$$\int_{-\infty}^{\infty} J_0(b\xi) \, e^{i\xi x} \, d\xi = 2(b^2 - x^2)^{-\frac{1}{2}} H(b - |x|),$$

which is found in Erdélyi, et al (1954, vol. 1, p. 43), and with the definition of $\bar{U}$ given in (3.7), we obtain (3.6b).

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TABLE 1

COMPARISON OF TRANSMISSION COEFFICIENT

$\Delta M = 0.05$

incidence angle ϕ_1 (degrees)	transmission coefficient T (Miles-Ribner)	transmission coefficient A (present)
90	1.000	1.000
80	.996	.996
70	.992	.993
60	.992	.992
50	.996	.995
40	1.011	1.008
30	1.059	1.044
20	1.324	1.166
17.75*	2.000	1.232

*critical incidence

$\Delta M = -0.05$

incidence angle ϕ_1 (degrees)	transmission coefficient T (Miles-Ribner)	transmission coefficient A (present)
90	1.000	1.000
80	1.004	1.004
70	1.007	1.007
60	1.008	1.008
50	1.005	1.005
40	.994	.992
30	.967	.958
20	.896	.857
10	.691	.464
0	0	0

Figure Captions

Figure 1. Definition Sketch.

Figure 2. Examples of Shear Profile.

Figure 3. Coordinate in Ducts.

Figure 4. Lines of Constant Phase.

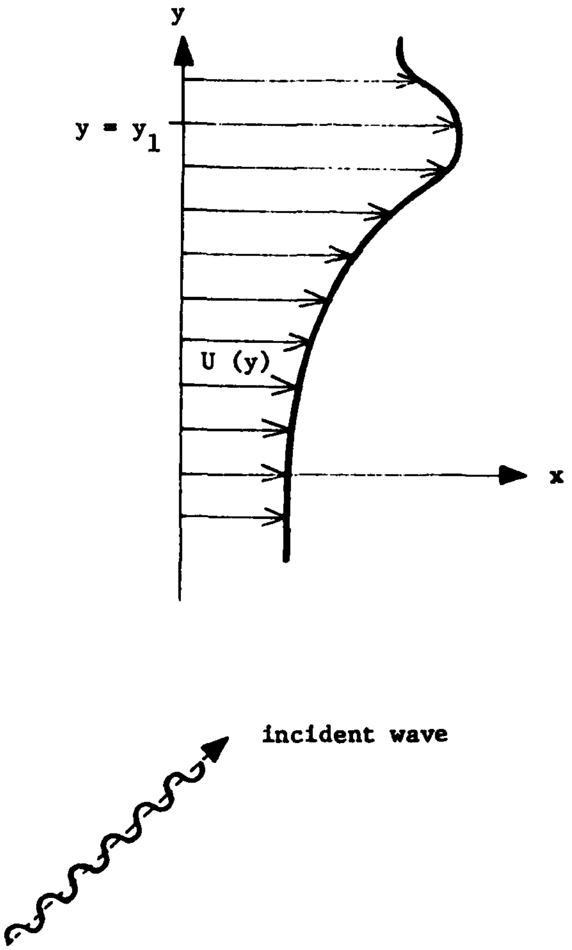

Figure 1. Definition Sketch.

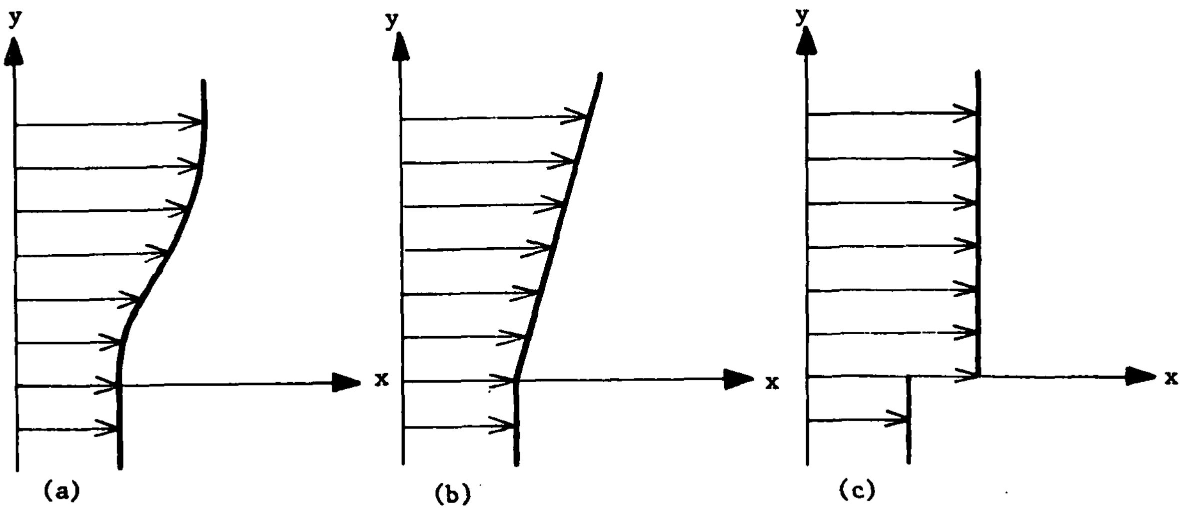

Figure 2. Examples of Shear Profile.

