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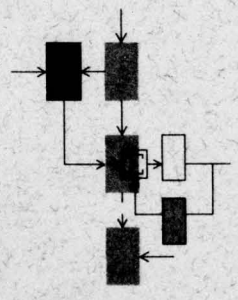

THE MULTIPLE ACCESS BROADCAST CHANNEL: PROTOCOL AND CAPACITY CONSIDERATIONS

John Ippocratis Capetanakis

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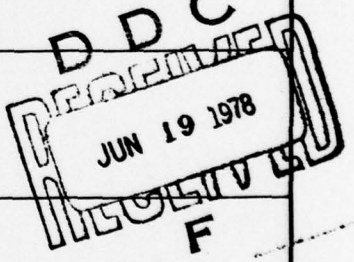

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Our work is directly applicable to packet switching broadcast networks, in which packets might contain data from such sources as computer, teletype terminals and vocoders. However, our results may also apply to more general systems, in which a central facility is accessible by a number of independent users. If the number of users that can be serviced simultaneously is less than the number that can demand service, the techniques developed here can be used to resolve the resulting contentions.

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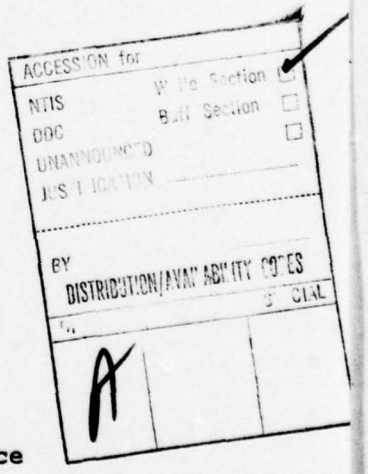
THE MULTIPLE ACCESS BROADCAST CHANNEL: PROTOCOL AND CAPACITY CONSIDERATIONS

by

John Ippocratis Capetanakis

This report is based on the unaltered thesis of John Ippocratis Capetanakis, submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the Massachusetts Institute of Technology August, 1977. The research was conducted at the M.I.T. Electronic Systems Laboratory, with support provided in part by the Office of Naval Research under Contract ONR-N00014-64-C-1183.



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THE MULTIPLE ACCESS BROADCAST CHANNEL: PROTOCOL AND CAPACITY CONSIDERATIONS

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THE MULTIPLE ACCESS BROADCAST CHANNEL: PROTOCOL AND CAPACITY CONSIDERATIONS

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Submitted to the Department of Electrical Engineering and Computer Science on August 25, 1977, in partial fulfillment of the requirements for the Degree of Doctor of Science.

ABSTRACT

The multi-accessing of a broadcast communication channel by independent sources is considered. Present accessing techniques suffer from long message delays, low throughput and/or congestion instabilities. The objective of this research, therefore, is to develop and analyze high speed, high throughput, stable, multi-accessing algorithms.

Contention resolving tree algorithms are introduced, and they are analyzed for specific probabilistic source models. It is shown that these algorithms are stable (in that all moments of delay exist) and are optimal in certain sense. Furthermore, they have a maximum throughput of .430 packets/slot and have good delay properties. It is also shown that under heavy traffic, the optimally controlled tree algorithm adaptively changes to the conventional TDMA protocol.

Our work is directly applicable to packet switching broadcast networks, in which packets might contain data from such sources as computer, teletype terminals and vocoders. However, our results may also apply to more general systems, in which a central facility is accessible by a number of independent users. If the number of users that can be serviced simultaneously is less than the number that can demand service, the techniques developed here can be used to resolve the resulting contentions.

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CHAPTER 1

INTRODUCTION

1.1 Problem Definition and Issues

A broadcast channel is a communications channel where a signal generated by one transmitter can be received by many receivers. Some examples of these channels are: a satellite channel where a satellite acts as a transponder; a ground radio network where a number of terminals have access to common frequencies; and coaxial cables where a number of terminals use a single cable for communicating by time multiplexing. When a number of spatially isolated independent sources have access to a common broadcast channel, a problem usually arises in allocating the channel capacity to the various users. This is so because the only means these independent users have of communicating their requests for channel capacity is through the channel itself.

There are three main issues that are involved in multi-accessing a common channel: a) the percentage of the capacity used in accessing, b) the time it takes to access the channel, and c) the stability of the multi-accessing system, i.e., how likely is it for the system to be in a state where many sources are attempting to access the channel and very few are actually succeeding. These three properties (throughput, delay, and stability) will be defined more precisely later on; however, it should be clear at this point that a desirable multi-accessing system is one that is stable, has high throughput and low accessing delay. As we will see in Section 1.3, when the number of sources is large and the message lengths short, present multi-accessing schemes suffer in at least one of these attributes. In this thesis,

therefore, we will be mainly concerned with the development of stable multi-accessing techniques that have high throughput and low delay.

More specifically, we will be concerned with the time domain multi-accessing of a common broadcast channel by having the sources transmit their data in packet form. A packet is a block of fixed length digital data that contains the information to be transmitted, along with the source and destination addresses and any other overhead information that might be necessary, such as error correcting and error detecting bits. We will assume that if more than one transmitter transmits simultaneously, then, they will interfere with each other, and all the packets will be received incorrectly. If, on the other hand, no packet collisions occur, then we assume that error free transmission results.

It is important for each transmitter (also to be referred to as a source) to be able to determine whether there are zero, one, or more than one packets in the channel at any one time. More than one packet corresponds to a collision, i.e., a channel contention. This channel state information may be obtained directly by listening to the channel or by some other means such as a central observer along with an auxiliary feedback channel. How the transmitters obtain this information is not important for our work. Once it is determined that a collision occurred, then the sources must take action to resolve this conflict. The resolution of this conflict is the heart of the multi-accessing problem, and it is here that we will focus our attention by developing and analyzing a new class of confict resolving algorithms. What makes this problem interesting is that, when an initial collision occurs, each of the contending sources knows that its packet collided; however, it does not know the identities or the number of the other contending sources.

The multiple access system to which our work is directly applicable is the so called packet switching broadcast network where the packets might contain data from such sources as computers, teletype terminals, or vocoders. The results of this thesis, however, are equally applicable to the various dynamic reservation multi-accessing systems in which there are two channels, a data channel and a reservation channel. In this system the reservation channel is used to make reservation requests for the data channel. Here the channel contention difficulties arise in the reservation channel and again one must resolve the conflicts between simultaneous requests on this channel.

Our results, however, need not be restricted to communication systems. They may be extended to more general systems in which a central facility is accessible by a number of independent users. If the number of users that the facility can service simultaneously is less than the maximum that can place demands upon it, then contentions will arise that might be solvable with the techniques developed here.

From the preceding discussion, we see that a multi-access system may be decomposed into three major components: the sources and messages, the channel and the multi-accessing protocol. These are discussed in more detail in the following sections. In Section 1.2 we present the channel and source models that will be used in this thesis and in Section 1.3 we consider the multi-accessing algorithms. In Section 1.4 we present an outline of the thesis, the analysis, and the main results. Finally in Section 1.5 we present the history and the work conducted by others in this field.

1.2 Channel and Source Models

In this section we will present the channel and source models that will be used in this thesis.

1.2.1 Channel Model

The channel is assumed to be slotted, that is, the channel time is divided into equal segments called slots. The length of each slot equals that of a packet, and it is assumed that a source, which is synchronized to the channel time, transmits a packet within only one slot. Furthermore, the channel is such that the sources can determine whether there are zero, one, or multiple packets in any one slot. Multiple packets per slot correspond to a collision and under such circumstances no one gets through. If, however, a packet does not collide with other packets, then it is assumed that the S/N is high enough or enough forward error correction is applied so that the packet is successfully transmitted. This last assumption is made so as to allow us to focus on the multi-access properties of the channel.

1.2.2 Source Models

We will consider two source models in this thesis. They will be designated as the Poisson and the finite source models.

i. Poisson Source Model

The Poisson source model assumes the existence of an infinite number of independent sources that collectively generate k packets per slot, where k is a Poisson random variable with constant mean λ. A source can have at

most two packets, one that has undergone a collision and is in the process of being retransmitted and one that may have arrived since the collision of the first packet occurred. If a second packet arrives, it is not transmitted until after the first is successfully transmitted. This last assumption will be discussed in more detail in Section 1.3 where the algorithms are considered.

ii. Finite Source Model

Here we assume that there are 2^N independent sources. This model is similar to the Poisson in that a source can have at most one packet in the process of being transmitted or retransmitted and at most one waiting to be processed. If a source has at most one packet then the probability that it will receive a packet in the next round trip interval is constant and it is given by ρ. It can be shown that the Poisson source model is the limiting case of the finite source model. That is, if we let $\rho 2^N$ = constant and let $N \to \infty$ then the finite model approaches the Poisson model.

1.3 Conflict Resolving Algorithms

Here we will consider the third component of the multiple access system, the conflict resolving algorithm. This section is organized into four subsections as follows. In Subsection 1.3.1, we first consider the TDMA and Aloha protocols, the two algorithms presently in use, and then present a third alternative, the tree algorithm. In Subsections 1.3.2 and 1.3.3 we expand upon the tree algorithm and in 1.3.4 we present an information theoretic discussion of the multiple access channel.

1.3.1 TDMA, Aloha and the Tree Algorithm

There are three basic techniques for accessing a communications channel in the time domain, the TDMA, the Aloha and the tree algorithm. In the TDMA scheme, contention is avoided by allocating a portion of the channel to each of the sources. Although TDMA is effective in situations where the number of sources is small and the message lengths are long, it suffers from low throughput and large delays when the number of sources is large and the duty cycle is short.

In the Aloha algorithm, when a source has a new packet, it transmits it, and then listens to the channel to determine whether or not the packet collided with packets from other sources. If a packet collision is detected, then the source retransmits the packet at a randomly selected time. The retransmission takes place at a randomly selected time so that conflicting packets will not surely collide again. It has been shown (see Section 1.5) that when the sources satisfy the Poisson source model, then the maximum throughput for the Aloha system is 1/e. However, a multi-access Aloha type system is unstable, and it eventually overflows. Therefore, although the

delay and throughput properties might be satisfactory in the short term they are quite poor when observed over a long interval of time.

There are numerous of dynamic reservation schemes in the literature (see Section 1.5). Most of these schemes use two channels; a reservation and a data channel. Before data can be transmitted, the data channel is reserved by TDMA in Aloha techniques on the reservation channel. It follows then, that the disadvantages of the TDMA and Aloha algorithms apply to the reservation channel.

The tree algorithm that we are about to introduce has (when used in conjunction with Poisson source model) a maximum average throughput of .430 packets/slot, is stable in that all the moments of the delay are finite if the arrival rate is less than .430 packets/slot and it has good delay properties. (The results of the analysis are presented in more detail in Section 1.4.) Below in Table 1.3.1.1 we present a qualitative comparison of the three algorithms when the number of sources is large and the message lengths are short.

	TDMA	Aloha	Tree Algr.
Delay	Poor	Fair	Good
Throughput	Poor	Good	Good
Stability	Good	Poor	Good

Table 1.3.1.1. A Qualitative Comparison of TDMA, Aloha and Tree Algorithm when the number of sources is large and the message lengths are short.

It is easiest to introduce the tree algorithm by an example. However, before we do this, we need to make the following definitions. Figure 1.3.1.1 should be helpful

Root node - the initial node of a tree; in Fig. 1.3.1.1 it is n_{00}.

Depth of a node - correponds to the tier at which the node is found. The root node is at depth zero.

Degree of a node - is the number of branches that emanate from a node.

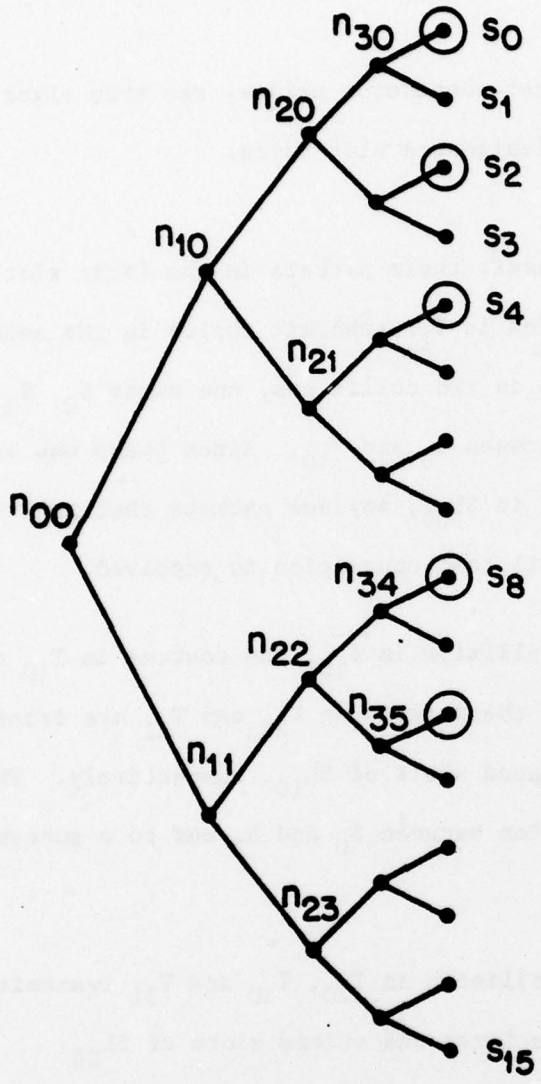
Subtree T_{ij} - the subtree whose root node is n_{ij}. Note that in a binary tree, j corresponds to the particular node of depth i, and that there are 2^i nodes of depth i.

Symmetry - a tree is symmetric if all nodes of equal depth have equal degrees.

Example: The Binary Tree Algorithm

Let there be 16 sources $\{S_0, S_1, \ldots, S_{15}\}$ and let each correspond to a leaf of a 16-leaf binary tree as shown in Fig. 1.3.1.1. The tree may be considered as an addressing procedure, in which each source has a 4-bit address depending on its location on the tree. In Fig. 1.3.1.1 we also present the slotted satellite time. For convenience, we assume that the round trip delay is zero; the effect of a nonzero round trip delay is considered in Section 1.3.3. Note that the slots are paired and that a slot pair is designated by SL_{ij}. There is a relationship between the subscripts of the nodes and those of the slot pairs; as we will see, T_{ij} transmits its packets in SL_{ij}.

Now assume that no collisions have occurred until the beginning of SL_{00}, when sources S_0, S_2, S_4, S_8, and S_{10}, each has a packet to transmit. Then



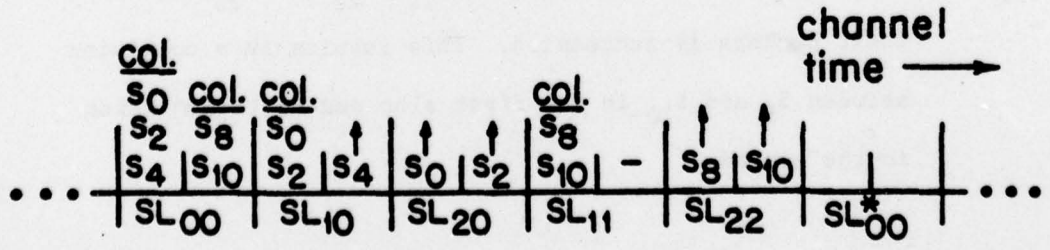


Figure 1.3.1.1 An Example of the Binary Tree Algorithm

beginning with SL_{00}, where the first contention arises, the tree algorithm takes the following steps in the designated slot pairs.

SL_{00} Sources in T_{10} transmit their packets in the first slot of SL_{00}, and the sources in T_{11} transmit theirs in the second slot. This results in two collisions, one among S_0, S_2, and S_4 and the other between S_8 and S_{10}. Since there was at least one collision in SL_{00}, any new packets that arrive are not transmitted until that contention is resolved.

SL_{10} Since there was a collision in T_{10}, the sources in T_{10} are divided in half and the packets in T_{20} and T_{21} are transmitted in the first and second slots of SL_{10}, respectively. This results in a collision between S_0 and S_2 and to a successful transmission by S_4.

SL_{20} Since there was a collision in T_{20}, T_{30} and T_{31} transmit their packets in the first and second slots of SL_{20}, respectively. This results in two successful transmissions by S_0 and S_2.

SL_{11} Since there was a collision in T_{11}, T_{22} and T_{23} transmit their packets is succession. This results in a collision between S_8 and S_{10} in the first slot and no transmission in the second.

SL_{22} Since there was a collision in T_{22}, T_{34} and T_{35} transmit. This results is two successful transmissions by S_8 and S_{10}.

SL^*_{00} Since all the sources were involved in some transmission in which no collisions occurred, we know that the original contention has been resolved. Any new packets that may have arrived to T_{10}, during this conflict resolution interval, are transmitted in the first slot of SL^*_{00}, and packets that arrived to T_{11} are transmitted in the second slot. The process continues on, as described above.

Note that in this example we used 10 slots to transmit 5 packets. Next we will state and discuss the binary tree algorithm.

The Binary Tree Algorithm

Let each source correspond to a leaf in a binary tree. If the number of sources is infinite, then the tree extends to infinity. The slots are paired into odd and even slots, and until a collision occurs the sources in T_{10} transmit their packets in the odd slots, whereas, the sources in T_{11} transmit theirs in the even slots.

Now let T_{x1} and T_{x2} be two variables and assume that no collision occurred up to the beginning of the present pair of slots. Thus, the binary tree algorithms is as follows:

1. $T_{x1} = T_{10}$; $T_{x2} = T_{11}$.
2. T_{x1} transmits in the first slot of the present pair of slots, and T_{x2} transmits in the second.

3. If any collisions occur in the preceding step, then

   a. Until they are resolved, no new packets are transmitted.

   b. Resolve the first collision before resolving the second.

   A collision in T_{xi} (i = 1,2) is resolved by dividing T_{xi} in two halves (say A and B), letting T_{x1} = A, T_{x2} = B and then repeating steps 2 and 3.

This algorithm is equivalent to the following tree search. Beginning with the root node and at each succeeding node, one asks whether there are zero, one, or more than one packet in each of the two emanating branches. If the answer for any of the branches is more than one, then proceed to the base nodes of those branches and repeat the question. This continues until all the leaves are separated into sets such that each set contains at most one packet. Note that the number of slots used in a conflict resolution interval equals twice the number of nodes visited.

The tree search may be carried out in one of two ways, serially or in parallel. In the serial search, two branches transmit their packets in two consecutive slots, and the results of those two transmissions are resolved before another two subtrees are allowed to transmit. In the parallel search, all the branches at some depth, whose parents have had a collision, transmit their packets in consecutive slots. It should be clear that the number of slots needed to process any particular set of sources is the same for both schemes.

The question arises as to whether some other tree besides the binary tree might not be more efficient, in that it requires fewer slots to resolve a conflict. The determination of the optimum tree is one of the major problems

that is solved in this thesis. More specifically, we will develop the optimum dynamic tree algorithm, in which the tree that is used to process a conflict is that which minimizes the average number of slots used, given the history of the transmission process; more will be said about this in Section 1.4.

The control for the tree algorithm may be centralized or distributed. If a central control is used, then an observer observes the transmission process and notifies, via a feedback channel, the sources that may transmit in the next slot. If, on the other hand, a distributed control is used, then all the sources must observe the channel and each must execute the algorithm by itself.

We will present an interesting variation to the above algorithm, before concluding this section. Step 3a of the algorithm may be changed somewhat, so that some packets that arrive after a collision occurs may be transmitted before that initial contention is resolved. In the example above, for instance, packets, that arrive to T_{23} before SL_{11}, could be transmitted in SL_{11} and they would be treated similarly to the way they would have been, had they arrived in SL_{00}. This variation to the binary tree algorithm appears to be more efficient then the original algorithm itself. However, it is more complicated and we have not analyzed it.

1.3.2 Deterministic and Random Source Addressing

In the tree algorithm, as it is given in the preceding section, each source is preassigned deterministically an address, i.e., a position on the tree. A variation to the deterministic addressing which has certain imple-

mentation advantages, is the random address assignment scheme. Here, the sources are not preassigned addresses on the tree. However, when a collision occurs, a contending source conducts the tree search by independently and with equal probability deciding to take the upper or the lower branch that emanate from each node, beginning with the root node. In general, if any collisions occur at nodes at one tier, the contending sources move to the next tier by randomly deciding which branch to take. This process continues until no collisions occur. As can be seen, the objective of this search is the same as that of the deterministic addressing scheme; it is to divide the sources into sets such that each set contains at most one active source.

Note that in the random addresssing scheme, the tree is infinite whether the number of sources is finite or infinite. Therefore, it can be shown that when the number of sources is finite, random addressing is slightly inferior to deterministic addressing in terms of delay and throughput. However, when the number of sources is infinite, such as in the Poisson model, then the two schemes are identical. To see this, note that in random addressing, the sequential random choice decisions that each contending source makes are equivalent to allowing each of the contending sources to choose an infinite dimensional address, and then to execute the tree algorithm as it is given in the preceding section. The only difference, therefore, between the two schemes is the way the addresses are chosen. Note, however, that the statistics of the addresses of the contending sources are the same under both schemes. This is so because in one scheme, the contending sources randomly choose them and in the other the addresses are preassigned but the sources are chosen randomly. Since the address statistics are identical and since the algorithm

is the same in both cases, we conclude that in the Poisson source model the throughput, delay and stability properties are the same for both addressing procedures.

The implementation advantage of the random addressing scheme is due to the fact that the addresses are not preassigned. Therefore, sources may enter or leave the system with much greater ease. Note that the tree search for the random addressing scheme may be carried out serially or in parallel, and that the tree need not be binary.

1.3.3 Sample Transmission Process and Definitions

Here, we will define some of the quantities that we will be using, and we will consider the effects of the round trip delay by presenting a sample transmission process for the serial search binary tree (SSBT) algorithm.

In the execution of the SSBT algorithm, two branches transmit their packets in a pair of consecutive slots; following this, no action is taken until the results of these two transmissions are received, when two more branches are allowed to transmit. An example of such a transmission process, in which the round trip delay equals four slots, is illustrated in Fig. 1.3.3.1. In this example, since one algorithm uses only 1/3 of the channel capacity, one may either divide the sources into three groups and process each group independently on 1/3 of the channel by a tree algorithm or use one tree algorithm on 1/3 of the channel to reserve the other 2/3's in a dynamic reservation scheme.

In any case, we will focus our attention on one algorithm. Therefore, we will assume that our channel consists only of those slots that are used

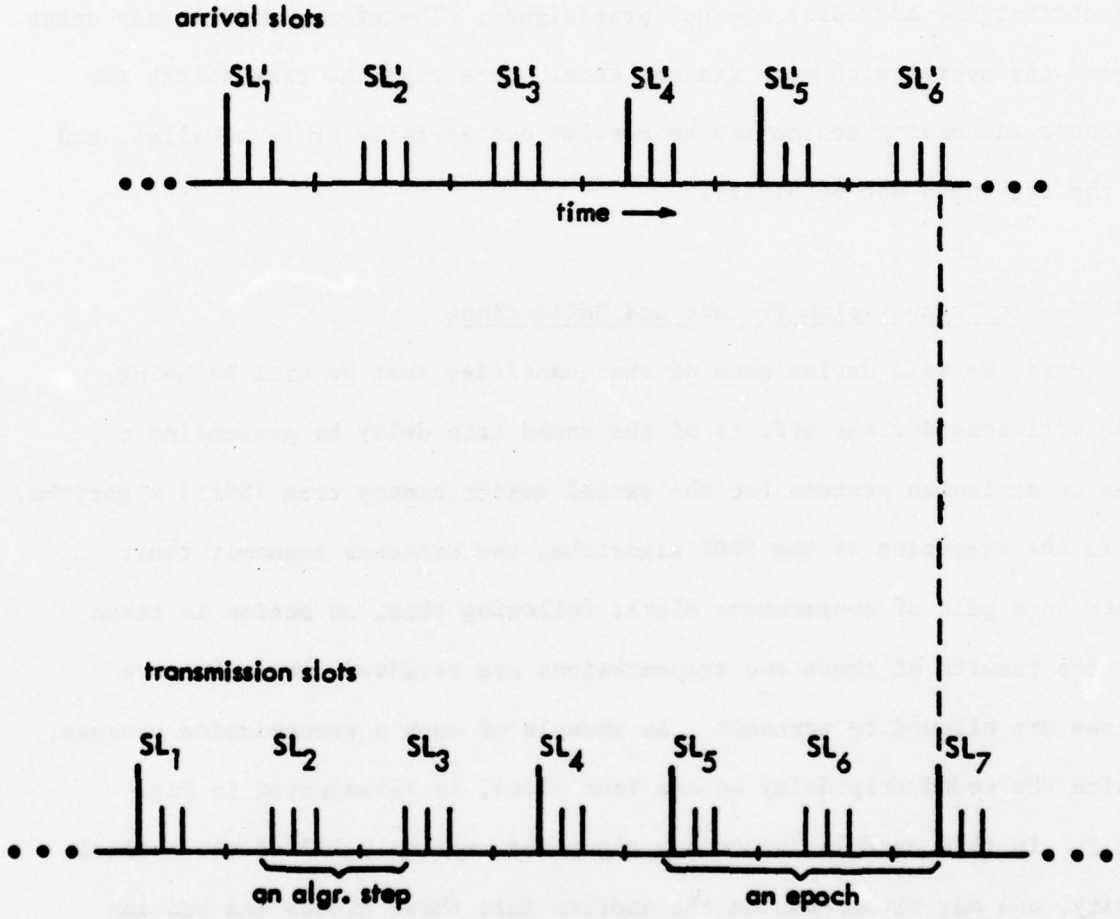


Figure 1.3.3.1 A Sample Transmission Process

by that algorithm. In Fig. 1.3.3.1 the channel that we are interested in is that composed of SL_1, SL_2, SL_3, ..., etc.

At this point, we will make several definitions.

Algorithm Step - It consists of the transmissions taken in a pair of slots, the observation of the results of those transmissions, and the decision as to what action to take in the next pair of slots. The time span of a step, therefore, equals the round trip delay plus the length of two slots. In Fig. 1.3.3.1, a step equals all the actions taken from the beginning of SL_1 to the beginning of SL_2.

Epoch - an interval of conflict resolution, if a conflict exists; otherwise, it is a pair of slots. In Fig. 1.3.3.1, [SL_1, SL_2, SL_3], [SL_4], [SL_5, SL_6] are three consecutive epochs.

ℓ_j - the length of the j'th epoch in algorithmic steps.

h_j - the number of slots used in the j'th epoch. This equals $2\ell_j$.

λ - the average of the total number of packets arriving in one slot.

μ_j - the average number of packets arriving in the j'th epoch. In the text, it is shown that for the Poisson source model $\mu_j = \lambda h_j$.

δ - packet delay; i.e., the time spent in the system by a packet. Normally we will express the delay in terms of algorithm steps.

average delay - the delay of a randomly chosen packet.

average throughput - the fraction of the slots, over a very long interval, that contain exactly one packet each.

stability - the system is K'th order stable if the K'th moment of the delay is finite.

In terms of the preceding definitions, in packet switching networks, packets that arrive in one epoch are transmitted in the following epoch. Note, however, that since only a portion of the channel is used by the algorithm, only those packets arriving in slots corresponding to those used by the algorithm in one epoch will be processed by the same algorithm in the following epoch. In the example of Fig. 1.3.3.1, the packets that arrive in slot pairs SL_1', SL_2', and SL_3' are transmitted in SL_4. Those that arrive in SL_4' are transmitted to SL_5 and those that arrive in SL_5' and SL_6' are transmitted in SL_7.

An interesting and a very useful property of the transmission process is that under the finite, as well as the Poission source model, ℓ_j is an embedded Markov chain. That is, ℓ_j can be considered to be the state of a Markov chain after the j'th transition. To see this, first note that ℓ_{j+1}, given ν_j, is independent of the transmission process up to the end of the j'th epoch. (ν_j is the number of packets arriving in the j'th epoch.) This observation coupled with the following probabilities (which are developed in the text) prove the Markovian property.

$$p(\nu_j|\ell_j,\ell_{j-1},\ldots) = \frac{(2\lambda\ell_j)^{\nu_j} e^{-(2\lambda\ell_j)}}{\nu_j!} \quad \text{for the Poisson model.}$$

$$= p(\nu_j|\ell_j)$$

$$p(\text{a packet arrives to a particular source in the j'th epoch}|\ell_j,\ell_{j-1},\ldots)$$

$$= 1 - (1-\rho)^{\ell_j} \quad \text{for the finite model.}$$

1.3.4 An Information Theoretic Approach to Multi-Accessing

As we will see, the maximum average throughput of the optimum dynamic tree algorithm is .430 packets/slot when the sources satisfy the Poisson model; whereas, the maximum throughput for the Aloha system is $1/e = .368$. Since the performance of these two algorithms is different, the question arises as to whether there are any other algorithms which are better, or, more interestingly, what is the maximum possible throughput to a particular multiple access system under all possible algorithms. We have attempted to answer these questions but were unsuccessful. However, our approach to this problem offers some insight into the multi-accessing problem and since it might be the basis for further research, we outline it here.

We have pointed out in the preceding section that the basic problem in a multiple access system is the resolution of the conflicts. So, let us examine this a little more carefully. Let us assume that we have a set of independent sources, and ν of them are active, i.e., have packets to transmit. Furthermore, let the probability measure on ν be $p(\nu)$.

Now, note that the conflict is resolved iff the sources are subdivided into sets such that each set contains at most one active source. Therefore, this partitioning of the sources must be the objective of any conflict resolving algorithm, whether it be the Aloha, the tree, or any other that might be proposed in the future. In other words, the execution of the algorithm must supply enough information so that one can partition the sources into sets such that each set contains at most one active source. Let $H_{min}(\text{source})$ be the minimum average information required to do this.

Next, by observing the contents of a slot, we learn whether there are zero, one, or more than one active sources in the set that is transmitting in that slot. Let H_{max} (trans.) be the maximum average information (maximized over all partitions) that can be obtained from any one slot; certainly, $H_{max}(\text{trans.}) \leq \log_2(3)$. Therefore, the minimum average number of slots required to resolve the conflict under any algorithm must be equal to or greater than $H_{min}(\text{source})/H_{max}(\text{trans.})$.

We did not proceed beyond this formulation because we were unable to obtain good bounds to $H_{min}(\text{source})$.

1.4 Thesis Outline and Results

The thesis is organized into four chapters. Chapter 1 is this introduction. The analysis is carried out in Chapters 2, 3 and 4 and in the appendices that accompany these chapters. Here we will consider the objectives and results of each of these chapters.

In Chapter 2, we analyze the serial search binary tree algorithm when it is used in conjunction with the Poisson source model. There are two main results in this chapter. The first concerns the average delay vs. average throughput trade-off; we obtain upper and lower bounds to the average delay, as a function of λ. These bounds are given in Eqs. (2.2.4.2) and (2.2.4.6) and they are illustrated in Fig. 2.2.4.1. It is shown that these results may be interpreted as average delay vs. average throughput, and we show that the maximum average throughput is .347 packets/slot. Furthermore, we show that it is possible to obtain a throughput of up to .430 packets/slot, but only for a limited time.

The second main result of Chapter 2 concerns the stability of the binary tree algorithm. Here, we prove that if $\lambda < 1/3$ packets/slot, then all the moments of the delay are finite. However, we observe that this is an overly conservative result, and point out that indications are that all the moments of the delay are finite for $\lambda < .347$ packets/slot.

In Chapter 3, we determine and analyze the optimum dynamic tree algorithm and examine a suboptimum algorithm that has certain implementation advantages. This algorithm is called optimum dynamic, because the tree is allowed to vary from epoch to epoch optimally depending on the traffic. The source model that is assumed in this chapter is the Poisson.

More specifically, we show that the tree which minimizes the expected number of slots needed to process ν packets, where ν is a Poisson random variable, is binary everywhere except for the root node whose degree g_0 depends on μ and is given by Eq. (3.1.0.2).

We point out that $\mu = h\lambda$; therefore, by observing the number of slots in the preceding epoch, we can determine μ and hence from Eq. (3.1.0.2) determine the optimum tree to be used in the next epoch. In order to simplify the analysis, we restricted the degree of the root node to be 2r. Thus, the root node corresponds to r algorithm steps and each of the other nodes corresponds to one step. For this dynamic algorithm we have obtained upper and lower bounds to the average delay as a function of λ. These results are given in Eqs. (3.3.2.13) and (3.3.2.14) and displayed in Fig. 3.3.2.2. We also show that the maximum average throughput is .430 packets/slot, and we prove that all the moments of the delay are finite for $\lambda < .430$ packets/slot.

In Chapter 3, we also consider the more easily implemented algorithm in which the root node degree is restricted to be 2^K ($K > 0$), and all other nodes are binary. Subject to the above constraints, first, we determine K^*, the K which minimizes $E\{h_{j+1}|\mu_j,K\}$; this is given in Eq. (3.1.0.3). Next we determine upper and lower bounds to the average delay vs. λ; these are given in Eqs. (3.4.2.13) and (3.4.2.14) and displayed in Fig. 3.4.2.1. The maximum average throughput is shown to be greater than .420 but less than .430 packets/slot.

In Chapter 4 we consider the static and dynamic tree algorithms when they are used in conjunction with the finite source model. For the static binary algorithm we obtain an average delay upper bound vs. average throughput lower bound curve. This is shown in Fig. 4.2.2.2 for 64 sources. The maximum throughput for the 64 source model is .507 packets/slot.

The dynamic tree was restricted to be binary everywhere except for the root node whose degree was restricted to be $g_0=2^K$. This algorithm was optimized, as before, over K and the results are given by Eqs. (4.3.1.1) and (4.3.1.6). An interesting observation, on the optimum dynamic algorithm, is that, under low traffic, it is identical to the binary tree algorithm. However, as the traffic increases this algorithm adaptively changes to a tree that has only one node with 2^N branches, which is recognized to be the TDMA protocol. (Note 2^N equals the number of sources.)

The delay-throughput characteristics of this optimum dynamic tree are determined, and are illustrated in Fig. 4.3.2.5 for $2^N = 64$. The maximum average throughput for this algorithm is one packet/slot. Chapter 4 is concluded with a theorem proving that the average delay of the optimum dynamic tree is less than or equal to that of the TDMA protocol.

1.5 History

The types of multiple access problems that have been considered by others fall into two main categories, circuit switching and packet switching. A third category contains the various dynamic reservation techniques, but this can be considered to be a hybrid of the first two classes.

With circuit switching, the channel is partitioned and allocated (leased) before hand. The sharing of the channel this way has been accomplished either by FDMA (Frequency Domain Multiple Access), or by TDMA (Time Domain Multiple Access) or by a combination of FDMA and TDMA. These techniques have been the only ones that were in use up to about 1970 [1] and they were quite effective for the communication needs of that time. If, however, there is a large number of sources that do not require continuous full use of the channel or if the data is bursty, i.e., the ratio of peak data rate to average data rate is high, then circuit switching can lead to long transmission delays and inefficient use of the channel.

In recent years attention has shifted to packet switching forms of multiple access. With this form of accessing, as was pointed out earlier, the transmitter formats the data into a packet of constant length along with source and destination addresses and error detecting bits. The source then transmits the packet to all the receivers including the desired one. If a packet is destroyed in transmission, the originating source learns about it either through a feedback channel or by listening to its own transmission. When a source determines that its packet has been destroyed it retransmits that packet at a randomly selected time. Most of the work

that has been done to date on packet switching broadcast systems assumes that the channel is noise free and that all the packets involved in a collision are destroyed.

There have been two approaches to packet switching broadcast systems; pure Aloha and slotted Aloha. In pure Aloha, packets are transmitted or retransmitted asynchronously; in slotted Aloha the sources are synchronized so that packets are transmitted in phase. In dynamic reservations the sources first dynamically reserve the channel capacity via an Aloha or a TDMA channel and then transmit their data.

Pure or classical Aloha has been studied by Abramson [1]. He assumed that 1) the starting times of the packets that are offered to the channel for the first time from all the sources comprise a Poisson point process, and 2) the starting times of all the packets (new plus retransmitted packets) comprise another Poisson point process. Given these two assumptions which taken together imply an equilibrium condition, he proves that the capacity of the system is $\frac{1}{2e}$ or approximately .184.

Roberts [2] pointed out that considerable improvement can be made in the capacity of the Aloha channel by synchronizing the sources so that all the packets arrive at the channel in phase. He proposed that the channel time be divided into slots and sources be allowed to use at most one slot per packet. By doing this he showed that capacity of the slotted system for the Poisson source model is $\frac{1}{e}$ or twice that of pure Aloha.

Metcalf [3] considered Aloha systems with blocking (that is a source may not generate a second packet until the first is successfully transmitted) and examined several random retransmission policies. More important he showed

that Aloha channels may be unstable in that the number of blocked sources can become very large. He also proposed control techniques by varying the retransmission probabilities.

Kleinrock and Lam [4], [5] quantified and extended Metcalf's results. They modeled the slotted Aloha system with blocking by a Markov Chain whose state corresponded to the number of blocked sources. They showed that the slotted Aloha channel with an infinite but independent population (i.e., the Poisson source model) is unstable. They further showed that if the number of sources is finite then the Aloha system may be bistable in that it is possible to have two stable operating points - one with a small number and one with a large number of blocked sources. Random perturbation in the channel traffic will cause the system to vacillate between these two stable points. They also proposed controlling the system through the retransmission probabilities and derived an optimum control policy based on exact knowledge of the state of the channel. Carleial and Hellman [6] also modeled the Aloha system as a Markov chain and examined its bistable behavior.

Several dynamic reservation schemes have been proposed. Basically, these techniques use either slotted Aloha or TDMA to make the reservations. Two protocols that use Aloha techniques to reserve the channel are Reservation-Aloha introduced by Crowther et.al. [7] and Interleaved Reservation-Aloha suggested by Roberts [8]. In Reservation-Aloha the channel slots are grouped into frames that are at least a round-trip delay long. A source that has successfully used a particular slot in one frame has access to that slot in the following frame. If a slot is unused then it is up for grabs and any one can contend for it by random Aloha techniques. In the Interleaved Reservation-Aloha

the channel is divided into two states. Reservation and Aloha. On the Reservation state the sources attempt to reserve the Aloha state through slotted Aloha techniques. As traffic increases the percentage of the channel being in the Aloha state increases allowing for greater utilization of the channel. In the Interleaved Reservation Aloha as the ratio of message length to reservation packet length increases the maximum throughput approaches 100%. Roberts also showed (in an example where the average message length was 27 times that of the reservation packet) that the transmission delay for this scheme is better than slotted Aloha at large throughput but not as good for throughput less than 20%. Crowther, et.al. do not present an analysis of their scheme but one would expect a behavior similar to Robert's algorithm. Stability is not considered in either paper but as long as random accessing techniques are used for the reservation, stability is an issue.

