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AN ALGORITHMIC SOLUTION TO THE GI/M/C QUEUE WITH GROUP ARRIVALS--ETC(U)

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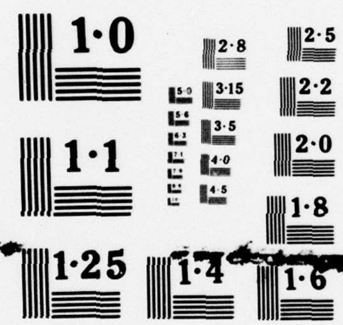
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Department of

STATISTICS AND COMPUTER SCIENCE

UNIVERSITY OF DELAWARE
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AN ALGORITHMIC SOLUTION TO THE GI/M/c QUEUE WITH GROUP ARRIVALS

by

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We study the c-server queue with general interarrival times, exponential service times and bounded group arrivals. It is shown that the stationary density of the queue length before arrivals is of a matrix-geometric form, provided that the queue is stable. The essential step in the computation of that stationary density is the evaluation of a positive square matrix R as the unique solution to a nonlinear matrix equation. The order of the matrix R is given by the upper bound K on the sizes of the arrival groups. → next page

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20. Abstract

Various other stationary distributions of waiting times, times in system and the queue length at an arbitrary time can be expressed in terms of the matrix R by means of formulas, which may readily be computationally implemented.

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Abstract

We study the c-server queue with general interarrival times, exponential service times and bounded group arrivals. It is shown that the stationary density of the queue length before arrivals is of a matrix-geometric form, provided that the queue is stable. The essential step in the computation of that stationary density is the evaluation of a positive square matrix R as the unique solution to a nonlinear matrix equation. The order of the matrix R is given by the upper bound K on the sizes of the arrival groups.

Various other stationary distributions of waiting times, times in system and the queue length at an arbitrary time can be expressed in terms of the matrix R by means of formulas, which may readily be computationally implemented.

Key Words

Queueing theory, computational probability, general interarrival times, group arrivals, multi-server queue, exponential services, matrix-geometric solution.

1. Introduction

We discuss the queueing model $GI^X/M/c$ in which customers arrive in groups at a c-server unit at time points which form a renewal process. The c servers have independent exponential processing times with the same rate μ. The times between group arrivals have the same distribution $F(\cdot)$ with finite mean α and satisfying $F(0+) = 0$. The successive group sizes are independent, bounded random variables with probability density $\{\theta_\nu, 1 \leq \nu \leq K\}$ and mean η. Without loss of generality, we require that $\theta_K > 0$.

It is well-known that in the case of single arrivals, the stationary queue length density immediately prior to arrivals exists if and only if $c\mu\alpha > 1$ and is then a modified geometric probability density in which the first c terms are obtained by solving a system of linear equations. We refer to the book by D. Gross and C. M. Harris [6] or several other texts on queueing theory for a discussion of this classical result.

The extension of this result to the case of bounded group arrivals appears to be new. We shall obtain the stationery queue length density in a modified matrix-geometric form. This main theorem leads to a highly efficient algorithm and yields most of the other stationary distributions in elegant computable forms.

The queue lengths immediately prior to arrivals form a Markov chain on the nonnegative integers. The transition probability matrix P of that chain has a form which depends on whether $K \geq c$ or $1 \leq K \leq c-1$. The remainder of this section is devoted to the detailed definition of the matrix P.

We define the quantities

$$(1) \qquad \phi_j(t) = \sum_{\nu=\max(1,-j)}^{K} \theta_\nu e^{-c\mu t} \frac{(c\mu t)^{\nu+j}}{(\nu+j)!}, \qquad \text{for } j \geq -K,\ t \geq 0,$$

and

(2) $$a_j = \int_0^\infty \phi_j(t)\, dF(t), \qquad \text{for } j \geq -K.$$

The probability generating function $A^*(z) = \sum_{\nu=0}^{\infty} a_{\nu-K} z^{\nu}$, is given by

(3) $$A^*(z) = f^*(c\mu - c\mu z) \sum_{\nu=1}^{K} \theta_\nu z^{K-\nu},$$

where $f^*(\cdot)$ is the Laplace - Stieltjes tranform of $F(\cdot)$. The mean $A^{*\prime}(1-)$ is given by