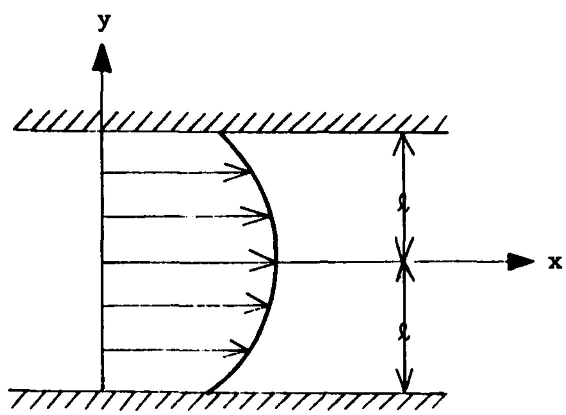

Figure 3. Coordinate in Ducts.

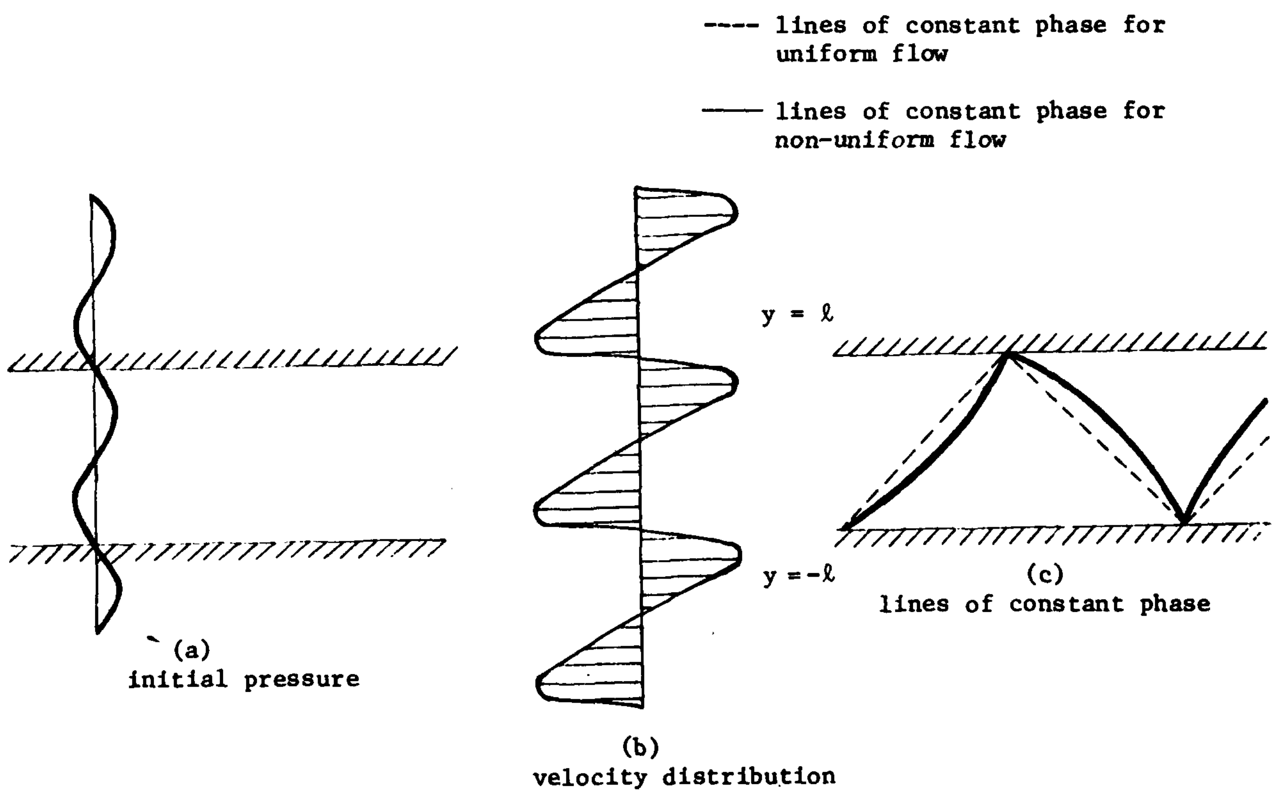

Figure 4. Lines of Constant Phase.

APPENDIX 7

"Three-Dimensional, Refraction Effect upon Unsteady Oscillating Airfoils in Supersonic Flow"

Three-Dimensional, Refraction Effect Upon Unsteady Oscillating Airfoils in Supersonic Flow

*M. Kurosaka**

The University of Tennessee Space Institute
Tullahoma, Tennessee 37388

I. Introduction.

On the subject of three-dimensional effect upon the supersonic flutter in aircraft engines, Ref. 1 identified the following, two key issues which could significantly modify the results based upon two-dimensional approximation: (1) reflection of waves from the encasing walls and (2) refraction arising from the propagation of disturbances through steady flow with strong velocity gradient in the radial direction existing in turbomachines. Of these two, Ref. 1 specifically restricted itself to the first problem or its variant of practical significance where the sound absorbent material is installed upon the wall surface.