Binder [9] offers a dynamic reservation scheme that uses TDMA techniques. Essentially what he proposes is that the slots be grouped into frames and each source be allocated a slot to which it has first priority. If a slot is not used by its owner then it is available to the other sources on a round-robin basis. Unfortunately, here again there is no analysis. Limited simulations indicate that this protocol is better than slotted Aloha at high traffic and worse at low traffic.

In the work that has been discussed up to now the authors assumed that if more than one packet is transmitted simultaneously then a collision occurs and all the packets are destroyed. Roberts [2] points out, however, that this need not be the case. FM receivers will track the strongest of many signals as long as the next strongest is down by 1.5 to 3 dB. He examines the problem where one receiver is surrounded by an equal density population of equal power trans-

mitters and shows that it is possible for the capacity to increase to .60 with FM capture. His example is for ground radio systems, but one should be able to take advantage of FM capture in satellite systems where the sources are essentially equidistant from the satellite by regulating the power of the various transmitters.

Another technique that offers considerable improvement to Aloha systems that have a small maximum time delay is CSMA (Carrier Sense Multiple Access). In CSMA a terminal with a packet to transmit first listens for the carrier of other users to determine whether the channel is busy. If it is busy then the source refrains from transmitting, whereas if the channel is empty of other carriers the source will transmit its packet with some probability. A collision will occur if the time between transmissions of two packets is less than the time delay between the corresponding sources. In case of a collision the contending sources retransmit at randomly selected times. Kleinrock and Tobagi [10], [11] have examined CSMA and have shown that if a) there are no hidden terminals, i.e., every terminal can hear every other terminal, b) the time to detect the carrier is zero, and c) the distance between terminals is not large than considerable improvement can be made. For example, if the ratio of the maximum time delay between terminals to packet length is .01 then the channel capacity is .86. CSMA can be quite effective in ground packet switching channels, but it is ineffective in satellite systems.

CHAPTER 2

STATIC BINARY TREE ALGORITHM WITH POISSON SOURCE MODEL

2.1 Introduction

In this chapter we will examine the delay, throughput and stability properties of the static binary tree algorithm when it is used in conjunction with the Poisson source model. We will restrict the analysis to the serial search and to the deterministic address assignment. As was pointed out in Chapter 1, under the Poisson assumption, the random and deterministic source address assignment schemes are identical in terms of delay, throughput and stability. However, the parallel and serial search algorithms have the same throughput, but the average delay of the parallel search scheme is less than that of the serial.

The average packet delay is defined to be the delay that a randomly selected packet undergoes, and it is the topic of Section 2.2. There, we obtain upper and lower bounds to the mean packet delay as a function of the arrival rate λ. These bounds are given by Eqs. (2.2.4.2) and (2.2.4.6) and are displayed in Fig. 2.2.4.1.

The average throughput is defined to be the fraction of slots over a very long interval of time that contain exactly one packet each. It is considered in Section 2.3, where we argue that if the average delay is finite, then the average arrival rate equals the average throughput. In that section we also show that the maximum average throughput is .347 packets/slot and that it is possible to attain a throughput of up to .430 packets/slot but only for a limited time.

The system is considered to be k'th order stable if the first k moments of the delay are finite. Stability is considered in Section 2.4. There, first, we show that if $\lambda < .347$ packets/slot, then the system is first order stable. Secondly, we prove that if $\lambda < 1/3$ then all the moments of the delay are finite. This result, however, seems to be overly conservative, and indications are that all the moments of the delay are finite if $\lambda < .347$ packets/slot. Accompanying the main text is Appendix A2 where some crucial results of this chapter are developed. A detailed outline of Appendix A2 is given in the following section after several definitions are made.

2.2 Average Delay

The objective of this section is to develop a characterization of the average packet delay in a multi-access communication system that uses the static binary tree protocol. More specifically, we will determine upper and lower bounds to the average delay, $E\{\delta\}$, that are functions of the packet arrival rate λ. (See Eqs. (2.2.4.2) and (2.2.4.6) and Fig. 2.2.4.1.)

The unit of measure for the delay is the time between two successive steps of the algorithm. This is constant and it normally equals the round trip delay plus the transmission time of two packets. The delay, therefore, that a packet undergoes is directly proportional to the number of steps that are executed by the algorithm from the time when a packet arrives to the time when it is received correctly.

The expected delay is defined to be the delay that a randomly chosen packet will undergo. Note that inherent in the definition of $E\{\delta\}$ is the concept of random incidence, this will play a central role in the analysis.

As has been discussed previously, packets that arrive in one epoch (an epoch is an interval of conflict resolution if a conflict exists or simply a pair of slots if there is no conflict) wait until that epoch ends and are transmitted in the following epoch. With this in mind, several definitions are presented below. All lengths are in algorithmic steps.

ε_1 = the epoch in which a randomly chosen packet arrives.

ε_2 = the epoch following ε_1.

y_1 = the length of ε_1. This is a random incidence random variable.

ℓ_j = a random variable that equals the length ε_j. The probability that $\ell_j = L$ is the steady state probability that an epoch of length L occurs.

$\overline{\ell}_u$ = upper bound to $E\{\ell\}$

$\overline{\ell}_\ell$ = lower bound to $E\{\ell\}$

$\overline{\ell_u^2}$ = upper bound to $E\{\ell^2\}$

$\overline{\ell_\ell^2}$ = lower bound to $E\{\ell^2\}$

h_j = the number of slots used by the algorithm in ε_j.

Note that $h_j = 2\ell_j$ for the binary tree algorithm.

d_j = the time spent by a packet in ε_j.

ν_j = the number of packets that arrive in ε_j.

λ = the average number of packets that arrive in any one slot.

μ_j = the average number of packets that arrive in ε_j given ℓ_j. Note that $\mu_j = E\{\nu_j|\ell_j\} = 2\lambda\ell_j = \lambda h_j$.

δ = the packet delay, or the time spent in the system by a packet. This quantity equals the sum of d_1 and d_2.

$\overline{\delta}_\ell$ = lower bound $E\{\delta\}$

$\overline{\delta}_u$ = upper bound $E\{\delta\}$

As we proceed, we will further expand upon some of the relationships of the above quantities.

Next we will present an outline of the analysis that follows. In Section 2.2.1 we develop the relationship between μ_j, ℓ_j and λ. In Section 2.2.2 upper and lower bounds to $E\{\delta\}$ are developed that are functions of $E\{\ell\}$ and $E\{\ell^2\}$. In Section 2.2.3 upper and lower bounds to $E\{\ell\}$ and $E\{\ell^2\}$ are developed that are functions of λ. Finally, in Section 2.2.4 we combine the results of Sections 2.2.2 and 2.2.3 to obtain upper and lower bounds to $E\{\delta\}$ that are functions of λ.

In Appendix A2 we prove some of the more tedious but still very crucial results of this chapter. This appendix is organized into eight sections. In Section A2.1, we derive an expression for $E\{\ell_{j+1}|\mu_j\}$ and in A2.2 some important properties of $E\{\ell_{j+1}|\mu_j\}$ are developed. In Section A2.3 an expression for $E\{\ell_{j+1}^2|\mu_j\}$ is derived and its properties are developed in A2.4. In Section A2.5 it is proved that $E\{\ell_j \ell_{j+1}\} \geq E\{\ell_j\}E\{\ell_{j+1}\}$. In A2.6 $E\{d_{j+1}|\mu_j\}$ is derived and in A2.7 we develop the relationship between $E\{d_{j+1}|\mu_j\}$ and $E\{\ell_{j+1}|\mu_j\}$. Finally in A2.8 we obtain upper bounds to $E\{e^{s\ell_{j+1}}|\mu_j\}$ and $\partial E\{e^{2\ell_{j+1}}|\mu_j\}/\partial s$.

2.2.1 Relationship Among μ_j, ℓ_j and λ.

As defined above, λ is the average number of packets that arrive in any one slot, μ_j is the average number of packets that arrive in ε_j and ℓ_j is the number of steps executed by the algorithm in ε_j. For the binary tree which is being considered here, ℓ_j also equals the number of nodes visited by the algorithm. It follows from the above definitions that the length of ε_j in seconds is given by $\ell_j(\tau_r + 2\tau_s)$ where τ_r is the round trip delay and τ_s is the length of one slot.

Another relationship which will be proved here is the following:

$$\mu_j = 2\lambda\ell_j \tag{2.2.1.1}$$

To see this, first note that the binary tree algorithm uses two slots for each step; therefore

$$h_j = 2\ell_j \tag{2.2.1.2}$$

Next note that the total number of packets that arrive in ε_j equals the sum of the packets that arrive in each of the h_j slots, and since the expectation of a sum equals the sum of the expectations, it follows that

$$\mu_j = h_j\lambda \tag{2.2.1.3}$$

Equation (2.2.1.1) follows from Eqs. (2.2.1.2) and (2.2.1.3).

2.2.2 Characterization of E{delay} in terms of E{ℓ} and E{ℓ²}

In this section we will develop upper and lower bounds to $E\{\delta\}$ that are functions of $E\{\ell\}$ and $E\{\ell^2\}$. More specifically if we let $\overline{\delta}_u$ and $\overline{\delta}_\ell$ be the upper and lower bounds to $E\{\delta\}$, respectively, then we will prove that

$$\overline{\delta}_u = 1.05\,\frac{E\{\ell^2\}}{E\{\ell\}} + .321 \tag{2.2.2.1}$$

and

$$\overline{\delta}_\ell = \frac{1}{2}\left[\frac{E\{\ell^2\}}{E\{\ell\}} + E\{\ell\}\right] \tag{2.2.2.2}$$

Now we will begin with the analysis. The total delay that a randomly chosen test packet will undergo can be decomposed into two parts; d_1, the time spent in ε_1 (the epoch in which it arrived) and d_2, the time spent in ε_2 (the epoch in which it is transmitted).

$$\delta = d_1 + d_2 \tag{2.2.2.3}$$

and

$$E\{\delta\} = E\{d_1\} + E\{d_2\} \tag{2.2.2.4}$$

Next we will derive expressions for $E\{d_1\}$ and $E\{d_2\}$. We begin with $E\{d_1\}$.

If we let y_1 be the length of the epoch that the test packet entered, then y_1 is a random incidence random variable. It is well known [Ref. 12, p. 149] that the density of y_1 can be expressed in terms of the density of ℓ_1 as follows,

$$p_{y_1}(y_1) = \frac{y_1 p_{\ell_1}(y_1)}{E\{\ell_1\}} \tag{2.2.2.5}$$

Since the slot in which the test packet enters ε_1 can occur with equal probability anywhere in ε_1, we have

$$E\{d_1\} = \frac{1}{2} E\{y_1\} \tag{2.2.2.6}$$

From Eqs. (2.2.2.5) and (2.2.2.6) follows

$$E\{d_1\} = \frac{1}{2} \sum_{y_1} \frac{y_1^2 \, p_{\ell_1}(y_1)}{E\{\ell\}}$$

or

$$E\{d_1\} = \frac{1}{2} \frac{E\{\ell^2\}}{E\{\ell\}} \tag{2.2.2.7}$$

The expectation of d_2 can be written as follows:

$$E\{d_2\} = \sum_{y_1} E\{d_2|y_1\}\, p_{y_1}(y_1) \qquad (2.2.2.8)$$

and substituting Eq. (2.2.2.5) into Eq. (2.2.2.8), we have

$$E\{d_2\} = \frac{1}{E\{\ell\}} \sum_{y_1} y_1 E\{d_2|y_1\}\, p_{\ell_1}(y_1) \qquad (2.2.2.9)$$

Combining Eqs. (2.2.2.4), (2.2.2.7) and (2.2.2.9), we have the following result.

$$E\{\delta\} = \frac{1}{2}\frac{E\{\ell^2\}}{E\{\ell\}} + \frac{1}{E\{\ell\}} \sum_{\ell_1} \ell_1 E\{d_2|\ell_1\} p_{\ell_1}(\ell_1) \qquad (2.2.2.10)$$

In Appendix A2.6, the following expression for $E\{d_2|\ell_1\}$ is derived.

$$E\{d_2|\mu\} = 1 + \sum_{j=1}^{\infty} \theta(\mu/2^i)(1 + \tfrac{1}{2}D(\mu/2^i)) \qquad (2.2.2.11)$$

where

$$\theta(\mu) = \frac{1-e^{-\mu}-\mu e^{-\mu}}{1-e^{-\mu}} \qquad (2.2.2.12)$$

$$D(\mu) = \sum_{i=0}^{\infty} 2^i \xi(\mu/2^i) \qquad (2.2.1.13)$$

$$\xi(\mu) = 1 - e^{-\mu} - \mu e^{-\mu} \qquad (2.2.2.13a)$$

and

$$\mu = 2\lambda\ell_1 \tag{2.2.2.14}$$

Note that the subscript of μ has been dropped for convenience.

Equations (2.2.2.10) through (2.2.2.14) is as far as we can go in obtaining an exact closed form expression for $E\{\delta\}$. Since this expression of $E\{\delta\}$, as it stands, is quite complex, we will turn our attention to deriving upper and lower bounds.

In Appendix A2.1, the following expression for $E\{\ell_2|\mu\}$ is derived.

$$E\{\ell_2|\mu\} = 1 + 2D(\mu/2) \tag{2.2.2.15}$$

and in Appendix A2.7 it is proved that

$$E\{d_2|\mu\} \leq .55\ E\{\ell_2|\mu\} + .321 \text{ for } \mu \geq 0 \tag{2.2.2.16}$$

But because of Eq. (2.2.2.14) it follows that

$$E\{d_2|\ell_1\} \leq .55\ E\{\ell_2|\ell_1\} + .321 \tag{2.2.2.17}$$

Now combining Eqs. (2.2.2.10) and (2.2.2.17) we have

$$E\{\delta\} \leq \frac{1}{2}\frac{E\{\ell^2\}}{E\{\ell\}} + \frac{.55}{E\{\ell\}}\sum_{\ell_1} \ell_1 E\{\ell_2|\ell_1\}p(\ell_1) + .321 \tag{2.2.2.18}$$

It can easily be shown that

$$\sum_{\ell_1} \ell_1 E\{\ell_2|\ell_1\}p(\ell_1) = E\{\ell_1\ell_2\}$$

$$\leq \frac{1}{2} E\{\ell_1^2\} + \frac{1}{2} E\{\ell_2^2\} \qquad (2.2.2.19)$$

But since $\overline{\ell_1^2} = \overline{\ell_2^2} = \overline{\ell^2}$ we have from Eqs. (2.2.2.18) and (2.2.2.19)

$$E\{\delta\} \leq 1.05 \frac{E\{\ell^2\}}{E\{\ell\}} + .321 \qquad (2.2.2.20)$$

This is the desired upper bound. Next we will derive an equivalent lower bound to $E\{\delta\}$.

In Appendix A2.7 it is also shown that

$$E\{d_2|\mu_1\} \geq \frac{1}{2} E\{\ell_2|\mu\} \quad \text{for } \mu \geq 0 \qquad (2.2.2.21)$$

Here again, from Eqs. (2.2.1.1) and (2.2.2.21) we have

$$E\{d_2|\ell_1\} \geq \frac{1}{2} E\{\ell_2|\ell_1\} \text{ for } \ell_1 \geq 1 \qquad (2.2.2.22)$$

Substituting this into Eq. (2.2.2.10) and then summing we have

$$E\{\delta\} \geq \frac{1}{2} \frac{E\{\ell^2\}}{E\{\ell\}} + \frac{1}{2} \frac{E\{\ell_1\ell_2\}}{E\{\ell\}} \qquad (2.2.2.23)$$

In Appendix A2.5 it is proved that

$$E\{\ell_1\ell_2\} \geq E\{\ell_1\}E\{\ell_2\} \tag{2.2.2.24}$$

Substituting this into Eq. (2.2.2.23) we have the following lower bound.

$$E\{\delta\} \geq \frac{1}{2}\frac{E\{\ell^2\}}{E\{\ell\}} + \frac{1}{2}E\{\ell\} \tag{2.2.2.25}$$

Another lower bound to $E\{\delta\}$ is

$$E\{\delta\} \geq E\{\ell\} \tag{2.2.2.26}$$

This follows from Eq. (2.2.2.25) by noting that $E\{\ell^2\} \geq E^2\{\ell\}$.

This concludes the first part of our analysis. In summary, the main results up to this point are the derivations of the upper and lower bounds [Eqs. (2.2.2.20), (2.2.2.25) and (2.2.2.26)] to $E\{\delta\}$ that depend only on $E\{\ell\}$ and $E\{\ell^2\}$. Our goal, however, as stated above is the characterization of $E\{\delta\}$ in terms of λ. Therefore, in the following section, we will determine upper and lower bounds to $E\{\ell\}$ and $E\{\ell^2\}$ as functions of λ.

2.2.3 Characterization of $E\{\ell\}$ and $E\{\ell^2\}$ in terms of λ

In Appendices A2.1 and A2.3 we derive expressions for $E\{\ell_{j+1}|\mu_j\}$ and $E\{\ell^2_{j+1}|\mu_j\}$, where ℓ_{j+1} is the length of ε_{j+1} and μ_j is the average number of packets that arrived in ε_j. More precisely, it is shown that,

$$E\{\ell_{j+1}|\mu\} = 1 + 2D(\mu/2) \tag{2.2.3.1}$$

$$E\{\ell_{j+1}^2|\mu\} = 1 + 2D(\mu/2) + [2D(\mu/2)]^2 + F(\mu) \tag{2.2.3.2}$$

where

$$D(\mu) = \sum_{i=0}^{\infty} 2^i \xi(\mu/2^i) \tag{2.2.3.3}$$

$$F(\mu) = \sum_{i=1}^{\infty} 2^i [2D(\mu/2^i)(1 - \xi(\mu/2^i)) + \xi^2(\mu/2^i)] \tag{2.2.3.4}$$

$$\xi(\mu) = 1 - e^{-\mu} - \mu e^{-\mu} \tag{2.2.3.5}$$

Note that the above expressions are independent of λ. The arrival rate λ enters the analysis through the following relationship.

$$\mu = 2\lambda\ell_j \tag{2.2.3.6}$$

The quantities $E\{\ell_{j+1}|\mu\}$ and $\sqrt{E\{\ell_{j+1}^2|\mu\}}$ are plotted in Figs. A2.2.1 and A2.4.1.

Equations (2.2.3.1) through (2.2.3.6) present the first and second moments of the length of one epoch conditioned on the length of the previous epoch with λ being a parameter. In this section, we will derive upper and lower bounds to $E\{\ell\}$ and $E\{\ell^2\}$, the steady state first and second moments of ℓ_j, in terms of λ and the above conditional moments.

This section is divided into four subsections. In Subsection-i, first we derive an upper bound to $E\{\ell\}$ based on a general set of conditions, next we demonstrate how these conditions relate to the arrival rate and to the system parameters and finally compute the upper bound to $E\{\ell\}$ for various values of λ. In Subsections ii, iii, and iv, we develop a lower bound to $E\{\ell\}$ and upper and lower bounds to $E\{\ell^2\}$ respectively. The procedure of the analysis in these subsections is the same as that of Subsection i. That is, first, we develop a bound based on a general set of conditions, next we relate these conditions to the system parameters and finally compute these bounds for various λ's.

i. Upper Bound to $E\{\ell\}$

Here we will develop an upper bound to $E\{\ell\}$ that is a function of λ. We begin by proving the following theorem.

Theorem 2.2.3.1: Let ℓ_j be a positive integer corresponding to the state of a Markov chain after the j'th transition. Also assume that for some constants b and α_u, $0 \leq \alpha_u < 1$

$$E\{\ell_{j+1} | \ell_j\} \leq \alpha_u(\ell_j - 1) + b \text{ for } \ell_j \geq 1 \qquad (2.2.3.7)$$

then

$$\lim_{j\to\infty} E\{\ell_j\} \leq \frac{b - \alpha_u}{1 - \alpha_u} \equiv \overline{\ell}_u \qquad (2.2.3.8)$$

Proof: Multiply both sides of Eq. (2.2.3.7) by $p(\ell_j)$ and then sum over ℓ_j to obtain

$$E\{\ell_{j+1}\} \leq \alpha_u E\{\ell_j\} + b - \alpha_u \tag{2.2.3.9}$$

Since $0 \leq \alpha < 1$, Eq. (2.2.3.9) is solved recursively to obtain the following steady state solution.

$$\lim_{j\to\infty} E\{\ell_j\} \leq \frac{b - \alpha_u}{1 - \alpha_u} \tag{2.2.3.10}$$

QED

It can be shown that the above bound is the optimum bound over a class of bounds. That is, if $E\{\ell_{j+1}|\ell_j\}$ is nondecreasing, and it is upperbounded as shown in Fig. 2.2.3.1 then the tightest upper bound over all σ occurs at $\sigma = 1$. Note that the upper bound to $E\{\ell_{j+1}|\ell_j\}$ at $\sigma = 1$ is given by Eq. (2.2.3.7).

Next we will relate the above theorem to the system parameters. By applying Property 3C (Appendix A2.2) it can be shown that for any $\mu_0 \geq 0$

$$E\{\ell_{j+1}|\mu\} \leq 1.44(\mu-\mu_0) + E\{\ell_{j+1}|\mu=\mu_0\} \text{ for } \mu \geq \mu_0 \tag{2.2.3.11}$$

Since $\mu = 2\lambda\ell_j$ and $\ell_j \geq 1$, it follows that $\min\mu_0 = 2\lambda$. Therefore, comparing Eqs. (2.2.3.7) and (2.2.3.11) and noting that $\mu = 2\lambda\ell_j$ we have that

$$\alpha_u = 2.88\lambda \tag{2.2.3.12}$$

and

$$b = E\{\ell_{j+1}|\mu= 2\lambda\}$$

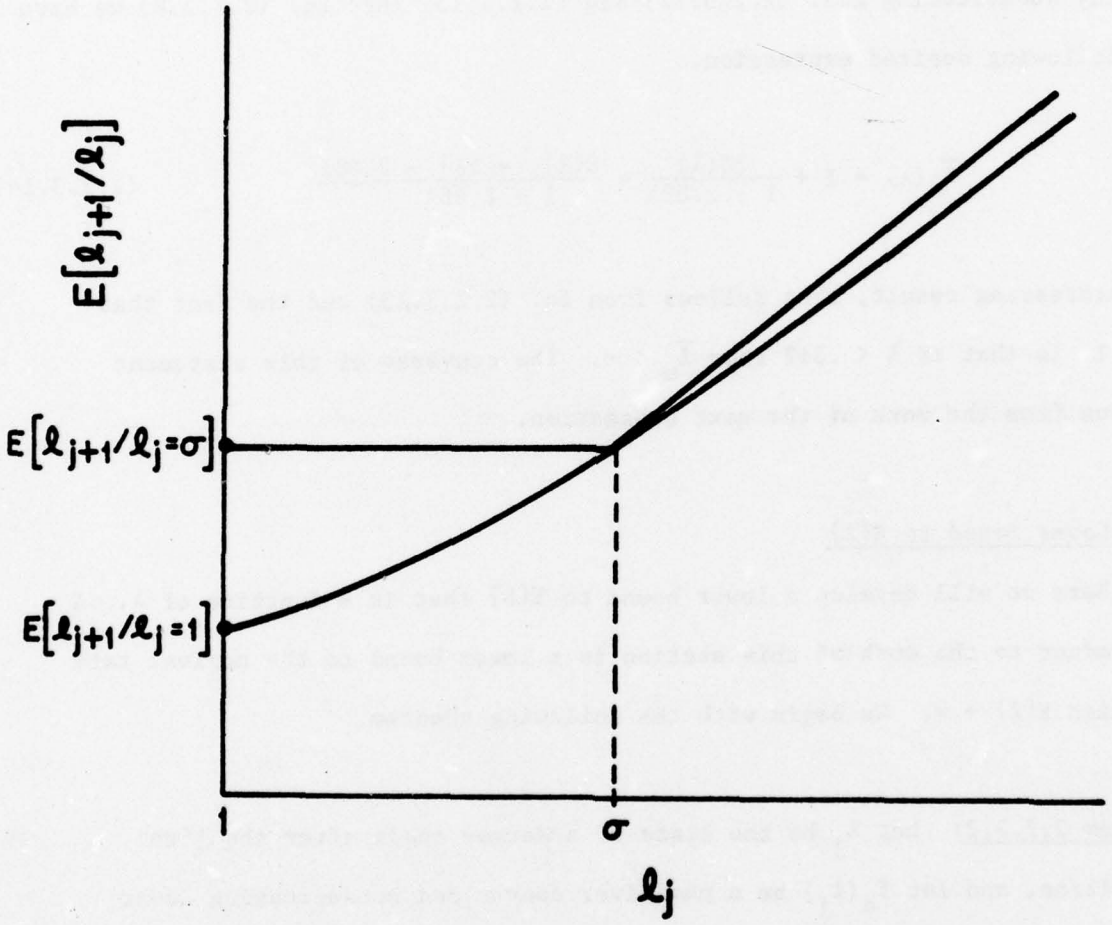


Figure 2.2.3.1 Illustrating the Form of the Upper Bound to $E\{\ell_{j+1}|\ell_j\}$

or from Eq. (2.2.2.15)

$$b = 1 + 2D(\lambda) \tag{2.2.3.13}$$

Finally substituting Eqs. (2.2.3.12) and (2.2.3.13) into Eq. (2.2.3.8) we have the following desired expression,

$$\bar{\ell}_u(\lambda) = 1 + \frac{2D(\lambda)}{1 - 2.88\lambda} = \frac{E\{\ell|\mu = 2\lambda\} - 2.88\lambda}{1 - 2.88\lambda} \tag{2.2.3.14}$$

An interesting result, that follows from Eq. (2.2.3.13) and the fact that $\alpha_u < 1$, is that if $\lambda < .347$ then $\bar{\ell}_u < \infty$. The converse of this statement follows from the work of the next subsection.

ii. Lower Bound to $E\{\ell\}$

Here we will develop a lower bound to $E\{\ell\}$ that is a function of λ. A by-product to the work of this section is a lower bound to the arrival rate at which $E\{\ell\} \to \infty$. We begin with the following theorem.

Theorem 2.2.3.2: Let ℓ_j be the state of a Markov chain after the j'th transition, and let $f_c(\ell_j)$ be a positive, convex and nondecreasing lower bound to $E\{\ell_{j+1}|\ell_j\}$. Then if

$$\ell_j = f_c(\ell_j) \tag{2.2.3.15}$$

has only one solution at $\ell_j = \ell^*$ and $f_c(\ell_j) < \ell_j$ for $\ell_j > \ell^*$ then

$$\lim_{j\to\infty} E\{\ell_j\} \geq \ell^* \tag{2.2.3.16}$$

If, on the other hand, Eq. (2.2.3.15) has two solutions, then there exists an initial state ℓ_0 to the Markov chain such that

$$\lim_{j\to\infty} E\{\ell_j\} = \infty \tag{2.2.3.17}$$

Proof:

$$E\{\ell_j\} = \sum_{\ell_j, \ell_{j-1}} \ell_j p(\ell_j|\ell_{j-1}) p(\ell_{j-1}) \tag{2.2.3.18}$$

$$= \sum_{\ell_{j-1}} E\{\ell_j|\ell_{j-1}\} p\{\ell_{j-1}\} \tag{2.2.3.19}$$

$$\geq \sum_{\ell_{j-1}} f_c(\ell_{j-1}) p(\ell_{j-1}) \tag{2.2.3.20}$$

$$\geq f_c[E\{\ell_{j-1}\}] \tag{2.2.3.21}$$

Equation (2.2.3.20) follows because $f_c(\ell_j) \leq E\{\ell_{j+1}|\ell_j\}$ and Eq. (2.2.3.21) follows from the convexity of f_c and Jenssen's inequality.

Applying Eq. (2.2.3.21) to ℓ_{j-1}, ℓ_{j-2},....,ℓ_1 recursively and using the fact that f_c is nondecreasing yields

$$E\{\ell_j\} \geq f_c^j\{\ell_0\} \tag{2.2.3.22}$$

where

$$f_c^j\{\ell_0\} \equiv \underbrace{f_c[f_c[\ldots[f_c}_{j \text{ times}}\{\ell\}]]] \tag{2.2.3.23}$$

Now assume that $\ell_j = f_c(\ell_j)$ at only one point, ℓ^*, and that $f_c(\ell_j) < \ell_j$ for $\ell_j > \ell^*$. Then $f_c(\ell_j)$ is as shown in Fig. 2.2.3.2 and since f_c is convex it follows that

$$\lim_{j\to\infty} f_c^j\{\ell_0\} = f_c\{\ell^*\} = \ell^* \tag{2.2.3.24}$$

Now if $\ell_j = f(\ell_j)$ has two solutions ℓ_1^* and ℓ_2^* ($\ell_1^* > \ell_2^*$), then $f(\ell_j)$ and ℓ_j are as shown in Fig. 2.2.3.3. Here, as can be seen from the figure, if $\ell_0 > \ell_2^*$ then

$$\lim_{j\to\infty} f_c^j\{\ell\} = \infty \tag{2.2.3.25}$$

Equations (2.2.3.22), (2.2.3.24) and (2.2.3.25) together conclude the proof.

QED

Our next step will be to relate this theorem to our system, and to obtain a lower bound to $E\{\ell\}$ as a function of λ.

It follows from Property 4 (Appendix A2.2) that $f_\ell(\mu)$ (given below) is a convex increasing postive lower bound to $E\{\ell_{j+1}|\mu\}$.

$$f_\ell(\mu) = 1 + \begin{cases} 2D(\mu/2) & \text{for } \mu \leq 8 \\ 2.88(\frac{\mu}{2} - 4) + 2D(4) & \mu \leq 8 \end{cases} \tag{2.2.3.26}$$

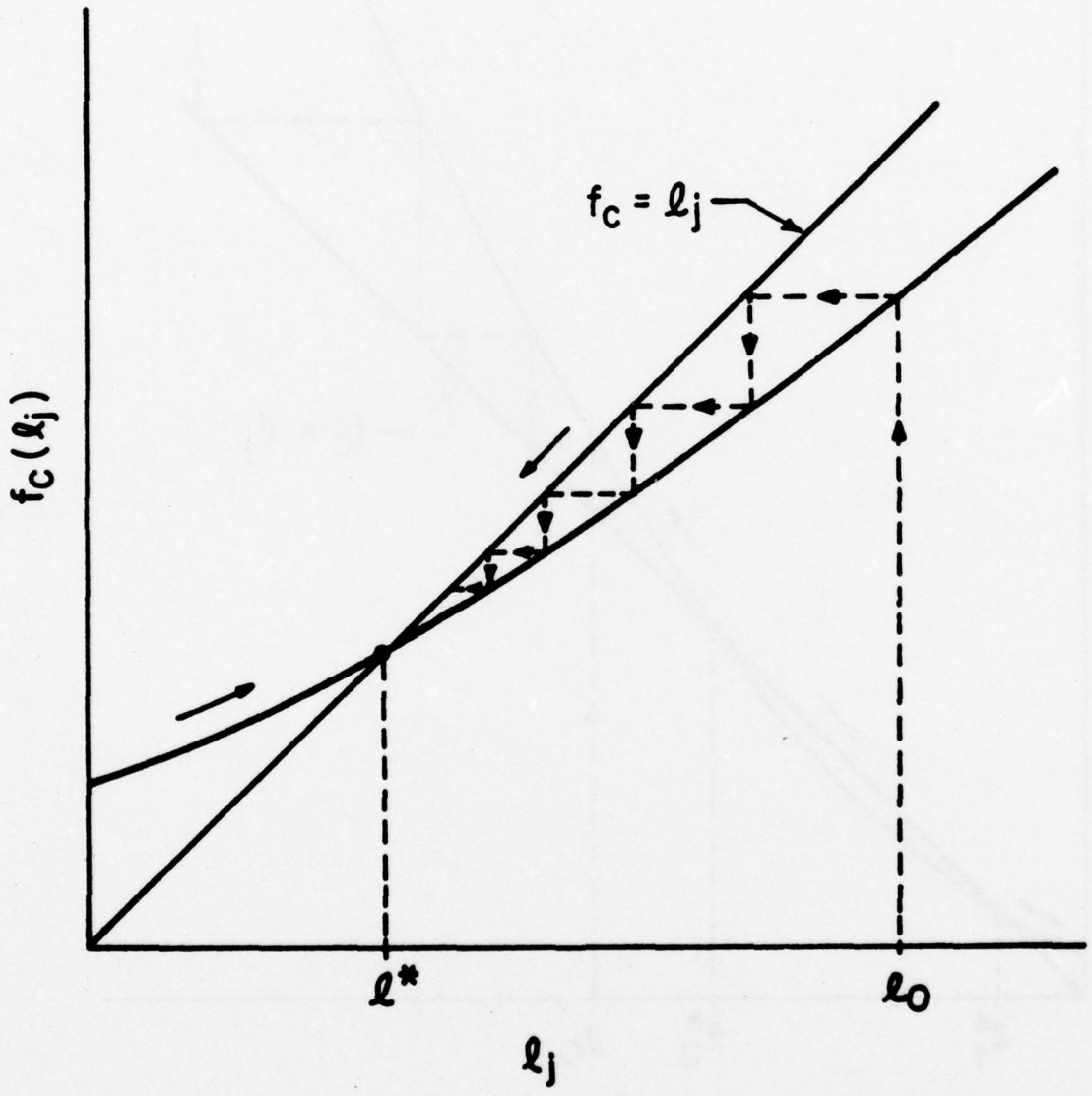


Figure 2.2.3.2 Illustrating the Conditions for Convergence of the Lower Bound to $\lim_{j\to\infty} E\{\ell_i\}$

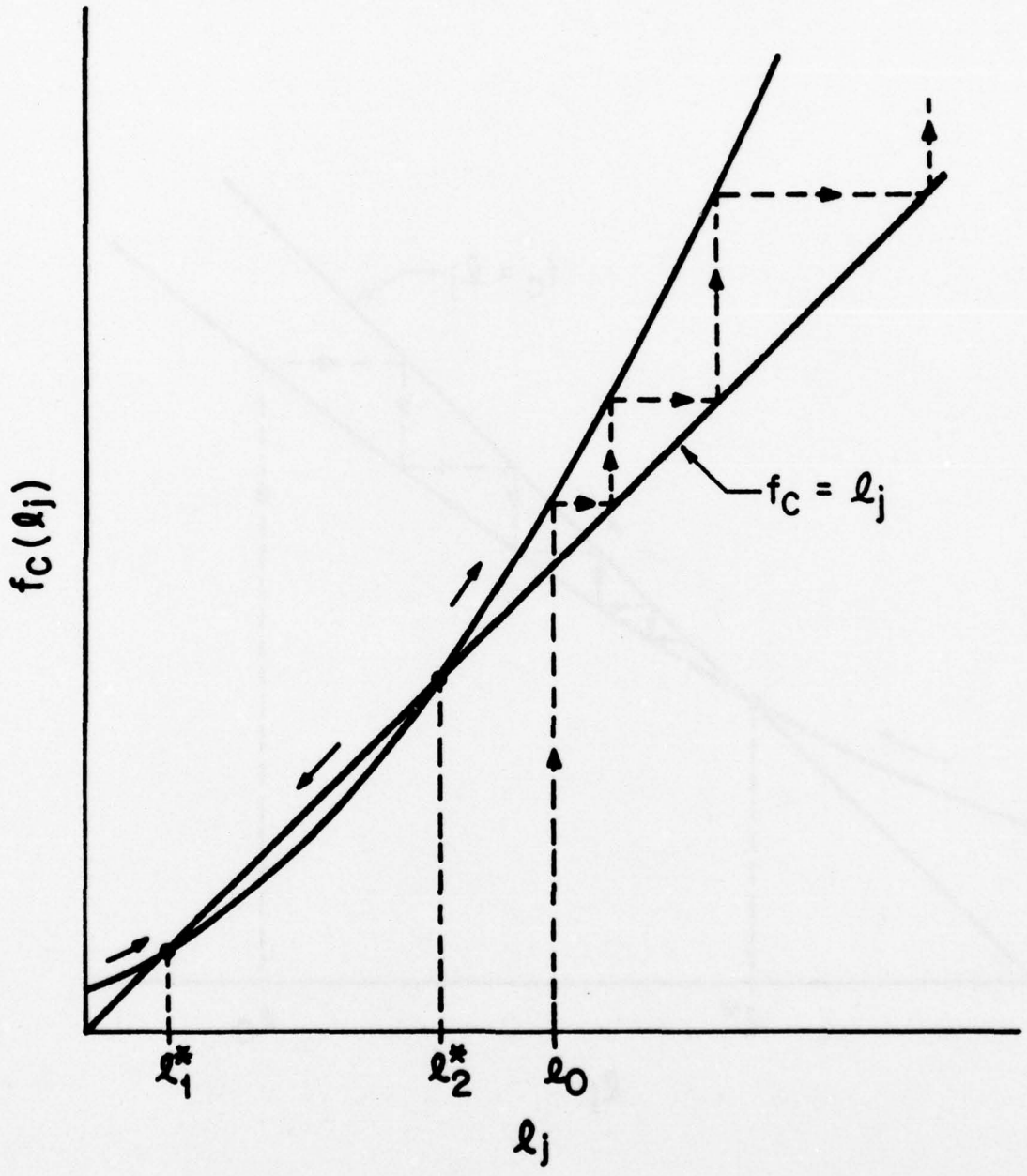


Figure 2.2.3.3 Illustrating the Conditions for Divergence of the Lower Bound to $\lim_{j\to\infty} E\{\ell_j\}$

Since $\mu = 2\lambda\ell$, we have that

$$f_c(\ell) = f_\ell(2\lambda\ell) \tag{2.2.3.27}$$

Therefore, an equivalent statement to Theorem 2.2.3.2 is the following. Let μ^* be defined by

$$f_\ell(\mu^*) = \frac{\mu^*}{2\lambda} \tag{2.2.3.28}$$

then if μ^* is unique and

$$f_\ell(\mu) < \frac{\mu}{2\lambda} \text{ for } \mu > \mu^* \tag{2.2.3.29}$$

we have that

$$E\{\ell\} \geq \frac{\mu^*}{2\lambda} \tag{2.2.3.30}$$

otherwise

$$E\{\ell\} = \infty \tag{2.2.3.31}$$

It follows from Eq. (2.2.3.26), that Eq. (2.2.3.28) has a unique solution satisfying Eq. (2.2.3.29) for $\mu \geq 0$, iff

$$\frac{\mu}{2\lambda} > 1.44\mu - .98 \tag{2.2.3.32}$$

Since Eq. (2.2.3.32) holds for all $\mu \geq 0$ iff $0 \leq \lambda \leq .347$, we conclude that if $\lambda > .347$ then

$$\overline{\ell}_{\ell}(\lambda) = \infty \tag{2.2.3.33}$$

and if $0 \leq \lambda < .347$ then $\overline{\ell}_{\ell}(\lambda)$ is given by Eq. (2.2.3.30). In summary, we have the following lower bound to $E\{\ell\}$,

$$\overline{\ell}_{\ell}(\lambda) = \begin{cases} \frac{\mu^*}{2\lambda} & \lambda < .347 \\ \infty & \lambda \geq .347 \end{cases} \tag{2.2.3.34}$$

iii. Upper Bound to $E\{\ell^2\}$

Here we will derive an upper bound to $E\{\ell^2\}$ that is a function of λ. We begin with the following theorem.