(4) $$A^{*\prime}(1-) = K - \eta + c\mu\alpha.$$

We further define the quantities $\hat{\phi}_{rj}(t)$, $t \geq 0$, by

(5) $$\hat{\phi}_{rj}(t) = \binom{r}{j} e^{-\mu j t} (1-e^{-\mu t})^{r-j}, \qquad \text{for } j \leq r \leq c,$$

$$= 0, \qquad \text{for } r < j \leq c,$$

$$= \int_0^t e^{-c\mu t} \frac{(c\mu t)^{r-c-1}}{(r-c-1)!} c\mu \binom{c}{j} e^{-\mu j(t-\tau)} [1-e^{-\mu(t-\tau)}]^{c-j} d\tau,$$

$$\text{for } r > c,\ 0 \leq j \leq c-1,$$

and

(6) $$P_{ij} = \sum_{\nu=1}^{K} \theta_\nu \int_0^\infty \hat{\phi}_{i+\nu,j}(t)\, dF(t), \qquad \text{for } i \geq 0,\ 0 \leq j \leq c-1.$$

The transition probability matrix P of the embedded Markov chain, obtained by considering the queue length immediately prior to arrivals, is given by

(7) For $K \geq c$:

$$P = \begin{vmatrix} P_{00} & \cdots & P_{0,c-1} & a_{-c} & \cdots & a_{1-K} & a_{-K} & 0 & 0 & 0 & \cdots \\ P_{10} & \cdots & P_{1,c-1} & a_{1-c} & \cdots & a_{2-K} & a_{1-K} & a_{-K} & 0 & 0 & \cdots \\ P_{20} & \cdots & P_{2,c-1} & a_{2-c} & \cdots & a_{3-K} & a_{2-K} & a_{1-K} & a_{-K} & 0 & \\ P_{30} & \cdots & P_{3,c-1} & a_{3-c} & \cdots & a_{4-K} & a_{3-K} & a_{2-K} & a_{1-K} & a_{-K} & \cdots \\ P_{40} & \cdots & P_{4,c-1} & a_{4-c} & \cdots & a_{5-K} & a_{4-K} & a_{3-K} & a_{2-K} & a_{1-K} & \cdots \\ P_{50} & \cdots & P_{5,c-1} & a_{5-c} & \cdots & a_{6-K} & a_{5-K} & a_{4-K} & a_{3-K} & a_{2-K} & \cdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

For $1 \leq K \leq c-1$:

$$P = \left|\begin{array}{cccccccc}
P_{00} & \cdots & P_{0,c-1} & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \\
P_{c-K-1,0} & \cdots & P_{c-K-1,c-1} & 0 & 0 & \cdots & 0 & \cdots \\
P_{c-K,0} & \cdots & P_{c-K,c-1} & a_{-K} & 0 & \cdots & 0 & \cdots \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \\
P_{c-1,0} & \cdots & P_{c-1,c-1} & a_{-1} & a_{-2} & \cdots & a_{-K} & \cdots \\
P_{c,0} & \cdots & P_{c,c-1} & a_{0} & a_{-1} & \cdots & a_{1-K} & \cdots \\
P_{c+1,0} & \cdots & P_{c+1,c-1} & a_{1} & a_{0} & \cdots & a_{2-K} & \cdots \\
P_{c+2,0} & \cdots & P_{c+2,c-1} & a_{2} & a_{1} & \cdots & a_{3-K} & \cdots
\end{array}\right|$$

The first c columns are given by Formula (6). The remaining columns are all obtained by successively shifting the column $[a_{-K}, a_{1-K}, \ldots]$ downward one place at a time.

2. The Main Theorem

Theorem 1

The Markov chain P is irreducible and aperiodic. It is positive recurrent if and only if $\eta < c\mu\alpha$.

For $K \geq c$, the invariant probability vector $\underline{x} = (\underline{x}_0, \underline{x}_1, \ldots)$, partitioned into K-vectors, is given by $\underline{x}_k = \underline{x}_0 R^k$, for $k \geq 0$. The matrix R is strictly positive and its spectral radius is less than one.