In the present paper, we turn our attention towards the second one and examine to what extent the unsteady pressure on the airfoils will be affected by the non-uniformity of surrounding, steady flow. As is

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*Associate Professor, Mechanical and Aerospace Engineering.

well known, the sheared velocity distribution always refracts the sound waves; for example, according to a geometrical acoustics picture convenient for high frequencies, the rays of sound will be bent while propagating through any stratified media. The refraction of disturbances traversing a flow field with steep radial velocity gradient present in turbomachinery environment may considerably alter the unsteady pressure signals, when compared to analysis based upon two-dimensional approximations. In order to focus our attention solely on the refractive aspect of the problem, we shall attempt to extricate ourselves from the other unneccessary complications. Therefore we pose and study a model problem, which seems to capture the central nature of the problem, and consider an isolated oscillating airfoil which is placed in a supersonic duct with non-uniform velocity. The steady, sheared velocity is in the postive x direction (Figure 1) and is function of y only, i.e. $U(y)$. The walls of the walls located at $y = \ell$ and $y = -\ell$ are rigid and the airfoil is harmonically oscillating with small amplitude in the z direction.

Our aim is to compare the effect of sheared flow upon the unsteady pressure acting on the surface of the airfoil with the uniform flow. In analyzing the problem, we rely exclusively on the results of Ref. 2, where a simplified representation of unsteady disturbance propagating through non-uniform flow is obtained.

II. *Formulation.*

The linearized governing equations can be written

$$\left.\begin{aligned}
&\frac{1}{a^2}\left(\frac{\partial p'}{\partial t} + U(y)\frac{\partial p'}{\partial x}\right) + \rho\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) = 0\ ,\\
&\frac{\partial u'}{\partial t} + U(y)\frac{\partial u'}{\partial y} + v'\frac{dU(y)}{dy} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}\ ,\\
&\frac{\partial v'}{\partial t} + U(y)\frac{\partial v'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial y}\ ,\\
&\frac{\partial w'}{\partial t} + U(y)\frac{\partial w'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial z}
\end{aligned}\right\} \tag{1}$$

where ρ and a are constant ambient density and acoustic speed, respectively and primes denote perturbed quantities with conventionally defined meanings. As usual, in the above we write all the perturbed quantities as the product of two factors, one for the spatially dependent amplitude and the other for time-dependent factor such as $p' = \bar{p}'\ e^{i\omega t}$ and obtain

$$\left.\begin{aligned}
&\frac{1}{a^2}\left(i\omega\bar{p}' + U(y)\frac{\partial\bar{p}'}{\partial x}\right) + \rho\left(\frac{\partial\bar{u}'}{\partial x} + \frac{\partial\bar{v}'}{\partial y} + \frac{\partial\bar{w}'}{\partial z}\right) = 0,\\
&i\omega\,\bar{u}' + U(y)\frac{\partial\bar{u}'}{\partial x} + v'\frac{dU(y)}{dy} = -\frac{1}{\rho}\frac{\partial\bar{p}'}{\partial x}\ ,\\
&i\omega\,\bar{v}' + U(y)\frac{\partial\bar{v}'}{\partial x} = -\frac{1}{\rho}\frac{\partial\bar{p}'}{\partial y}\ ,\\
&i\omega\,\bar{w}' + U(y)\frac{\partial\bar{w}'}{\partial x} = -\frac{1}{\rho}\frac{\partial\bar{p}'}{\partial z}\ .
\end{aligned}\right\} \tag{2}$$

The boundary condition on walls is that the normal velocities are equal to zero, i.e.,

$$v'(x, y = \pm\ell\ , z) = 0\ . \tag{3}$$

On the airfoil, the velocity normal to it is prescribed by the given movement of the airfoil and if its amplitude is equal to W(x,y), then

$$w'(x, y,\ z = 0) = W\ (x, y)\ . \tag{4}$$

Upstream of the initial Mach cone, the flow is quiescent. Because of the supersonic nature of the flow, the flow fields above and below the airfoil is unrelated and hereafter we consider only the flow above the airfoil, $z > 0$.

III. *Solution*.

In order to obtain a solution of (2), we first eliminate the x dependence by taking the Fourier transform in that direction; next, the application of the Fourier cosine transform in the z direction removes the z dependence and the resultant equation involving only y is solved with the aid of the results obtained in Ref. 2; the description of these successive steps will be given below.

A. *Fourier transform in the x direction*.

We take the Fourier transform of (2) defined, for example, by

$$\tilde{p} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{p'} \, e^{i\alpha x} \, dx,$$

and eliminating the resulting $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$, one obtains the following equations in terms of $\tilde{p}$:

$$\frac{\partial^2 \tilde{p}}{\partial y^2} + \frac{\partial^2 \tilde{p}}{\partial z^2} - \frac{2\alpha}{\alpha M(y) - k} \frac{dM(y)}{dy} \frac{\partial \tilde{p}}{\partial y} + [(\alpha M(y) - k)^2 - \alpha^2] \, \tilde{p} = 0 \, , \tag{5a}$$

where $k = \frac{\omega}{a}$ and with the boundary conditions

$$\left.\frac{\partial \tilde{p}}{\partial y}\right|_{y = \pm \ell} = 0 \, , \tag{5b}$$

$$\left.\frac{\partial \tilde{p}}{\partial z}\right|_{z = 0} = -i\rho(\omega - U(x)\alpha) \, \tilde{W}(y) \, . \tag{5c}$$