Theorem 2.2.3.3: Let ℓ_j be the state of a Markov chain after the j'th transition and assume that

$$E\{\ell_{j+1}^2 | \ell_j\} \leq (\alpha_u \ell_j + c)^2 \quad \text{for } \ell_j \geq 1 \tag{2.2.3.35}$$

where

$$0 \leq \alpha_u < 1 \tag{2.2.3.36}$$

and

$$\lim_{j \to \infty} E\{\ell_j\} = E\{\ell\}$$

then

$$\lim_{j\to\infty} E\{\ell_j^2\} \leq \frac{2\alpha_u c E\{\ell\} + c^2}{1 - \alpha_u^2} \tag{2.2.3.37}$$

Proof: Expand the right side of Eq. (2.2.3.35), multiply by $p(\ell_j)$ and then sum over all ℓ_j to obtain

$$E\{\ell_{j+1}^2\} \leq \alpha_u^2 E\{\ell_j^2\} + 2\alpha_u c E\{\ell_j\} + c^2 \tag{2.2.3.38}$$

Equation (2.2.3.38) is a difference unequality in $E\{\ell_j^2\}$ where $2\alpha_u c E\{\ell_i\} + c^2$ is the driving function. Since from the hypothesis $\lim_{j\to\infty} E\{\ell_j\} = E\{\ell\}$ and $0 \leq \alpha_u < 1$, it can be solved recursively to obtain the following steady state solution.

$$E\{\ell^2\} \leq \frac{2\alpha_u c E\{\ell\} + c^2}{1 - \alpha_u^2} \tag{2.2.3.39}$$

QED

Next we will relate this theorem to our system. In Appendix A2.4 we proved that $E\{\ell^2|\mu\} \leq (1 + 1.44\,\mu)^2$. Since, as can be seen from Fig. A2.4.1,

$$\max \frac{\partial}{\partial\mu} \sqrt{E\{\ell_{j+1}^2|\mu\}} = 1.44$$

this bound can be generalized as in the following expression

$$\sqrt{E\{\ell_{j+1}^2|\mu\}} \leq 1.44(\mu-\mu_0) + \sqrt{E\{\ell_{j+1}^2|\mu=\mu_0\}} \text{ for } \mu \geq \mu_0 \tag{2.2.3.40}$$

where μ_0 is an arbitrary parameter $\mu_0 \geq 0$

Now since $\mu = 2\lambda\ell_j$ and $\ell_j \geq 1$, it follows that min $\mu_0 = 2\lambda$. Therefore, comparing Eqs. (2.2.3.35) and (2.2.3.40) we have that

$$c = \sqrt{E\{\ell_{j+1}^2 | \mu=2\lambda} - \alpha_u \tag{2.2.3.41}$$

and

$$\alpha_u = 2.88\lambda \tag{2.2.3.42}$$

Finally substituting Eq. (2.2.3.42) into Eq. (2.2.3.39) we have

$$E\{\ell^2\} \leq \overline{\ell_u^2}\ (\lambda) \tag{2.2.3.43}$$

where

$$\overline{\ell_u^2}(\lambda) = \frac{5.76c\lambda\ E\{\ell\} + c^2}{1 - (2.88\lambda)^2} \tag{2.2.3.44}$$

and c is given by Eq. (2.2.3.41).

This is the desired result. Note that as $\lambda \to .347$, $\overline{\ell_u^2}$, as well as, $\overline{\ell}_u$ and $\overline{\ell}_\ell$ approach infinity.

Next we will turn our attention to the derivation of $\overline{\ell_\ell^2}(\lambda)$. This is performed in the following subsection.

iv. Lower Bound to $E\{\ell^2\}$

Here we will develop a lower bound to $E\{\ell^2\}$ as a function of λ. Since the work here is similar to that of the previous subsection, the details will be omitted.

Theorem 2.2.3.4: Let ℓ_j be as given in the statement of Theorem 2.2.3.3 except for

$$E\{\ell_{j+1}^2|\ell_j\} \geq (\alpha_\ell \ell_j + c_\ell)^2 \tag{2.2.3.45}$$

then,

$$\lim_{j\to\infty} E\{\ell_j^2\} \geq \frac{2c_\ell\alpha_\ell\, E\{\ell\} + c_\ell^2}{1 - \alpha_\ell^2} \tag{2.2.3.46}$$

Proof: This proof is identical to the proof of Theorem 2.2.3.3 with the inequalities reversed.

QED

Finally in relating Theorem 2.2.3.4 to the system parameters, first substitute $\mu = 2\lambda\ell$ into Eq. (2.2.3.45) to obtain

$$E\{\ell^2|\mu\} \geq (\frac{\alpha_\ell \mu}{2\lambda} + c_\ell)^2 \tag{2.2.3.47}$$

Comparing this to the results of Appendix A2.4 where it is shown that

$$E\{\ell^2|\mu\} \geq (1.44\mu + .25)^2 \tag{2.2.3.48}$$

we have from Eq. (2.2.3.46) that

$$E\{\ell^2\} \geq \frac{1.44E\{\ell\}\lambda + .125}{1 - (2.88\lambda)^2} \qquad (2.2.3.49)$$

Note here again that as $\lambda \to .347$ then $\overline{\ell_\ell^2} \to \infty$.

2.2.4 E{delay} versus Arrival Rate

In Section 2.2.2 we derived upper and lower bound to $E\{\delta\}$ in terms of $E\{\ell\}$ and $E\{\ell^2\}$, and in Section 2.2.3 we determined upper and lower bounds to $E\{\ell\}$ and $E\{\ell^2\}$ in terms of λ. In this section we will combine these results to obtain upper and lower bound to $E\{\delta\}$ in terms of λ. We begin with the upper bound to $E\{\delta\}$. Equation (2.2.2.20) which is the upper bound to $E\{\delta\}$ in terms of $E\{\ell\}$ and $E\{\ell^2\}$ is rewritten below.

$$E\{\delta\} \leq 1.05 \frac{E\{\ell^2\}}{E\{\ell\}} + .321 \qquad (2.2.4.1)$$

Now from Eqs. (2.2.4.2), (2.2.3.34) and (2.2.3.44) we have

$$E\{\delta\} \leq \frac{6.05c\lambda}{1 - (2.88\lambda)^2} + \frac{1.05c^2}{(1 - (2.88\lambda)^2)\overline{\ell_\ell}(\lambda)} + .321 \qquad (2.2.4.2)$$

where c and $\overline{\ell}_\ell(\lambda)$ are given in Eqs. (2.2.3.41) and (2.2.3.34).

Equation (2.2.4.2) is the desired lower upper bound to $E\{\delta\}$. This upper bound was computed and the result is presented in Fig. 2.2.4.1.

The derivation of the lower bound to $E\{\delta\}$ is similar to that of the upper bound. From Eqs. (2.2.2.25) and (2.2.3.49) we have

$$E\{\delta\} \geq \frac{.72\lambda}{1 - (2.88\lambda)^2} + \frac{1}{32E\{\ell\}(1 - (2.88\lambda)^2)} + \frac{1}{2} E\{\ell\} \qquad (2.2.4.3)$$

$$\geq \frac{.72\lambda}{1 - (2.88\lambda)^2} + \frac{1}{2} \overline{\ell_\ell}(\lambda) \qquad (2.2.4.4)$$

Equation (2.2.4.4) is one lower bound to $E\{\delta\}$, another follows from Eq. (2.2.2.26); that is

$$E\{\delta\} \geq \overline{\ell}_\ell(\lambda) \qquad (2.2.4.5)$$

The best lower bound is the maximum of the two bounds given in Eqs. (2.2.4.4) and (2.2.4.5). This is given below.

$$E\{\delta\} \geq \max\left[\frac{.72\lambda}{1 - (2.88\lambda)^2} + \frac{1}{2} \overline{\ell_\ell}(\lambda) \quad , \quad \overline{\ell_\ell}(\lambda)\right] \qquad (2.2.4.6)$$

Equation (2.2.4.6) has been computed and it is plotted in Fig. 2.2.4.1 along with the upper bound.

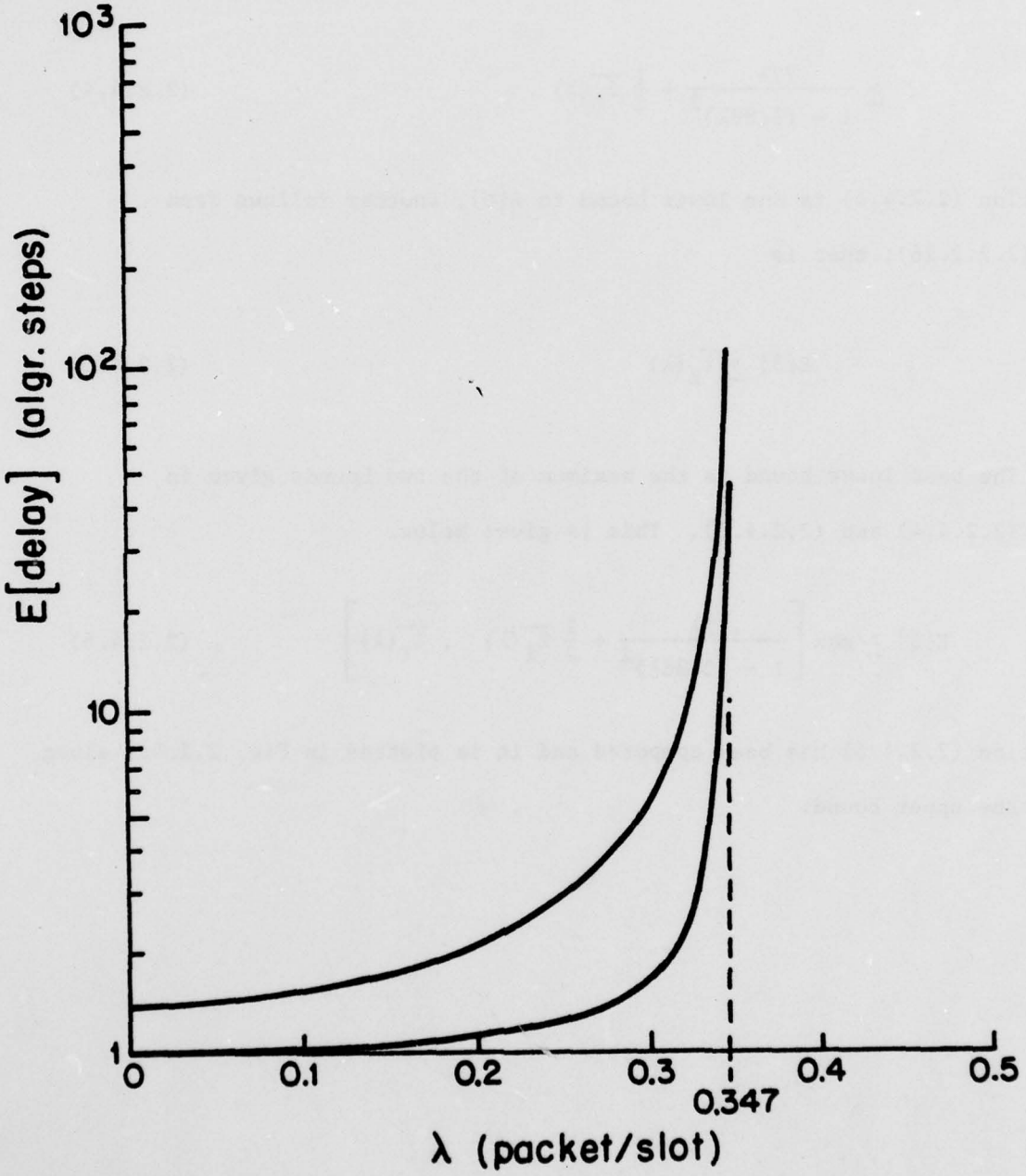


Figure 2.2.4.1 Upper and Lower Bounds to the Average Delay versus Average Arrival Rate for the Binary Tree/Poisson Source System

Note: An algr. step equals one round trip delay plus two slots

2.3 Average Throughput

The average throughput of the system is defined to be the fraction of the time that the channel contains valid data, i.e., exactly one packet per slot. Now if the average delay is finite, it follows that over a long interval of time, all the packets that arrive will be successfully transmitted. Therefore, the average throughput equals the average arrival rate if the average delay is finite. From the above discussion, it follows that the E{delay} vs λ results of Section 2.2 may be interpreted as E{delay} vs E{throughput}, (see Fig. 2.2.4.1), and the maximum average throughput is .347 packets/slot.

In the preceding paragraph we defined and determined the maximum average throughput. It turns out, however, that the system may be operated at a throughput up to .43 packets/slot but only for a limited time.

In Fig. 2.3.0.1 we have $E\{\ell|\mu\}$ versus μ. We know from the results of Section 2.2 that the average delay is finite for $\lambda < .347$. However, if $.347 < \lambda < .430$ then $\frac{\mu}{2\lambda}$ intersects $E\{\ell|\mu\}$ in two places, at μ_s and μ_c which corresponds to say, ℓ_s and ℓ_c. Now, if $.347 \leq \lambda \leq .430$ and $\ell_j \leq \ell_c$ then

$$E\{\ell_{j+1}|\ell_j\} < \ell_j \text{ and}$$

the system will operate around ℓ_s. However, statistical perturbations will eventually cause ℓ_j to become greater than ℓ_c, in which case

$$E\{\ell_{j+1}|\ell_j\} > \ell_j$$

Here, the system is expected to overflow; this is accompanied by an increase in delay and a drop in throughput.

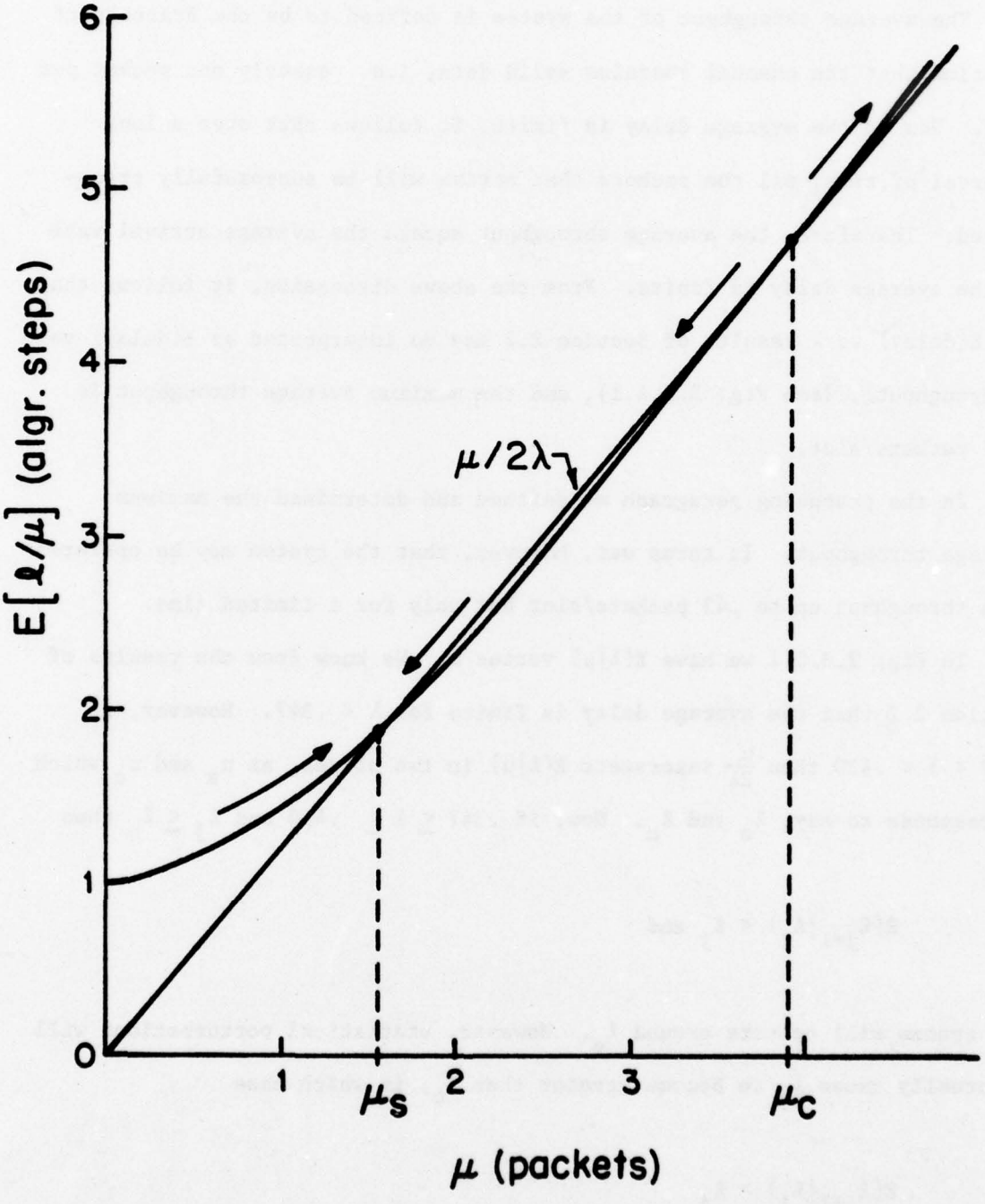


Figure 2.3.0.1 Illustrating the Attainment of the Maximum Throughput of .430 pck/slot, for the Binary Tree/Poisson Source System

2.4 System Stability

Heurestically, an unstable multi-access system is one where the following scenario is possible. Originally, several channel packet collisions reduce the number of packets being successfully transmitted. This results in a packet backlog which further increases the number of channel collisions, this in turn increases the backlog, etc., until a total breakdown occurs when essentially everybody is trying to get through with very few actually succeeding.

A more precise definition of stability of a multi-access system is the following. The system is defined to be k'th order stable if the k'th moment of the delay is finite. In Section 2.2, we showed that if $\lambda < .347$, then the average delay is finite, therefore, our system is at least first order stable. In the remainder of this section, we will show that if $\lambda < 1/3$, then all the moments of the delay are finite. As has been pointed out previously, this stability result is overly conservative, and indications are that all moments are finite for $\lambda < .347$ packets/slot. In Appendix A2.8, a way is suggested for determining a larger lower bound to λ max.

From the results of Section 2.3, we know that if $E\{\ell^k\} < \infty$ then $E\{\delta^{k-1}\} < \infty$. Therefore, it is sufficient to show that $E\{\ell^k\} < \infty$ for $\lambda < 1/3$ and for all k. This is accomplished by showing that $\lim_{j\to\infty} E\{e^{s\ell_j}\} \equiv E\{e^{s\ell}\}$ exists for $0 \leq s \leq s_0$, $\lambda < 1/3$, $s_0 > 0$. The analysis is carried out by first obtaining an upper bound to $E\{e^{s\ell}\}$ under a general set of conditions (this is accomplished in Theorem 2.4.0.1), and then relating these conditions to the multi-access system. Note that upperbounding a moment generating function assures its existance.

Theorem 2.4.0.1: Let ℓ_j be the state of a Markov chain after the j'th transition. Furthermore, let

$$E\{e^{s\ell_{j+1}} | \ell_j\} \leq e^{s(a\ell_j + b)} \tag{2.4.0.1}$$

where a and b are constant, $0 \leq a < 1$, and ℓ_o, the initial state, is finite Then

$$\lim_{j\to\infty} E\{e^{s\ell_j}\} \leq e^{\frac{b}{1-a}s} \tag{2.4.0.2}$$

Proof: Multiply both sides of Eq. (2.4.0.1) by $p(\ell_j)$ and then sum over ℓ_j to obtain

$$E\{e^{s\ell_{j+1}}\} \leq E\{e^{sa\ell_j}\}e^{sb}$$

Solving the above equation recursively, we have

$$E\{e^{s\ell_j}\} \leq E\{e^{a^j s\ell_0}\}\, e^{sb(a^{j-1}+\ldots+1)} \tag{2.4.0.3}$$

Since $\ell_0 < \infty$ and $0 \leq a < 1$, Eq. (2.4.0.2) follows by letting $j \to \infty$ in Eq. (2.4.0.3).

QED

Next we will show that the conditional generating function of ℓ_j can be upper bounded as in Eq. (2.4.0.1) where $a < 3\lambda$. In Appendix A2.8 we proved that

$$E\{e^{s\ell}|\mu\} \leq (1+\mu)e^{s-\mu} - (A/B^2 + A\mu/B)e^{-\mu} + (A/B^2)e^{(B-1)\mu} \qquad (2.4.0.4)$$

where

$$A = \frac{e^s}{2 - e^s}, \quad B = \frac{(x-1)e^s}{x - e^s}, \quad 0 \leq s \leq s_0, \; s_0 > 0 \qquad (2.4.0.5)$$

and x is an arbitrary parameter $1 < x < 3$. Since $\mu e^{-\mu}$ and $e^{-\mu} \leq 1$ for $\mu \geq 0$, we have from Eq. (2.4.0.4) that

$$E\{e^{s\ell}|\mu\} \leq 2e^s - A/B^2 - A/B + (A/B^2)e^{(B-1)\mu} \qquad (2.4.0.6)$$

Now for

$$a_0 > \left.\frac{\partial B}{\partial s}\right|_{s=0} = \frac{x}{x-1} \qquad (2.4.0.7)$$

$s_1 > 0$ exists such that

$$B-1 < a_0 s \text{ for } 0 \leq s \leq s_1$$

Therefore, from Eq. (2.4.0.6) we have

$$E\{e^{s\ell}|\mu\} \leq 2e^{s} - A/B^2 - A/B + (A/B^2)\, e^{a_0\mu s} \tag{2.4.0.8}$$

$$\leq e^{s(a_0\mu + b)} \quad \text{for } b > \frac{x}{x-1},\ 0 \leq s \leq s_2,\ s_2 \geq 0 \tag{2.4.0.9}$$

Equation (2.4.0.9) follows from Eq. (2.4.0.8, because at $s = 0$, both equal 1, both are continuous and $\frac{\partial}{\partial s}$ of Eq. (2.4.0.9) is greater than that of Eq. (2.4.0.8).

Finally by substituting $\ell = \ell_{j+1}$ and $\mu = 2\lambda\ell_j$ into Eq. (2.4.0.9) we obtain

$$E\{e^{s\ell_{j+1}}|\ell_j\} \leq e^{s(2a_0\lambda\ell_j + b)} \tag{2.4.0.10}$$

From Theorem 2.4.0.1, we know that for convergence of $\lim_{j\to\infty} E\{e^{s\ell_j}\}$ we need

$$\lambda < \frac{1}{2a_0}$$

and from Eq. (2.4.0.7)

$$\lambda < \frac{x-1}{2x} \tag{2.4.0.11}$$

Since $1 < x < 3$, we have

$$\lambda < \sup_{1<x<3} \left[\frac{x-1}{2x}\right] = \frac{1}{3} \qquad (2.4.0.12)$$

This concludes this section.

APPENDIX A2

PROPERTIES OF THE BINARY TREE/POISSON SOURCE SYSTEM

A2.1 Derivation of $E\{\ell|\mu\}$

If we let the number of packets that are to be processed in an epoch be a Poisson random variable with mean μ and let ℓ be the number of algorithmic steps required to process these packets by the binary tree algorithm, then in this appendix we will show that

$$E\{\ell|\mu\} = 1 + 2D(\mu/2) \tag{A2.1.1}$$

where

$$D(\mu) \equiv \sum_{i=0}^{\infty} 2^i \xi(\mu/2^i) \tag{A2.1.2}$$

and

$$\xi(\mu) \equiv 1 - e^{-\mu} - \mu e^{-\mu} \tag{A2.1.3}$$

Before proving this result several definitions will be given. (Fig. A.2.1.1 might be helpful in visualizing these definitions)

n_{ij} = the j'th node of level i. There are 2^i nodes at level i. (A2.1.4)

T_{ij} = subtree whose root node is n_{ij} (A2.1.5)

x_{ij} = a random variable that equals one if n_{ij} is visited by the algorithm and zero otherwise. Note that n_{00} is always visited by the algorithm, therefore, $x_{00} = 1$. (A2.1.6)

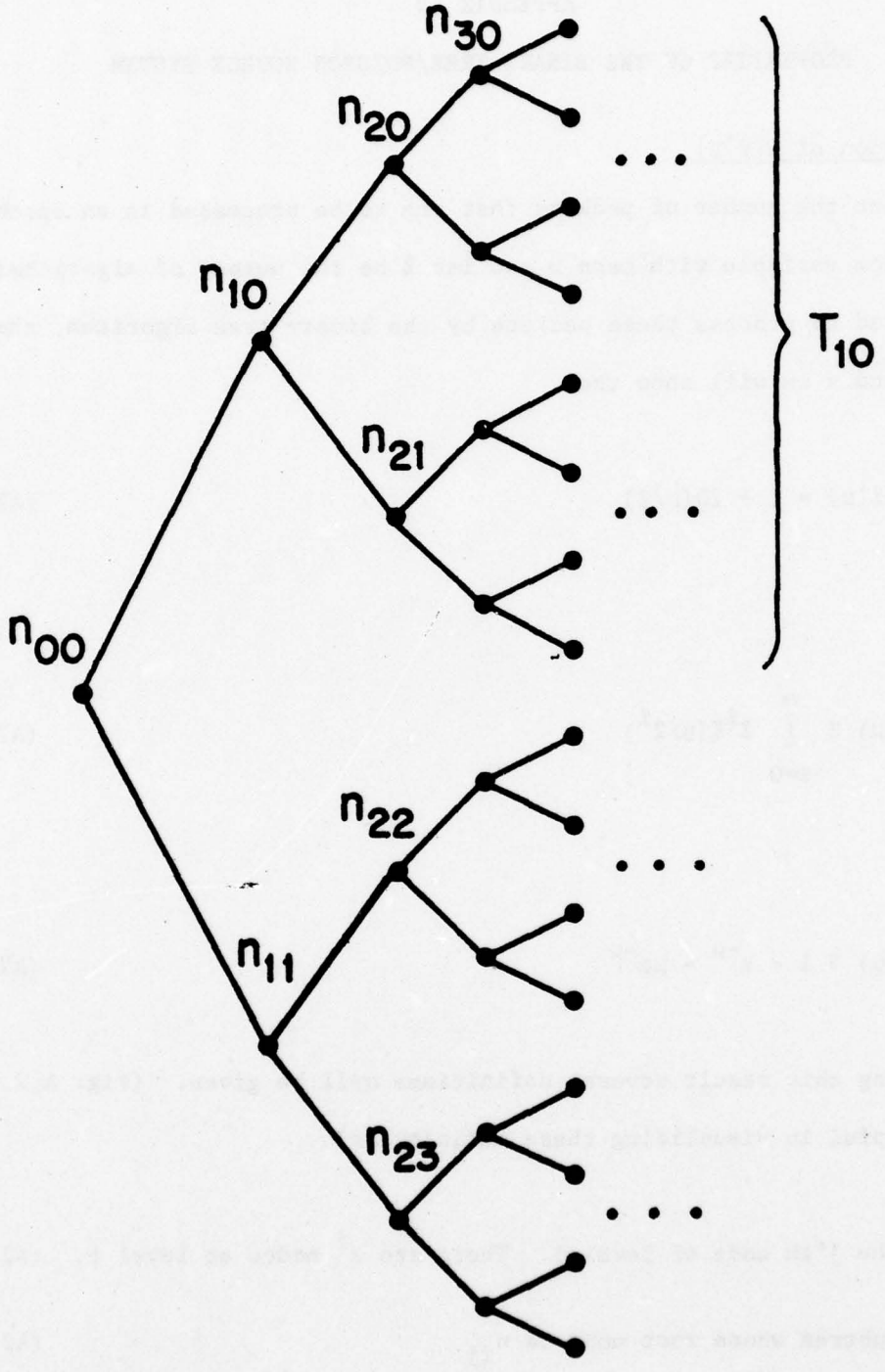


Figure A2.1.1 The Infinite Binary Tree

As has been pointed out an algorithmic step is equivalent to visiting one node. Therefore, the total number of algorithmic steps equals the total number of nodes visited. Motivated by this observation, we prove the following lemma which is useful in determining the probability that a particular node is visited.

Lemma A2.1.1. A node n_{ij} $i \neq 0$ is transversed in the binary tree algorithm if and only if there are at least two active sources in T_{ij}.

Proof: Let $[n_{mr}: m = 0, 1, \ldots, i]$ be the set of all the nodes that lie on the path from n_{00} to n_{ij}. Since T_{ij} is included in $[T_{mr} : m = 0, 1, \ldots, i]$, it follows that if T_{ij} contains at least two active sources so does $[T_{mr} : m \leq i]$. Now if the protocol is on node $[n_{mr} : m < i]$ and asks for the number of active sources in $T_{m+1,r}$ the answer will be greater than one, and it will move to node $n_{m+1,r}$. Since this holds for at least $m = 0, 1, \ldots, i-1$ we conclude that node n_{ij} will be transversed by the algorithm. If, on the other hand, there is at most one source in T_{ij} and the algorithm got at least as far as $n_{i-1,r}$ and asked for the number of active sources in T_{ij}, the answer will be 0 or 1 and it will not continue to n_{ij}.

QED

Theorem A2.1.1: Let there be ν active sources (where ν is a Poisson random variable with mean μ) and let ℓ be the number of nodes transversed by the binary tree algorithm in resolving the conflict, i.e., the process of separating all the sources into subsets such that each subset contains at most one active source. Then,

$$E\{\ell|\mu\} = 1 + 2D(\mu/2) \tag{A2.1.7}$$

where

$$D(\mu) = \sum_{i=0}^{\infty} 2^i \xi(\mu/2^i) \tag{A2.1.8}$$

and

$$\xi(\mu) = 1 - e^{-\mu} - \mu e^{-\mu} \tag{A2.1.9}$$

Proof: From the definitions in Eqs. (A2.1.4) to (A2.1.6) we have that

$$\ell = \sum_{i=0}^{\infty} \sum_{j=0}^{2^j-1} x_{ij} \tag{A2.1.10}$$

From Lemma A2.1.1 we have that

$$p(x_{ij}=1|\mu) = \begin{cases} p(\text{At least 2 active sources in } T_{ij}|\mu) \text{ for } i\neq 0 \\ 1 \text{ if } i=0 \end{cases} \tag{A2.1.11}$$

But since the number of active sources is Poisson distributed, and the sources are independent of each other, it follows that the number of active sources in T_{ij} is also Poisson distributed with parameter $\mu/2^i$. Therefore, from Eq. (A2.1.11) follows that

$$p(x_{ij}=1|\mu) = 1 - e^{-\mu/2^i} - \mu/2^i\, e^{-\mu/2^i};\ i \neq 0 \qquad \text{(A2.1.12)}$$

$$\equiv \xi(\mu/2^i) \qquad \text{(A2.1.13)}$$

and from Eq. (A2.1.10) we have that

$$E\{\ell|\mu\} = \sum_{i=0}^{\infty} \sum_{j=0}^{2^i-1} p(x_{ij}=1|\mu) \qquad \text{(A2.1.14)}$$

and from Eq. (A2.1.11) and (A2.1.13)

$$E\{\ell|\mu\} = 1 + \sum_{i=1}^{\infty} 2^i \xi(\mu/2^i) \qquad \text{(A2.1.15)}$$

QED

A2.2 Properties of $E\{\ell|\mu\}$

Here we will derive several properties of $E\{\ell|\mu\}$. Some of these properties are trivial, whereas, others are more involved. However, all of them are presented here for completeness. The quantity $E\{\ell|\mu\}$ which was derived in Appendix A2.1, is rewritten below and plotted in Fig. A2.2.1.

$$E\{\ell|\mu\} = 1 + 2D(\mu/2) \tag{A2.2.1}$$

$$D(\mu) = \sum_{i=0}^{\infty} 2^i \xi(\mu/2^i) \tag{A2.2.2}$$

$$\xi(\mu) = 1 - e^{-\mu} - \mu e^{-\mu} \tag{A2.2.3}$$

Since $E\{\ell|\mu\}$ depends trivially on $D(\mu)$ and since $D(\mu)$ is a basic quantity that appears in other expressions, we will develop the properties of $D(\mu)$. The corresponding properties of $E\{\ell|\mu\}$ follow from those of $D(\mu)$ in a straight forward manner. The properties of $D(\mu)$ that are derived here are listed on page 84. In order to motivate these properties we calculated $D(\mu)$, $D'(\mu)$ and $D''(\mu)$ for $\mu\varepsilon[0,16]$. The results are shown in Figs. A2.2.2, A2.2.3 and A2.2.4.