Defining the $K \times K$ matrices A_n, $n \geq 0$, by

$$(8) \quad A_0 = \left|\begin{array}{cccc}
a_{-K} & 0 & \cdots & 0 \\
a_{1-K} & a_{-K} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
a_{-1} & a_{-2} & \cdots & a_{-K}
\end{array}\right|, \quad A_n = \left|\begin{array}{ccc}
a_{nK-K} & \cdots & a_{nK-2K+1} \\
a_{nK-K+1} & \cdots & a_{nK-2K+2} \\
\vdots & & \vdots \\
a_{nK-1} & \cdots & a_{nK-K}
\end{array}\right| \quad n \geq 1,$$

the matrix R is the minimal solution of the matrix equation

(9) $$R = \sum_{n=0}^{\infty} R^n A_n,$$

in the set X of nonnegative matrices of order K with spectral radius less than one. The matrix R is the unique solution of (9) in the set X.

Defining the $K \times K$ matrices B_n, $n \geq 0$, by partitioning the first K columns of the matrix P for $K \geq c$ into $K \times K$ blocks, the matrix

(10) $$B[R] = \sum_{n=0}^{\infty} R^n B_n,$$

is a strictly positive, <u>stochastic</u> matrix. The vector $\underline{x}_0$ is the unique vector which satisfies

(11) $$\underline{x}_0 = \underline{x}_0 B[R], \qquad \underline{x}_0 (I-R)^{-1} \underline{e} = 1,$$

where $\underline{e} = (1,1,\ldots,1)'$.

For $1 \leq K \leq c-1$, we partition the invariant vector $\underline{x}$ as $(\underline{x}_0, \underline{x}_1, \underline{x}_2, \ldots)$, where $\underline{x}_k$, $k \geq 1$, has K components and $\underline{x}_0$ has $c-K$ components. The vectors $\underline{x}_k$, $k \geq 1$, are given by $\underline{x}_k = \underline{x}_1 R^{k-1}$, where the matrix R is defined in the same manner as above.

We partition the first $c-K$ columns of P for $K \leq c-1$, into a $(c-K) \times (c-K)$ block B_{00} and the remainder into $K \times (c-K)$ blocks B_{n0}, for $n \geq 1$. The next K columns are partitioned into a first $(c-K) \times K$ block B_{01} and the remainder into $K \times K$ blocks B_{n1}, for $n \geq 1$.

The $c \times c$ matrix B[R], given by

(12) $$B[R] = \begin{vmatrix} \overset{B_{00}}{\sum_{n=1}^{\infty} R^{n-1} B_{n0}} & \overset{B_{01}}{\sum_{n=1}^{\infty} R^{n-1} B_{n1}} \end{vmatrix},$$

is irreducible stochastic. The c-vector $(\underline{x}_0, \underline{x}_1)$ is the unique vector satisfying

(13) $(\underline{x}_0, \underline{x}_1) = (\underline{x}_0, \underline{x}_1)\, B[R], \qquad \underline{x}_0\underline{e} + \underline{x}_1\, (I-R)^{-1}\underline{e} = 1.$

Proof

The matrix equation (9) was examined in detail for a more general situation in [11]. We shall draw on results proved there, without repeating the lengthy proofs.

The matrix $A = \sum_{n=0}^{\infty} A_n$ is readily seen to be a strictly positive circulant stochastic matrix. Its invariant probability vector $\underline{\pi}$ is given by $\underline{\pi} = K^{-1}\underline{e}'$. The vector $\underline{\beta} = \sum_{n=1}^{\infty} nA_n\underline{e}$ satisfies $\underline{\pi}\ \underline{\beta} = K^{-1}\ A^{*\prime}(1-) = K^{-1}(c\mu\alpha + K-\eta)$. This equality was proved in [12], Lemma 1.

In [11], we showed that a unique matrix R with the stated properties exists if and only if $\underline{\pi}\ \underline{\beta} > 1$, or equivalently $c\mu\alpha > \eta$. If $c\mu\alpha \leq \eta$, we know that the queue cannot be stable. Since the matrix P is clearly irreducible and aperiodic, we establish positive recurrence by proving that P has a strictly positive invariant probability vector if $c\mu\alpha > \eta$.