B. Fourier cosine transform in the z direction.

Taking the Fourier cosine transform of $\tilde{p}$ defined by

$$F_c(\tilde{p}) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{p} \cos \beta z dz,$$

(5) becomes

$$\frac{d^2 F_c(\tilde{p})}{dy^2} - \frac{2\alpha}{\alpha M(y) - k} \frac{dM}{dy} \frac{dF_c(\tilde{p})}{dy} + [(\alpha M(y) - k)^2 - (\alpha^2 + \beta^2)] F_c(\tilde{p})$$

$$= i \rho a [\alpha M(y) - k] \sqrt{\frac{2}{\pi}} \cdot \tilde{W}(y), \tag{6a}$$

with the boundary condition

$$\left. \frac{dF_c(\tilde{p})}{dy} \right|_{y = \pm \ell} = 0 . \tag{6b}$$

In order to solve this inhomogeneous equation, we first consider, according to the standard method, eigenfunctions of a corresponding homogeneous equation given by

$$\frac{d^2 F}{dy^2} - \frac{2\alpha}{\alpha M(y) - k} \frac{dM}{dy} \frac{dF}{dy} + [(\alpha M(y) - k)^2 - (\alpha^2 + \beta^2)] F = 0 , \tag{7a}$$

with the boundary condition

$$\left. \frac{dF}{dy} \right|_{y = \pm \ell} = 0 . \tag{7b}$$

To solve (7), one assumes that the non-uniformity of the flow inside the duct is small, i.e.,

$$M(y) = M_0 + \varepsilon g(y), \tag{8a}$$

where M_0 is the averaged velocity in the duct defined by

$$M_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} M(y)\, dy, \tag{8b}$$

and $\varepsilon \ll 1$. According to (8a) and (8b), it immediately follows that

$$\int_{-\ell}^{\ell} g(y)\, dy = 0 . \tag{8c}$$

Under this assumption, the eigenfunctions of (7) are derived in Ref. 2 and they are given by

$$u_0\,(\alpha,\, y) = \exp\,[-2\,\varepsilon\alpha(\alpha M_0 - k) \int_0^y d\eta \int_{-\ell}^{\eta} g(\xi)\, d\xi\,], \tag{9a}$$

$$u_n\,(\alpha,y) = \frac{1}{2}\,(\exp\, z_n + \exp\, \bar{z}_n),\ n = 1,\ 2,\ --- \tag{9b}$$

where $\bar{z}_n$ is the complex conjugate of z_n defined by

$$z_n = i\,\gamma_n y + i\varepsilon\, \frac{2\alpha\gamma_n{}^2}{\alpha M_0 - k} \int_{-\ell}^{y} g(\eta)\, d\eta$$

$$+\,\varepsilon\,[\frac{4\alpha\gamma^2{}_n}{\alpha M_0 - k} - 2\alpha\,(\alpha M_0 - k)]$$

$$\times \left[\int_{-\ell}^{y} d\eta \int_{-\ell}^{\eta} g(\xi)\, e^{-2i\gamma_n(\eta - \xi)}\, d\xi + \frac{i}{(2\gamma_n)^3} g'(\ell) - \frac{i}{(2\gamma_n)^5} g'''(\ell) + --- \right]$$

$$+ \frac{i\epsilon}{\gamma_n} \chi_n (y - \ell) [M_0(\alpha M_0 - k) - \alpha], \tag{9c}$$

and where

$$\gamma_n = \frac{n\pi}{\ell}, \quad n = 1, 2, --- \tag{9d}$$

$$\chi_n = \frac{1}{\ell[M_0(\alpha M_0 - k) - \alpha]} \left[\frac{\alpha \gamma_n^2}{\alpha M_0 - k} - \frac{1}{2} (\alpha M_0 - k)\right]$$

$$\times \left\{ \frac{1}{(2\gamma_n)^2} [g'(\ell) - g'(-\ell)] - \frac{1}{(2\gamma_n)^4} [g'''(\ell) - g'''(-\ell)] + --- \right\}, \tag{9e}$$

and $(\alpha M_0 - k)^2 - \alpha^2 - \beta^2$ takes the following eigenvalues:

$$(\alpha M_0 - k)^2 - \alpha^2 - \beta^2 = \left(\frac{n\pi}{\ell}\right)^2 + 2\chi_n \epsilon [M_0(\alpha M_0 - k) - \alpha] . \tag{9f}$$