Before beginning the derivations we will comment on one of the techniques used in this appendix as well as several other places in the thesis. At times we need to show that a function is positive over some finite range of its argument. This is accomplished here simply by calculating the function at several points. An example of the use of this technique can be found in Property 4 where it is shown that $\frac{\partial^2 D}{\partial \mu^2} \geq 0$ for $0 \leq \mu \leq 4$. We are aware of the fact that calculating the function at a finite number of points does

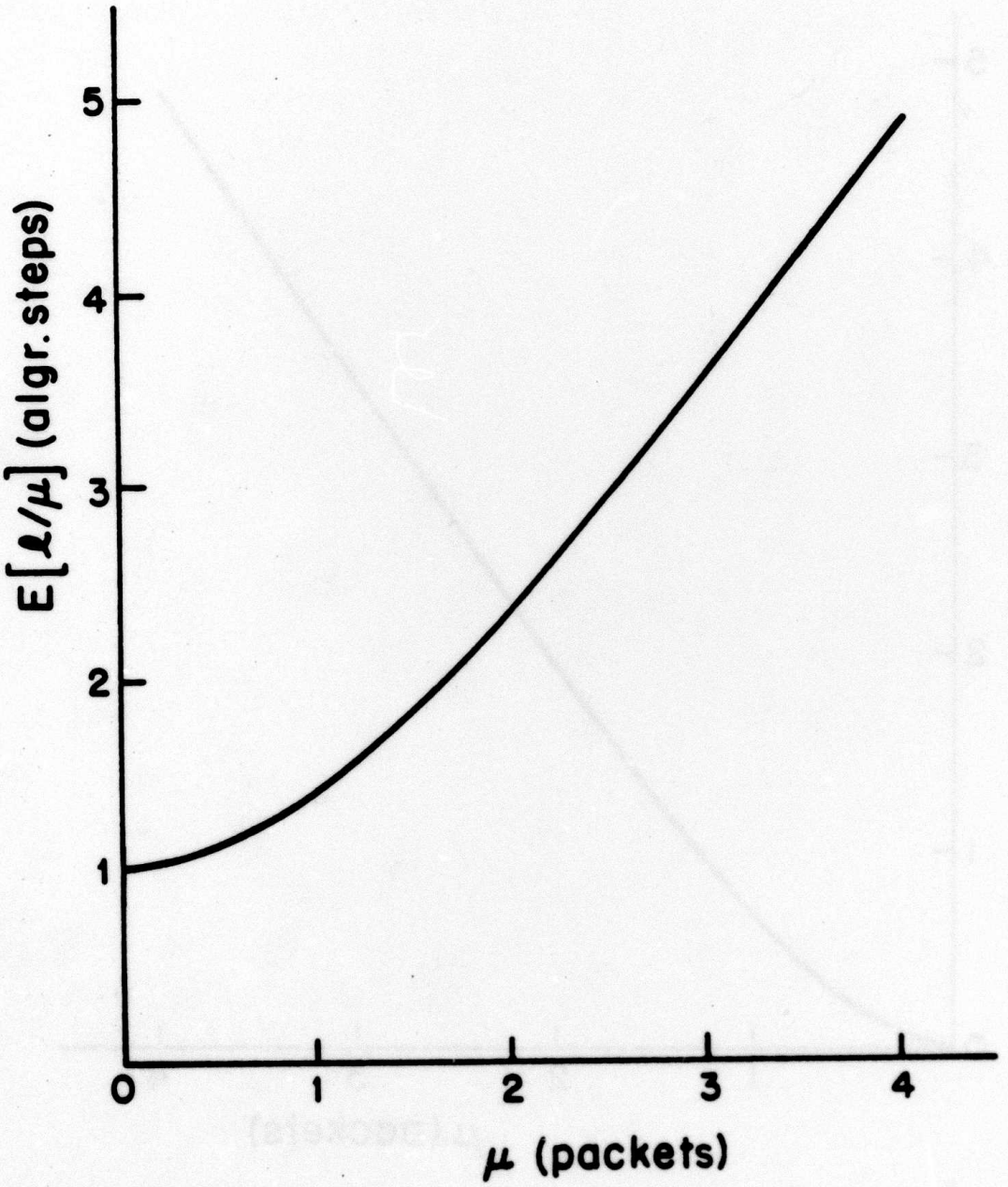


Figure A2.2.1 $E\{\ell|\mu\}$ versus μ for the Binary Tree/Poisson Source System

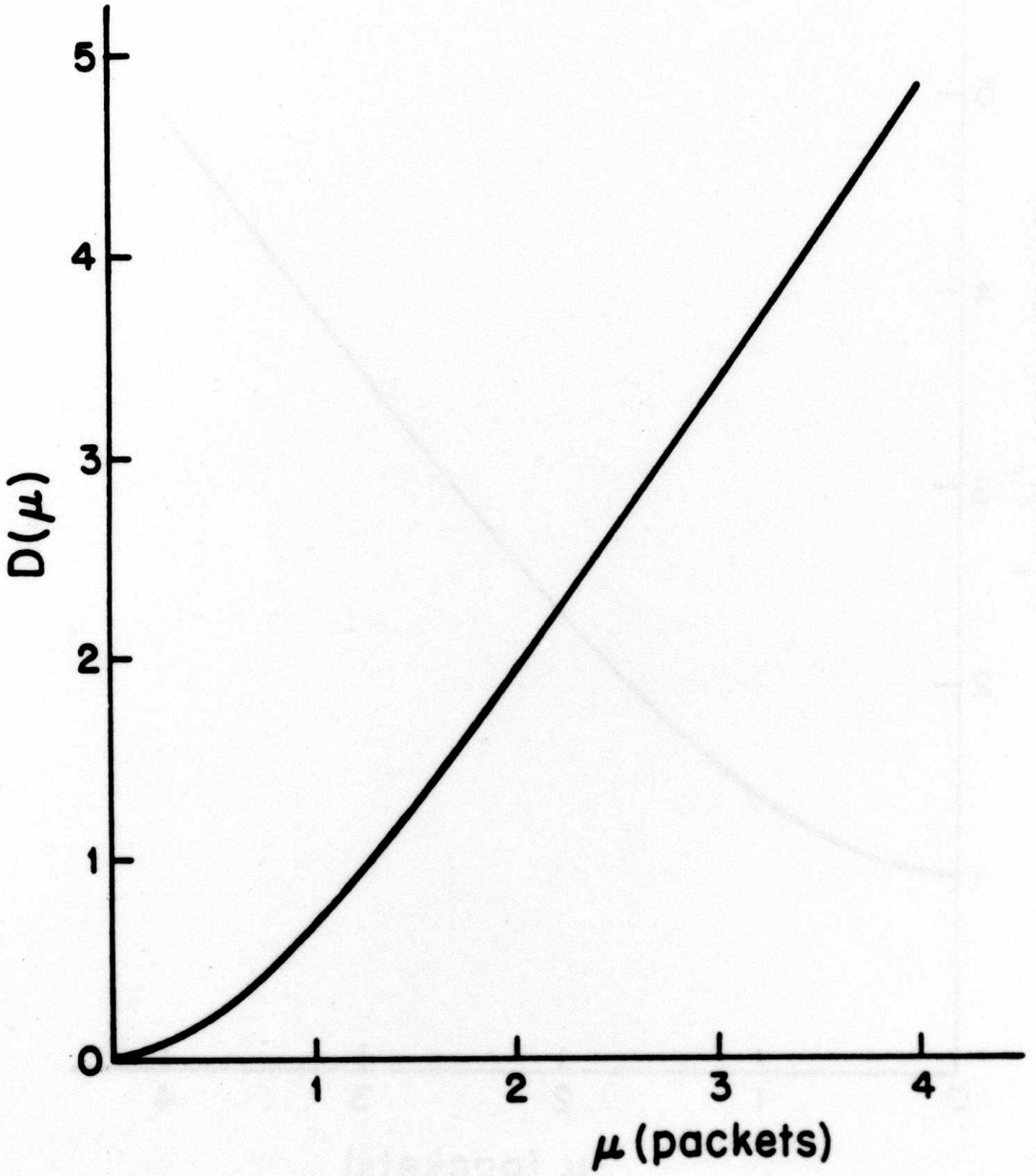


Figure A2.2.2 D(μ) for the Binary Tree/Poisson Source System

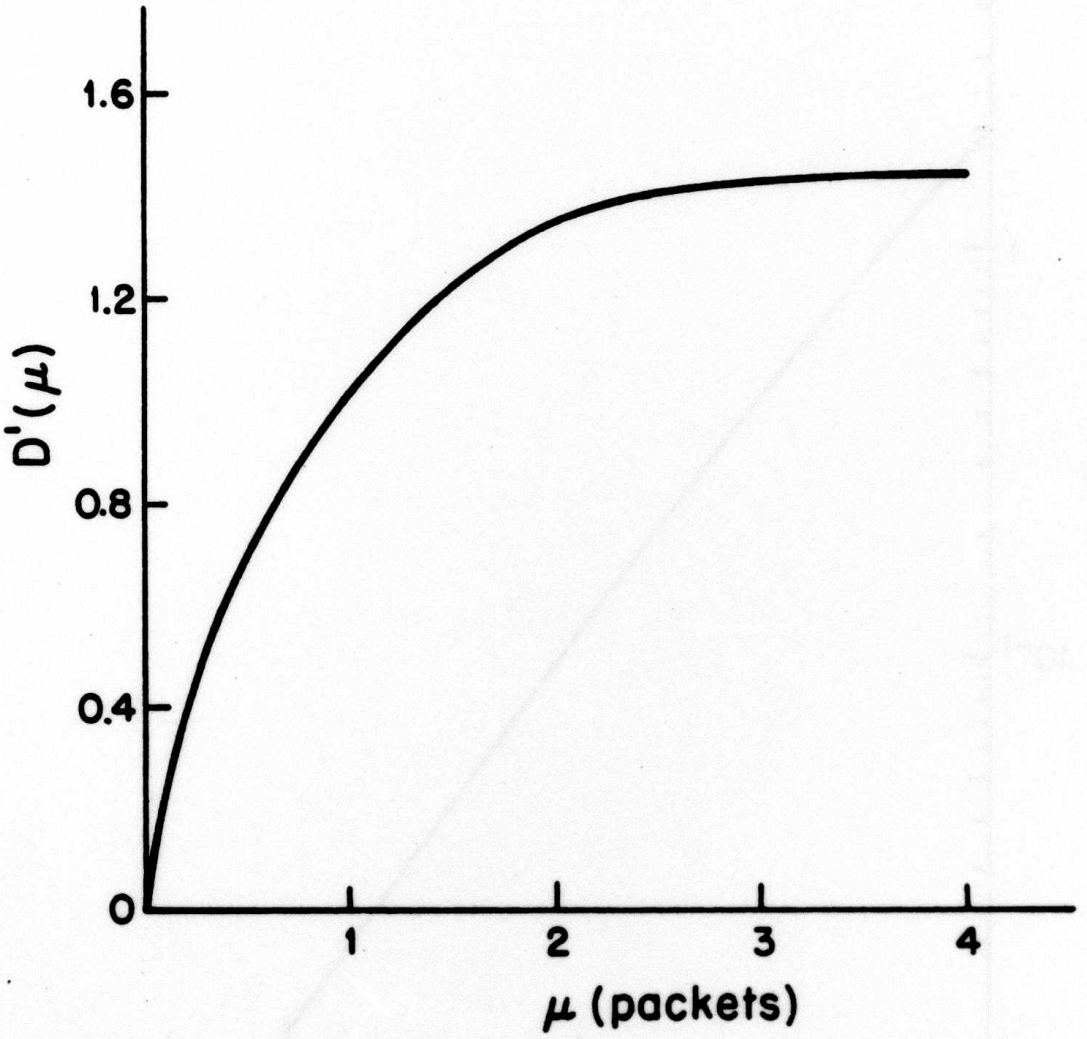


Figure A2.2.3 $\partial D/\partial \mu$ **for the Binary Tree/Poisson Source System**

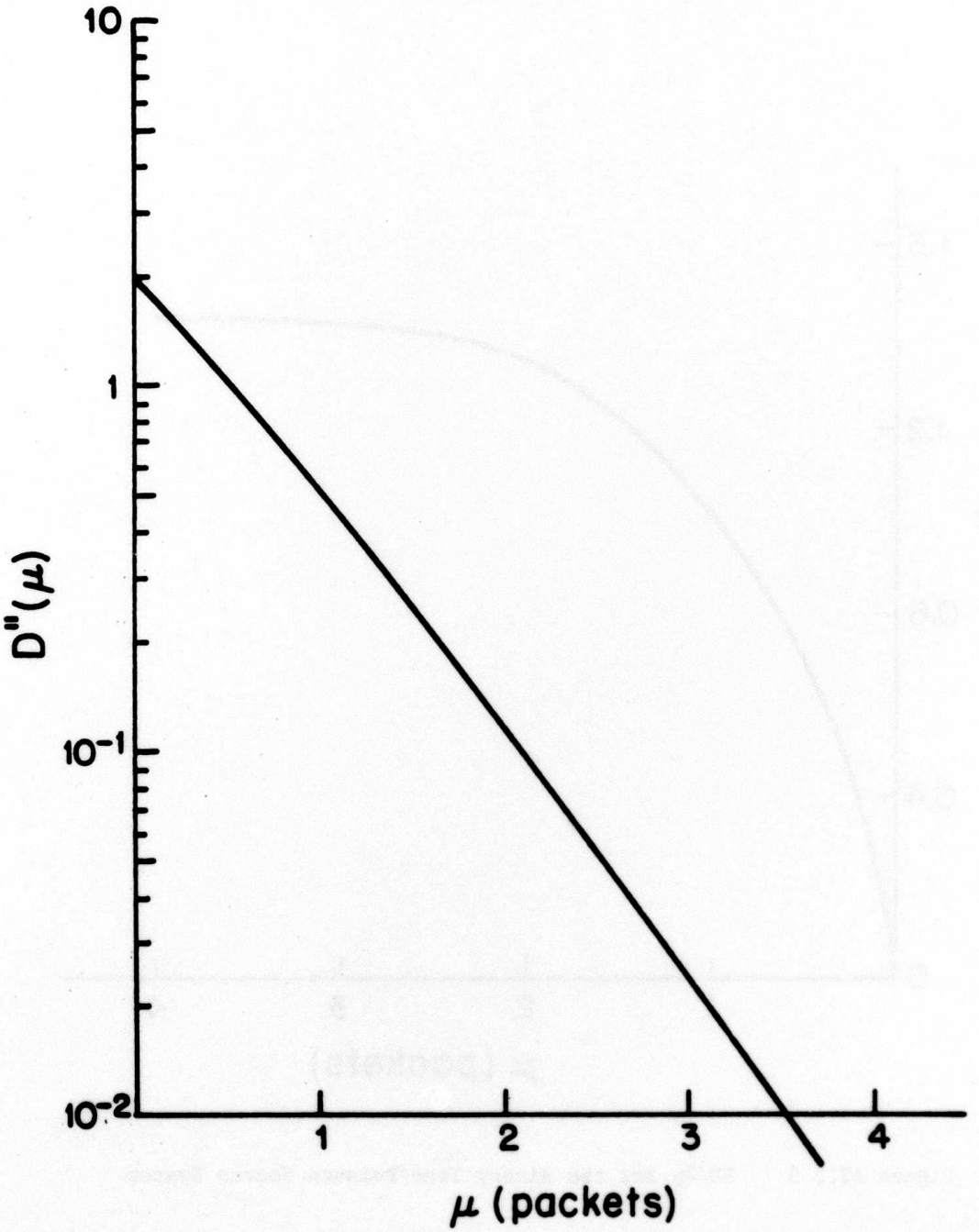


Figure A2.2.4 $\partial^2 D/(\partial\mu)^2$ for the Binary Tree/Poisson Source System

not constitute a proof of its properties over a range. However, given the smoothness of the functions involved (Properties 2A, 2B and 2C), it is felt that this technique is justified. A more rigorous approach would be to find a lower bound that is valid over a small finite region and then show by computation that this lower bound is positive over the desired range. For example, by taking the second derivative of $D(\mu)$ we have

$$\frac{\partial^2 D}{\partial \mu^2} = \sum_{i=0}^{\infty} \left[\frac{1}{2^i} \left(1 - \frac{\mu}{2^i}\right) e^{-\mu/2^i} \right]$$

It follows easily that

$$D''(\mu) \geq \sum_{i=0}^{\infty} \frac{e^{-(\mu_0+\Delta\mu)2^i}}{2^i} - \frac{\mu_0+\Delta\mu}{2^i} e^{-\mu_0/2^i}$$

$$\text{for } \mu_0 \leq \mu \leq \mu_0 + \Delta\mu$$

So to show rigorously that $D''(\mu) \geq 0$ for $0 \leq \mu \leq 4$ one need simply show by computation that the above lower bound is positive at

$$\mu_0 = K\Delta\mu \text{ for some } \Delta\mu \text{ and } K = 0,1,2,\ldots,\left|\frac{4}{\Delta\mu}\right|$$

As was pointed out above, it is felt that this more rigorous technique is not necessary for our purpose, therefore, when needed to show properties, such as positiveness, of a single variable function over some small range it will be accomplished by computing the function.

The properties that will be proved are listed below.

Property 1A $$\xi(\mu) = \sum_{k=1}^{\infty} k\frac{(-\mu)^{k+1}}{(k+1)!}$$

1B $$\frac{\mu^2}{2} - \frac{\mu^3}{3} \leq \xi(\mu) \leq \frac{\mu^2}{2} \text{ for } 0 \leq \mu \leq 1.2$$

1C $$D(\mu) = \sum_{k=1}^{\infty} \frac{k}{(k+1)!}(-\mu)^{k+1}\frac{2^k}{2^k-1}$$

1D $$\mu^2 - \frac{4}{9}\mu^3 \leq D(\mu) \leq \mu^2 \quad 0 \leq \mu \leq 1.2$$

2A $$D(2\mu) = 2D(\mu) + \xi(2\mu)$$

2B $$D'(2\mu) = D'(\mu) + \xi'(2\mu)$$

2C $$D''(2\mu) = \frac{1}{2}D''(\mu) + \xi''(2\mu)$$

2D $$D(\mu) = \sum_{i=0}^{\infty} \xi(\mu/2^i) + \sum_{i=1}^{\infty} D(\mu/2^i)$$

3A $$\min_{\mu \geq \mu_\ell} D'(\mu) = \min_{2\mu_\ell > \mu \geq \mu_\ell} D'(\mu) \equiv \beta_\ell(\mu_\ell)$$

$$\beta_\ell(4) = 1.440$$

3B $\quad D'(\mu) \geq 0$ for $\mu \geq 0$

3C $\quad \max_{\mu \geq 0} D'(\mu) \equiv \beta_u = 1.443$

3D $\quad D(\mu) \leq \beta_u \mu$

4 $\quad$ Let $f = \begin{cases} D(\mu) & 0 \leq \mu \leq 4 \\ D'(4)(\mu-4) + D(4) & \mu \geq 4 \end{cases}$

where $D'(4) = 1.44$ and $D(4) = 4.77$, then f is a convex increasing, positive lower bound to $D(\mu)$.

For the main text, we will use values of $\beta_\ell(4)$ and β_u that are rounded off to three significant figures. Therefore we will assume that $\beta_\ell(4) = \beta_u = 1.44$. The error caused by this approximation is insignificant.

Properties 1A and 1B. Here we will show that

$$\xi(\mu) = \sum_{k=1}^{\infty} k \frac{(-\mu)^{k+1}}{(k+1)!} \qquad \text{(A2.2.4)}$$

$$\frac{\mu^2}{2} - \frac{\mu^3}{3} \leq \xi(\mu) \leq \frac{\mu^2}{2} \quad \text{for } 0 < \mu \leq 1.2 \qquad \text{(A2.2.5)}$$

Proof: Substitute the Taylor series expansion of $e^{-\mu}$ into Eq. (A2.2.3) to obtain

$$\xi(\mu) = 1 - \sum_{k=0}^{\infty} \frac{(-\mu)^k}{k!} - \mu \sum_{k=0}^{\infty} \frac{(-\mu)^k}{k!} \qquad \text{(A2.2.6)}$$

Equation (A2.2.6) is then rearranged to obtain (A2.2.4). Equation (A2.2.5) follows from (A2.2.4) becuase for $\mu \leq 1.2$ the sum in Eq. (A2.2.4) is the sum of an alternating sequence whose magnitude is decreasing.

Properties 1C and 1D. Here we will prove that

$$D(\mu) = \sum_{k=1}^{\infty} \frac{k}{(k+1)!} (-\mu)^{k+1} \frac{2^k}{2^k-1} \qquad \text{(A2.2.7)}$$

and

$$\mu^2 - \frac{4}{9}\mu^3 \leq D(\mu) \leq \mu^2 \qquad \text{for } 0 < \mu \leq 1.2 \qquad \text{(A2.2.8)}$$

Proof: To obtain Eq. (A2.2.7), first substitute Eq. (A2.2.4) into Eq. (A2.2.2).

$$D(\mu) = \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} 2^i \frac{k(-\mu/2^i)^{k+1}}{(k+1)!} \qquad \text{(A2.2.9)}$$

Equation (A2.2.7) follows from (A2.2.9) by first interchanging the order of summation and then summing over the index i.

Equation (A2.2.8) follows from Eq. (A2.2.7) because the sum is Eq. (A2.2.7) is that of a decreasing alternating sequence.

QED

Properties 2A, 2B, and 2C. Here it will be proved that

$$D(2\mu) = 2D(\mu) + \xi(2\mu) \tag{A2.2.10}$$

$$D'(2\mu) = D'(\mu) + \xi'(2\mu) \tag{A2.2.11}$$

$$D''(2\mu) = \frac{1}{2} D''(\mu) + \xi''(2\mu) \tag{A2.2.12}$$

Proof: From Eq. (A2.2.2) follows that

$$D(2\mu) = \sum_{i=0}^{\infty} 2^i \, \xi(\mu/2^{i-1}) \tag{A2.2.13}$$

$$= 2 \sum_{i=0}^{\infty} 2^{i-1} \, \xi(\mu/2^{i-1})$$

$$= 2[\tfrac{1}{2} \xi(2\mu) + D(\mu)]$$

therefore

$$D(2\mu) = 2D(\mu) + \xi(2\mu) \tag{A2.2.14}$$

Equations (A2.2.11) and (A2.2.12) follow from Eq. (A2.2.14) by differentiating with respect to μ.

Property 2D. Here, we will prove that

$$D(\mu) = \sum_{i=0}^{\infty} \xi(\mu/2^i) + \sum_{i=1}^{\infty} D(\mu/2^i) \qquad \text{(A2.2.15)}$$

Proof: From Eq. (A2.2.2) we have that

$$D(\mu) = \sum_{i=0}^{\infty} \xi(\mu/2^i) + \sum_{i=0}^{\infty} (2^i-1)\xi(\mu/2^i) \qquad \text{(A2.2.16)}$$

Now,

$$2^i-1 = \sum_{k=0}^{i-1} 2^k$$

so

$$\sum_{i=0}^{\infty} (2^i-1)\xi(\mu/2^i) = \sum_{i=1}^{\infty} \sum_{k=0}^{i-1} 2^k \xi(\mu/2^i) \qquad \text{(A2.2.17)}$$

Interchange order of summation

$$= \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} 2^k \xi(\mu/2^i)$$

$$= \sum_{k=0}^{\infty} 2^k \sum_{i=1}^{\infty} \xi(\mu/2^{i+k})$$

Interchange order of summation once again to obtain

$$= \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} 2^k \xi(\mu/2^{i+k}) \qquad \text{(A2.2.18)}$$

But

$$\sum_{k=0}^{\infty} 2^k \xi(\mu/2^{i+k}) = D(\mu/2^i) \qquad \text{(A2.2.19)}$$

Equation (A2.2.15) follows by substituting Eq. (A2.2.19) into Eq. (A2.2.18) and Eq. (A2.2.18) into Eq. (A2.2.16).

Property 3A. Let

$$\beta_\ell(\mu_\ell) = \min_{\mu_\ell \leq \mu \leq 2\mu_\ell} D'(\mu) \qquad \text{(A2.2.20)}$$

Then Property 3A states that

$$D'(\mu) \geq \beta_\ell(\mu_\ell) \text{ for } \mu \geq \mu_\ell \qquad \text{(A2.2.21)}$$

and Property 3B is

$$D'(\mu) \geq 0 \text{ for } \mu \geq 0 \qquad \text{(A2.2.22)}$$

Proof: From Eqs. (A2.2.3) and (A2.2.11) we have

$$D'(2\mu) = D'(\mu) + 2\mu e^{-2\mu} \qquad \text{(A2.2.23)}$$

since

$$2\mu e^{-2\mu} > 0$$

it follows that

$$D'(2\mu) \geq D'(\mu)$$

therefore

$$\underset{\mu_\ell \leq \mu < 2\mu_\ell}{\text{Min}} \; D'(\mu) \leq \underset{2\mu_\ell \leq \mu < 4\mu_\ell}{\min} \; D'(\mu) \qquad \text{(A2.2.24)}$$

Equation (A2.2.21) follows by induction from Eq. (A2.2.24).

QED

The quantity $\beta_\ell(4)$ was calculated and it is given by

$$\beta_\ell(4) = 1.440 \qquad \text{(A2.2.25)}$$

Property 3B. Prove that

$$D'(\mu) \geq 0 \text{ for } \mu \geq 0 \tag{A2.2.26}$$

Proof: Differentiating Eq. (A2.2.2) we have

$$D'(\mu) = \sum_{i=0}^{\infty} (\mu/2^i)e^{-\mu/2^i} \tag{A2.2.27}$$

which is positive for $\mu \geq 0$.

Property 3C. Let

$$D'(\mu) = \sum_{i=0}^{\infty} (\mu/2^i)e^{-\mu/2^i} \qquad \mu \geq 0 \tag{A2.2.28}$$

and

$$D'_{\infty}(c) = \sum_{i=-\infty}^{\infty} \frac{c}{2^i} e^{-c/2^i}$$

then

$$\sup_{\mu \geq 0} D'(\mu) = \max_{1 \leq c \leq 2} D'_{\infty}(c) \tag{A2.2.29}$$

$$\equiv \beta_u \tag{A2.2.30}$$

First we will prove Eq. (A2.2.29) and then we will calculate β_u.

Proof: Substituting into Eq. (A2.2.28)

$$\mu = c2^k \text{ for } 1 \leq c \leq 2 \text{ and } k=0,1,2,\ldots$$

and then rearranging we have

$$D'(c2^k) = \sum_{i=-k}^{\infty} \frac{c}{2^i} e^{-c/2^i} \qquad \text{(A2.2.31)}$$

Since the summand in Eq. (A2.2.31) is positive we have

$$\lim_{k\to\infty} D'(c2^k) = D'_\infty(c)$$

$$\geq D'(c2^j) \text{ for } 1 \leq c \leq 2,\ j=0,1,2,\ldots \qquad \text{(A2.2.32)}$$

QED

The maximum of $D'_\infty(c)$ over $1 \leq c \leq 2$ was calculated and it equals 1.443. Therefore

$$\beta_u = 1.443 \qquad \text{(A2.2.33)}$$

<u>Property 3D</u> $\qquad D(\mu) \leq \beta_u \mu \qquad$ (A2.2.34)

This follows from Property 3C and the fact that $D(0) = 0$.

Property 4. A postive, convex non-decreasing lower bound to $D(\mu)$ is $f(\mu)$ where

$$f(\mu) = \begin{cases} D(\mu) & \text{for } 0 \leq \mu \leq 4 \\ D'(4)[\mu-4] + D(4) & \text{for } \mu \geq 4 \end{cases} \tag{A2.2.35}$$

Proof: From Property 3B we have that $D'(\mu) \geq 0$ and since $D(0) = 0$ it follows that $f(\mu)$ is positive and non-decreasing. That $D(\mu)$ is convex for $0 \leq \mu \leq 4$ follows from Fig. A2.2.4 and from a more detailed computer printout of $\frac{\partial^2 D}{\partial \mu^2}$. To show that f is convex for $\mu \geq 0$, note that

$$\frac{\partial f^2}{\partial \mu^2} = 0 \text{ for } \mu \geq 4 \tag{A2.2.36}$$

and that $f(\mu)$ is continuous and has a continuous first derivative at $\mu = 4$. Therefore $f(\mu)$ is convex.

Next we will show that

$$f(\mu) \leq D(\mu) \text{ for } \mu \geq 0 \tag{A2.2.37}$$

Since $f(\mu) = D(\mu)$ for $\mu \leq 4$, Eq. (A2.2.37) would be true if $f'(\mu) \leq D'(\mu)$ for $\mu \geq 4$. But

$$f'(\mu) = D'(4) \text{ for } \mu \geq 4 \tag{A2.2.38}$$

so we need to show that

$$D'(4) \leq D'(\mu) \text{ for } \mu \geq 4 \qquad \text{(A2.2.39)}$$

From Property 3a

$$\min_{4 \leq \mu \leq 8} D'(\mu) \leq D'(\mu) \text{ for } \mu \geq 4 \qquad \text{(A2.2.40)}$$

$D'(\mu)$ was computed for $4 \leq \mu \leq 8$ (see Fig. A2.2.2). The minimum did occur at 4 where

$$D'(4) = 1.440$$

QED

A2.3 Derivation of $E\{\ell^2|\mu\}$

In this section we will derive an expression for $E\{\ell^2|\mu\}$. This is accomplished in Theorem A2.3.1. However, before this theorem is proved it will be necessary to prove the following Lemma.

Lemma A2.3.1. Let x_{ij} be a random variable as defined in Eq. (A2.1.6) with the probability density given in Eq. (A2.1.11). Then for $i \geq m \geq 1$

$$E[x_{ij}x_{mk}|\mu] = \begin{cases} \xi(\mu/2^i) & \text{if } n_{ij} \text{ and } n_{mk} \text{ lie on the same path} \\ \xi(\mu/2^i)\xi(\mu/2^m) & \text{otherwise} \end{cases} \quad \text{(A.2.3.1)}$$

Note, two nodes (n_{ij} and n_{mk} : $i \geq m$) lie on the same path if n_{mk} lies on the line connecting n_{ij} and n_{00}.

Proof: If n_{ij} and n_{mk} lie on the same path, then

$$P(x_{mk}=1|x_{ij}=1) = 1 \quad \text{(A2.3.2)}$$

and

$$p(x_{ij}=1|\mu) = \xi(\mu/2^i) \quad \text{(A2.3.3)}$$

therefore

$$p(x_{ij}=1 \text{ and } x_{mk}=1|\mu) = \xi(\mu/2^i) \tag{A2.3.4}$$

Next note that

$$x_{ij}x_{mn} = \begin{cases} 1 \text{ iff } x_{ij}=1 \text{ and } x_{mk}=1 \\ 0 \text{ otherwise} \end{cases} \tag{A2.3.5}$$

Therefore, from Eqs. (A2.3.4) and (A2.3.5)

$$E[x_{ij}x_{mk}|\mu; n_{ij} \text{ and } n_{mk} \text{ on same path; } i \geq m] = \xi(\mu/2^i) \tag{A2.3.6}$$

If, on the other hand, n_{ij} and n_{mk} do not lie on the same path, then x_{ij} and x_{mk} are independent because the sources of T_{ij} and T_{mk} are independent. Therefore

$$E[x_{ij}x_{mk}|\mu; n_{ij} \text{ and } n_{mk} \text{ not on same path}] = \xi(\mu/2^i)\xi(\mu/2^m) \tag{A2.3.7}$$

QED

Now we are ready to proceed with Theorem A2.3.1. where $E[\ell^2|\mu]$ is derived.

Theorem A2.3.1. Let ν be a Poisson random variable with mean μ corresponding to the number of contending sources. Also let ℓ (given in Eq. (A2.2.10) and rewritten below) be the number of nodes transversed by the binary tree algorithm.

$$\ell = 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{2^i-1} x_{ij} \tag{A2.3.8}$$

Where x_{ij} is defined be Eqs. (A2.1.6) and (A2.1.11), then

$$E[\ell^2|\mu] = 1 + 2D(\mu/2) + [2D(\mu/2)]^2 + F(\mu) \tag{A2.3.9}$$

where

$$D(\mu) = \sum_{i=0}^{\infty} 2^i \xi(\mu/2^i) \tag{A2.3.10}$$

and

$$F(\mu) = 2 \sum_{i=1}^{\infty} 2^i D(\mu/2^i)[1 - \xi(\mu/2^i)] + \sum_{i=1}^{\infty} 2^i \xi^2(\mu/2^i) \tag{A2.3.11}$$

Proof: Square both sides of Eq. (A2.3.8) and then take the expectation to obtain

$$E[\ell^2|\mu] = 1 + 2E\left[\sum_{i=1}^{\infty} \sum_{j=0}^{2^i-1} x_{ij}|\mu\right] + E\left[\left(\sum_{i=1}^{\infty} \sum_{j=0}^{2^i-1} x_{ij}\right)^2|\mu\right] \tag{A2.3.12}$$

From Theorem A2.1.1 and Eq. (A2.3.10), the above expression can be simplified to

$$E[\ell^2|\mu] = 1 + 4D(\mu/2) + E\left[\left(\sum_{i=1}^{\infty}\sum_{j=0}^{2^i-1} x_{ij}\right)^2|\mu\right] \qquad \text{(A2.3.13)}$$

Next note that

$$\left(\sum_{i=1}^{\infty}\sum_{j=0}^{2^i-1} x_{ij}\right)^2 = \sum_{i=1}^{\infty}\sum_{j=0}^{2^i-1}\left[2\sum_{m\geq i}\sum_{n=\eta(i,m)}^{2^m-1} x_{ij}x_{mn} - x_{ij}^2\right] \qquad \text{(A2.3.14)}$$

where $$\eta(i,m) = \begin{cases} j \text{ for } m=i \\ 0 \text{ otherwise} \end{cases}$$

Now the sums over m and n will be decomposed into three sums Y_{ij}, Z_{ij} and W_{ij} : Y_{ij} is the sum over all $(m,n) \in T_{ij}$, Z_{ij} is the sum over all $(m,n) \notin T_{ij}$ and $m > i$, and W_{ij} is the sum over all $(m,n) \notin T_{ij}$ such that $m=i$, $n>j$. Figure A2.3.1 might be helpful in visualizing the preceding.

In terms of the above definitions we have

$$Y_{ij} = \sum_{(m,n)\in T_{ij}} x_{ij}x_{mn} \qquad \text{(A2.3.15)}$$

$$Z_{ij} = \sum_{k\in K}\sum_{(m,n)\in T_{i+1,k}} x_{ij}x_{mn} \qquad \text{(A2.3.16)}$$

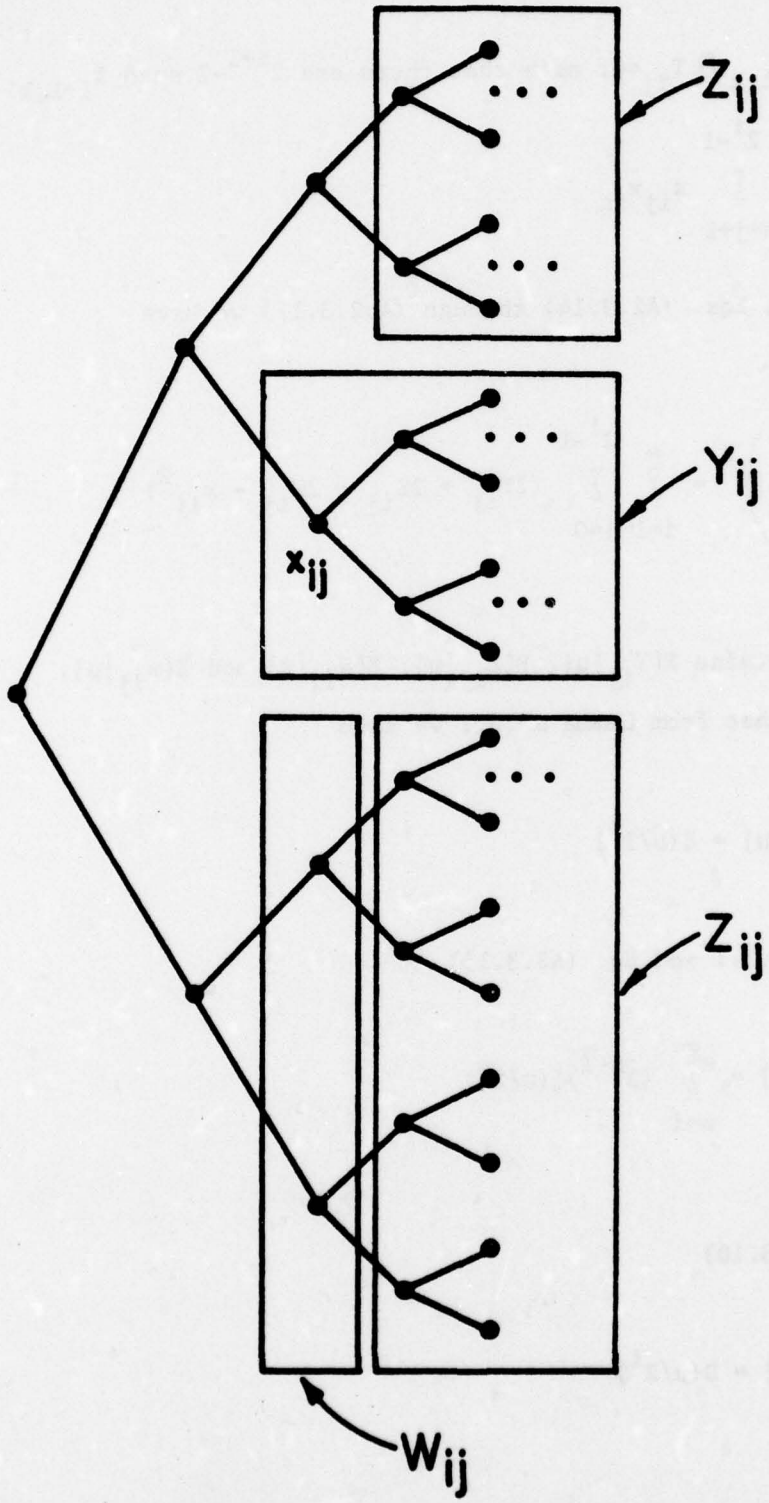


Figure A2.3.1 Partitioning the Binary Tree for the Derivation of $E\{\ell^2|\mu\}$

Where $k \in K$ if $T_{i+1,k} \cap T_{ij} = 0$; note that there are $2^{i+1}-2$ such $T_{i+1,k}$.

$$W_{ij} = \sum_{n=j+1}^{2^i-1} x_{ij}x_{in} \tag{A2.3.17}$$

Finally from Eqs. (A2.3.14) through (A.2.3.17) we have

$$\left(\sum_{i=1}^{\infty} \sum_{j=0}^{2^i-1} x_{ij} \right)^2 = \sum_{i=1}^{\infty} \sum_{j=0}^{2^i-1} (2Y_{ij} + 2Z_{ij} + 2W_{ij} - x_{ij}^2) \tag{A2.3.18}$$

Next we will determine $E[Y_{ij}|\mu]$, $E[Z_{ij}|\mu]$, $E[W_{ij}|\mu]$ and $E[x_{ij}^2|\mu]$.

First note that from Lemma A2.3.1 we have

$$E[x_{ij}^2|\mu] = \xi(\mu/2^i) \tag{A2.3.19}$$

Now from Lemma A2.3.1 and Eq. (A2.3.15)

$$E[Y_{ij}|\mu] = \sum_{m=i}^{\infty} (2^{m-i})\xi(\mu/2^m)$$

and from Eq. (A2.3.10)

$$E[Y_{ij}|\mu] = D(\mu/2^i) \tag{A2.3.20}$$

Next apply Lemma A2.3.1 to Eq. (A2.3.16) and use symmetry to obtain,

$$E[Z_{ij}|\mu] = (2^{i+1}-2) \sum_{m=i+1}^{\infty} 2^{m-i-1}\xi(\mu/2^i)\xi(\mu/2^m) \qquad \text{(A2.3.21)}$$

Simplifying

$$E[Z_{ij}|\mu] = 2(2^i-1)\xi(\mu/2^i)D(\mu/2^{i+1}) \qquad \text{(A2.3.22)}$$

Similarly

$$E[W_{ij}|\mu] = (2^i-1-j)\xi^2(\mu/2^i) \qquad \text{(A2.3.23)}$$

Next take expectations of both sides of Eq. (A2.3.18), substitute Eqs. (A2.3.19), (A2.3.20), (A2.3.22) and (A2.3.23) into (A2.3.18) sum over j and then substitute this into Eq. (A2.3.13) to obtain,

$$E[\ell^2|\mu] = 1+2D(\mu/2) + \sum_{i=1}^{\infty} \Big(2(2^i)D(\mu/2^i)+4(2^i)(2^i-1)\xi(\mu/2^i)D(\mu/2^{i+1}) + 2^i(2^i-1)\xi^2(\mu/2^i)\Big) \qquad \text{(A2.3.24)}$$

Now from Property 2A we have

$$D(\mu/2^{i+1}) = \tfrac{1}{2} D(\mu/2^i) - \tfrac{1}{2} \xi(\mu/2^i) \qquad \text{(A2.3.25)}$$

Substituting this into Eq. (A2.3.24) and then rearranging we have,

$$E[\ell^2|\mu] = 1+2D(\mu/2) + \sum_{i=1}^{\infty} 2^i[2D(\mu/2^i)+2(2^i-1)D(\mu/2^i)\xi(\mu/2^i)-(2^i-1)\xi^2(\mu/2^i)] \tag{A2.3.26}$$

Next note that

$$(2D(\mu/2))^2 = \left[\sum_{i=1}^{\infty} 2^i\,\xi(\mu/2^i)\right]^2 \tag{A2.3.27}$$

$$= 2\sum_{i=1}^{\infty}\sum_{j=i}^{\infty} 2^{i+j}\xi(\mu/2^i)\xi(\mu/2^j) - \sum_{i=1}^{\infty}(2^i)^2\xi^2(\mu/2^i)$$

Summing over j

$$= \sum_{i=1}^{\infty} 2(2^i)^2\xi(\mu/2^i)D(\mu/2^i) - (2^i)^2\xi^2(\mu/2^i) \tag{A2.3.28}$$

Substituting Eq. (A2.3.28) into Eq. (A2.3.26) concludes the proof.

QED

Next in Appendix A2.4 we will derive upper and lower bounds to $E[\ell^2|\mu]$.