For $K \geq c$, the remaining statements all follow by application of Theorem 2 in [11]. Since the matrices R and B_n, $n \geq 0$, are positive, so is B[R] and hence the vector $\underline{x}_0$.

For $1 \leq K \leq c-1$, the matrix B[R] is positive, except for a triangular corner on the upper right hand side. Such a matrix is clearly irreducible. In order to verify that B[R] is stochastic, it suffices to check that

$$\sum_{n=1}^{\infty} R^{n-1}\ B_{n0}\underline{e} + \sum_{n=1}^{\infty} R^{n-1}\ B_{n1}\underline{e} = \underline{e}.$$

The left hand side may successively be written as

$$\sum_{n=1}^{\infty} R^{n-1}\ (B_{n0}\underline{e} + B_{n1}\underline{e}) = \sum_{n=1}^{\infty} R^{n-1}\ (\underline{e} - \sum_{\nu=0}^{n-1} A_\nu\underline{e}) =$$

$$(I-R)^{-1}\underline{e} - (I-R)^{-1} \sum_{\nu=0}^{\infty} R^{\nu} A_\nu\underline{e} = \underline{e}.$$

The left invariant vector $(\underline{x}_0, \underline{x}_1)$ of B[R], normalized so that $\underline{x}_0\underline{e} + \underline{x}_1(I-R)^{-1}\underline{e} = 1$, is uniquely determined and positive. It is now a routine matter to verify that the positive probability vector $\underline{x} = (\underline{x}_0, \underline{x}_1, \underline{x}_1 R, \underline{x}_1 R^2, \ldots)$ is the invariant vector of P.

<u>Corollary 1</u>

For $K \geq c$, the stationary mean queue length L_1, including the customers in service, prior to arrivals is given by

$$(14) \qquad L_1 = \sum_{\nu=0}^{\infty} \underline{x}_0 R^{\nu} (\nu K\underline{e} + \underline{u}) = \underline{x}_0 (I-R)^{-1} \underline{u} + K\underline{x}_0 (I-R)^{-2} R\underline{e},$$

where $\underline{u} = (0, 1, 2, \ldots, K-1)'$.

For $1 \leq K \leq c-1$, the corresponding formula is

$$(15) \qquad L_1 = \underline{x}_0\underline{u}' + \sum_{\nu=1}^{\infty} \underline{x}_1 R^{\nu-1} [(c-K)\underline{e} + (\nu-1) K\underline{e} + \underline{u}] =$$

$$\underline{x}_0\underline{u}' + \underline{x}_1 (I-R)^{-1} [\underline{u} + (c-K)\underline{e}] + K\underline{x}_1 (I-R)^{-2} R\underline{e},$$

where $\underline{u}' = (0, 1, \ldots, c-K-1)'$.

<u>Corollary 2</u>

If the customers in arriving groups are numbered 1, 2, ..., so as to specify the order in which they will enter service, the stationary waiting time distribution $W_r(\cdot)$, $1 \leq r \leq K$, of the r^{th} customer in a group is given by

$$(16) \qquad W_r(x) = \sum_{i \leq c-r} x_i \, U(x) + \sum_{i=\max(0,c-r+1)}^{\infty} x_i \, E_{i+r-c}(c\mu, x),$$

for $x \geq 0$, where $U(\cdot)$ is the degenerate distribution and $E_k(c\mu, \cdot)$ is the Erlang distribution of order k with parameter $c\mu$.

<u>Proof</u>

With probability x_i, the r^{th} customer in the arriving group finds $i+r-1$ customers ahead of him. If $i+r-1 \leq c-1$, he does not have to wait.

If $i+r-1 \geq c$, he waits until the number of customers ahead of him drops to $c-1$, or equivalently until $i+r-c$ services have been completed. Since during that time all c servers are busy, the distribution of the time to complete $i+r-c$ services has an $E_{i+r-c}(c\mu,\cdot)$ distribution.

Corollary 3

The stationary distribution $W^*(\cdot)$ of the server backlog, immediately after the arrival of a group is given by

$$W^*(x) = \sum_{r=1}^{K} \theta_r \sum_{i \leq c-r} x_i G_{i+r}(x) + \tag{17}$$

$$\sum_{r=1}^{K} \theta_r \sum_{i=\max(0,c-r+1)}^{\infty} x_i E_{i+r-c}(c\mu,\cdot) * G_c(x),$$

where $G_j(x) = (1-e^{-\mu x})^j$, for $x \geq 0$.