(Contrary to the situation of Ref. 2, where the eigenvalues are assigned only to α, here we ascribe it to the constant consisting of combination of α, β, M_0 and k, which appears on the left hand side.) Since (7) is a Sturm-Liouville equation, the eigenfunctions u_n are orthogonal and they are indeed complete. Hence we expand $F_c(\tilde{p})$ of the inhomogeneous equation of (6a) and also its right hand side into series of eigenfunctions, i.e.,

$$F_c(\tilde{p}) = \sum_{n=0}^{\infty} a_n u_n (\alpha , y) , \tag{10a}$$

and

$$\tilde{W} (y) = \sum_{n=0}^{\infty} \tilde{b}_n (\alpha , \beta) u_n (\alpha , y), \tag{10b}$$

$$g(y) \tilde{W} (y) = \sum_{n=0}^{\infty} \tilde{c}_n (\alpha , \beta) u_n (\alpha , y), \tag{10c}$$

and where

$$\tilde{b}_n (\alpha , \rho) = \frac{\int_{-\ell}^{\ell} \tilde{W}(y) u_n(\alpha, \beta, y) dy}{\int_{-\ell}^{\ell} [u_n(\alpha, \beta,y)]^2 dy} , \tag{10d}$$

and

$$\tilde{c}_n(\alpha, \beta) = \frac{\int_{-\ell}^{\ell} g(y) \tilde{W}(y) u_n(\alpha, \beta,y) dy}{\int_{-\ell}^{\ell} [u_n(\alpha, \beta,y)]^2 dy} . \qquad \text{10e)}$$

Substituting these into (6a) and making use of the fact that u_n satisfies the homogeneous equation (7a), one can determine a_n of (10a) and $F_c(\tilde{p})$ becomes

$$F_c(\tilde{p}) = i\rho a\sqrt{\frac{2}{\pi}}\left[\frac{(\alpha M_0 - k)\tilde{b}_0}{(\alpha M_0 - k)^2 - \alpha^2 - \beta^2} + \frac{\alpha\varepsilon\,\tilde{c}_0}{(\alpha M_0 - k)^2 - \alpha^2 - \beta^2}\right] u_0(\alpha , y)$$

$$+ \sum_{n=1}^{\infty} i\rho a \sqrt{\frac{2}{\pi}}$$

$$\times \left\{ \frac{(\alpha M_0 - k)\tilde{b}_n}{(\alpha M_0 - k)^2 - \alpha^2 - \beta^2 - (\frac{n\pi}{\ell})^2 - 2\chi_n\varepsilon[M_0(\alpha M_0 - k) - \alpha]} \right.$$

$$\left. + \alpha\varepsilon \frac{\tilde{c}_n}{(\alpha M_0 - k)^2 - \alpha^2 - \beta^2 - (\frac{n\pi}{\ell})^2} \right\} u_n(\alpha , \beta). \tag{11}$$

If the velocity on the surface of the airfoil, (4) is such that

$$W(x,y) = H(x)X(x)\;(y + \ell)\;\frac{\phi}{2\ell} \tag{12}$$

where H is a step function and ϕ is the amplitude of oscillation and for linear duct flow distribution represented by

$$g(y) = Ay , \tag{13}$$

then $\chi_n = 0$ and (10d) and (10e) become

$$\tilde{b}_0 = \tilde{X}\frac{\phi}{2\ell}\left[\ell + \frac{4}{15}A\ell^4\alpha\varepsilon(\alpha M_0 - k)\right],$$

$$\tilde{b}_n = \varepsilon\tilde{X}\frac{\phi}{2\ell}\frac{A(-1)^n}{\left(\frac{n\pi}{\ell}\right)^4}\left[\frac{4\alpha\gamma_n^2}{\alpha M_0 - k} - 2\alpha(\alpha M_0 - k)\right], \quad n = 1,2,--- \tag{14a}$$

$$\tilde{c}_0 = \tilde{X}\frac{\phi}{2\ell}\frac{1}{3}A\ell^2$$

$$\tilde{c}_n = \tilde{X}\frac{\phi}{2\ell}4A\ell^2\frac{(-1)^n}{(n\pi)^2}, \quad n = 1, 2, --- \tag{14b}$$

Inverting (11), we obtain

$$\tilde{p} = \rho a[(\alpha M_0 - k)\tilde{b}_0 + \alpha\varepsilon\tilde{c}_0]\frac{1}{\sqrt{(\alpha M_0 - k)^2 - \alpha^2}}u_0(\alpha, y)$$

$$\times\ i\cdot\sin\left[z\sqrt{(\alpha M_0 - k)^2 - \alpha^2}\right]$$

$$+\sum_{n=1}^{\infty}\rho a\left[(\alpha M_0 - k)\tilde{b}_n + \alpha\varepsilon\tilde{c}_n\right]\frac{1}{\sqrt{(\alpha M_0 - k)^2 - \alpha^2 - \left(\frac{n\pi}{\ell}\right)^2}}u_n(\alpha, y)$$

$$\times\ i\sin\left[z\sqrt{(\alpha M_0 - k)^2 - \alpha^2 - \left(\frac{n\pi}{\ell}\right)^2}\right]. \tag{15}$$

One can show that if one replaces i·sine of (15) by cosine, it also satisfies (5a and b); and (5c) becomes, in such a case

$$\left.\frac{\partial\tilde{p}}{\partial z}\right|_{z=0} = 0 .$$

Hence the solution is a sum of these two expressions, i.e.