A2.4 Upper and Lower Bounds to $E[\ell^2|\mu]$

In Appendix A2.3 we showed that

$$E[\ell^2|\mu] = 1 + 2D(\mu/2) + [2D(\mu/2)]^2 + F(\mu) \tag{A2.4.1}$$

where

$$D(\mu) = \sum_{i=0}^{\infty} 2^i \xi(\mu/2^i) \tag{A2.4.2}$$

$$F(\mu) = 2 \sum_{i=1}^{\infty} 2^i D(\mu/2^i)[1-\xi(\mu/2^i)] + \sum_{i=0}^{\infty} 2^i \xi^2(\mu/2^i) \tag{A2.4.3}$$

The quantity $\sqrt{E[\ell^2|\mu]}$ was calculated and it is plotted in Fig. A2.4.1. In this Appendix we will derive upper and lower bounds to $E[\ell^2|\mu]$. More specifically we will show that

$$E[\ell^2|\mu] \leq (1 + 1.44\mu)^2 \tag{A2.4.4}$$

$$E[\ell^2|\mu] \geq (.25 + 1.44\mu)^2 \tag{A2.4.5}$$

We begin with the proof of Eq. (A2.4.4). Two other quantities will be needed in the analysis that follows. These are

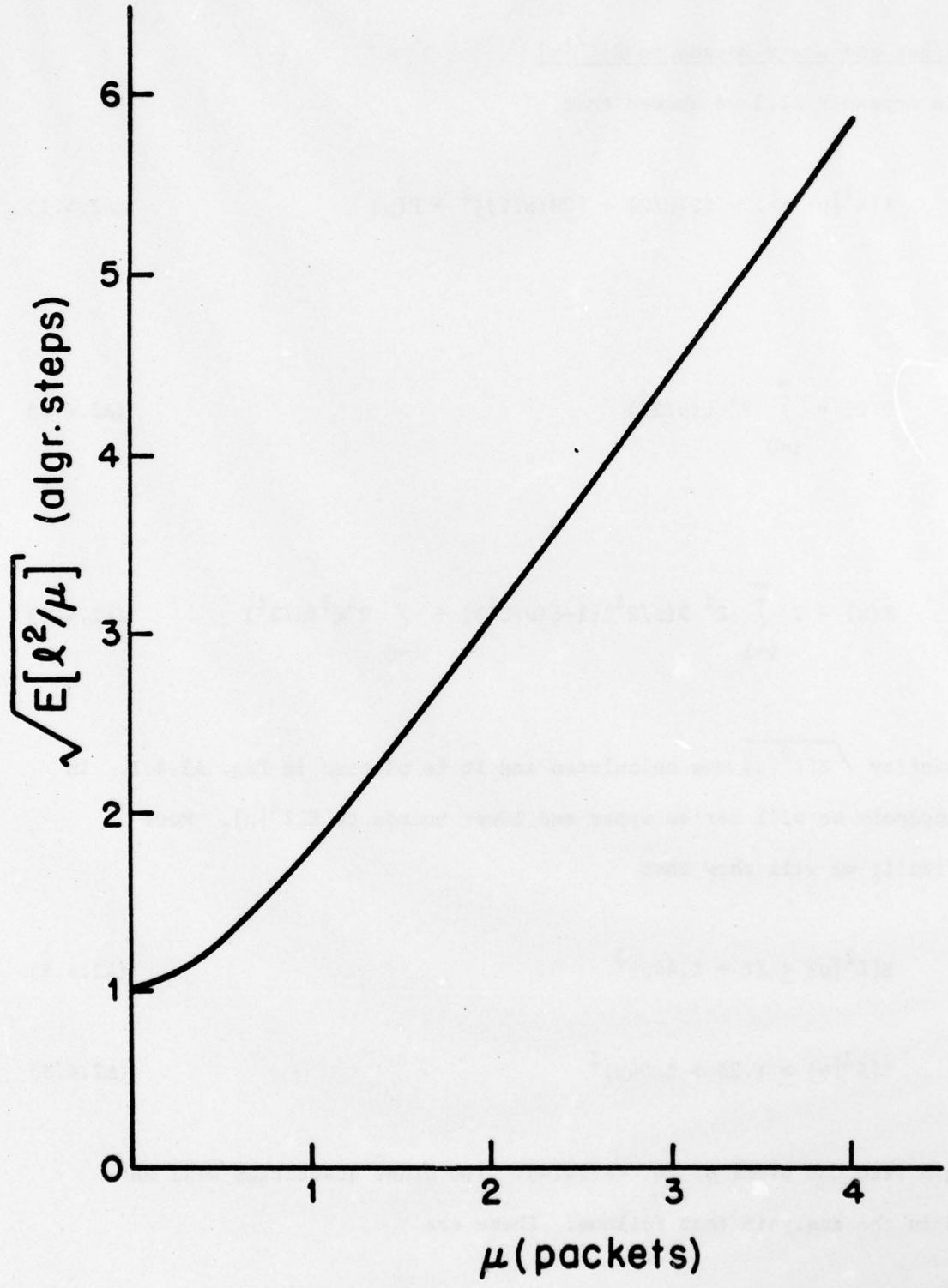


Figure A2.4.1 $\sqrt{E\{\ell^2|\mu\}}$ versus μ for the Binary Tree/Poisson Source System

$$F'_{mx}(16) = \max_{\mu \geq 16} F'(\mu) = 4.369 \qquad \text{(A2.4.6)}$$

$$D(8) = 10.54 \qquad \text{(A2.4.7)}$$

The computations of $F'_{mx}(16)$ is somewhat involved and it is performed at the end of this Appendix. The value for $F'_{mx}(16)$ given in Eq. (A2.4.6) is calculated in Eqs. (A2.4.11) through (A2.4.22). The computation of D(8) is straightforward and no special explanation for its computation will be given.

From Fig. A2.4.1 and from a more detailed computer printout, it can be seen that Eq. (A2.4.4) is true for $0 \leq \mu \leq 16$. The validity of this equation for $\mu > 16$ will follow if we prove that

$$\frac{\partial}{\partial \mu} E[\ell^2|\mu] \leq \frac{\partial}{\partial \mu} (1 + 1.44\mu)^2 \text{ for } \mu \geq 16 \qquad \text{(A2.4.8)}$$

We begin by differentiating Eq. (A2.4.1)

$$\frac{\partial}{\partial \mu} E[\ell^2|\mu] = D'(\mu/2) + 4D(\mu/2)D'(\mu/2) + F'(\mu) \qquad \text{(A2.4.9)}$$

Now by applying Property 3C we can upper bound the right side of Eq. (A2.4.9) as follows:

$$\frac{\partial}{\partial\mu} E[\ell^2|\mu] \leq \beta_u + 4\beta_u[\frac{\beta_u}{2}(\mu-16)+D(8)] + F'_{mx}(16) \text{ for } \mu \geq 16$$

or

$$\frac{\partial}{\partial\mu} E[\ell^2|\mu] \leq 4.147\mu + .164 \text{ for } \mu \geq 16 \qquad \text{(A2.4.10)}$$

Equation (A2.4.8) follows by carrying out the indicated differentiation on its right side and then comparing the result to Eq. (A2.4.10).

The above argument was based on Eq. (A2.4.6). Next, we will prove this equation. Now from Eq. (A2.4.3) we have

$$F(\mu) = \sum_{i=1}^{\infty} 2^i f(\mu/2^i) \qquad \text{(A2.4.11)}$$

where

$$f(\mu) \equiv 2D(\mu)(1-\xi(\mu)) + \xi^2(\mu) \qquad \text{(A2.4.12)}$$

Differentiating Eq. (A2.4.11) we have

$$F'(\mu) = \sum_{i=1}^{\infty} f'(\mu/2^i) \qquad \text{(A2.4.13)}$$

Next any $\mu \geq 16$ can be expressed as

$$\mu = x2^k \text{ where } 8 \leq x \leq 16 \text{ and } k = 1,2,3,... \qquad \text{(A2.4.14)}$$

Substituting Eq. (A2.4.14) into Eq. (A2.4.13) and then rearranging we have

$$F'(x2^k) = F'(x) + \sum_{i=1}^{k} f'(2^i x) \qquad \text{(A2.4.15)}$$

Next we will show that

$$\sum_{i=1}^{k} f'(2^i x) \leq 10^{-4} \text{ for } k \geq 1 \text{ and } 8 \leq x \leq 16 \qquad \text{(A2.4.16)}$$

Differentiating Eq. (A2.4.12) and then substituting $D(\mu) = 2D(\mu/2)+\xi(\mu)$ (Property 2A) we have

$$f'(\mu) = 2D'(\mu)(1-\xi(\mu))-4D(\mu/2)\xi'(\mu)$$

$$= 2e^{-\mu}[D'(\mu)(1+\mu) - 2D(\mu/2)\mu] \qquad \text{(A2.4.17)}$$

A simple upper to $f'(\mu)$ is

$$f'(\mu) \leq 2e^{-\mu}D'(\mu)(1+\mu) \qquad \text{(A2.4.18)}$$

Equation (A2.4.18) can be, further, upper bounded by applying Property 3C. Doing this we have

$$f'(\mu) \leq 2\beta_u(1+\mu)e^{-\mu} \qquad \text{(A2.4.19)}$$

therefore

$$\sum_{i=1}^{k} f'(2^i x) \leq \sum_{i=1}^{\infty} 2\beta_u (1+2^i x) e^{-2^i x} \leq 10^{-4} \text{ for } x\varepsilon[8,16] \qquad \text{(A2.4.20)}$$

Finally from Eq. (A2.4.15) we have that

$$\max_{\mu \geq 16} F'(\mu) \leq \max_{8 \leq \mu \leq 16} F'(\mu) + 10^{-4} \qquad \text{(A2.4.21)}$$

The right side of Eq. (A2.4.21) was calculated and is given by

$$F_{mx}(16) = 4.369 \qquad \text{(A2.4.22)}$$

This concludes the proof of Eq. (A2.4.4).

The proof of Eq. (A2.4.5) is similar to that of Eq. (A2.4.4) and will not be given here. That Eq. (A2.4.5) is true should be obvious from Fig. A2.4.1.

A2.5 $E[\ell_1\ell_2] \geq E[\ell_1]E[\ell_2]$

In this Appendix we will prove that $E[\ell_1\ell_2] \geq E[\ell_1]E[\ell_2]$ where ℓ_1 and ℓ_2 are the lengths of two consecutive epochs. This will be accomplished in two steps; first, in Lemma A5.1 it is shown that $E[\ell_2|\ell_1]$ is a non-decreasing function of ℓ_1, and then in Theorem A5.1, the main result is proved.

Lemma A5.1. Let

$$E[\ell_2|\mu] = 1 + 2D(\mu/2) \text{ (See Eq. (A2.1.7))} \tag{A2.5.1}$$

where

$$\mu = 2\lambda\ell_1 \text{ (See. Eq. (2.2.1.1))} \tag{A2.5.2}$$

and $\lambda \geq 0$ is the packet arrival rate. Then

$$\frac{\partial}{\partial \ell_1} E[\ell_2|\ell_1] \geq 0 \tag{A2.5.3}$$

Proof: From Eqs. (A2.5.1) and (A2.5.2) we have

$$\frac{\partial}{\partial \ell_1} E[\ell_2|\ell_1] = 2\lambda D'(\lambda\ell_1) \tag{A2.5.4}$$

From Property 3B, $D'(\mu) \geq 0$ and since $\lambda \geq 0$ it follows that $\frac{\partial}{\partial \ell_1} E[\ell_2|\ell_1] \geq 0$.

QED

Now we are ready to prove the main result.

Theorem A2.5.1. Let $E[\ell_2|\ell_1]$ be non-decreasing in ℓ_1. Then, assuming all expectations to exist,

$$E[\ell_1\ell_2] \geq E[\ell_1]E[\ell_2] \tag{A2.5.5}$$

Proof: An equivalent statement to Eq. (A2.5.5) is

$$E[(\ell_1-\bar{\ell}_1)(\ell_2-\bar{\ell}_2)] \geq 0 \tag{A2.5.6}$$

Now,

$$E[(\ell_1-\bar{\ell}_1)(\ell_2-\bar{\ell}_2) = \sum_{\ell_1} (\ell_1-\bar{\ell}_1)[E[\ell_2|\ell_1)-\bar{\ell}_2]p(\ell_1)$$

$$= \sum_{\ell_1 \leq \bar{\ell}_1} (\bar{\ell}_1-\ell_1)[\bar{\ell}_2-E(\ell_2|\ell_1)]p(\ell_1) + \sum_{\ell_1 > \bar{\ell}_1} (\ell_1-\bar{\ell}_1)[E(\ell_2|\ell_1)-\bar{\ell}_2]p(\ell_1) \tag{A2.5.7}$$

$$\geq \sum_{\ell_1 \leq \bar{\ell}_1} (\bar{\ell}_1-\ell_1)[\bar{\ell}_2-E(\ell_2|\ell_1)]p(\ell_1) + \sum_{\ell_1 > \bar{\ell}_1} (\ell_1-\bar{\ell}_1)[E(\ell_2|\bar{\ell}_1)-\bar{\ell}_2]p(\ell_1) \tag{A2.5.8}$$

$$= 0 \tag{A2.5.9}$$

Equation (A2.5.8) follows from (A2.5.7) because $E[\ell_2|\ell_1]$ is non-decreasing. Carrying out the summation in Eq. (A2.5.8) results in Eq. (A2.5.9).

QED

This concludes Appendix A2.5.

A2.6 Derivation of E[d|μ]

Assume that there are $\nu \geq 1$ packets to be processed by the binary tree algorithm where ν is a Poisson random variable with mean μ, and let d correspond to the number of nodes visited by the algorithm before a randomly selected packet from the set of the ν active ones is successfully transmitted. In this Appendix we will determine an expression for $E[d|\mu]$. Note that $E[\ell|\mu]$, as derived in Appendix A2.1, is the expected number of nodes visited given μ, whereas $E[d|\mu]$ equals the expected number of nodes visited before a randomly selected packet is transmitted. The relationship between $E[\ell|\mu]$ and $E[d|\mu]$ is examined in Appendix A2.7.

The binary tree is shown in Fig. A2.6.1. As has been pointed out previously, the sources correspond to the leaves of this tree. Another representation of the sources which will be more convenient for the work in this Appendix is to represent each source by the binary number S where,

$$S = s_0 s_1 s_2 \ldots . \qquad \text{(A2.6.1)}$$

The equivalence between the two representations is established through the following convention.

> If s_i=0 take the upper branch emanating from the node of level-i, and if s_i=1 take the lower branch. This is performed consecutively beginning with i=0,1,....etc.

In Fig. A2.6.1, for example, the circled nodes correspond to S=0110...

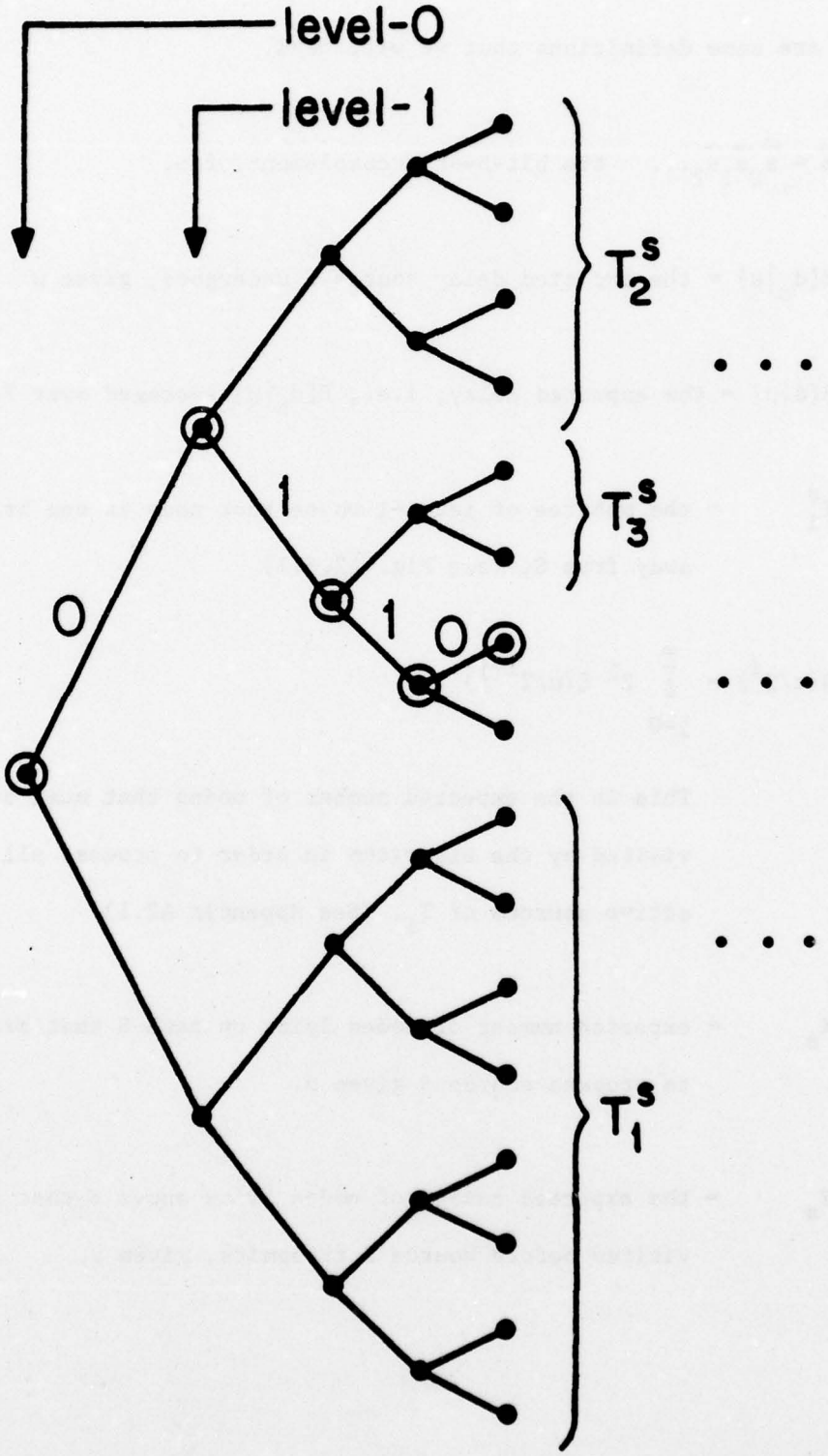


Figure A2.6.1 The Binary Tree: Definitions for the Derivation of $E\{d|\mu\}$

Below are some definitions that we will need.

$\bar{S} = \bar{s}_0\bar{s}_1\bar{s}_2\ldots$ the bit-by-bit complement of S. (A2.6.2)

$E[d_s|\mu]$ = the expected delay source S undergoes, given μ. (A2.6.3)

$E[d|\mu]$ = the expected delay, i.e., $E[d_s|\mu]$ averaged over S. (A2.6.4)

T_i^s = the subtree of level-i whose root node is one branch away from S, (see Fig. A2.6.1) (A2.6.5)

$$D(\mu/2^i) = \sum_{j=0}^{\infty} 2^j \xi(\mu/2^{i+j}) \quad \text{(A2.6.6)}$$

This is the expected number of nodes that must be visited by the algorithm in order to process all the active sources of T_i. (See Appendix A2.1)

X_s = expected number of nodes lying on path-S that are used to process source S given μ. (A2.6.7)

Y_s = the expected number of nodes lying above S that are visited before source S transmits, given μ. (A2.6.8)

$$\theta(\mu/2^i) = \frac{\xi(\mu/2^i)}{1-e^{-\mu/2^i}} \qquad \text{(A2.6.9)}$$

This is the probability that there are at least two active sources in T_i, given that there is at least one active source.

In terms of the above definitions we have

$$E[d_s|\mu] = X_s + Y_s \qquad \text{(A2.6.10)}$$

Next we will calculate $E[d_s|\mu]$ and $E[d_{\bar{s}}|\mu]$ and show that $E[d_s|\mu] + E[d_{\bar{s}}|\mu]$ is independent of S. This result will lead us to conclude that

$$E[d|\mu] = \frac{1}{2}\,[E[d_s|\mu] + E[d_{\bar{s}}|\mu]] \qquad \text{(A2.6.11)}$$

First we will calculate X_s. A node of level i will be visited by the algorithm if there are at least two active sources in the subtree that emanate from it (see Lemma A2.1.1). Now any subtree whose root node lies on S contains at least one active source, i.e., source S. Therefore, the probability that a node of level-i, that is on S will be used is $\xi(\mu/2^i)$. Now as in Theorem A2.1.1 use the fact that expectation of a sum equals the sum of the expectation to obtain,

$$X_s = 1 + \sum_{i=1}^{\infty} \theta(\mu/2^i) \qquad \text{(A2.6.12)}$$

Now we will make several observations before deriving Y_s. Figure A2.6.1 should be helpful in conceptualizing the following observations.

A) T_i^s lies above S iff $s_{i-1} = 1$.

B) The set of all the T_i^s that lie above S equals all the nodes that lie above S.

C) The average number of nodes in T_i^s that are used (not necessarily before S is transmitted) is $D(\mu/2^i)$. This follows from Theorem A2.1.1.

D) A tree T_i^s lying above S will be processed before S if there are more than one active source in the subtree of level-i that contains S. This is so because if S is the only active source in T_i, then that source would have been transmitted at some node at level less than i. Therefore, the probability that T_i^s that lies above S will be processed before S is $\theta(\mu/2^i)$.

Next we will combine the above observations so as to derive an expression for Y_s.

From observations A and D we have

$$P_r[T_{i+1}^s \text{ lies above S and it is processed before S}] = s_i \xi(\mu/2^{i+1}) \quad \text{(A2.6.13)}$$

From Eq. (A2.6.13) and observation C we have

E[number of nodes in T^s_{i+1} that are processed before S]

$$= S_i \theta(\mu/2^{i+1}) D(\mu/2^{i+1}) \tag{A2.6.14}$$

From Eq. (A2.6.14) and observation B we have

$$Y_s = \sum_{i=0}^{\infty} s_i \theta(\mu/2^{i+1}) D(\mu/2^{i+1}) \tag{A2.6.15}$$

Therefore substituting Eqs. (A2.6.12) and (A2.6.15) into Eq. (A2.6.10) and then rearranging we have

$$E[d_s|\mu] = 1 + \sum_{i=1}^{\infty} (1+s_{i-1} D(\mu/2^i)) \theta(\mu/2^i) \tag{A2.6.16}$$

similarly

$$E[d_s|\mu] = 1 + \sum_{i=1}^{\infty} (1+\bar{s}_{i-1} D(\mu/2^i)) \theta(\mu/2^i) \tag{A2.6.17}$$

and

$$\tfrac{1}{2}[E[d_s|\mu]+E[d_s|\mu]] = 1 + \sum_{i=1}^{\infty} \left(1+\frac{s_{i-1}+\bar{s}_{i-1}}{2} D(\mu/2^i)\right) \theta(\mu/2^i) \tag{A2.6.18}$$

Since $s_i + \bar{s}_i = 1$, $\frac{1}{2}[E[d_s]+E[d_s]]$ is independent of S, therefore,

$$E[d|\mu] = 1 + \sum_{i=1}^{\infty} \theta(\mu/2^i)[1+\tfrac{1}{2}D(\mu/2^i)] \qquad \text{(A2.6.19)}$$

This concludes this Appendix. In Appendix A2.7 that follows, we develop the relationship between $E[\ell|\mu]$ and $E[d|\mu]$.

A2.7 Relationship Between E[d|μ] and E[ℓ|μ]

In Appendices A2.1 and A2.6 we derived expressions for E[ℓ|μ] and E[d|μ]. These expressions are rewritten below.

$$E[\ell|\mu] = 1 + 2D(\mu/2) \tag{A2.7.1}$$

$$E[d|\mu] = 1 + \sum_{i=1}^{\infty} \theta(\mu/2^i)[1 + \tfrac{1}{2} D(\mu/2^i)] \tag{A2.7.2}$$

where

$$D(\mu) = \sum_{i=0}^{\infty} 2^i \xi(\mu/2^i) \tag{A2.7.3}$$

$$\xi(\mu) = 1 - e^{-\mu} - \mu e^{-\mu} \tag{A2.7.4}$$

$$\theta(\mu) = \frac{\xi(\mu)}{1-e^{-\mu}} \tag{A2.7.5}$$

In this Appendix we will prove in Theorems A2.7.1 and A2.7.2, respectively that

$$E[d|\mu] \geq \tfrac{1}{2} E[\ell|\mu] \qquad \text{for } \mu \geq 0 \tag{A2.7.6}$$

and

$$E[d|\mu] \leq a\, E[\ell|\mu] + b \text{ for } \mu \geq 0 \qquad \text{(A2.7.7)}$$

where $(a,b) = (.55,.321)$

Equation (A2.7.7) is specifically proved for the constants $(a,b) = (.55,.321)$. However, it should be noted that there are other (a,b) pairs, that satisfy Eq. (A2.7.7).

Now we are ready to prove Eqs. (A2.7.6) and (A2.7.7). In the Theorems A2.7.1 and A2.7.2 that follow, we will refer extensively to the properties of $D(\mu)$ that were developed in Appendix A2.2. We will also need the following Lemma.

Lemma A2.7.1. Let $E[d|\mu]$ be as given in Eq. (A2.7.2). Then

$$\frac{\partial}{\partial \mu} E[d|\mu] \geq 0 \qquad \text{(A2.7.8)}$$

Proof: Differentiating Eq. (A2.7.2) and then rearranging we have

$$\frac{\partial}{\partial \mu} E[d|\mu] = \sum_{i=1}^{\infty} \frac{1}{2^i}\left[\frac{1}{2} D'(\mu/2^i)\, \frac{\xi(\mu/2^i)}{1-e^{-\mu/2^i}} + \frac{\left(1+\frac{1}{2}D\left(\frac{\mu}{2^i}\right)\right)\left(-1+\frac{\mu}{2^i}+e^{-\frac{\mu}{2^i}}-e^{-\frac{\mu}{2^i}}\right)}{\left(1-e^{-\mu/2^i}\right)^2}\right] \qquad \text{(A2.7.9)}$$

Now, it can easily be shown that $-1 + \mu + e^{-\mu} \geq 0$ for $\mu \geq 0$. And since $D(\mu) \geq 0$ and $D'(\mu) \geq 0$ (See Property 3B), it follows that all the terms in Eq. (A2.7.9) are positive.

QED

Theorem A2.7.1. Let $E[d|\mu]$ and $E[\ell|\mu]$ be as given in Eqs. (A2.7.1) and (A2.7.2) then,

$$E[d|\mu] \geq \frac{1}{2} E[\ell|\mu] \qquad \text{for } \mu \geq 0 \tag{A2.7.10}$$

Proof: By computation it can be shown that the maximum of $\frac{1}{2} E[\ell|\mu] - E[d|\mu]$ in the range of $\mu \epsilon [0,4]$ occurs at $\mu=0$ and it equals 0. Therefore,

$$\frac{1}{2} E[\ell|\mu] - E[d|\mu] \leq 0 \qquad \text{for } 0 \leq \mu \leq 4 \tag{A2.7.11}$$

For $\mu \geq 4$ we will first prove that for any $\mu \geq 4$ there exists a $\mu^* \epsilon [2,4]$ such that

$$\frac{1}{2} E[\ell|\mu] - E[d|\mu] \leq \frac{1}{2} E[\ell|\mu^*] - E[d|\mu^*] \quad \text{for } \mu \geq 4 \tag{A2.7.12}$$

The conclusion to the proof will follow by using Eq. (A2.7.11) to upper bound the right side of Eq. (A2.7.12).

From Eqs. (A2.7.1) and (A2.7.2) and Properties 2A and 2D we have

$$\frac{1}{2}\,E[\ell|\mu]-E[d|\mu] = -\frac{1}{2} + \frac{1}{2}\sum_{i=1}^{\infty} f_1(\mu/2^i) \qquad \text{(A2.7.13)}$$

where

$$f_1(\mu) = \left(\frac{1+e^{-\mu}}{1-e^{-\mu}}\right)\xi(\mu) + \left(\frac{e^{-\mu}}{1-e^{-\mu}}\right) D(\mu) \qquad \text{(A2.7.14)}$$

Now for any $\mu > 4$ there exists a k such that $2 < \mu/2^k \leq 4$. For that k Eq. (A2.7.13) can be rewritten as follows.

$$\frac{1}{2}[E[\ell|\mu]-E[d|\mu]] = \frac{1}{2}[E[\ell|\mu 2^{-k})-E[d|\mu 2^{-k}]] + \sum_{i=1}^{k-1} f_1(\mu/2^i) \qquad \text{(A2.7.15)}$$

From the definition of k it follows that $\mu/2^i > 4$ for $i < k$. Next, therefore, we will show that $f_1(x) \leq 0$ for $x > 4$, our ultimate objective being to show that the sum in Eq. (A2.7.15) is negative for $\mu \geq 4$.

From Property 3D we have

$$f_1(x) \leq \left(\frac{1+e^{-x}}{1-e^{-x}}\right)\xi(x) + (\beta_u x)\,\frac{xe^{-x}}{x-e^{-x}} \qquad \text{(A2.7.16)}$$

It is a straightforward exercise to show that the right side of Eq. (A2.7.16) is negative for $x \geq 4$. So

$$f_1(x) < 0 \quad \text{for} \quad x \geq 4 \tag{A2.7.17}$$

And from Eq. (A2.7.15) we have

$$\tfrac{1}{2}[E[\ell|\mu]-E[d|\mu]] \leq \tfrac{1}{2}[E[\ell|\mu 2^{-k}]-E[d|\mu 2^{-k}]] \tag{A2.7.18}$$

where

$$\mu \geq 4 \text{ and } 2 \leq \mu 2^{-k} < 4$$

QED

Theorem A2.7.2. Let $E[\ell|\mu]$ and $E[d|\mu]$ be as given in Eqs. (A2.7.1) and (A2.7.2). Then

$$E[d|\mu] \leq .55\ E[\ell|\mu] + .321 \tag{A2.7.19}$$

Proof: The structure of this proof is similar to that of Theorem A2.7.1. By computation it can be shown that

$$E[d|\mu] - .55\ E[\ell|\mu] \leq .321 \text{ for } 0 \leq \mu \leq 8 \qquad \text{(A2.7.20)}$$

To prove Eq. (A2.7.19) for $\mu > 8$ we have from Eqs. (A2.7.1) and (A2.7.2) and Properties 2A and 2D that

$$E[d/\mu] - .55\ E[\ell/\mu] = .45 + \sum_{i=1}^{\infty} f_2(\mu/2^i) \qquad \text{(A2.7.21)}$$

where

$$f_2(\mu) = \xi(\mu) \left(\frac{.45 + .55e^{-\mu}}{1 - e^{-\mu}} \right) + D(\mu) \left[\frac{1}{2} \frac{\xi(\mu)}{1 - e^{-\mu}} - .55 \right] \qquad \text{(A2.7.22)}$$

As in Theorem A2.7.1 we conclude this proof by showing that $f_2(\mu) \leq 0$ for $\mu \geq 8$. This follows by substituting the following lower bound for $D(\mu)$ into Eq. (A2.7.22).

$$D(\mu) \geq \beta_\ell(4)[\mu-4] + D(4)$$

$$= 1.44(\mu-4) + 4.77 \qquad \text{(A2.7.23)}$$

QED

This concludes Appendix A2.7.

A2.8 An Upper Bound to $E\{e^{s\ell}|\mu\}$

Let $G(s,\mu) = E\{e^{s\ell}|\mu\}$ be the generating function for the number of steps required to resolve a conflict by the binary tree algorithm, given that the number of contending packets is a Poisson random variable with mean μ. In this appendix we will upper bound $G(s,\mu)$ and $\partial G(s,\partial)/\partial s$. More specifically if we will let

$$G_u(s,\mu,x) = (1+\mu)e^{s-\mu} - (A/B^2 + A\mu/B)e^{-\mu} + (A/B^2)\, e^{-\mu+B\mu} \quad (A2.8.1)$$

then

$$G(s,\mu) \leq G_u(s,\mu,x) \quad (A2.8.2)$$

and

$$\frac{\partial}{\partial s} G(s,\mu) \leq \frac{\partial}{\partial s} G_u(s,\mu,2) \quad (A2.8.3)$$

where

$$A = \frac{e^s}{2-e^s}\ ,\ B = \frac{(x-1)e^s}{x-e^s},\ 1 < x < 3,\ 0 \leq s \leq s_o,\ s_o > 0. \quad (A2.8.4)$$

s_o depends on x and is positive for $1 < x < 3$; at $x=2$, $s_o = \ln 2$. x is a variable which is chosen so as to minimize the right side of Eq. (A2.8.1). Note that the right side of Eq. (A2.8.1) is decreasing in x, therefore, its greatest lower bound occurs at $x=3$.

Now we are ready to begin with the proof of Eq. (A2.8.2). Let,

$$f_j(s) = E\{e^{s\ell}|j \text{ packets to resolve}\} \quad (A2.8.5)$$

$$= \sum_{i=1}^{j-1} e^s 2^{-j} \binom{j}{i} f_i(s) f_{j-i}(s) + 2^{-j+1} e^s f_j(s) \qquad \text{(A2.8.6)}$$

We have divided j into i packets in the upper subtree and j-i in the lower. Note that e^s is the contribution of the root node, $2^{-j}\binom{j}{i}$ is the probability of the assumed division, and $f_i(s)$ and $f_{j-i}(s)$ are the generating functions for the number of steps to resolve each subtree. Furthermore, note that $f_o(s) = f_1(s) = 1$; this is because exactly one step is required when there are either zero or one packet.

Now solving Eq. (A2.8.6) for $f_j(s)$ we have,

$$f_j(s) = \frac{\sum_{i=1}^{j-1} e^s 2^{-j} \binom{j}{i} f_i(s) f_{j-i}(s)}{1-2^{-j+1}e^s} \qquad \text{for } j > 1 \qquad \text{(A2.8.7)}$$

Solving the above equations recursively we have

$$f_1(s) = 1$$

$$f_2(s) = \frac{e^s}{2-e^s}$$

$$f_3(s) = \left(\frac{e^s}{2-e^s}\right)\left(\frac{3e^{-s}}{4-e^s}\right)$$

Motivated by the above expressions, we will show that

$$f_j(s) \leq \frac{e^s}{2-e^s} \left(\frac{(x-1)e^s}{x-e^s}\right)^{j-2} \quad \text{for } j \geq 2,\ 1 < x < 3 \text{ and } s \leq s_o \qquad \text{(A2.8.8)}$$

where $s_o > 0$ for $1 < x < 3$.

Assuming that the above is true for j-1, then

$$f_j(s) \leq \left[\sum_{i=2}^{j-2} e^s 2^{-j} \binom{j}{i} \left(\frac{e^s}{2-e^s}\right)^2 \left(\frac{(x-1)e^s}{x-e^s}\right)^{j-4} + 2e^s j 2^{-j} \frac{e^s}{2-e^s} \left(\frac{(x-1)e^s}{x-e^s}\right)^{j-3} \right] \frac{1}{1-2^{-j+1}e^s} \qquad \text{(A2.8.9)}$$

$$= \frac{e^s}{2-e^s} \left(\frac{(x-1)e^s}{x-e^s}\right)^{j-4} \left[\frac{e^s}{2-e^s} \left(1-j2^{-j+1}-2^{-j+1}\right) + \frac{(x-1)e^s}{x-e^s} \left(j2^{-j+1}\right) \right] \frac{e^s}{1-2^{-j+1}e^s} \qquad \text{(A2.8.10)}$$

$$\leq \frac{e^s}{2-e^s} \left(\frac{(x-1)e^s}{x-e^s}\right)^{j-2} \quad \text{for } j > 2 \quad 1 < x < 3 \text{ and } 0 \leq s < s_o, \text{ where } s_o(x) > 0 \text{ for } 1 < x < 3. \qquad \text{(A2.8.11)}$$

Equation (A2.8.11) follows from Eq. (A2.8.10), by noting that at s=0; both equal one, both are continuous, and $\frac{\partial}{\partial s}$ of Eq. (A2.8.9) is greater than that of Eq. (A2.8.8).

Finally

$$E\{e^{s\ell}|\mu) = e^s p(\ell=0,1|\mu) + \sum_{j=2}^{\infty} f_j(s) p(\ell=j|\mu) \qquad \text{(A2.8.12)}$$

$$\leq e^s(1+\mu)e^{-\mu} + \sum_{j=2}^{\infty} AB^{j-2} \left(\frac{\mu^j e^{-\mu}}{j!}\right) \qquad \text{(A2.8.13)}$$

where A and B are given in Eq. (A2.8.4). Carrying out the indicated summation in Eq. (A2.8.13), we obtain Eq. (A2.8.2).

A tighter bound to $G(s,\mu)$ should be possible if we use for an upper bound to $f_i(s)$ the following

$$f_j(s) \leq \left(\frac{e^s}{2-e^s}\right)\left(\frac{3e^{-s}}{4-e^s}\right)\left(\frac{(x-1)e^s}{x-e^s}\right)^{j-3} \quad \text{for } j \geq 3,\ 0 \leq s \leq s_o. \tag{A2.8.14}$$

The variable x is chosen so that the induction step is satisfied for all $j \geq 3$ and for $s_0 > 0$. Although we have not carried out the derivation of this upper bound, it is felt that the resulting lower bound to the maximum arrival rate at which the moments of the delay are finite, will be greater than that obtained from the bound given in Eq. (A2.8.2) (see Section 2.4).

Now we will prove Eq. (A2.8.3). From Eq. (A2.8.12), we see that it is sufficient to show that

$$\frac{\partial f_j(s)}{\partial s} \leq \frac{\partial}{\partial s}\left(\frac{e^s}{2-e^s}\right)^{j-1} = \frac{2(j-1)}{2-e^s}\left(\frac{e^s}{2-e^s}\right)^{j-1} \quad \text{for } j > 1 \tag{A2.8.15}$$

Differentiating Eq. (A2.8.7), we have

$$\frac{\partial}{\partial s} f_j(s) = \frac{f_j(s) + \sum_{i=1}^{j-1} e^s 2^{-j}\binom{j}{i} \frac{\partial}{\partial s}\left[f_i(s)\, f_{j-i}(s)\right]}{1 - 2^{-j+1}e^s} \tag{A2.8.16}$$

Assuming Eq. (A2.8.15) to be true for j-1, we have from Eqs. (A2.8.11) and (A2.8.16) that

$$\frac{\partial}{\partial s} f_j(s) \leq \left(\frac{1 + 2(j-2)(1-2^{-j+1})}{1 - 2^{-j+1}e^s}\right)\left(\frac{e^s}{2-e^s}\right)^{j-1}$$

$$\leq \frac{2(j-1)}{2-e^{s}} \left(\frac{e^{s}}{2-e^{s}}\right)^{j-1} \qquad \text{(A2.8.17)}$$

This concludes Appendix A2.8.

CHAPTER 3

OPTIMUM DYNAMIC TREE ALGORITHMS WITH POISSON SOURCE MODEL

3.1 Introduction

In Chapter 2 we presented and analyzed the static binary tree algorithm, a protocol where the tree is fixed to be binary and independent of traffic condition. In this chapter we will consider a dynamic algorithm where the tree is allowed to vary from epoch to epoch depending on traffic conditions. Even though the tree may vary from one epoch to another, it is held fixed within any one epoch. The tree search is the same as that of the static algorithms that were considered in Chapter 2 and may be carried out serially or in parallel, deterministically or randomly. The source model that will be assumed in this chapter is the Poisson Source Model.

Now we will state the issues of this chapter more precisely. Three main problems will be considered here. The first is the determination of the optimum tree to be used in ε_{j+1}. The optimum tree for ε_{j+1} is defined to be that tree which minimizes the expected number of slots used in ε_{j+1}, given the observation of the transmission process up to the end of ε_j. The second problem will be the analysis of this optimal dynamic protocol, i.e., the determination of the delay, throughput and stability properties. The third is the optimization and analysis of a suboptimum algorithm where the tree is restricted to have binary nodes everywhere except for the root node; the degree of the root node is constrained to be a power of two. As will be seen, this algorithm does have certain implementation advantages.