Proof

Conditioning on the size of the arriving group, we see that if r customers join and there are at most $c-r$ customers present, the backlog consists of the time until $i+r$ customers are processed. The distribution of that time is $G_{i+r}(\cdot)$.

If r customers join and there are already i present, so that $i + r > c$, the backlog consists of the time until only c customers remain, plus the time required to serve those c customers by c parallel servers. This leads readily to the second term.

3. The Queue Length in Continuous Time

In this section, we assume that $F(\cdot)$ is not a lattice distribution. Analogous, but cumbersome formulas hold in the lattice case.

By $\underline{y}$, we denote the vector of probabilities y_j, $j \geq 0$, where y_j is the probability that j customers are present at time t in the stationary version of the queue. We shall find it convenient to partition the vector $\underline{y}$ in the same manner as the vector $\underline{x}$ was partitioned above. We shall again need to distinguish between the formulas for the cases $K \geq c$ and $1 \leq K \leq c-1$.

It is also convenient to introduce a stochastic matrix $\tilde{P}$ which has the same general form as the matrix P, defined in Formula (7), except that the corresponding entries are defined by

(18)
$$\tilde{a}_j = \int_0^\infty \phi_j(t) \frac{1-F(t)}{\alpha} dt, \qquad \text{for } j \geq -K$$
$$\tilde{P}_{ij} = \sum_{\nu=1}^{K} \theta_\nu \int_0^\infty \phi_{i+\nu,j}(t) \frac{1-F(t)}{\alpha} dt, \qquad \text{for } i \geq 0,\ 0 \leq j \leq c-1.$$

We shall partition the matrix $\tilde{P}$ in the same manner as in the proof of Theorem 1 and the corresponding matrices in the partition will be denoted by the same letters, but distinguished by tildes.

By a classical application of the Key renewal theorem, the probabilities y_j, $j \geq 0$, are related to the probabilities x_j, $j \geq 0$, by

$$(19) \quad y_j = \sum_{r=1}^{K} \theta_r \sum_{i=\max(0,j-r)}^{\infty} \frac{x_i}{\alpha} \int_0^\infty f_{i+r,j}(u)[1-F(u)]du, \quad j \geq 0,$$

where for $u \geq 0$,

$$f_{ij}(u) = 0 \qquad \text{for } j > i \geq 0,$$
$$= e^{-c\mu u} \frac{(c\mu u)^{i-j}}{(i-j)!}, \qquad \text{for } i \geq j \geq c,$$
$$= \binom{i}{j} (1-e^{-\mu u})^{i-j} e^{-\mu j u}, \qquad \text{for } 0 \leq j \leq i \leq c,$$
$$= \int_0^u e^{-c\mu\tau} \frac{(c\mu\tau)^{i-c-1}}{(i-c-1)!} c\mu \binom{c}{j} [1-e^{-\mu(u-\tau)}]^{c-j} e^{-\mu j(u-\tau)} d\tau,$$
$$\text{for } i > c > j \geq 0.$$

Formula (19) is not particularly enlightening, but shows upon inspection that

$$(20) \qquad \underline{y} = \underline{x}\,\tilde{P}.$$

By considering the partitioned forms of the matrix $\tilde{P}$, we obtain computationally useful formulas.

Theorem 2

For $K \geq c$, we obtain

$$\underline{y}_0 = \underline{x}_0 \tilde{B}[R] = \underline{x}_0 \sum_{n=0}^{\infty} R^n \tilde{B}_n, \tag{21}$$

$$\underline{y}_i = \underline{x}_0 R^{i-1} \tilde{A}[R], \qquad \text{for } i \geq 1,$$

where $A[R] = \sum_{n=0}^{\infty} R^n \tilde{A}_n$.