$$\tilde{p} = \rho a\,[(\alpha M_0 - k)\,\tilde{b}_0 + \alpha\varepsilon\,\tilde{c}_0]\,\frac{1}{\sqrt{(\alpha M_0 - k)^2 - \alpha^2}}\;u_0\,(\alpha, y)$$

$$\times\;\exp\,[iz\sqrt{(\alpha M_0 - k)^2 - \alpha^2}]$$

$$+\sum_{n=1}^{\infty}\rho a\,[(\alpha M_0 - k)\,\tilde{b}_n + \alpha\varepsilon\,\tilde{c}_n]\,\frac{1}{\sqrt{(\alpha M_0 - k)^2 - \alpha^2 - (\frac{n\pi}{\ell})^2}}\;u_n(\alpha, y)$$

$$\times\;\exp\,[iz\sqrt{(\alpha M_0 - k)^2 - \alpha^2 - (\frac{n\pi}{\ell})^2}]\;. \tag{16}$$

C. Inversion of Fourier Transform.

Inverting (16) with respect to α and by deforming the contour of integration, we obtain the pressure on the airfoil, $z = 0$, as

$$\bar{p}' = \frac{\phi}{2\ell}\frac{\rho a}{\sqrt{2\pi}}\left\{X\,(x)\,[a_1(0) + a_2'(0) + a_3\,(0)]\right.$$

$$-X'(x)\,[a^2\,(0) + 2a_3'(0)]$$

$$+X''(x)\,a_3\,(0)$$

$$\left.+\int_0^x X\,(x)\,a_0(x - t)\,dt\right\}, \tag{17a}$$

where

$$a_0(x) = -k\ell\ \theta_0(x) + \mu i\theta_0'(x)$$

$$+ \ \nu\,\theta_0''(x) - \frac{1}{3}\,\xi\ i\theta_0''(x)$$

$$+ \varepsilon A \sum_{n=1}^{\infty} \frac{(-1)^n}{(\frac{n\pi}{\ell})^4} \left\{ [4\ (\frac{n\pi}{\ell})^2 - 2k^2]\ i\theta_n'(x) \right.$$

$$\left. -4k\ M_0\ \theta_n''(x) + 2i\ M_0^2\ \theta_A'''(x) \right\}, \tag{17b}$$

$$a_1(x) = \mu i\ \theta_0(x)$$

$$+ 2\ \theta_0'(x) - \xi i\ \theta_0''(x)$$

$$+ \varepsilon A \sum_{n=1}^{\infty} \frac{(-1)^n}{(\frac{n\pi}{\ell})^4} \left\{ [4\ (\frac{n\pi}{\ell})^2 - 2k^2] i\,\theta_n(x) \right.$$

$$\left. -8k\ M_0\ \theta_n'(x) + 6i\ M_0^2\ \theta n''(x) \right\}, \tag{17c}$$

$$a_2(x) = \nu\theta_0(x) - \xi i\theta_0'(x)$$

$$+ \varepsilon A \sum_{n=1}^{\infty} \frac{(-1)^n}{(\frac{n\pi}{\ell})^4} \left\{ -4k\ M_0\ \theta_n(x) + 6i\ M_0^2\ \theta_n'(x) \right\}, \tag{17d}$$

$$a_3(x) = -\ \xi i\ \theta_0(x)$$

$$+ \varepsilon A \sum_{n=1}^{\infty} \frac{(-1)^n}{(\frac{n\pi}{\ell})^4}\ 2i\ M_0{}^2\ \theta_n(x)\ , \tag{17e}$$

where

$$\mu = M_0\ell + \frac{4}{15} A\ell^4\ \varepsilon k^4 + \frac{\varepsilon}{3} A\ell^3\ ,$$

$$\nu = \frac{8}{15} A\ell^4\ \varepsilon k\ M_0\ ,$$

$$\xi = \frac{4}{5} A\ell^4\ \varepsilon M_0{}^2\ ,$$

and where

$$\theta_n(x) = \frac{-2i}{m\sqrt{2\pi}} \exp\left(-ix \frac{M_0 k}{m^2}\right)$$

$$x \int_0^{\pi} \cos h\ (m\lambda_n z \sin\theta) \exp(-i\chi\lambda_n \cos\theta)\ v_n(\theta, y)\ d\theta\ , \tag{17f}$$

and $\lambda_n = \left[\frac{k^2}{m^4} + \frac{1}{m^2} (\frac{n\pi}{\ell})^2 \right]^{1/2}$,

$$v_n(\theta,y) = u_n\ (\alpha, y)\ ,$$

which is given in (9) and where

$$\alpha = \lambda_n \cos\theta + \frac{M_0 k}{m^2}\ .$$

If the amount of shear, ε, is equal to zero, one can readily show that the above expression is reduced to the standard solution of two-dimensional airfoil oscillating in uniform supersonic flow.

IV. *Discussion of Results.*

Figure 2 and 3 compare $c_p = \bar{p}'/\frac{1}{2}\rho u^2$, computed from (17) and evaluated on the surface of airfoils placed in a linearly sheared duct with the one corresponding to uniform flow. The pressure is calculated at the top of the duct and the airfoil is oscillating in torsional motion at mid-chord with unit amplitude. Additional parameters are as follows:

$\frac{\ell}{c} = 1.5$, where ℓ is half-duct height, c is airfoil chord,

$$\frac{\omega c}{a} = 0.7 ,$$

and

$$M = 1.3 + 0.0666 \left(\frac{y}{\ell}\right) .$$

It clear that the spanwise velocity distribution corresponding to the three dimensional, radial velocity gradient indeed induces significant modification to the one for radially uniform or two-dimensional approximation.