Given that the number of arrivals in any one slot is a Poisson random variable which is independent of the arrivals in any other slot, it follows that the only quantity from the transmission process that is needed to fully characterize the packets to be processed during ε_{j+1} is h_j - the number of slots in ε_j. This is because, given h_j, the number of packets that arrived in ε_j is a Poisson random variable with mean μ where,

$$\mu = \lambda h_j \qquad (3.1.0.1)$$

If we let g_i^* be the degree of the optimum tree at depth-i, then in Section 3.2 it is proved that

$$g_0^* = \begin{cases} 1 & \mu \leq 1.70 \\ n & 1.70 + 1.15(n-2) < \mu \leq 1.70 + 1.15(n-1) \end{cases}$$

$$g_i^* = 2 \text{ for } i \geq 1; \quad \text{all } \mu \qquad (3.1.0.2)$$

In terms of Eqs. (3.1.0.1) and (3.1.0.2), the optimum dynamic algorithm may be stated as follows:

1. Observe h, the number of slots in the previous epoch
2. Calculate μ from Eq. (3.1.0.1)
3. Determine the optimum tree to be used in the following epoch from Eq. (3.1.0.2)
4. Execute the search in the following epoch using one of the tree search algorithms of Chapter 2.

In Section 3.3 we will analyze the slightly suboptimum but easier to implement tree, where g_0 is restricted to be an even positive integer. There, we will calculate upper and lower bounds to the average delay as a function of the arrival rate (see Fig. 3.3.2.2), prove that the maximum average throughput is .430 packets/slot, and show that system is stable for $\lambda < .430$, in the sense that all the moments of the delay exist for $\lambda < .430$.

The third problem that was posed above will be considered in Section 3.4. There we will show that if the root node degree of a binary tree is constrained to be 2^K; then K^*, the optimum K, is given by

$$K^* = \begin{cases} 1 \text{ for } \mu \leq 3.40 \\ K \quad 3.40\ (2^{K-2}) < \mu \leq 3.40(2^{K-1}) \end{cases} \qquad (3.1.0.3)$$

We will also determine upper and lower bounds to the E[delay] (see Fig. 3.4.2.1) and we will show that for this dynamic algorithm the maximum E[thr] is less than .430 but greater than .420 packets/slot.

In Appendix A3 that accompanies this chapter, we prove several of the theorems. This appendix is consulted frequently as we proceed through this chapter.

3.2 The Optimum Tree

In this section we will prove that if the number of contending packets is a Poisson random variable with mean μ, then the symmetric tree that minimizes the expected number of slots required to process the contending packets is given by Eq. (3.1.0.2). Although we do not prove that symmetric trees are optimum, that this is so should be obvious from the symmetry of the problem.

It can be shown, by a procedure analogous to that of Appendix A2.1.1, that the expected number of slots required to process the packets is given by

$$E\{h|\mu\} = g_0 + \mu \sum_{i=0}^{\infty} \frac{g_{i+1}}{M_i} \xi(M_i) \qquad (3.2.0.1)$$

where

$$M_i \equiv \frac{\mu}{g_0 g_1 \cdots g_i} \qquad (3.2.0.2)$$

and

$$\xi(M) = 1 - e^{-M} - Me^{-M} \qquad (3.2.0.3)$$

In terms of the above three equations, our goal in this section is to determine that $\{g_i;\ i = 0,1,2,\ldots\}$ which minimizes $E\{h|\mu\}$ subject to the constraint that g_i is an integer and

$$g_i \geq \begin{cases} 1 \text{ if } i = 0 \\ 2 \text{ otherwise} \end{cases} \qquad (3.2.0.4)$$

The solution will be carried out in three steps. First, we will show that g_i^*, the optimum g_i, equals 2 for sufficiently large i. This is accomplished in Section 3.2.1. Secondly, in Section 3.2.2, we will prove that the minimum of $E\{h|\mu\}$ over g_{k-1} and g_k, subject to the condition that $g_i = 2$ for $i > k$, occurs at then $g_k^* = 2$. This result applied recursively with the result of Section 3.2.1 as the boundary condition proves that $g_i^* = 2$ for $i \geq 1$. Finally, in Section 3.2.3, g_0^* is calculated as a function of μ; this result is given by Eq. (3.1.0.2).

3.2.1 Optimum Tail End of the Tree

We begin this section with the following definitions:

$$G_k \equiv [g_k, g_{k+1}, \ldots.]; \; k > 0 \tag{3.2.1.1}$$

$$G_k^* \equiv \text{the } G_k \text{ which minimizes } E\{h|\mu\} \text{ subject to } g_i \; (i \geq k) \text{ being an integer and greater than 1} \tag{3.2.1.2}$$

$$\underline{2} \equiv [2,2,2,\ldots.] \tag{3.2.1.3}$$

$$\hat{h}(G_k, M_{k-1}) \equiv \frac{1}{M_{k-1}} \sum_{i=0}^{\infty} \frac{g_{k+i}}{M_{k-1+i}} \xi(M_{k-1+i}) \tag{3.2.1.4}$$

$$\hat{h}(G_k) \equiv \frac{1}{2} \sum_{i=0}^{\infty} g_{k+i} \left(\prod_{j=0}^{i-1} g_{k+i} \right)^{-1} ; \text{ where } \prod_{j=0}^{-1} \equiv 1 \tag{3.2.1.5}$$

In this section we will prove that if $k > 0$ and if $M_{k-1} < \frac{3}{26}$ then $G_k^* = \underline{2}$. It should be observed that since $M_i = \mu/g_0 g_1 \ldots g_i$ and $g_i \geq 2$ for $i > 0$,

then for any finite μ, a k exists such that $M_{k-1} < \frac{3}{26}$. In other words, the main result of this section states that beyond some finite depth the optimum tree has binary nodes.

The solution will be carried out as follows:

a) Show that if G_k minimizes $\tilde{h}(G_k, M_{k-1})$ it will also minimize $E\{h|\mu\}$. This is Theorem 3.2.1.1.

b) Determine upper and lower bounds to $\tilde{h}(G_k, M_{k-1})$ in terms of $\hat{h}(G_k)$. This is accomplished in Theorem 3.2.1.2.

c) Show that the upper bound of $\tilde{h}$ evaluated at $G_k = \underline{2}$ is less than the lower bound at $G_k \neq \underline{2}$. This is accomplished in Theorems 3.2.1.3 and 3.2.1.4 and Corollary 3.2.1.1.

Theorem 3.2.1.1: If, G_k minimizes $\tilde{h}(G_k, M_{k-1})$ for all M_{k-1} then it will also minimize $E\{h|\mu\}$.

Proof: By combining Eqs. (3.2.0.1) and (3.2.1.4), $E\{h|\mu\}$ may be written as follows:

$$E\{h|\mu\} = g_0 + \mu \sum_{i=0}^{k-2} \frac{g_{i+1}}{M_i} \xi(M_i) + \mu M_{k-1} \tilde{h}(G_k, M_{k-1}) \qquad (3.2.1.6)$$

Since of the three terms on the right side of Eq. (3.2.1.6), only $\tilde{h}$ depends on G_k and since $\tilde{h}$ and μM_{k-1} are greater than zero, it follows that if G_k^* minimizes $\tilde{h}$ then it will also minimize $E\{h|\mu\}$.

QED

Theorem 3.2.1.2: Let $\tilde{h}(G_k,M_{k-1})$, $\hat{h}(G_k)$, M_k, and G_k be as defined in Eqs. (3.2.1.1) through (3.2.1.5) and (3.2.0.2). Furthermore, let $0 \leq M_{k-1} < 1$ and g_i be integer and greater than one. Then,

$$(1 - \frac{2}{3} M_{k-1})\hat{h}(G_k) < \tilde{h}(G_k,M_{k-1}) < \hat{h}(G_k) \tag{3.2.1.7}$$

Proof: First substitute the Taylor series expansion of e^{-M_i} into Eq. (3.2.0.3) and then substitute Eq. (3.2.0.3) into Eq. (3.2.1.4) and rearrange to obtain

$$\tilde{h}(G_k,M_{k-1}) = \frac{1}{2M_{k-1}} \sum_{i=0}^{\infty} g_{k+i}M_{k-1+i} - \frac{1}{M_{k-1}} \sum_{i=0}^{\infty} g_{k+i}R(M_{k-1+i}) \tag{3.2.1.8}$$

where

$$R(M_{k-1+i}) = \sum_{n=2}^{\infty} (-M_{k-1+i})^n \frac{n}{(n+1)!} \tag{3.2.1.9}$$

If $0 \leq M_{k-1+i} < 1$, then $R(M_{k-1+i})$ is the sum of an alternating sequence of decreasing magnitude and it follows that,

$$0 \leq R(M_{k-1+i}) \leq \frac{1}{3} M_{k-1+i}^2 \tag{3.2.1.10}$$

From Eqs (3.2.1.8) and (3.2.1.10) we have

$$\tilde{h}(G_k,M_{k-1}) < \frac{1}{2M_{k-1}} \sum_{i=0}^{\infty} g_{k+i}M_{k-1+i} \tag{3.2.1.11}$$

and

$$\hat{h}(G_k, M_{k-1}) > \frac{1}{2M_{k-1}} \sum_{i=0}^{\infty} g_{k+i} M_{k-1+i} - \frac{1}{M_{k-1}} \sum_{i=0}^{\infty} \frac{g_{k+i}}{3} M_{k-1+i}^2 \qquad (3.2.1.12)$$

Since $0 < M_{k+i} < M_k$ for $i \geq 1$, it follows that

$$\frac{1}{3} \sum_{i=0}^{\infty} g_{k+i} \frac{M_{k-1+i}^2}{M_{k-1}} \leq \frac{1}{3} \sum_{i=0}^{\infty} g_{k+i} M_{k-1+i} \qquad (3.2.1.13)$$

Substituting Eq. (3.2.1.13) into Eq. (3.2.1.12) we have

$$\hat{h}(G_k, M_{i-1}) > (1 - \frac{2}{3} M_{k-1}) \frac{1}{2M_{k-1}} \sum_{i=0}^{\infty} g_{k+i} M_{k-1+i} \qquad (3.2.1.14)$$

Finally from Eqs. (3.2.0.2) and (3.2.1.5) we have

$$\hat{h}(G_k) = \frac{1}{2M_{k-1}} \sum_{i=0}^{\infty} g_{k+i} M_{k-1+i} \qquad (3.2.1.15)$$

and by substituting Eq. (3.2.1.15) into Eqs. (3.2.1.11) and (3.2.1.14), we obtain the desired result.

QED

Theorem 3.2.1.3: Let $\hat{n}(G_k)$ be as given in Eq. (3.2.1.5) and let g_i be integer and greater than one. Then the minimum of $\hat{n}(G_k)$ is 2 and it occurs at $G_k = \underline{2}$.

Proof: For $n \geq k$ we have from Eq. (3.2.1.5)

$$\hat{n}(G_k) = \frac{1}{2} \sum_{i=0}^{n-k-1} g_{k+i} \left(\prod_{i=0}^{i-1} g_{k+i} \right)^{-1} + \left(\prod_{i=0}^{n-k-1} g_{k+i} \right)^{-1} \hat{n}(G_n) \qquad (3.2.1.16)$$

From Eq. (3.2.1.16) it follows that if g_n minimizes $\hat{n}(G_n)$, it will also minimize $\hat{n}(G_k)$. Therefore we will determine g_n^*.

From Eq. (3.2.1.5) we have

$$\hat{n}(G_n) = \frac{g_n}{2} + \frac{\hat{n}(G_{n+1})}{g_n} \qquad (3.2.1.17)$$

Now assume that G_{n+1}^* has been determined in Eq. (3.2.1.17) and we would like to know g_n^*. But $\hat{n}(G_n)$ is convex in g_n, therefore the optimum g_n^* is the smallest integer greater than one such that

$$\hat{n}(g_n, G_{n+1}^*) \leq \hat{n}(g_n+1, G_{n+1}^*) \qquad (3.2.1.18)$$

Substituting Eq. (3.2.1.17) into Eq. (3.2.1.18) and rearranging, we have the following equivalent expression that g_n^* must satisfy,

$$g_n^*(g_n^* + 1) \geq 2\, \hat{n}(G_{n+1}^*) \qquad (3.2.1.19)$$

But since

$$\hat{n}(G^*_{n+1}) \leq \hat{n}(\underline{2}) = 2$$

it follows that the smallest integer g^*_n greater than one that satisfies Eq. (3.2.1.19) is

$$g^*_n = 2$$

QED

Corollary 3.2.1.1: Let G_k be such that $g_k \geq 3$. Then $\hat{n}(G_k) \geq 13/6$.

Proof: From Eq. (3.2.1.17) we have

$$\hat{n}(G_k) = \frac{g_k}{2} + \frac{\hat{n}(G_{k+1})}{g_k} \qquad (3.2.1.20)$$

and from Theorem 3.2.1.3 we have

$$\hat{n}(G_{k+1}) \geq 2 \qquad (3.2.1.21)$$

therefore

$$\hat{n}(G_k) \geq \frac{g_k}{2} + \frac{2}{g_k} \qquad (3.2.1.22)$$

Since the derivative of the right side of Eq. (3.2.1.22) is positive for $g_k > 2$, it follows that for $g_k \geq 3$

$$\frac{g_k}{2} + \frac{2}{g_k} \geq \frac{3}{2} + \frac{2}{3}$$

or

$$\hat{h}(G_k) \geq \frac{13}{6} \tag{3.2.1.23}$$

QED

Now we are ready to prove the main result of this section in the following theorem.

Theorem 3.2.1.4: If $M_{k-1} < \frac{3}{26}$, then $G_k = 2$ minimizes $\tilde{h}(G_k, M_{k-1})$.

Proof: Take any $G_k \neq 2$ and let g_n for $n \geq k$ be the first element of G_k that is greater than 2. Then from Eq. (3.2.1.4) we have,

$$\tilde{h}(G_k M_{k-1}) = \frac{1}{M_{k-1}} \sum_{i=0}^{n-k-1} \frac{g_{k+i}}{m_{k-1+i}} \xi(M_{k-1+i}) + \frac{M_{n-1}}{M_{k-1}} \tilde{h}(G_n, M_{n-1}) \tag{3.2.1.24}$$

First, note that the G_n that will minimize $\tilde{h}(G_n, M_{n-1})$ will also minimize $\tilde{h}(G_k, M_{k-1})$. Then from Theorems 2 and 3 and Corollary 1, we have

$$\tilde{h}(2, M_{n-1}) \leq 2 \tag{3.2.1.25}$$

$$\tilde{h}(G_n, M_{n-1}) \geq (1 - \frac{2}{3} M_{n-1}) \frac{13}{6} \tag{3.2.1.26}$$

But for $n \geq k$, $0 \leq M_{n-1} < M_{k-1} < 3/26$, therefore from Eq. (3.2.1.26),

$$\tilde{h}(G_n, M_{n-1}) > 2 \tag{3.2.1.27}$$

and it follows from Eqs. (3.2.1.25) and (3.2.1.27) that,

$$\tilde{h}(\underline{2}, M_{n-1}) < \tilde{h}(G_n, M_{n-1}) \tag{3.2.1.28}$$

Finally since n was chosen so that $g_i = 2$ for $k \leq i \leq n-1$, we have that

$$\tilde{h}(G_k, M_{k-1}) - \tilde{h}(\underline{2}, M_{k-1}) = \frac{M_{n-1}}{M_{k-1}} [\tilde{h}(G_n, M_{n-1}) - \tilde{h}(\underline{2}, M_{n-1})] \tag{3.2.1.29}$$

and from Eq. (3.2.1.28) we conclude that

$$\tilde{h}(G_k, M_{k-1}) - \tilde{h}(\underline{2}, M_{k-1}) > 0 \text{ for } M_{k-1} < \frac{3}{26} \tag{3.2.1.30}$$

QED

Theorems 3.2.1.1 and 3.2.1.4 taken together prove the main result of this section, i.e., the tail end of the optimum tree is binary. In the next section we will be concerned with the degrees of the nodes of the optimum tree from the tail end to the root node.

3.2.2 A Recursive Optimization Technique

Whereas in Section 3.2.1 we were concerned with the tail end of the optimum tree, in this section we will develop a recursive relationship

which in conjuction with the results of the previous section proves that $g_i^* = 2$ for $i > 0$. The optimum degree g_0^* of the root node does depend on μ and this will be considered in Section 3.2.3.

The recursive relationship that will be proved here is as follows. Let g_k and g_{k+1} be integer and greater than one (for k=0,1...) and let $g_i = 2$ for $i > k + 1$, then for any $\mu \geq 0$, the point that minimizes $E\{h|\mu\}$ satisfies,

$$g_{k+1} = 2 \qquad (3.2.2.1)$$

The minimization problem has a slightly different objective for k=0 then it does for $k > 0$. Therefore, the analysis of this section is organized into two subsections. In Subsection i the case where $k > 0$ is considered and in Subsection ii the case where k=0 is considered.

i. Recursive Relationship for $k > 0$

Now we begin the analysis for $k > 0$. Setting $g_i = 2$ for $i \geq k + 2$ in Eq. (3.2.0.1) we have

$$E\{h|\mu\} = g_0 + \mu \sum_{i=0}^{k-2} \frac{g_{i+1}}{M_i} \xi(M_i) + \left(\prod_{i=0}^{k-1} g_i \right) L(g_k, g_{k+1}, M_{k-1}) \qquad (3.2.2.2)$$

where

$$L(g_k, g_{k+1}, M_{k-1}) \equiv g_k \xi(M_{k-1}) + g_k g_{k+1} \xi\left(\frac{M_{k-1}}{g_k}\right) + 2 g_k g_{k+1} D\left(\frac{M_{k-1}}{g_k g_{k+1}}\right) \qquad (3.2.2.3)$$

and

$$D(x) = \sum_{i=0}^{\infty} 2^i \xi(x/2^i) \qquad (3.2.2.4)$$

As can be seen from Eq. (3.2.2.2), the (g_k, g_{k+1}) that minimizes L will also minimize $E\{h|\mu\}$. Therefore, we will be concerned with the minimization of $L(g_k, g_{k+1}, M_{k-1})$ over g_k, g_{k+1}. In order to simplify the notation somewhat we will drop the subscript of M and also use g_1 and g_2 instead of g_k and g_{k+1}. In what follows, therefore, we will prove that for any $M \geq 0$, the minimum of $L(g_1, g_2, M)$ occurs at $g_2^* = 2$. This will be accomplished in two parts. In Part 1 it will be proved for $M \leq 8$ and in Part 2 for $M > 8$.

Part 1. Minimization of $L(g_1, g_2, M)$ for $M \leq 8$

In this part we will prove that if $M \leq 8$, and if g_1 and g_2 are integers and greater than one, then the $\min_{g_1, g_2} L(g_1, g_2, M)$ occurs at $g_2 = 2$. The results of Section 3.2.1 prove this statement for $M \leq 3/26$; therefore, we will be concerned with its proof for $3/26 \leq M \leq 8$.

First divide the $(g_1 \times g_2)$ space into three regions, R_1, R_2, and R_3 as follows.

$$R_1 = [(g_1, g_2): \ 2 \leq g_1 \leq 9, \ 2 \leq g_2 \leq 5] \qquad (3.2.2.5)$$

$$R_2 = [(g_1, g_2): \ g_1 \geq 10, \ g_2 \geq 3] \qquad (3.2.2.6)$$

$$R_3 = [(g_1, g_2): \ 2 \leq g_1 \leq 9, \ g_2 \geq 6] \qquad (3.2.2.7)$$

(See Fig. 3.2.2.1).

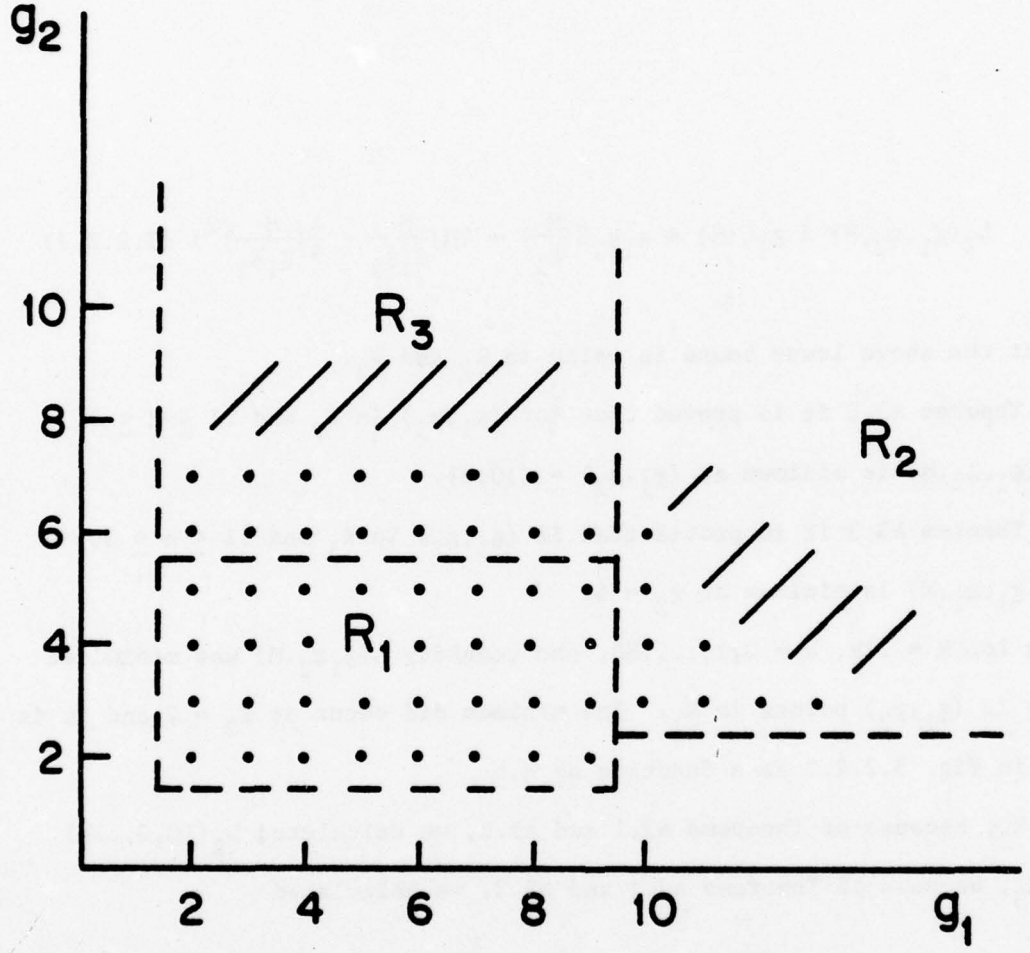


Figure 3.2.2.1 The Subdivision of the ($g_1 \times g_2$) Space into R_1, R_2 and R_3

In the Appendix to this chapter we prove in Theorem A3.1 that

$$L_\ell(g_1,g_2,M) \leq L(g_1,g_2,M) \text{ for } M/(g_1g_2) \leq 1.2 \qquad (3.2.2.8)$$

where

$$L_\ell(g_1,g_2,M) \equiv g_1\xi(M) + g_1g_2\xi(\frac{M}{g_1}) + 2M(\frac{M}{g_1g_2} - \frac{4}{9}(\frac{M}{g_1g_2})^2) \qquad (3.2.2.9)$$

Note that the above lower bound is valid in R_2 and R_3.

In Theorem A3.2 it is proved that for (g_1,g_2) in R_2 and $.1 \leq M \leq 8$, then $L_\ell(g_1,g_2,M)$ is minimum at $(g_1,g_2) = (10,3)$.

In Theorem A3.3 it is proved that if (g_1,g_2) in R_3 and $.1 \leq m \leq 8$, then $L_\ell(g_1,g_2,M)$ is minimum at $g_2 = 6$.

Now for $M = .1k$, $k = 0,1,....80$, the quantity $L(g_1g_2,M)$ was minimized over the 32 (g_1,g_2) points in R_1. The minimum did occur at $g_2 = 2$ and it is plotted in Fig. 3.2.2.2 as a function of M.

In R_2, because of Theorems A3.1 and A3.2, we calculated $L_\ell(10,3,.1K)$ and in R_3, because of Theorems A3.1 and A3.3, we calculated

$$\min_{2\leq g_1\leq 9} L_\ell(g_1,6,.1K) \text{ for } k = 0,1,...,80$$

These results are also included in Fig. 3.2.2.2. As can be seen from this figure,

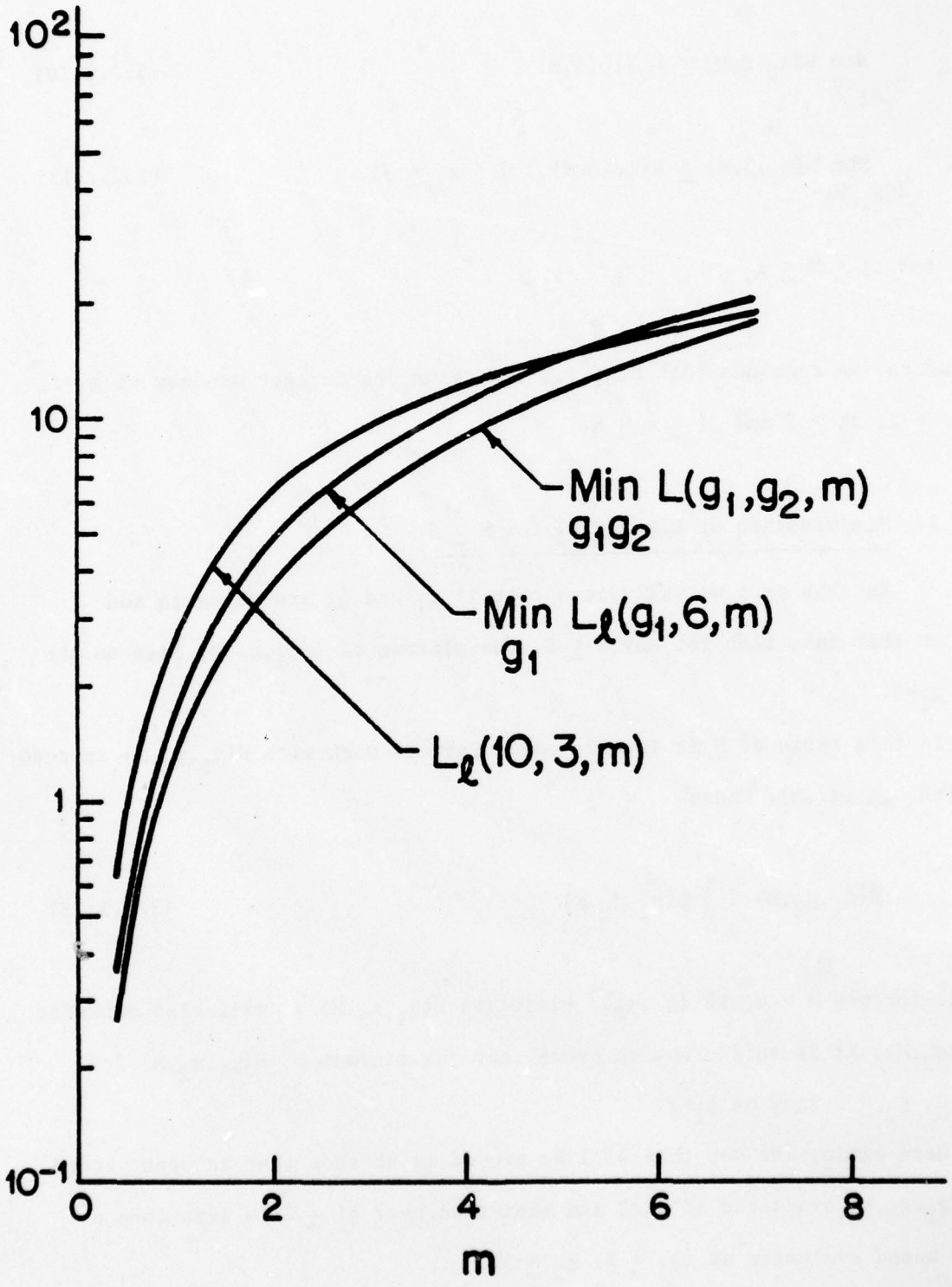


Figure 3.2.2.2 The Minimization of $L(g_1, g_2, m)$ over g_1 and g_2

$$\underset{2\leq g_1\leq 9}{\text{Min}}\ L(g_1,2,M) \leq L_\ell(10,3,M) \qquad (3.2.2.10)$$

$$\underset{2\leq g_1\leq 9}{\text{Min}}\ L(g_1,2,M) \leq L(g_1,6,M)\ ;\ 2 \leq g_1 \leq 9 \qquad (3.2.2.11)$$

for $.1 \leq M \leq 8$

Therefore, we conclude that $L(g_1,g_2,M)$ attains its integer minimum at $g_2=2$, if $g_1 \geq 2$, $g_2 \geq 2$ and $.1 \leq m \leq 8$.

Part 2. Minimization of $L(g_1,g_2,M)$ for $M \geq 8$

In this part we will prove that if g_1 and g_2 are integers and greater than one, then for any $M \geq 8$, the minimum of $L(g_1,g_2,M)$ lies on the line $g_2=2$.

In this range of M it is more convenient to work with $H(g_1,g_2,M)$ instead of with $L(g_1,g_2,M)$, where

$$H(g_1,g_2,M) \equiv \frac{1}{M} L(g_1,g_2,M) \qquad (3.2.2.12)$$

Since, for any $M > 0$, if $(g_1,g_2)^*$ minimizes $H(g_1,g_2,M)$ it will also minimize $L(g_1,g_2,M)$, it is sufficient to prove that the minimum of $H(g_1,g_2,M)$ for $g_1 \geq 2$, $g_2 \geq 2$ lies on $g_2=2$.

Here again, the way this will be proved is to show that an upper bound to $H(g_1,g_2,M)$ evaluated at $g_2=2$ and minimized over $g_1 \geq 2$ is less than a lower bound evaluated at $\{g_1 \geq 2,\ g_2 \geq 3\}$.

Since $\xi(M)$ is an increasing function bounded by 1, we have from Eqs. (3.2.2.3) and (3.2.2.12) that,

$$H_\ell(g_1,g_2,M) \leq H(g_1,g_2,M) \leq H_u(g_1,g_2,M) \ ; \text{ for } M \geq 8 \qquad (3.2.2.13)$$

where

$$H_\ell(g_1,g_2,M) = \xi(8)\frac{g_1}{M} + \frac{g_1g_2}{M}\,\xi\left(\frac{M}{g_1}\right) + 2\sum_{i=0}^{\infty}\frac{2^i g_1 g_2}{M}\xi\left(\frac{M}{2^i g_1 g_2}\right) \qquad (3.2.2.14)$$

$$H_u(g_1,g_2,M) = \frac{g_1}{M} + \frac{g_1g_2}{M}\,\xi\left(\frac{M}{g_1}\right) + 2\sum_{i=0}^{\infty}\frac{2^i g_1 g_2}{M}\xi\left(\frac{M}{2^i g_1 g_2}\right) \qquad (3.2.2.15)$$

Since

$$H_u(g_1,2,M) = \frac{g_1}{M} + 2\sum_{i=0}^{\infty}\frac{g_1 2^i}{M}\,\xi\left(\frac{M}{g_1 2^i}\right) \qquad (3.2.2.16)$$

we have from Theorem A3.4 that

$$\min_{g_1>2} H_u(g_1,2,M) \leq 2.34 \qquad (3.2.2.17)$$

In Theorem A3.5 it is proved that if $M/g_1 \leq .5$, $g_2 \geq 3$, and $M \geq 8$, then

$$H_\ell(g_1,g_2,M) \geq 2.498 \qquad (3.2.2.18)$$

And in Theorem A3.6 it is proved that

$$H_{\ell}(g_1,g_2,M) > 2.34 \text{ for } .1 \leq M/g_1 \leq 3 \text{ and } g_2 \geq 3 \qquad (3.2.2.19)$$

Therefore, from Theorems A3.4, A3.5 and A3.6 we conclude that if

$M/g_1 \leq 3$, $M \geq 8$, $g_1 \geq 2$, $g_2 \geq 2$ then

the minimum of $H(g_1,g_2,M)$ occurs on the line

$$g_2 = 2$$

In Theorem A3.7 it is proved that the minimum of $H(g_1,g_2,M)$ for $M/g_1 > 3$ also occurs on

$$g_2 = 2$$

This concludes Part 2.

ii. Determination of Optimum g_1

What we proved up to this point is that the minimum of $E\{h|\mu\}$ over g_k and g_{k+1} (subject to the constraints that g_k and g_{k+1} are integer and greater than one; $g_i = 2$ for $i \geq k+2$, and $k > 0$) occurs at $g_{k+1}=2$. We still have to prove the above statement for k=0 and this is what we will do next.

Setting $g_i = 2$ for $i \geq 2$ in Eq. (3.2.0.1) we have

$$E\{h|\mu\} = g_0 + g_0 g_1 \xi\left(\frac{\mu}{g_0}\right) + 2\, g_0 g_1 D\left(\frac{\mu}{g_0 g_1}\right) \qquad (3.2.2.20)$$

We are going to show that the minimum of the above expression subject to g_0 and g_1 being integer and $g_0 \geq 1$, $g_1 \geq 2$ occurs at $g_1 = 2$. By comparing Eq. (3.2.2.20) to Eq. (3.2.2.3) it can be seen that this problem is very similar to that solved in Subsection i. Therefore, one may go through a similar procedure in solving the above programming problem. Here, however, we are going to be less rigorous and simply show that $g_1^* = 2$ by computation. The procedure to be used here is as follows.

Divide Eq. (3.2.2.20) by μ and let $x = \mu/g_0$ to obtain $f(x,g_1)$.

$$f(x,g_1) = \frac{1}{x}\left(1 + g_1 \xi(x) + 2g_1 D\left(\frac{x}{g_1}\right)\right) \qquad (3.2.2.21)$$

Given the preceding definition of $f(x,g_1)$, it follows that if the minimum of $f(x,g_1)$ occurs at $g_1=2$ then the minimum of $E\{h|\mu\}$ will also occur at $g_1=2$. We computed $f(x,g_1)$ for various values of (x,g_1) and the results are shown in Fig. 3.2.2.3.

Now if $\mu < 3.08$ then $x = \frac{\mu}{g_0} < 1.54$, and as can be seen from this figure

$$f(x,2) \leq f(x,g_1) \text{ for } x < 1.54;\ g_1 > 2 \qquad (3.2.2.22)$$

If, on the other hand, $\mu \geq 3.08$ then it can be shown that a $g_1 \geq 7$ exists such that $.89 \leq \mu/g_1 \leq 1.54$. But here again as can be seen from Fig. 3.2.2.3

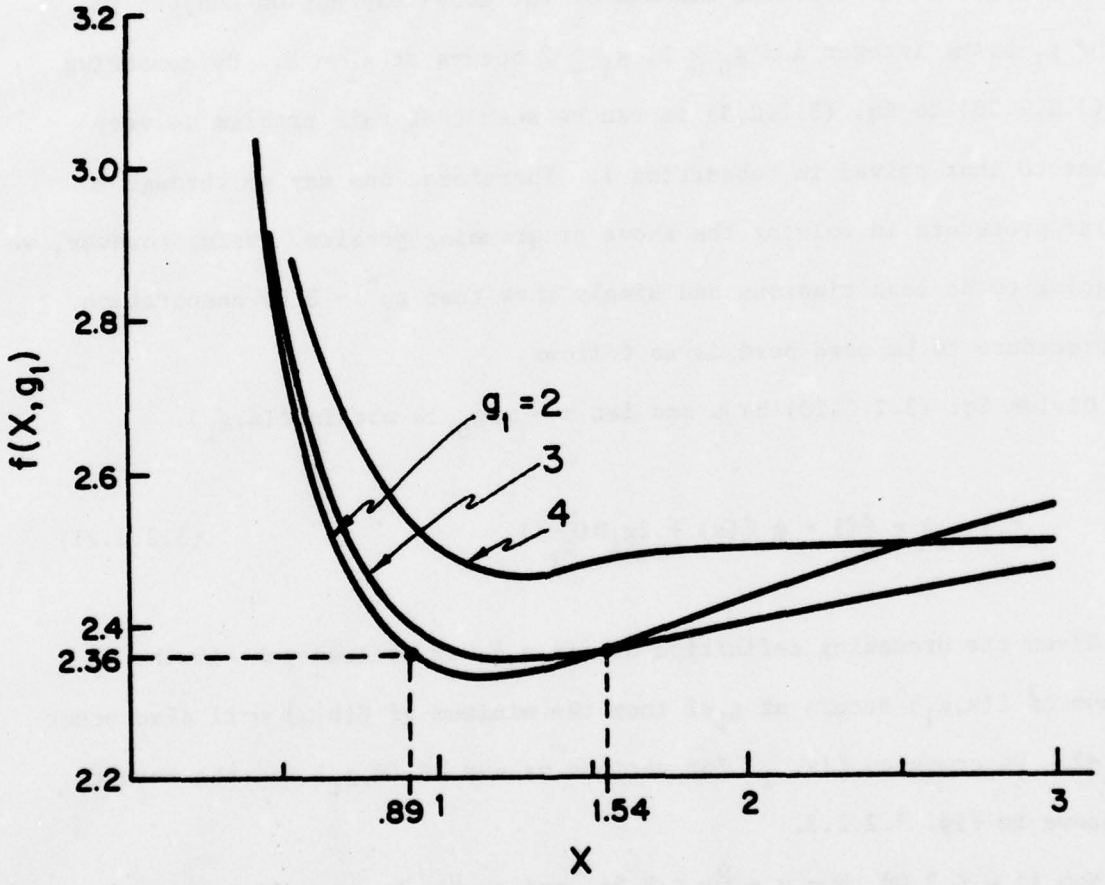


Figure 3.2.2.3 The Determination of the Optimum g_1

$$f(x,2) \leq f(x,g_1) \text{ for } .89 \leq x \leq 1.54;\ g_1 > 2 \qquad (3.2.2.23)$$

Therefore, we will assume that

$$g_1^* = 2 \qquad (3.2.2.24)$$

This concludes Section 3.2.2.

3.2.3 The Root Node of the Optimum Tree

In this section we will determine the optimum g_0, i.e., that g_0 which minimizes $E\{h|\mu,g_0\}$ subject to g_0 being integer, $g_0 \geq 1$ and $g_i = 2$ for $i \geq 1$.