For $1 \leq K \leq c-1$, we obtain

$$\underline{y}_0 = \underline{x}_0 \tilde{B}_{00} + \underline{x}_1 \sum_{n=1}^{\infty} R^{n-1} \tilde{B}_{n0} \tag{22}$$

$$\underline{y}_1 = \underline{x}_0 \tilde{B}_{01} + \underline{x}_1 \sum_{n=1}^{\infty} R^{n-1} \tilde{B}_{n1}$$

$$\underline{y}_i = \underline{x}_1 R^{i-2} A[R], \qquad \text{for } i \geq 2.$$

Proof

By direct computation and by application of Theorem 1.

Corollary 4

For $K \geq c$, the mean queue length L_2 at an arbitrary time point in the stationary version of the queue is given by

$$L_2 = \underline{x}_0 \left\{ \tilde{B}[R] + (I-R)^{-1} \tilde{A}[R] \right\} \underline{u} + K\underline{x}_0 (I-R)^{-2} \tilde{A}[R]\underline{e}, \tag{23}$$

where $\underline{u} = (0, 1, \ldots, K-1)'$.

The corresponding formula for the case $1 \leq K \leq c-1$ is

$$L_2 = \underline{x}_0 \tilde{B}_{00} \underline{u}' + (c-K) \underline{x}_0 \tilde{B}_{01}\underline{e} + \underline{x}_0 \tilde{B}_{01}\underline{u} + \underline{x}_1 \tilde{B}_0[R] \underline{u}' \tag{24}$$

$$+ (c-K) \underline{x}_1 \tilde{B}_1[R]\underline{e} + \underline{x}_1 \tilde{B}_1[R] \underline{u} + \underline{x}_1 (I-R)^{-1} \tilde{A}[R] \underline{u} +$$

$$K\underline{x}_1 (I-R)^{-2} \tilde{A}[R]\underline{e} + K\underline{x}_1 (I-R)^{-1} \tilde{A}[R]\underline{e},$$

where $\underline{u} = (0,1,\ldots,K-1)'$, $\underline{u}' = (0,1,\ldots,c-K-1)$, $\tilde{B}_0[R] = \sum_{n=1}^{\infty} R^{n-1}\tilde{B}_{n0}$, and $\tilde{B}_1[R] = \sum_{n=1}^{\infty} R^{n-1}\tilde{B}_{n1}$.

4. Numerical Methods

We first note that the matrices P and $\tilde{P}$, for a general value of K are simply related to the corresponding matrices for K = 1. Let Q and $\tilde{Q}$ be the transition probability matrices corresponding to the case K = 1, and let for any infinite stochastic matrix X with index set {0,1,2,...}, the "shifted" matrix ΔX be defined by

$$(25) \qquad (\Delta X)_{ij} = X_{i+1,j}, \qquad \text{for } i \geq 0,\ j \geq 0,$$

then it is easy to see that

$$(26) \qquad P = \theta_1 Q + \theta_2 \Delta Q + \ldots + \theta_K \Delta^{K-1} Q,$$

$$\tilde{P} = \theta_1 \tilde{Q} + \theta_2 \Delta\tilde{Q} + \ldots + \theta_K \Delta^{K-1}\tilde{Q}.$$

It is therefore sufficient to discuss the computation of the entries of the matrices Q and $\tilde{Q}$. Except for the first c columns, all entries of Q and $\tilde{Q}$ involve only the quantities

$$(27) \quad b_\nu = \int_0^\infty e^{-c\mu t} \frac{(c\mu t)^\nu}{\nu!} dF(t), \qquad \tilde{b}_\nu = \int_0^\infty e^{-c\mu t} \frac{(c\mu t)^\nu}{\nu!} \cdot \frac{1-F(t)}{\alpha} dt,$$

for $\nu \geq 0$.

A simple partial integration shows that

$$(28) \qquad \tilde{b}_\nu = \frac{1}{c\mu\alpha}\left(1 - \sum_{j=0}^{\nu} b_j\right), \qquad \text{for } \nu \geq 0,$$

so that only the quantitites $\{b_\nu\}$ remain to be found. In general, the latter require a careful numerical integration. If $F(\cdot)$ has a density, quadrature formulas using the properties of Laguerre polynomials may be profitably implemented. If $F(\cdot)$ is a discrete distribution, the probability density

$\{b_\nu\}$ is a mixture of Poisson distributions and the b_ν can be evaluated by a simple recursive scheme using the recurrence relation for the Poisson probabilities. In the important case, where $F(\cdot)$ is a distribution of phase type [8], the probabilities $\{b_\nu\}$ can also be evaluated by a numerically stable recursive method, which does not involve any numerical integrations [9,10].