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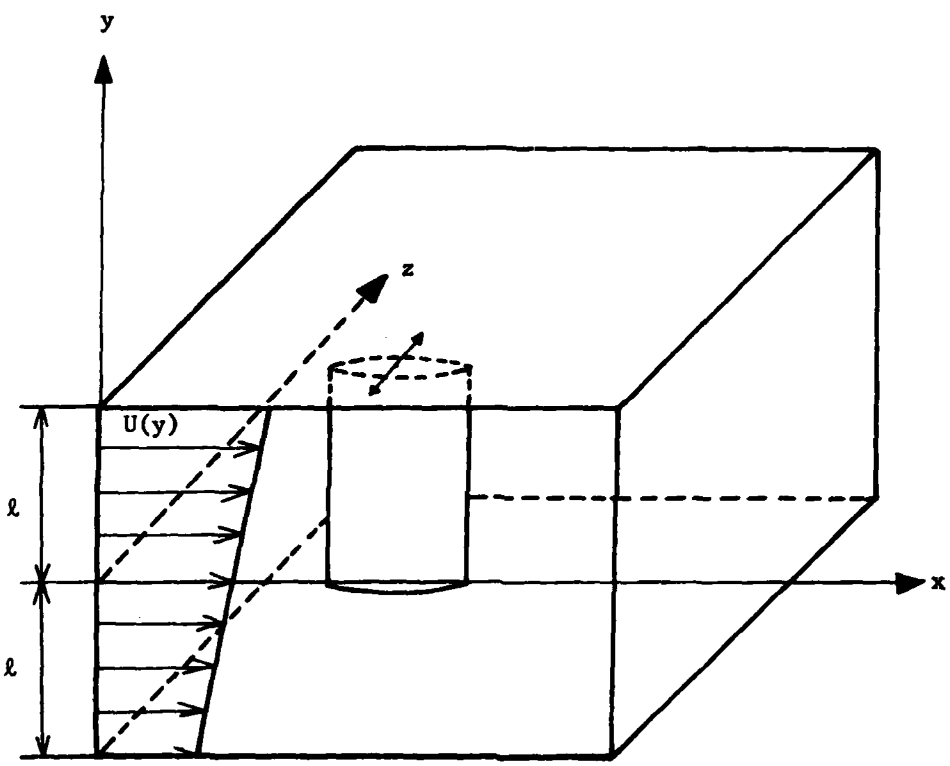


Figure 1. Definition Sketch.