We begin by setting $g_1=2$ in Eq. (3.2.2.2) to obtain

$$E\{h|\mu, g_0\} = g_0\{1 + 2\mathcal{D}(\mu/g_0)\} \qquad (3.2.3.1)$$

This equation was calculated and the results are illustrated in Fig. 3.2.3.1. The problem posed in this section is solved by determining $\hat{\mu}(n)$, a quantity defined by,

$$g_0^*(\mu) = n \text{ for } \hat{\mu}(n) < \mu \leq \mu(n+1) \qquad (3.2.3.2)$$

As can be seen from Fig. 3.2.3.1, $\hat{\mu}(g_0)$ is that μ which satisfies

$$E\{h|\mu,g_0-1\} = E\{h|\mu, g_0\} \qquad (3.2.3.3)$$

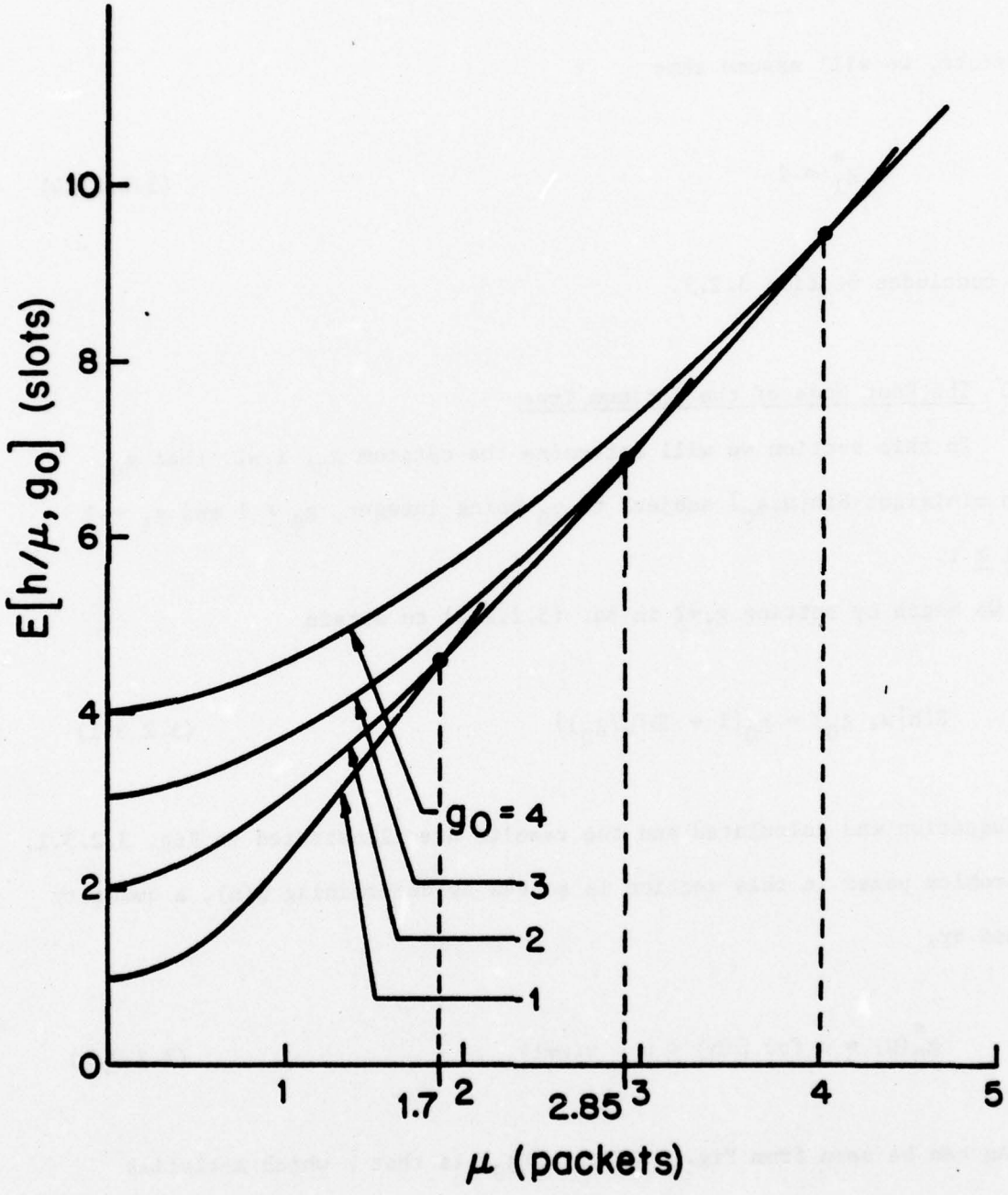


Figure 3.2.3.1 The Determination of the Optimum g_0

Equation (3.2.3.3) was solved, and $\hat{\mu}(n)$ to within two significant places to the right of the decimal point is given by,

$$\hat{\mu}(n) = 1.70 + 1.15(n-2) \qquad (3.2.3.4)$$

This concludes Section 3.2.3.

3.3 Analysis of the Dynamic Algorithm

In this section we will analyze (for the Poisson Source Model) the dynamic tree algorithm where all the nodes except for the root node are binary - the degree of the root is restricted to be even and it is chosen so as to minimize the expected number of slots used, given the expected number of contending sources. This algorithm should be recognized as being very similar to the optimum algorithm of Section 3.2, the only difference being that here the root node is restricted to taking on an even degree where in the optimum algorithm it is not. There are two reasons for considering this suboptimum algorithm. First, the even degree root node tree is easier to implement since here all nodes are even, and secondly the analysis is neater, even though it is not essentially less complex.

This algorithm will be executed serially by assigning two consecutive slots to it for each round trip interval. If, for example, the initial node has degree 2r, then the tree search is identical to r consecutive serial searches (of the type described in Chapter 2) where each of the r searches is over 1/r of the sources.

As in Chapter 2, an algorithmic step consists of the actions taken in two consecutive slots. Therefore, the root node correponds to r steps but all other nodes correspond to a single step each. It also follows that if we let δ be the number of steps between the arrival and the successful transmission of a packet, and τ_r and τ_s the round trip delay and length of one slot, then the delay (in seconds) experienced by that packet is given by

$$\text{packet delay} = \delta\{\tau_r + 2\tau_s\} \qquad (3.3.0.1)$$

Since in the above equation τ_r and τ_s are constant, it follows that in order to characterize the packet delay, one need only obtain the statistics of δ. In this section, therefore, we will obtain upper and lower bounds to $E\{\delta\}$ as a function of the packet arrival rate λ. A by-product of this analysis is the determination of the maximum throughput and the characterization of stability.

Since the expected delay with the parallel execution of the algorithm is less than that with the series execution, the delay results obtained here may be considered to be upper bounds to those of the parallel execution. The maximum throughput, however, is the same for both schemes.

The analysis is organized as follows. In Section 3.3.1, expressions for $E\{\ell|\mu,r\}$, $E\{\ell^2|\mu,r\}$ and $E\{d|\mu,r\}$ are derived. The quantities ℓ, μ and d have the same definition as they did in Chapter 2; that is, μ is the expected number of packet arrivals in the previous epoch, ℓ is the number of algorithmic steps in the present epoch, and d is the number of algorithmic steps until a randomly chosen packet from the set of conflicting packets is successfully transmitted. Furthermore, note that the above conditional moments with $r=1$ are identical to the corresponding ones of Chapter 2. In Section 3.3.2, we determine the optimum r and obtain upper and lower bound to $E\{\delta\}$ (see Eqs. (3.3.2.3), (3.3.2.13), and (3.2.2.14)). In Sections 3.3.3 and 3.3.4 the stability and throughput are considered, respectively. It should be noted that the analysis of the dynamic tree is very similar to that of the static binary tree which was considered in Section 2.2. Therefore, since many of the results of Section 2.2 carry over to the dynamic tree analysis, we will not be as detailed here.

3.3.1. Derivation of $E\{\ell|\mu,r\}$, $E\{\ell^2|\mu,r\}$ and $E\{d|\mu,r\}$

Consider a tree where the root node has degree 2r and where all other nodes have degree two. Furthermore, the slots are used in pairs, and the algorithm is executed serially as described above. In this section, expressions of $E\{\ell|\mu,r\}$, $E\{\ell^2|\mu,r\}$ and $E\{d|\mu,r\}$ will be derived in terms of $E\{\ell|\frac{\mu}{r},1\}$ and $E\{\ell^2|\frac{\mu}{r},1\}$ (the corresponding quantities of the binary tree that was analyzed in Section 2.2. These derivations are developed in Theorems 3.3.3.1 and 3.3.3.2 below.

Theorem 3.3.3.1: Let ℓ and μ be as defined above and let the source model be Poisson. Then

$$E\{\ell|\mu,r\} = rE\{\ell|\tfrac{\mu}{r},1\} \tag{3.3.1.1}$$

$$E\{\ell^2|\mu,r\} = r(r-1)E^2\{\ell|\tfrac{\mu}{r},1\} + rE\{\ell^2|\tfrac{\mu}{r},1\} \tag{3.3.1.2}$$

or

$$E\{\ell^2|\mu,r\} = r^2E^2\{\ell|\tfrac{\mu}{r},1\} + r\,\mathrm{Var}\{\ell|\tfrac{\mu}{r},1\} \tag{3.3.1.3}$$

Proof: Divide the 2r subtrees that emanate from the root node into r pairs and let ℓ_i be the number of nodes that are visited in the i'th pair. Then since the root node corresponds to r steps and all other nodes correspond to a single step, we have

$$\ell = \sum_{i=1}^{r} (1 + \ell_i) \tag{3.3.1.4}$$

Taking expectations of the above equation and then noting that $1 + \ell_i$ equals the number of nodes in a binary tree with mean μ/r we have Eq. (3.3.1.1).

To prove Eq. (3.3.1.2), square both sides of Eq. (3.3.1.4) to obtain,

$$\ell^2 = 2 \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} (1 + \ell_i)(1 + \ell_j) + \sum_{i=1}^{r} (1 + \ell_i)^2 \qquad (3.3.1.5)$$

Taking expectation of Eq. (3.3.1.5) and noting the independence of ℓ_i and ℓ_j for $i \neq j$ we have Eq. (3.3.1.2).

QED

Theorem 3.3.1.2: Let the source model be Poisson, and ℓ, d, μ be as defined above, (Note that d is the same as d_2). Then

$$E\{d|\mu,r\} \leq .55E\{\ell|\mu,r\} + .321 \qquad (3.3.1.6)$$

and

$$E\{d|\mu,r\} \geq \frac{1}{2} E\{\ell|\mu,r\} \qquad (3.3.1.7)$$

Proof: Here, again, divide the tree into r pairs of subtrees and observe that each pair is equivalent to a binary tree where the expected number of contending packets is $\frac{\mu}{r}$.

Assuming that each pair is processed in sequence, we have

$$E\{d|\mu,r, \text{ test packet is in the i'th pair}\} = (i-1)E\{\ell|\tfrac{\mu}{r},1\} + E\{d|\tfrac{\mu}{r},1\} \qquad (3.3.1.8)$$

Next multiply by $\frac{1}{r}$ and sum over i to obtain

$$E\{d|\mu,r\} = \frac{r-1}{2} E\{\ell|\frac{\mu}{r},1\} + E\{d|\frac{\mu}{r},1\} \qquad (3.3.1.9)$$

But from Appendix A2.4.2

$$E\{d|y,1\} \leq .55E\{\ell|y,1\} + .321 \qquad (3.3.1.10)$$

and

$$E\{d|y,1\} \geq \frac{1}{2} E\{\ell|y,1\} \qquad (3.3.1.11)$$

Substituting Eq. (3.3.1.10) into Eq. (3.3.1.9) we have

$$E\{d|\frac{\mu}{r},1\} \leq \frac{(r+.1)}{2r} r\, E\{\ell|\frac{\mu}{r},1\} + .321 \qquad (3.3.1.12)$$

But for $r \geq 1$,

$$\frac{r+.1}{2r} \leq .55 \qquad (3.3.1.13)$$

Therefore, substituting Eq. (3.3.1.1) into Eq. (3.3.1.12) and then applying Theorem 3.3.3.1, we get Eq. (3.3.1.6). To obtain Eq. (3.3.1.7), substitute Eq. (3.3.1.11) into Eq. (3.3.1.9) and apply Theorem 3.3.1.1.

QED

3.3.2 Upper and Lower Bounds to Average Delay

Here, upper and lower bounds to $E\{\delta\}$, that are functions of λ, will be obtained for the dynamic tree. This will be accomplished as follows. First, bounds to $E\{\delta\}$ that depend on $E\{\ell\}$ and $E\{\ell^2\}$ are obtained. Secondly $r^*(\mu)$, the optimum relationship between μ and r, is obtained. (Note: 2r is the degree of the root node.) And finally, $r^*(\mu)$ is substituted into $E\{\ell|\mu,r\}$, $E\{\ell^2|\mu,r\}$ and $E\{d|\mu,r\}$ so that techniques similar to those used in Section 2.2.1.3 may be applied here to obtain bounds to $E\{\ell\}$ and $E\{\ell^2\}$ that are functions of λ.

$E\{\delta\}$ vs $E\{\ell\}$ and $E\{\ell^2\}$

Since Eqs. (3.3.1.6) and (3.3.1.7) are identical to Eqs. (2.2.2.16) and (2.2.2.17), the results of Section 2.2.2 (Eqs. (2.2.2.20) and (2.2.2.25)) are applicable here. Therefore,

$$E\{\delta\} \leq 1.05 \frac{E\{\ell^2\}}{E\{\ell\}} + .321 \qquad (3.3.2.1)$$

$$E\{\delta\} \geq \frac{1}{2} \frac{E\{\ell^2\}}{E\{\ell\}} + \frac{1}{2} E\{\ell\} \qquad (3.3.2.2)$$

Determination of $r^*(\mu)$ for Even Degree Node Tree

The quantity $r^*(\mu)$ is the optimum relationship between the degree of the root node and the expected number of contending packets for a tree where the root node is restricted to be even and all others are binary. It was calculated by a procedure analogous to that used in Section 3.2.3 to obtain g_0^*; the result is,

$$r^*(\mu) = \begin{cases} 1 \text{ for } \mu \leq 3.40 \\ r \text{ for } 3.40 + (r-2)2.30 < \mu \leq 3.40 + (r-1)2.30 \end{cases} \tag{3.3.2.3}$$

Determination of $E\{\delta\}$ vs. λ

Next $r^*(\mu)$ is substituted into Eqs. (3.3.1.1) and (3.3.1.2) to obtain $E\{\ell|\mu,r^*(\mu)\}$ and $\sqrt{E\{\ell^2|\mu,r^*(\mu)\}}$. These two quantities are plotted in Fig. 3.3.2.1. The discontinuities of $E\{\ell^2|\mu,r^*\}$, which are evident in this figure, are due to changes in the degree of the root node.

As in Chapter 2, it can be shown that

$$E\{\ell_{j+1}|\ell_j,r^*\} \leq 2.325\lambda(\ell_j-1) + E\{\ell_{j+1}|\ell_j=1,1\} \tag{3.3.2.4}$$

$$\sqrt{E\{\ell_{j+1}^2|\ell_j,r^*\}} \leq 2.325\lambda(\ell_j-1) + \sqrt{E\{\ell_{j+1}^2|\ell_j=1,1\}} \tag{3.3.2.5}$$

$$\sqrt{E\{\ell_{j+1}^2|\ell_j,r^*\}} \geq 2.325\lambda\ell_j + .6 \tag{3.3.2.6}$$

$$f_\mu(\mu) = \begin{cases} E\{\ell|\mu,1\} & \text{for } \mu \leq 2 \\ 2.325(\mu-2) + E\{\ell|\mu=2,1\} & \mu > 2 \end{cases} \tag{3.3.2.7}$$

Where f_μ is a convex, increasing lower bound to $E\{\ell|\mu,r^*(\mu)\}$.

Continuing, as in Chapter 2, it can, further be shown that

$$E\{\ell\} \leq 1 + \frac{2D(\lambda)}{1-2.325\lambda} \tag{3.3.2.8}$$

$$E\{\ell\} \geq \ell^* \tag{3.3.2.9}$$

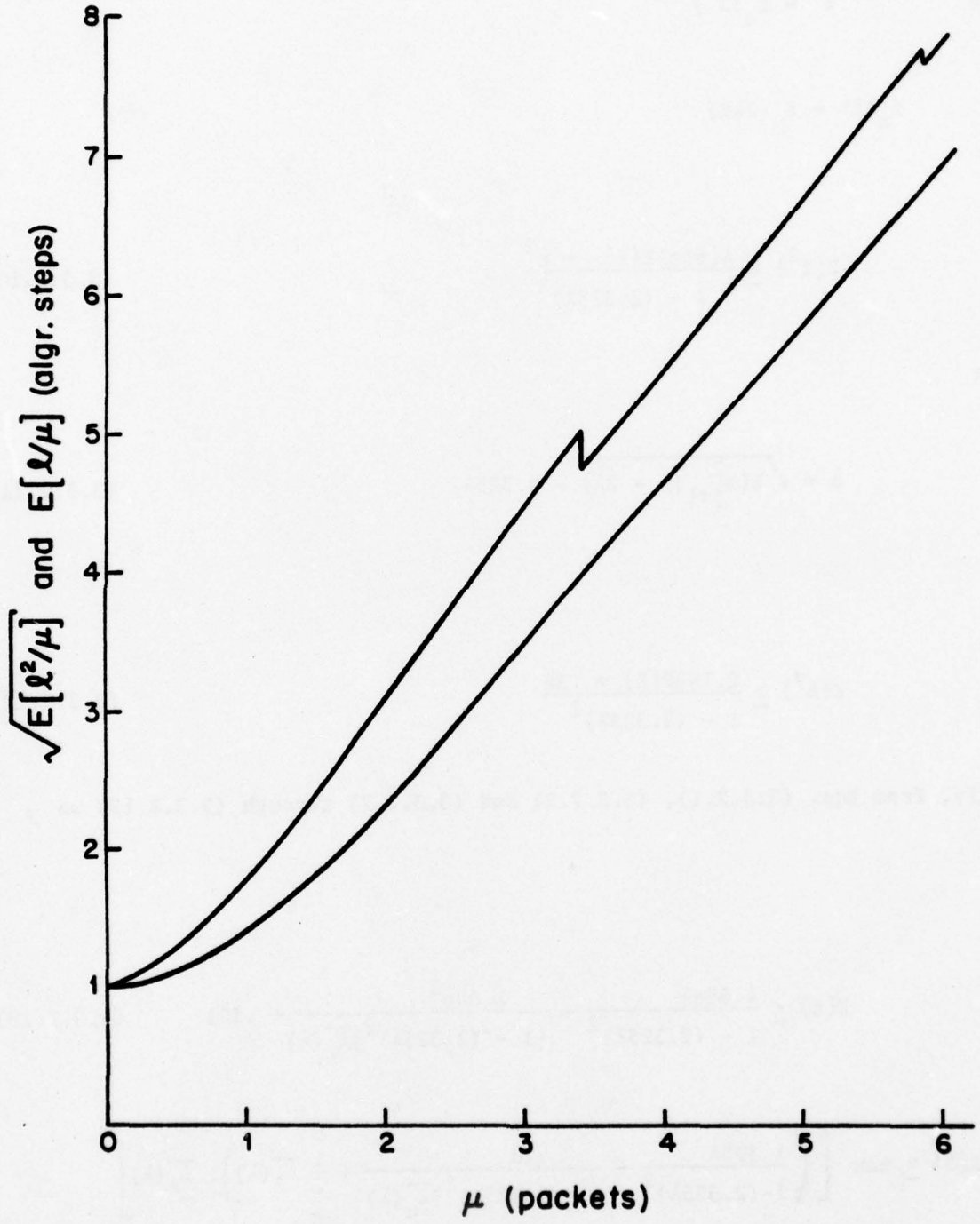


Figure 3.3.2.1 $E\{\ell|\mu\}$ and $\sqrt{E\{\ell^2|\mu\}}$ versus μ for the Optimum Dynamic Tree/Poisson Source System

where $\ell^* = f_c(\ell^*)$

and $f_c(\ell) = f_\mu(2\lambda\ell)$

$$E\{\ell^2\} \leq \frac{4.65c\lambda E\{\ell\} + c^2}{1 - (2.325\lambda)^2} \tag{3.3.2.10}$$

where

$$c = \sqrt{E\{\ell_{j+1}^2 | \mu = 2\lambda\}} - 2.325\lambda \tag{3.3.2.11}$$

and

$$E\{\ell^2\} \geq \frac{2.79\lambda E\{\ell\} + .36}{1 - (2.325\lambda)^2} \tag{3.3.2.12}$$

Finally, from Eqs. (3.3.2.1), (3.3.7.2) and (3.3.2.7) through (3.3.2.12) we have

$$E\{\delta\} \leq \frac{4.88c\lambda}{1 - (2.325\lambda)^2} + \frac{1.05c^2}{[1 - (2.325\lambda)^2]\overline{\ell_\ell}(\lambda)} + .321 \tag{3.3.2.13}$$

$$E\{\delta\} \geq \max\left[\left(\frac{1.395\lambda}{1-(2.325\lambda)^2} + \frac{.18}{[1-(2.325\lambda)^2]\overline{\ell_u}(\lambda)} + \frac{1}{2}\overline{\ell_\ell}(\lambda)\right); \overline{\ell_\ell}(\lambda)\right] \tag{3.3.2.14}$$

Equations (3.3.2.13) and (3.3.2.14) have been calculated and they are plotted in Fig. 3.3.2.2. Note that the maximum arrival rate is .430 packets/slot.

3.3.3 Average Throughput

Based on the results of the preceding section and Section 2.3, it follows that the maximum average throughput for the optimum dynamic algorithm is .430 packets/slot. Furthermore, the average delay vs. average arrival rate results of Section 3.3.2 may be interpreted as average delay vs. average throughput (see Fig. 3.3.2.2).

3.3.4 System Stability

In this section, we will consider the stability of the optimum dynamic tree where the root node has degree $2r^*$ and all other nodes are binary. The definition of stability of the multi-access system with the dynamic algorithm is the same as it is with the static algorithm. That is, the system is k'th order stable if the k'th moment of the delay is finite. In this section we will prove that if $E\{\ell\}$ exists, then the generating function of ℓ also exists (ℓ corresponds to the number of algorithm steps). Since in Section 3.3.2, we showed that $E\{\ell\}$ is finite iff $\lambda < .430$ and since $E\{[\text{delay}]^{k-1}\} < \infty$ if $E\{\ell^k\} < \infty$, the preceding statement allows us to conclude that all the moments of the delay are finite as long as $\lambda < .430$ packets/slot.

Let $G(s,\mu)$ and $G_d(s,\mu)$ be the generating functions of ℓ given μ for the binary and for the dynamic tree algorithms. The analysis of this section is organized as follows.

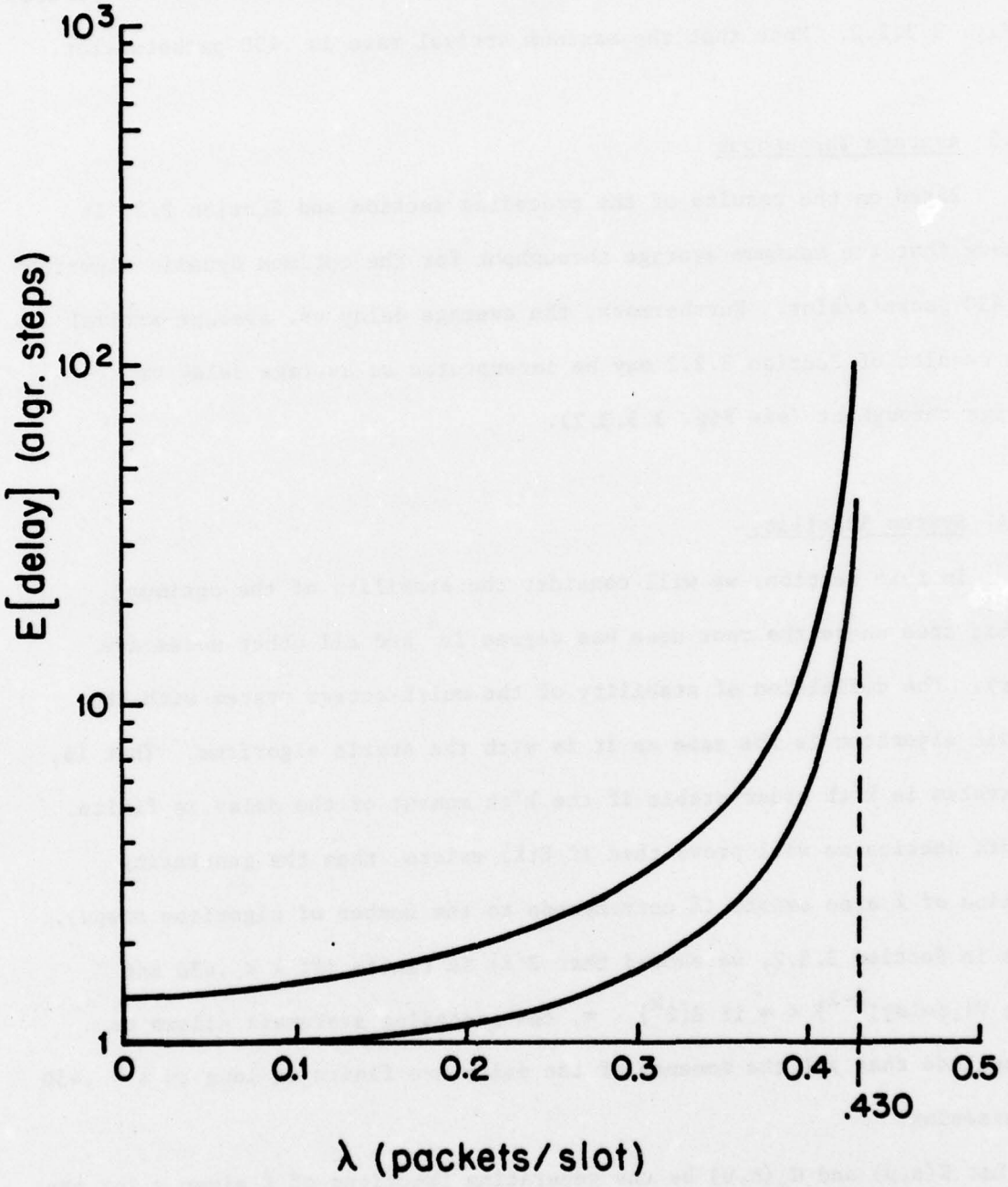


Figure 3.3.2.2 Upper and Lower Bounds to the E{delay} versus the Arrival Rate for the Optimum Dynamic Tree/Poisson Source System

Note: An algr. step equals one round trip delay plus two slots

i. Show that $G(s,\mu) \leq e^{s[E\{\ell|\mu\} + \varepsilon]}$ for $0 \leq s \leq s_0$, $0 \leq \mu \leq 3.4$, $s_0 > 0$, and ε arbitrary.

ii. Show that $G_d(s,\mu) \leq e^{s[E_d\{\ell|\mu\} + \varepsilon]}$ for all μ and $0 \leq s \leq s_0$, $s_0 > 0$. Note that $E_d\{\ell|\mu\}$ is the indicated expectation for the optimum dynamic tree algorithm.

iii. Argue that $E\{e^{s\ell}\}$ exists if $E\{\ell|\mu\}$ exists or equivalently if $\lambda < .430$ packets/slot.

i. <u>Upperbound to $G(s,\mu)$</u>

Let

$$G_u(s,\mu) = e^{-\mu}\left(e^s - \left(\frac{2-e^s}{e^s}\right)\right) + \mu e^{-\mu}(e^s-1) + \frac{2-e^s}{e^s}\, e^{\mu\left[\frac{e^s}{2-e^s} - 1\right]} \qquad (3.3.4.1)$$

$$\text{for } 0 \leq s < \ln 2$$

Then in Appendix A2.8 we show that

$$G(s,\mu) \leq G_u(s,\mu) \text{ for } s < \ln 2 \qquad (3.3.4.2)$$

and

$$\frac{\partial G}{\partial s}(s,\mu) \leq \frac{\partial}{\partial s} G_u(s,\mu) \text{ for } s < \ln 2 \qquad (3.3.4.3)$$

It can be shown that if a generating function is bounded, then it is continuous in s. The same is true for $\frac{\partial G}{\partial s}$. Therefore, it follows from Eqs. (3.3.4.2) and (3.3.4.3) that $G(s,\mu)$ and $\frac{\partial}{\partial s}G(s,\mu)$ are continuous in s for $s < \ln 2$. Note that $G(s,\mu)$ and $\frac{\partial}{\partial s}G(s,\mu)$ are also continuous in μ. This follows from the fact that $P\{j \text{ packets are active}\} = \frac{\mu^k e^{-\mu}}{k!}$.

Since $G(s,\mu)$ and $\frac{\partial}{\partial s}G(s,\mu)$ are continuous in s and μ, $\frac{\partial}{\partial s}\ln G(s,\mu)$ is continuous in s and μ for $s < \ln 2$ and μ. Then, in any closed bounded region, say $0 \leq s \leq \frac{1}{2}\ln 2$, $0 \leq \mu \leq 3.4$, $\frac{\partial}{\partial s}\ln G(s,\mu)$ is uniformly continuous. Thus for any $\varepsilon > 0$ we can choose $s_0 > 0$ so that

$$\frac{\partial}{\partial s}\ln G(s,\mu) \leq \frac{\partial}{\partial s}\ln G(s,\mu)\Big|_{s=0} + \varepsilon \text{ for } 0 \leq s \leq s_0,\ 0 \leq \mu \leq 3.4 \quad (3.3.4.4)$$

Therefore, since $\frac{\partial}{\partial s}\ln(G(s,\mu)\Big|_{s=0} = E\{\ell|\mu\}$,

$$G(s,\mu) \leq e^{s[E(\ell|\mu) + \varepsilon]} \quad \text{for } s \leq s_0,\ a \leq \mu \leq 3.4 \quad (3.3.4.5)$$

ii. Upperbound to $G_d(s,\mu)$

Let $2r^*(\mu)$ be the degree of the optimum root node. Note that this tree corresponds to $r^*(\mu)$ independent binary trees where the mean number of contending packets at each subtree is μ/r^*. Therefore, from Eq. (3.3.4.5) we have

$$G_d(s,\mu) \leq e^{sr^*[E\{\ell|\frac{\mu}{r^*}\} + \varepsilon]} \qquad s \leq s_0$$

But $rE\{\ell|\frac{\mu}{r}\} = E_d\{\ell|\mu\}$ and $\frac{\mu}{r^*} \leq 3.4$ (see Section 3.3.2), therefore

$$G_d(s,\mu) \leq e^{s[E_d\{\ell|\mu\} + \frac{\mu}{3.4}\varepsilon]} \qquad s \leq s_0 \tag{3.3.4.6}$$

iii. Existance of $E\{e^{s\ell}\}$

In Section 3.3.2 we showed that $E_d\{\ell|\mu\} \leq 1 + 1.163\mu$. Substituting this along with $\ell=\ell_{j+1}$ and $\mu=2\lambda\ell_j$ into Eq. (3.3.4.6) we have

$$E_d\{e^{s\ell_{j+1}}|\ell_j\} \leq e^{s[(1.163 + \varepsilon/3.4)2\lambda + 1]} \tag{3.3.4.7}$$

Finally, from Theorem 2.4.0.1 we have

$$E\{e^{s\ell}\} \text{ exists for } \lambda \leq \frac{1}{2(1.163 + \varepsilon/3.4)} \text{ and } s \leq s_0$$

but ε is arbitrary so $\lambda_{max} = .430$ packets/slot.

It can be shown that if $E\{e^{s\ell}\}$ exists for $s \leq s_0$ then so does $E\{\ell^k\}$ for any k.

This concludes this section.

3.4 An Efficient Suboptimum Dynamic Tree Algorithm

In this section we will consider a tree algorithm whose root node degree is constrained to be a power of 2, but whose all other nodes are binary. The advantage of such an algorithm is that it is relatively easy to implement. The reason for this is that as the root node degree is varied dynamically, it is not necessary to choose a different tree and a different addressing scheme each time; variations in the root node degree may be realized in a single binary tree simply by varying the depth of the nodes where the algorithm originates.

In Subsection 3.4.1, we determined the optimum tree given the above constraints. That is, if we let,

$$g_1 = \begin{cases} 2^K;\ K = 1,2,\ldots & \text{for } i = 0 \\ 2 & i > 0 \end{cases} \qquad (3.4.0.1)$$

Then we will show that K^*, the K which minimizes $E\{h/\mu,K\}$, is given by

$$K^* = \begin{cases} 1 \text{ for } \mu \leq 3.40 \\ K \text{ for } 3.40(2^{K-2}) < \mu \leq 3.40(2^{K-1}),\ K > 1 \end{cases} \qquad (3.4.0.2)$$

In Subsection 3.4.2, where the above algorithm is analyzed, we obtain upper and lower bounds to the E{delay}; these are illustrated in Fig. 3.4.2.1. In this section we also show that the maximum throughput is less than .430 but greater than .420 packets/slot.

3.4.1 Optimizing the Suboptimum Tree

Let a dyanmic algorithm have a tree with node degrees given by

$$g_i = \begin{cases} 2^K, \ K = 1,2,3,\ldots \text{ for } i = 0 \\ 2 \qquad\qquad\qquad\qquad i > 0 \end{cases} \tag{3.4.1.1}$$

Then, in this section, we will determine the K which minimizes $E\{\ell/\mu,K\}$. Note that since $h = 2\ell$, minimizing $E\{\ell/\mu,K\}$ is equivalent to minimizing $E\{h/\mu,K\}$.

By making the association that $2r = 2^K$, we have from Eq. (3.3.1.1) that

$$E\{\ell|\mu,K\} = \frac{2^K}{2} E\{\ell|\frac{2\mu}{2^K},1\} \tag{3.4.1.2}$$

As in Section 3.2.3, K^* may be determined by setting

$$E\{\ell|\mu,K-1\} = E\{\ell|\mu,K\} \tag{3.4.1.3}$$

and then solving for $\hat{\mu}(K)$ for $K = 2,3,4,\ldots$. Where $\hat{\mu}(K)$ is defined by

$$K^* = K \text{ for } \hat{\mu}(K) < \mu \leq \hat{\mu}(K+1) \tag{3.4.1.4}$$

This problem is considerably simplified if both sides of Eq. (3.4.1.3) are divided by μ before it is solved. (Note that this operation does not affect the solution $\hat{\mu}(k)$.) The simplification arises from the fact that

$E\{\ell|\mu,2^K\}/\mu$ is simply a function of $\mu/2^K$, therefore, as will be shown below, $\hat{\mu}(K) = \hat{\mu}(2)2^{K-2}$; and it follows that Eq. (3.4.1.3) need be solved only for $K = 2$. Let

$$H(\mu/2^K) \equiv \frac{2^K}{2\mu} E\{\ell|\frac{2\mu}{2^K},1\} \tag{3.4.1.5}$$

then if $\hat{\mu}(2)$ satisfies

$$H(\mu/2) = H(\mu/4) \tag{3.4.1.6}$$

it follows that

$$\hat{\mu}(K) = \hat{\mu}(2)2^{K-2} \tag{3.4.1.7}$$

will satisfy

$$H(\mu/2^{K-1}) = H(\mu/2^K) \tag{3.4.1.8}$$

Equation (3.4.1.6) was solved numerically; and the answer is

$$\mu(2) = 3.40 \tag{3.4.1.9}$$

The final result to this section follows from Eqs. (3.4.1.4), (3.4.1.7) and (3.4.1.9). It is given by

$$K^* = \begin{cases} 1 \text{ for } \mu \leq 3.40 \\ K \text{ for } 3.40(2^{K-2}) < \mu \leq 3.40(2^{K-1}),\ K > 1 \end{cases} \tag{3.4.1.10}$$

3.4.2 Analysis of the Suboptimum Algorithm

The first step of the analysis is to determine β_u and β_ℓ, the slopes of the straight lines that respectively upper and lower bound $E\{\ell/\mu,K^*\}$. These two quantities are given by,

$$\beta_u = \max_{\mu \geq 3.4} \frac{E\{\ell/\mu,K^*\}}{\mu} \tag{3.4.2.1}$$

$$\beta_\ell = \min_{\mu \geq 3.4} \frac{E\{\ell/\mu,K^*\}}{\mu} \tag{3.4.2.2}$$

From Eq. (3.4.1.10) and the fact that $E\{\ell|\mu,K\}/\mu$ is simply a function of $\mu/2^K$, it can be shown that

$$\max_{\mu \geq 3.4} \frac{E\{\ell|\mu,K^*\}}{\mu} = \max_{3.4 \leq \mu < 6.8} \frac{E\{\ell|\mu,K^*\}}{\mu} \tag{3.4.2.3}$$

and

$$\min_{\mu \geq 3.4} \frac{E\{\ell|\mu,K^*\}}{\mu} = \min_{3.4 \leq \mu < 6.8} \frac{E\{\ell|\mu,K^*\}}{\mu} \tag{3.4.2.4}$$

Equations (3.4.2.3) and (3.4.2.4) were solved numerically and the answers are

$$\beta_u = 1.189 \tag{3.4.2.5}$$

$$\beta_\ell = 1.164 \tag{3.4.2.6}$$

The corresponding β's of $\sqrt{E\{\ell^2|\mu,K^*\}}$ equal those of $E\{\ell|\mu\}$. To see this, first note that by setting $2r = 2^K$ in Eq. (3.3.1.2) we have

$$E\{\ell^2|\mu,K^*\} = 2^{K^*-1}(2^{K^*-1}-1)E^2\{\ell|\mu/2^{K^*-1},1\} + 2^{K^*-1}E\{\ell^2|\mu/2^{K^*-1},1\} \tag{3.4.2.7}$$

Next take the square root of this expression, divide by μ, let $\mu \to \infty$, and use Eq. (3.4.1.10) to obtain,

$$\lim_{\mu\to\infty} \frac{\sqrt{E\{\ell^2|\mu,K^*\}}}{\mu} = \frac{E\{\ell|\mu,K^*\}}{\mu} \begin{cases} \leq 1.189 \\ \geq 1.164 \end{cases} \tag{3.4.2.8}$$

Using the above results and numerical calculation of $E\{\ell|\mu,K^*\}$ and $E\{\ell^2|\mu,K^*\}$ it can be shown as in Section 3.3.2 that

$$E\{\ell\} \leq 1 + \frac{2D(\lambda)}{1 - 2.378\lambda} \tag{3.4.2.9}$$

$$E\{\ell\} \geq \ell^* \tag{3.4.2.10}$$

where $\ell^* = f_c(\ell^*)$

$$f_c(\ell) = f_\mu(2\lambda\ell)$$

$$f_\mu = \begin{cases} E\{\ell|\mu,K=1\} & \text{for } \mu \leq 2 \\ 2.328(\mu-2) + E\{\ell|\mu=2,K=1\} & \mu > 2 \end{cases}$$

Furthermore it can be shown that

$$E\{\ell^2\} \leq \frac{4.756c\lambda E\{\ell\} + c^2}{1 - (2.378\lambda)^2} \qquad (3.4.2.11)$$

where

$$c = \sqrt{E\{\ell_{j+1}^2|\mu=2\lambda\}} - 2.378\lambda$$

and

$$E\{\ell^2\} \geq \frac{2.328E\{\ell\} + .25}{1 - (2.328\lambda)^2} \qquad (3.4.2.12)$$

Now substitute Eqs. (3.4.2.9-3.4.2.12) into Eqs. (3.3.2.1) and (3.3.2.2) to obtain

$$E\{\delta\} \leq \frac{4.99\ c\ \lambda}{1 - (2.378\lambda)^2} + \frac{1.05c^2}{1 - [(2.378\lambda)^2 \mathcal{L}_\ell(\lambda)}} + .321 \qquad (3.4.2.13)$$

and

$$E\{\delta\} \geq \max\left[\left(\frac{1.164\,\lambda}{1-(2.328\lambda)^2}+\frac{.125}{1-(2.328\lambda)^2\,\overline{\ell_u}(\lambda)}+\frac{1}{2}\,\overline{\ell_\ell}(\lambda)\right);\ \overline{\ell_\ell}(\lambda)\right] \tag{3.4.2.14}$$

Equations (3.4.2.13) and (3.4.2.14) are the desired bounds; they are illustrated in Fig. 3.4.2.1. Note that the upper bound approaches infinity at $\lambda = .420$, whereas, the lower bound approaches infinity at $\lambda = .430$.