In order to compute the first c columns of $\acute{Q}$, we first note that the rows labeled 0,...,c-1, have 1,...,c positive elements respectively. Moreover the row-sums are equal to one. These entries may be computed by routine numerical integrations.

Next, we observe that

$$\text{(29)} \qquad \sum_{j=0}^{c-1} Q_{rj} = 1 - \sum_{\nu=0}^{r-c+1} b_\nu = c\mu\alpha\, \tilde{b}_{r-c+1}, \qquad \text{for } r \geq c.$$

Also, for $0 \leq j \leq c-1$.

$$\text{(30)} \qquad \sum_{r=c}^{\infty} \sum_{j=0}^{c-1} Q_{rj} = c\mu\alpha$$

These relations can be used as accuracy checks and also to define truncation rules in procedures to evaluate the quantities Q_{rj}, for $r \geq c$, $0 \leq j \leq c-1$.

In general, these quantities may be computed by solving the system of differential equations

$$\text{(31)} \qquad \hat{\phi}'_{r+1,j}(t) = -\mu j\, \hat{\phi}_{r+1,j}(t) + \mu(j+1)\, \hat{\phi}_{r+1,j+1}(t), \qquad r \geq c,\ 0 \leq j \leq c-1,$$

numerically, after truncation at a sufficently high index r. We note that the system is inhomogeneous, since $\hat{\phi}_{r+1,c}(t)$ is equal to $e^{-c\mu t}\dfrac{(c\mu t)^{r-c+1}}{(r-c+1)!}$, for $r \geq c$. The initial conditions are $\hat{\phi}_{r+1,j}(0) = 0$, for $r \geq c$, $0 \leq j \leq c-1$. The probabilities Q_{rj} are then obtained by a progressive numerical integration method, such as Simpson's rule.

A routine partial integration yields that

(32) $$\tilde{Q}_{rj} = \frac{1}{\mu j \alpha} \sum_{\nu=0}^{j-1} Q_{r\nu}, \qquad \text{for } 1 \le j \le c-1,\ r \ge c.$$

The quantities $\tilde{Q}_{r0}$ may be found by

(33) $$\tilde{Q}_{r0} = 1 - \sum_{j=1}^{c-1} \tilde{Q}_{rj} - \sum_{\nu=0}^{r-c} \tilde{b}_{\nu}, \qquad \text{for } r \ge c.$$

The preceding discussion applies to any distribution F(•) and lists computational shortcuts, which are useful in all cases. If F(•) is a probability distribution of phase type, it is possible to compute all entries of Q and $\tilde{Q}$ by recursive schemes, which do not require any numerical integrations whatever. The details of this, as well as extensive numerical examples will be discussed elsewhere.

The next major step in the algorithm is the computation of the matrix R. The equation (9) may be quite efficiently solved by successive substitution in

(34) $$R = \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} R^n A_n (I-A_1)^{-1},$$

starting with R = 0. Extensive numerical experience, with matrices of order as high as fifty, indicates that convergence is rapid, except for queues which are close to critical.

The matrix B[R] may be routinely computed. Since B[R] is stochastic, the row-sums of the computed matrix should be close to one. Row-sums exceeding $1 - 10^{-9}$ can usually be attained without excessive computational efforts.

A further accuracy check is obtained by verifying that the matrix $\tilde{B}[R] + (I-R)^{-1} \tilde{A}[R]$ is stochastic. In practice, a large number of terms of the probability densities $\underline{x}$ and $\underline{y}$, as well as the distribution of waiting times and a variety of moments may be computed at a small cost and to an adequate degree of accuracy.

5. Related Queueing Models

The literature on the GI/M/c queue and its various generalizations is fairly extensive. A useful set of references is given in the paper [2] by U.N. Bhat, which deals with transient results in the case of single arrivals. F. G. Foster [4] and F. G. Foster and A. G. A. D. Perera [5] deal with group arrivals of fixed group sizes for the single server case. It may be verified that the roots inside the unit disk of the equation (1) in [4] are the eigenvalues of the matrix R, corresponding to the $GI^k/M/1$ queue, discussed there.