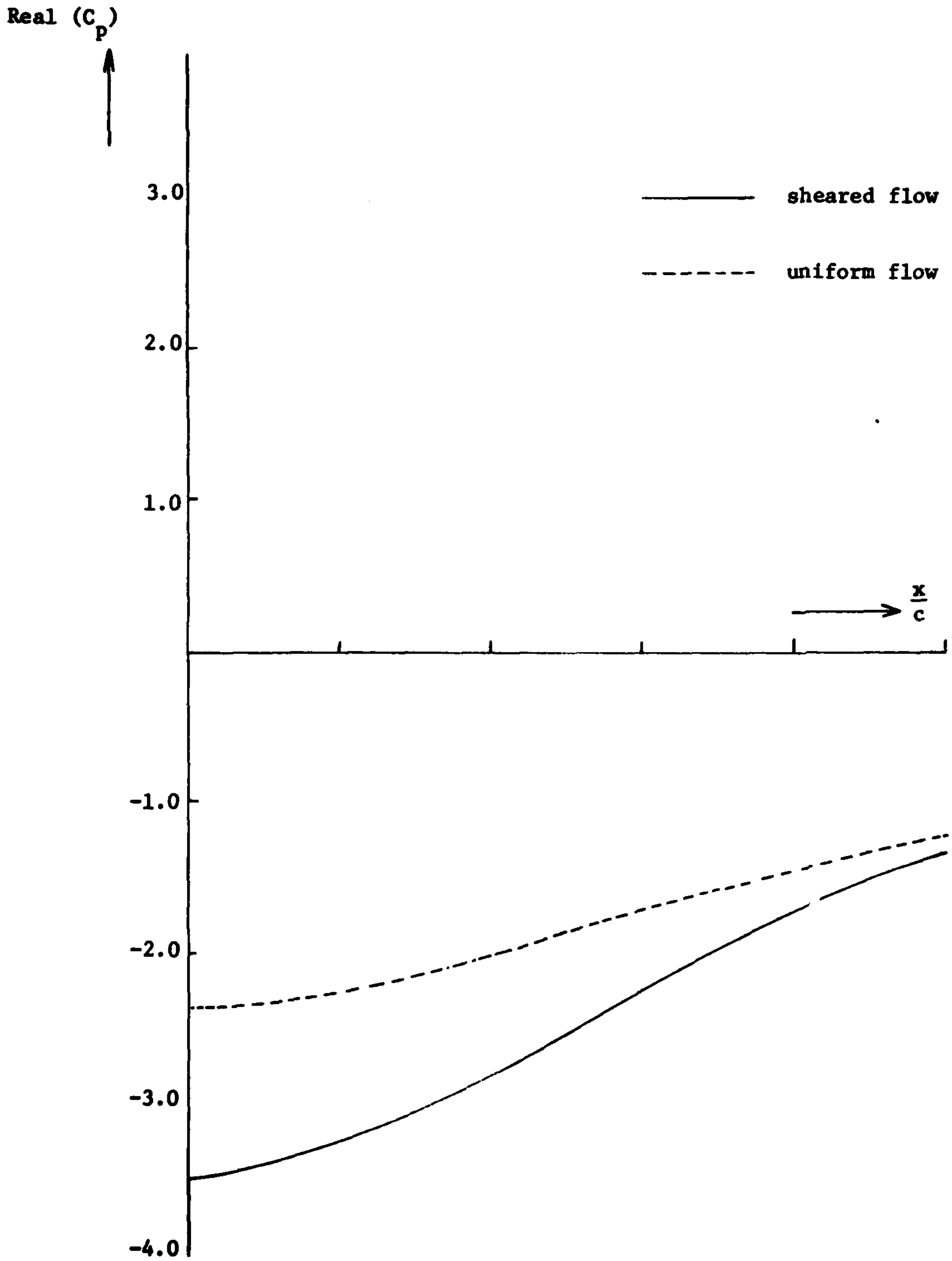
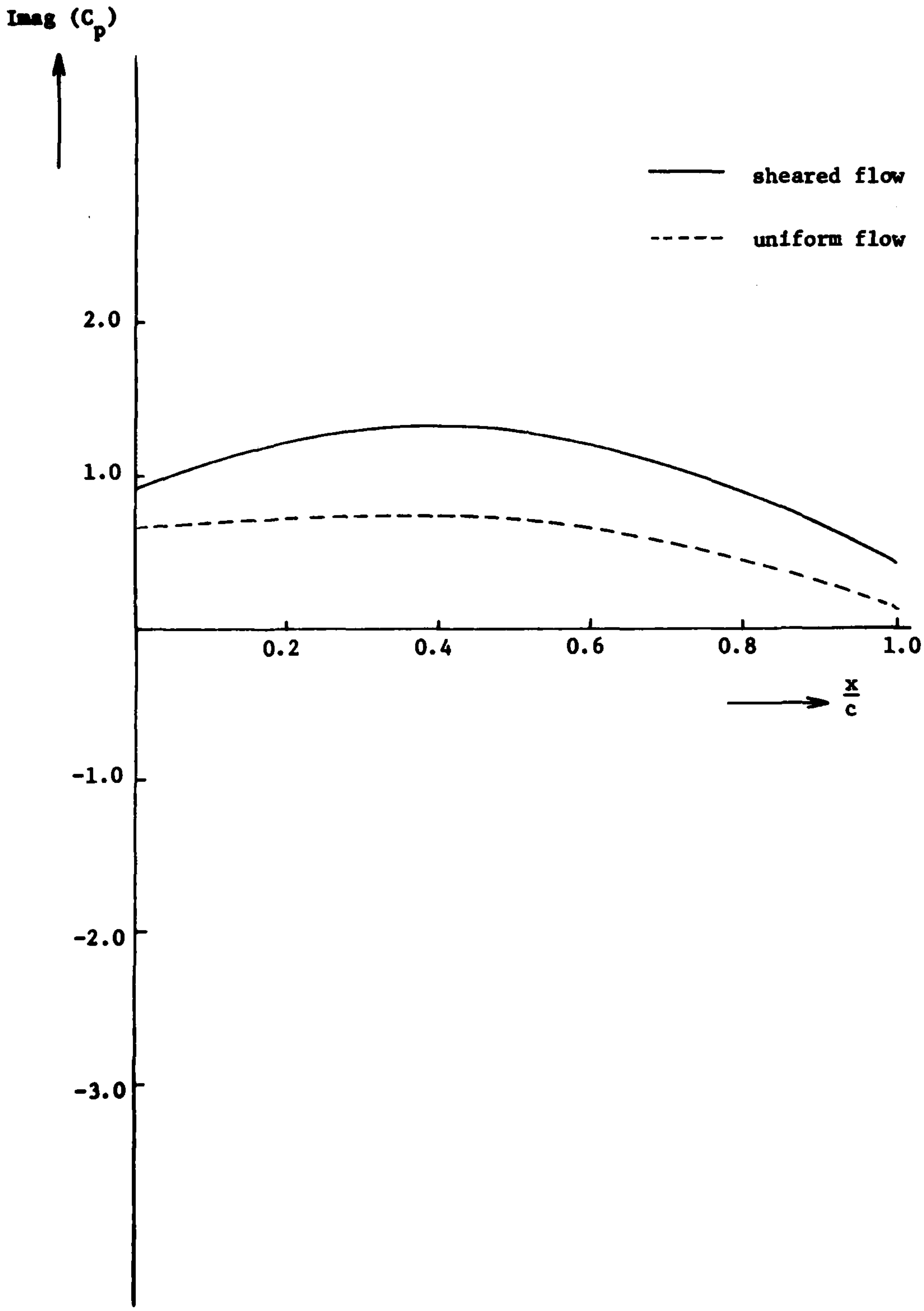


Figure 2.



Imag (C_p)
sheared flow
uniform flow
2.0
1.0
0.2
0.4
0.6
0.8
1.0
$\frac{x}{c}$
-1.0
-2.0
-3.0

Figure 3.