The extensions to bulk service, treated by P. B. M. Roes [13] and K. Shyu [14], affect only the analytic expressions for the entries of the matrix P, but not the general nature of the partitioning of P. The latter determines the form of the steady-state vector, so that the bulk arrival-bulk service extension of the GI/M/c queue may be treated by routinely adapting the method given here.

The algorithmic papers by F. S. Hillier and F. D. Lo [7], O. S. Yu [15], and D. M. Avis [1] deal with Erlang service times and heterogeneous servers, but involve single arrivals. Although also in this case, a matrix-geometric characterization of the steady-state vector may be given, the dimension of the matrix R is so large that it does not provide us with a feasible computational alternative to the method proposed by these authors.

Finally we note that the $GI^X/M/c$ queue with bounded group sizes for the arrivals, may also be considered as a particular case of an SM/M/c queue in which some of the sojourn time distributions of the semi-Markovian arrival process are degenerate. Details of this construction may be found in E.

Çinlar [3, p. 378]. The matrix-geometric characterization, given in Corollaries 3 and 4 of Neuts [11], of the steady-state queue length distribution for the SM/M/1 queue may be generalized in a direct manner to the SM/M/c queue. This approach would lead to the same computational results via different algorithmic steps and there may be some merit in an empirical comparison between the two methods.

REFERENCES

[1] Avis, D. M. (1977)
Computing Waiting Times in $GI/E_k/c$ Queueing Systems
in Algorithmic Methods in Probability, TIMS Studies in the Management Sciences, No. 7, 215 - 232, North-Holland Publ. Co.

[2] Bhat, U.N. (1968)
Transient Behavior of Multi-server Queues with Recurrent Input and Exponential Service Times
J. Appl. Prob., 5, 158 - 168

[3] Çinlar E. (1967)
Queues with Semi-Markovian Arrivals
J. Appl. Prob., 4, 365 - 379

[4] Foster, F. G. (1961)
Queues with Batch Arrivals I
Acta Math. Acad. Sci. Hung., 12, 1-10

[5] Foster, F. G., and Perera, A. G. A. D. (1965)
Queues with Batch Arrivals II
Acta Math. Acad. Sci. Hung., 16, 275 - 287.

[6] Gross, D. and Harris, C. M. (1974)
Fundamentals of Queueing Theory
John Wiley and Sons, New York

[7] Hillier, F. S. and Lo, F. D. (1971)
Tables for Multi-server Queueing Systems involving Erlang Distributions
Tech. Rept. No. 31, Dept. of O.R., Stanford University, Stanford, CA

[8] Neuts, M. F. (1975)
Probability Distributions of Phase Type
in - Liber Amicorum Professor Emeritus H. Florin
Department of Mathematics, University of Louvain, Belgium, 173 - 206.

[9] Neuts, M. F. (1975)
Computational Problems related to the Galton-Watson Process
Forthcoming in the Proceedings of an Actuarial Research Conference held at Brown University in 1975.

[10] Neuts, M. F. (1977)
Algorithms for the Waiting Time Distributions under Various Queue Disciplines in the M/G/1 Queue with Service Time Distributions of Phase Type
in Algorithmic Methods in Probability, TIMS Studies in the Management Sciences, No. 7, 177 - 197, North-Holland Publ. Co.

[11] Neuts, M. F. (1978)
Markov Chains with Applications in Queueing Theory, which have a Matrix-geometric Invariant Vector
Adv. Appl. Prob., 10, (forthcoming)

[12] Neuts, M. F. (1978)
Queues solvable without Rouché's Theorem
Operations Research (forthcoming)

[13] Roes, P. B. M. (1966)
A Many Server Bulk Queue
Operations Research, 14, 1037 - 1044

[14] Shyu, K. (1962)
On the queueing processes in the system GI/M/n with bulk service.
Chinese Math, 1, 196 - 204

[15] Yu, O. S. (1977)
The Steady-state Solution of a Heterogeneous-server Queue with
Erlang Service Times
in Algorithmic Methods in Probability, TIMS Studies in the Management
Sciences, No. 7, 199 - 213, North-Holland Publ. Co.