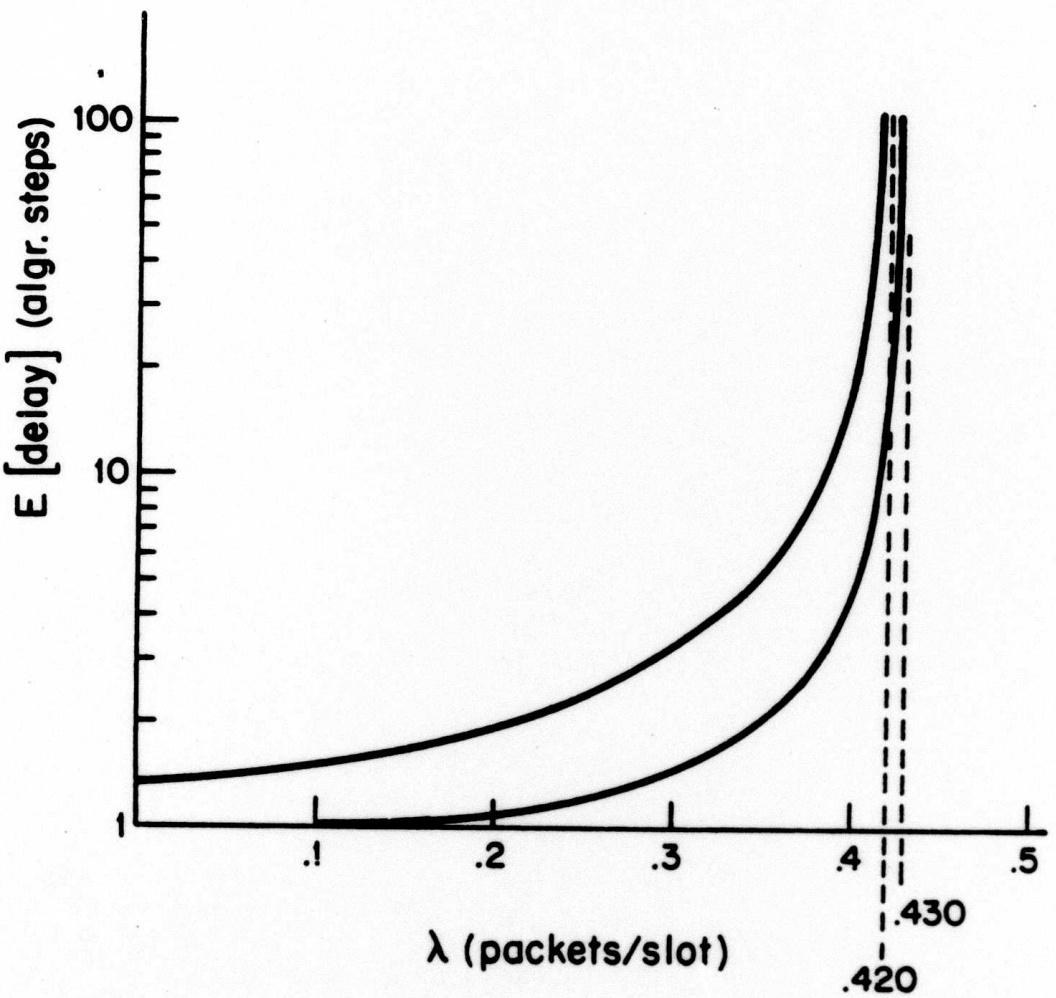


Figure 3.4.2.1 Upper and Lower Bounds to the Average Delay versus the Arrival Rate for the Suboptimum Dynamic Tree/ Poisson Source System

Note: An algr. step equals one round trip delay plus two slots

APPENDIX A3

SELECTED THEOREMS OF CHAPTER 3

Theorem A3.1: Let $.1 \leq \mu \leq 8$; $\frac{\mu}{g_1 g_2} \leq 1.2$;

$$L(g_1,g_2,\mu) = g_1\xi(\mu) + g_1g_2\xi(\frac{\mu}{g_1}) + 2\sum_{i=0}^{\infty} 2^i g_1g_2\xi(\frac{\mu}{g_1g_2 2^i}) \qquad (A.3.1)$$

and

$$L_\ell(g_1,g_2,\mu) = g_1\xi(\mu) + g_1g_2\xi(\frac{\mu}{g_1}) + 2\mu(\frac{\mu}{g_1g_2} - \frac{4}{9}(\frac{\mu}{g_1g_2})^2) \qquad (A.3.2)$$

then;

$$L(g_1,g_2,\mu) \geq L_\ell(g_1,g_2,\mu) \qquad (A.3.3)$$

Proof: This follows from Property 1D in Appendix A.2.2.

QED

Theorem A3.2: Let $L_\ell(g_1,g_2,\mu)$ be as in Eq. (A.3.2), furthermore, let $g_1 \geq 10$, $g_2 \geq 3$ and $.1 \leq \mu \leq 8$. Then the minimum of $L_\ell(g_1,g_2,\mu)$ occurs at $g_1 = 10$, $g_2 = 3$.

Proof: This theorem will be proved by showing that $\frac{\partial L_\ell}{\partial g_2} > 0$ and $\frac{\partial L_\ell}{\partial g_1}\Big|_{g_2=3} > 0$. Differentiating Eq. (A3.2),

$$\frac{\partial L_\ell}{\partial g_2} = g_1\left[\xi\left(\frac{\mu}{g_1}\right) - 2\left(\frac{\mu}{g_1 g_2}\right)^2 + \frac{4}{9}\left(\frac{\mu}{g_1 g_2}\right)^3\right] \tag{A3.4}$$

$$\geq \xi\left(\frac{\mu}{g_1}\right) - 2\left(\frac{\mu}{g_1 g_2}\right)^2 \qquad \text{since } g_1 \geq 10,\ \frac{\mu}{g_1 g_2} > 0 \tag{A3.5}$$

$$\geq \frac{1}{2}\left(\frac{\mu}{g_1}\right)^2 - \frac{1}{3}\left(\frac{\mu}{g_1}\right)^3 - 2\left(\frac{\mu}{g_1 g_2}\right)^2 \quad \text{from Property 1B in Appendix A2.2} \tag{A3.6}$$

$$\geq \frac{5}{18}\left(\frac{\mu}{g_1}\right)^2 - \frac{16}{18}\left(\frac{\mu}{g_1}\right)^3 \qquad \text{since } g_2 \geq 3 \tag{A3.7}$$

$$\geq 0 \qquad \text{since } \frac{\mu}{g_1} \leq \frac{5}{6} \tag{A3.8}$$

For the second part of this proof substitute the expression for $\xi(\mu)$ given in Property 1A in Appendix A2.2 into Eq. (A3.2) to obtain,

$$\frac{\partial L_\ell}{\partial g_1} = \xi(\mu) + \mu g_2 \sum_{k=1}^{\infty} \left(\frac{-\mu}{g_1}\right)^{k-1} \frac{1}{(k-1)!}\left(\frac{-k}{k+1}\right)\frac{\mu}{g_1^2} + 2\mu\left(\frac{-\mu}{g_1^2 g_2} + \frac{8}{9}\frac{\mu^2}{g_1^3 g_2^2}\right)$$

$$\geq \xi(\mu) - \frac{g_2}{2}\left(\frac{\mu}{g_1}\right)^2 + \frac{2g_2}{3}\left(\frac{\mu}{g_1}\right)^3 - \frac{2}{g_2}\left(\frac{\mu}{g_1}\right)^2 + \frac{16}{9g_2^2}\left(\frac{\mu}{g_1}\right)^3 \text{ for } \frac{\mu}{g_1} \leq 1$$

$$= \xi(\mu) - \frac{13}{6}\left(\frac{\mu}{g_1}\right)^2 + \frac{178}{81}\left(\frac{\mu}{g_1}\right)^3 \quad \text{at } g_2 = 3 \tag{A3.9}$$

Now $\mu \leq 1.2$ and $\xi(\mu) \geq \frac{1}{2}\mu^2 - \frac{1}{3}\mu^3$, therefore

$$\frac{\partial L_\ell}{\partial g_1} \geq \mu^2\left[\frac{1}{2} - \frac{13}{6}\frac{1}{g_1^2} - \mu\left(\frac{1}{3} - \frac{178}{81}\frac{1}{g_1^3}\right)\right] \tag{A3.10}$$

$$\geq 0 \qquad \text{for } \mu \leq 1.2 \tag{A3.11}$$

For $\mu \geq 1.2$

$$\xi(\mu) \geq \xi(1.2) = .337$$

But

$$\left(\frac{\mu}{g_1}\right)^2\left(\frac{13}{6}\right) - \left(\frac{\mu}{g_1}\right)^3\frac{178}{81} \leq .312$$

Therefore from Eq. (A3.9)

$$\frac{\partial L_\ell}{\partial g_1} \geq 0 \quad \text{for} \quad \mu \geq 1.2 \tag{A3.12}$$

Eqs. (A3.11) and (A3.12) taken together conclude the proof.

QED

Theorem A3.3: Let $L_\ell(g_1,g_2,\mu)$ be as given in Eq. (A3.2); furthermore, let $2 \leq g_1 \leq 9$, $g_2 \geq 6$ and $.1 \leq \mu \leq 8$. Then the minimum of $L_\ell(g_1,g_2,\mu)$ occurs at $g_2 = 6$.

Proof: We will prove this by showing that when g_1,g_2,μ are as specified in the above statement then $\frac{\partial L_\ell}{\partial g_2} > 0$. From Eq. (A3.5) we have that

$$\frac{\partial L_\ell}{\partial g_2} \geq \xi\left(\frac{\mu}{g_1}\right) - 2\left(\frac{\mu}{g_1 g_2}\right)^2$$

$$\geq \xi\left(\frac{\mu}{g_1}\right) - \frac{1}{18}\left(\frac{\mu}{g_1}\right)^2 \quad \text{for} \quad g_2 \geq 6 \tag{A3.13}$$

$$= \left(\frac{\mu}{g_1}\right)^2\left[\xi\left(\frac{g_1}{\mu}\right)^2 \xi\left(\frac{\mu}{g_1}\right) - \frac{1}{18}\right] \tag{A3.14}$$

Since $\frac{\mu}{g_1} \geq 0$, we need to show that $\left(\frac{g_1}{\mu}\right)^2 \xi\left(\frac{\mu}{g_1}\right) \geq \frac{1}{18}$.

Now since $2 \leq g_1 \leq 9$ and $.1 \leq \mu \leq 8$ it follows that

$$\frac{1}{90} \leq \frac{\mu}{g_1} \leq 4 \tag{A3.15}$$

And since $\frac{d}{dx}(\xi(x)/x^2) \leq 0$, we have that

$$\left(\frac{g_1}{\mu}\right)^2 \xi\left(\frac{\mu}{g_1}\right) \geq \left(\frac{1}{4}\right)^2 \xi(\mu) \text{ for } 0 \leq \frac{\mu}{g_1} \leq 4 \tag{A3.16}$$

$$= .05678 \tag{A3.17}$$

$$> 1/18 \tag{A3.18}$$

QED

Theorem A3.4: Define,

$$H_u(g,\mu) = \frac{g}{\mu} + 2 \sum_{i=0}^{\infty} \frac{2^i g}{\mu} \xi\left(\frac{\mu}{2^i g}\right) \tag{A3.19}$$

And let $\mu \geq 8$ and g be integer and greater than one. Then

$$\min_g H_u(g,\mu) \leq 2.34 \tag{A3.20}$$

Proof: This will be proved by first finding an interval X such that for any $\mu \geq 8$, an integer y $\geq$ exists such that $\mu/g \in X$ and then showing that $\max_{(\mu/g)\in X} H_u(\frac{\mu}{g}) \leq 2.34$. First, we prove that $X = \{\frac{\mu}{g} : 1.1 \leq \frac{\mu}{g} \leq 1.257\}$ satisfies the above criteria. It is sufficient to show that

$$\sup_{\mu\geq 8}\left[\min_{g\leq \mu/1.1} \left(\frac{\mu}{g}\right)\right] = 1.257 \tag{A3.21}$$

But $\min_g \frac{\mu}{g} = \frac{\mu}{7}$ for $8 \leq \mu < 8(1.1)$. Therefore,

$$\sup_{8\leq\mu<8.8} \left[\frac{\mu}{7}\right] = (1.1)\frac{8}{7} = 1.257 \tag{A3.22}$$

Similarly for $1.1k \leq \mu \leq 1.1(k+1)$; $\mu \geq 8$

$$\sup_{1.1k \leq \mu < 1.1(k+1)} \left[\min_{g_1 \leq \mu/1.1} \left(\mu/g \right) \right] = 1.1\frac{k+1}{k} \quad \text{for } k \geq 8 \tag{A3.23}$$

Equations (A3.22) and (A3.23) prove Eq. (A3.21)

Now let

$$x = \mu/g \tag{A3.24}$$

then from Eq. (A3.19) we have

$$H_u(x) = \frac{1}{x} + 2 \sum_{i=0}^{\infty} \frac{2^i}{x} \xi\left(\frac{x}{2^i}\right) \tag{A3.25}$$

Finally, $H_u(x)$ was maximized over $x \epsilon X$ and the result is

$$\max_{x \epsilon X} H_u(x) = 2.34 \tag{A3.26}$$

QED

Theorem A3.5: Define

$$H_\ell(x,g) = \frac{1}{x}\xi(8) + g\frac{\xi(x)}{x} + 2 \sum_{i=0}^{\infty} \frac{2^i g}{x} \xi\left(\frac{x}{2^i g}\right) \tag{A3.27}$$

and let $0 < x \leq .5$ and $g \geq 3$. Then

$$H_\ell(x,g) \geq 2.498 \tag{A3.28}$$

Proof: Since the sum in Eq. (A3.27) is positive we have that

$$H_\ell(x,g) \geq \frac{1}{x}\xi(8) + g\frac{\xi(x)}{x} \tag{A3.29}$$

$$\geq \frac{1}{x}\xi(8) + 3\frac{\xi(x)}{x} \text{ for } g \geq 3 \tag{A3.30}$$

$$\geq \frac{1}{x}\xi(8) + \frac{3}{2}x - x^2 \tag{A3.31}$$

The last equation follows from Property 1B of Appendix A2.1.2. Now for $0 \leq x \leq .5$, the right side of Eq. (A3.31) is decreasing in x. Therefore, evaluating it at $x = .5$ we have

$$H_\ell(x,g) \geq 2.498$$

QED

Theorem A3.6: Let $H_\ell(x,g)$ be as in Eq. (A3.27). Also let $.5 \leq x \leq 3$ and $g \geq 3$. Then

$$H_\ell(x,g) \geq 2.34 \tag{A3.32}$$

Proof: First we prove this for $g \geq 12$. It can easily be shown that,

$$\frac{d^2}{dx^2}\left[\frac{\xi(x)}{x}\right] = \frac{1}{x^3}[1 - (1 + x + x^3)e^{-x}] \leq 0 \text{ for } x \leq 3$$

therefore

$$\min_{.5 \leq x \leq 3} \left[\frac{\xi(x)}{x}\right] = \min\left[\frac{\xi(.5)}{.5} ; \frac{\xi(3)}{3}\right] = \frac{\xi(.5)}{.5} \tag{A3.33}$$

But from Eq. (A3.29)

$$H_\ell(x,g) \geq \frac{1}{3}\xi(8) + 12\frac{\xi(.5)}{.5} \text{ for } .5 \leq x \leq 3 \text{ and } g \geq 12 \tag{A3.34}$$

$$= 2.497 \text{ for } g \geq 12 \tag{A3.35}$$

Finally, $H_\ell(x,g)$ is minimized over $g = 3,4,\ldots,12$ and $.5 \leq x \leq 3$. The result is

$$H_\ell(x,g) \geq 2.34 \text{ for } .5 \leq x \leq 3 \text{ and } g = 2,3,\ldots,12 \tag{A3.36}$$

Theorem A3.7: Define $H(g_1,g_2,\mu)$ as

$$H(g_1,g_2,\mu) = \frac{g_1}{\mu}\xi(\mu) + \frac{g_1 g_2}{\mu}\xi\left(\frac{\mu}{g_1}\right) + 2\sum_{i=0}^{\infty} 2^i \frac{g_1 g_2}{\mu}\xi\left(\frac{\mu}{g_1 g_2}\right) \tag{A3.37}$$

and let g_1, g_2 be integer and greater than one. Furthermore, let $\mu \geq 8$ and $\frac{\mu}{g_1} \geq 3$.

then the $\min_{g_1 g_2} H(g_1, g_2, \mu)$ lies on $g_2 = 2$ (A3.38)

Proof: Since $\xi(\mu)$ is an increasing function we have for $\mu \geq 8$ that

$$H(g_1, g_2, \mu) \leq \frac{g_1}{\mu} + \frac{g_1 g_2}{\mu} \xi\left(\frac{\mu}{g_1}\right) + 2 \sum_{i=0}^{\infty} 2^i \frac{g_1 g_2}{\mu} \xi\left(\frac{\mu}{g_1 g_2 2^i}\right) \quad \text{(A3.39)}$$

$$H(g_1, g_2, \mu) \geq \frac{g_1}{\mu} \xi(8) + \frac{g_1 g_2}{\mu} \xi\left(\frac{\mu}{g_1}\right) + 2 \sum_{i=0}^{\infty} \frac{2^i g_1 g_2}{\mu} \xi\left(\frac{\mu}{g_1 g_2 2^i}\right) \quad \text{(A3.40)}$$

for $\mu \geq 8$

Now for $2 \leq g_2 \leq \frac{\xi(8) e^{\mu/g_1}}{1 + \mu/g_1}$ the following holds

$$\xi(8) + g_2 \xi(\mu/g_1) \geq g_2 \quad \text{(A3.41)}$$

therefore,

$$H(g_1, g_2, \mu) \geq \frac{g_1 g_2}{\mu} + 2 \sum_{i=0}^{\infty} \frac{2^i g_1 g_2}{\mu} \xi\left(\frac{\mu}{g_1 g_2 2^i}\right) \quad \text{(A3.42)}$$

and

$$H(g_1,g_2,\mu) \geq \min_{g\geq 2}[\frac{g}{\mu} + 2 \sum_{i=0}^{\infty} \frac{2^i g}{\mu}\xi(\frac{\mu}{g2^i})] \tag{A3.43}$$

But the right side of Eq. (A3.43) is the upper bound (Eq. (A3.39)) evaluated at $g_2 = 2$ and minimized over g_1. So we have proved that if $g_2 \leq \frac{\xi(8)e^{\mu/g_1}}{1 + \mu/g_1}$, then the minimum of $H(g_1,g_2,\mu)$ lies on $g_2 = 2$.

The proof is concluded by demonstrating that Eq. (A3.43) also holds if $g_2 > \frac{\xi(8)e^{\mu/g_1}}{1 + \mu/g_1}$. Letting $x = \mu/g$ in Eq. (A3.40) we have from Theorem A3.1 that

$$H(x,g_2) \geq \frac{1}{x}\xi(8) + \frac{g_2}{x}\xi(x) + \frac{2x}{g_2} - \frac{8}{9}(x/g_2)^2 \tag{A3.44}$$

$$> \frac{1}{x}\xi(8) + \frac{g_2}{x}\xi(x) + \frac{2x}{g_2} - \frac{8}{9}(\frac{x}{1+x}e^{-x})^2 \text{ for } g_2 > \frac{e^{-x}}{1+x} \tag{A3.45}$$

Now let y be defined by

$$y = g_2\frac{\xi(x)}{x} + \frac{2x}{g_2} \tag{A3.46}$$

Then it can easily be shown that y is convex in g_2 for $x \geq 0$ with the minimum occurring at

$$g_2^* = x\sqrt{\frac{2}{\xi(x)}} \tag{A3.47}$$

But for $x \geq 3$

$$g_2^* \leq \frac{e^x}{1+x} \tag{A3.48}$$

and since y is convex it follows that

$$y \geq \frac{e^x}{1+x} \frac{\xi(x)}{x} + 2x\left(\frac{1+x}{e^x}\right) \text{ for } x \geq 3$$

Therefore from the above equation and Eq. (A3.45) we have that

$$H(x,g_2) \geq \frac{\xi(8)}{x} + \frac{e^x}{1+x} \frac{\xi(x)}{x} + 2x\frac{(1+x)}{e^x} - \frac{8}{9}\left(\frac{x}{1+x}\, e^x\right)^2$$

$$\geq 2.55$$

and from Theorem A3.4 we conclude that

$$\geq \min_{g_1>2} \left[\frac{g}{\mu} + 2 \sum_{i=0}^{\infty} \frac{2^i g}{\mu} \xi\left(\frac{\mu}{2^i g}\right)\right]$$

QED

CHAPTER 4

THE TREE ALGORITHMS WITH A FINITE SOURCE MODEL

4.1 Introduction

In this chapter we will consider a multiple access system with 2^N independent sources. The transmissions process is essentially the same as that which is described in Chapter 2. That is, packets that arrive in one epoch are processed in the following epoch by a tree algorithm (both static and dynamic algorithms will be considered in this chapter). Here, as in Chapter 2, a group of sources that has undergone a conflict is divided in half and the two subgroups transmit their packets in two consecutive slots. The decision to divide a group in half, the transmission of the two halves and the observation of the results of those two transmissions constitute an algorithmic step. The number of algorithmic steps in one epoch will be designated by ℓ.

It will be assumed that a source may receive at most one new packet per epoch. The probability that a source will receive a packet in the next step, given that it has not yet received one in the present epoch, is a constant and will be designated by ρ. It follows then, that q, the probability that a source will receive a packet in an epoch of length ℓ, is given by

$$q = 1 - (1-\rho)^{\ell}. \qquad (4.1.0.1)$$

It should be noted that in the above source model, a source can have at most two packets at any one time; one that arrived in the previous epoch and which is in the process of being transmitted, and one that arrived in

the present epoch and which will be processed in the following epoch.

The channel model is the same as that which was considered in Chapters 2 and 3. That is, it is slotted and the sources have the means to determine whether there are 0, 1 or more than 1 packets in any one slot. If a slot contains more than one packet, then it is assumed that no one gets through.

The tree searches of the algorithms that we will consider here will be carried out serially and the source addresses are assigned deterministically. The parallel search and the random address assignment will not be considered here. It is important to point out however that, even though for the Poisson source model the random and deterministic source assignments had the same delay and throughput properties, in the finite source model it can be shown that the random address assignment has larger average delay than that of the deterministic.

The rest of this chapter is organized into two sections. In Section 4.2 we consider the static binary tree. Here we obtain an upper bound to the average delay and a lower bound to the average throughput in terms of ρ. We also combine these two bounds to obtain an E{delay} vs E{throughput} performance curve. This is illustrated in Fig. 4.2.2.2 for N=6. In Section 4.3 we consider the optimum dynamic tree algorithm. Here we restrict all nodes to be binary except for the root node which is allowed to have a degree that is a power of 2 but less than or equal to 2^N. Subject to the preceding restrictions, g_0 is chosen so as to minimize the expected number of slots needed to process the contending packets, given q. The optimum g_0 is given by Eqs. (4.3.1.1) and (4.3.1.6). Following the determination of the optimum tree, first, we obtain upper and lower bounds to the average

delay and average throughput respectively, and then obtain an E{delay} vs E{throughput} curve. The E{delay} vs E{throughput} curve is shown in Fig. 4.3.2.5. Section 4.3 is concluded with a theorem proving that the E{delay} for the optimum dynamic tree protocol is less than or equal to the E{delay} of the TDMA protocol.

4.2 Static Binary Tree Algorithm with Finite Source Model

This section contains two subsections: 4.2.1 where an upper bound to the E{delay} is obtained and 4.2.2 where a lower bound to E{throughput} is obtained. Both of these bounds are functions of ρ, the traffic parameter. They have been computed for N=6 and they are plotted in Figs. 4.2.1.4 and 4.2.2.1. The E{delay} vs. E{throughput} curve is given in 4.2.2.1.

4.2.1 Average Delay

We begin this section by presenting several definitions. We will be using the following quantities: ε_j, ℓ_j, d_j δ, $\bar{\ell}_u$, ρ, and q. The first five of these are defined in Section 2.2 and the last 2 are defined in Section 4.1. Furthermore, let

$$\psi(q,m) = 1 - (1-q)^{2^m} - 2^m q(1-q)^{(2^m-1)} \qquad (4.2.1.1)$$

$$\phi(q,m) = 1 - (1-q)^{(2^m-1)} \qquad (4.2.1.2)$$

$$D(q,m) = \sum_{i=0}^{m-1} 2^i \psi(q,m-i) \qquad (4.2.1.3)$$

$\psi(q,m)$ is the probability that there are at least two active sources in a branch with 2^m leaves. $\phi(q,m)$ is the probability that there is at least one other active source in a branch with 2^m leaves given that one particular source of that branch is active. $D(q,m)$ is the expected number of nodes visited in a branch with 2^m leaves. These quantitites are considered in more detail in Appendix A4.

The analysis is carried out as follows:

i. Calculate $E\{\delta|\ell_1,\rho\}$.

ii. Show that $E\{\delta|\ell_1,\rho\}$ is concave and increasing in ℓ_1.

iii. Calculate $\overline{\ell}_u$, an upper bound to $E\{\ell\}$.

iv. Apply Jenssen's inequality to the above three steps to prove that $E\{\delta|\rho\} \leq E\{\delta|\overline{\ell}_u,\rho\} \equiv \overline{\delta}_u$.

i. Derivation of $E\{\delta|\ell_1,\rho\}$

The delay that a packet undergoes can be decomposed into d_1 and d_2 where d_1 is the time spent by the packet in the epoch of arrival and d_2 is the time spent in the following epoch, i.e., the epoch where it is successfully transmitted. As a consequence of this observation we have that

$$E\{\delta|\ell_1,\rho\} = E\{d_1|\ell_1,\rho\} + E\{d_2|\ell_1,\rho\} \qquad (4.2.1.4)$$

The subscript of ℓ will not be used where ambiguities do not arise. Note that ℓ_1 refers to length of the epoch in which the packet arrived and when a random variable is conditioned on ℓ we will mean ℓ_1.

Next an expression for $E\{d_1|\ell,\rho\}$ will be determined; this is accomplished by noting that, given that ϵ_1 has length ℓ, the delay of a packet arriving in the j^{th} step of ϵ_1 is,

$$d_1 = \ell - j - 1 \qquad (4.2.1.5)$$

The probability of a given packet arriving in the j^{th} step is

$$P(j|\ell,\rho) = \frac{\rho(1-\rho)^j}{1-(1-\rho)^\ell} \quad \text{for } j=0,1,\ldots,\ell-1. \tag{4.2.1.6}$$

Multiplying Eqs. (4.2.1.5) and (4.2.1.6) and summing over j we have

$$E\{d_1|\ell,\rho\} = \ell - 1 - \frac{1-\rho-(1-\rho)^\ell(\ell\rho+1-\rho)}{\rho(1-(1-\rho)^\ell)} \tag{4.2.1.7}$$

This is the desired expression for $E\{d_1|\ell,\rho\}$. $E\{d_2|\ell,\rho\}$ is derived in Appendix A4 and it is rewritten below,

$$E\{d_2|\ell,\rho\} = 1 + \sum_{i=1}^{N-1} [1 + \tfrac{1}{2} D(q,N-i)]\, \phi(q,N-i) \tag{4.2.1.8}$$

where q, ϕ, and D are given by Eqs. (4.1.0.1), (4.2.1.2) and (4.2.1.3). The expression for $E\{\delta|\ell,\rho\}$ follows from Eqs. (4.2.1.4), (4.2.1.7) and (4.2.1.8).

ii. Properties of $E\{\delta|\ell,\rho\}$

For N=6 and selected values of ρ, the quantity $E\{\delta|\ell,\rho\}$ was calculated for $\ell=1,2,3,\ldots 63$. The results are shown in Fig. 4.2.1.1. From the computer printout, as well as from this figure, it is evident that $E\{\delta|\ell,\rho\}$ is concave and increasing in ℓ. These properties will be used in Subsection iv.

iii. Determination of Upper Bound to $E\{\ell\}$

In this subsection, first, we will derive $\bar{\ell}_u$, an upper bound to $E\{\ell\}$, and then compute that upper bound for N=6. The upper bound follows from the following theorem.

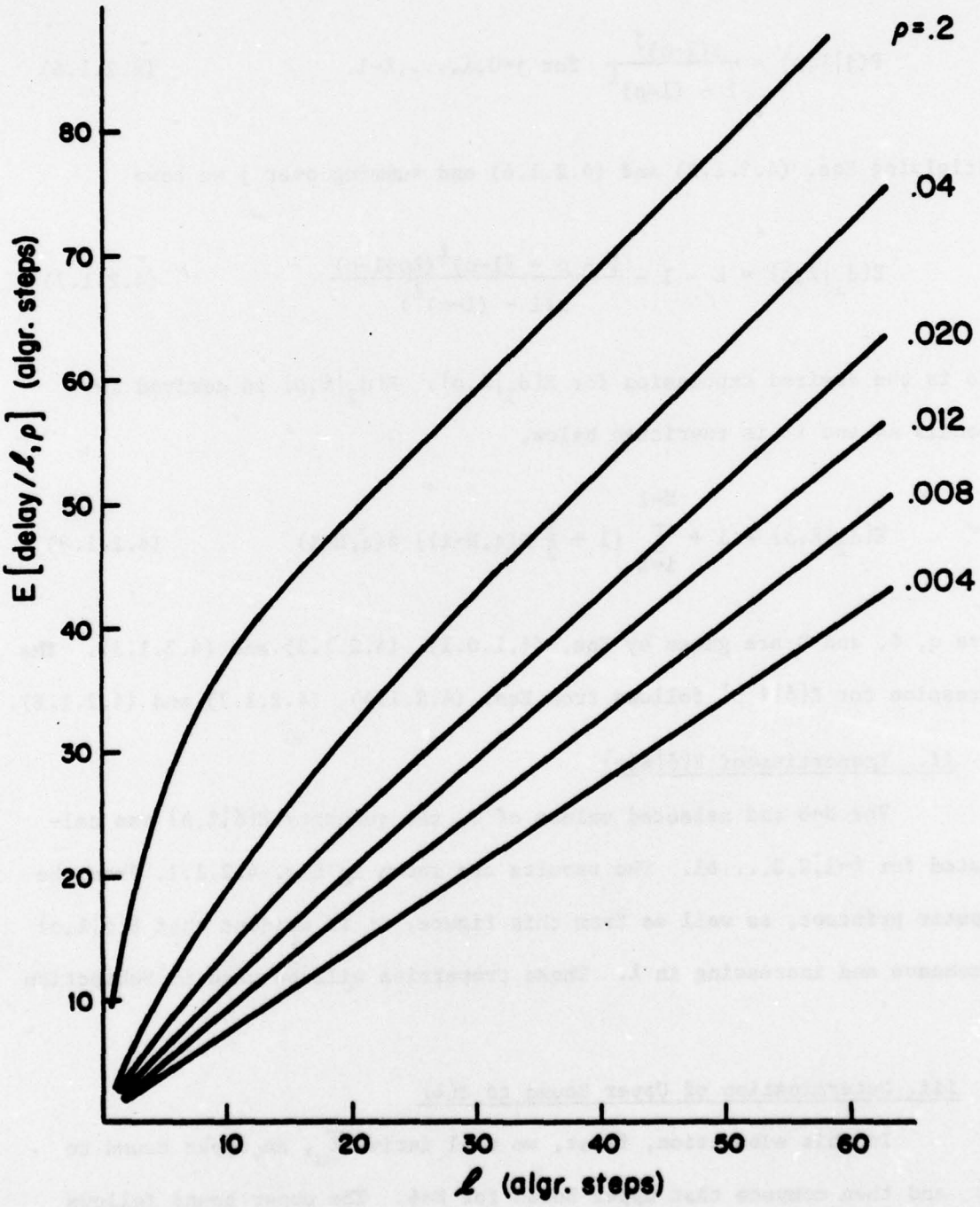


Figure 4.2.1.1 E{delay|ℓ,ρ} versus ℓ for the Binary Tree Algorithm with 64 Sources

Theorem 4.2.1.1: Let ℓ_j be the outcome of a Markov chain after the j^{th} transition and let $f_v(\ell_j)$ be a nondecreasing concave upper bound to $E\{\ell_{j+1}|\ell_j\}$. Furthermore, let ℓ^* be defined by

$$\ell^* = f_v(\ell^*) \tag{4.2.1.9}$$

and assume that $\ell_j > f_v(\ell_j)$ for $\ell_j > \ell^*$ then

$$\lim_{j\to\infty} E\{\ell_j\} \leq \ell^* \equiv \overline{\ell}_u \tag{4.2.1.10}$$

Proof: This proof is similar to that of Theorem 2.2.3.2. Therefore, it will not be given.

QED

Next we will calculate $\overline{\ell}_u$ for N=6. In Appendix A4 the following expression is derived.

$$E\{\ell_2|q\} = 1 + 2D(q,N-1) \tag{4.2.1.11}$$

Equation (4.2.1.11) is plotted in Fig. 4.2.1.2 for N=6, where for comparison we also plot on the same figure $E\{\ell_2|q\}$ for the optimum dynamic tree — this is derived in Section 4.3.1. $E\{\ell_2|\ell_1,\rho\}$ follows from Eqs. (4.2.1.11) and (4.1.0.1). This quantity is illustrated in Fig. 4.2.1.3. As can be seen from this figure, for $\rho \geq .016$, $f_v(\ell^*) = E\{\ell_2|\ell^*\}$. Equation (4.2.1.9) was solved for this f_v and for $\rho \geq .016$. The results are listed in Table 4.2.1.1. For $\rho = .004$, .008 and .012, the following three linear expressions upperbound $E\{\ell_2|\ell_1\}$ respectively.

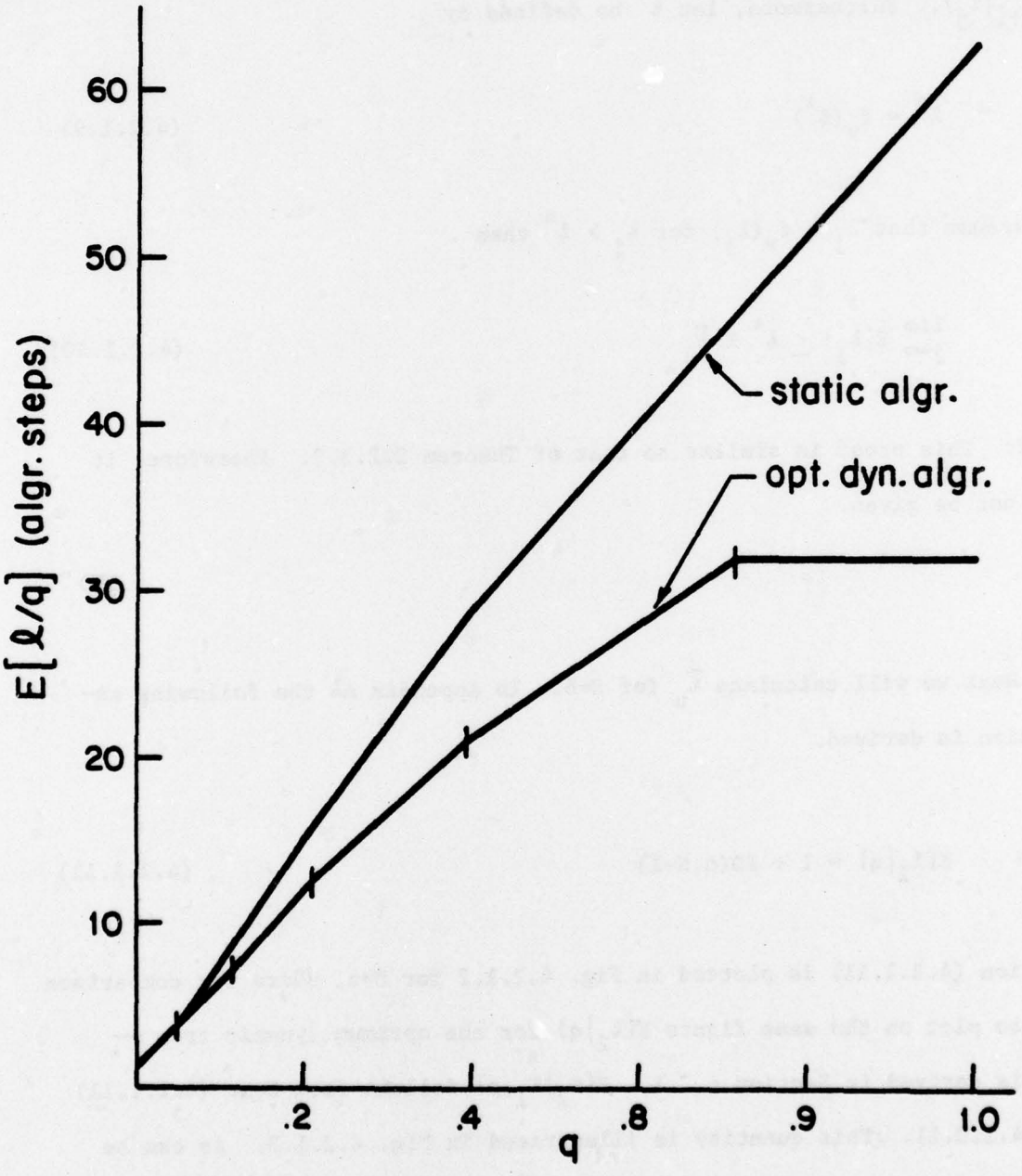


Figure 4.2.1.2 E{ℓ|q} versus q for the Binary and the Optimum Dynamic Tree Algorithms with 64 Sources

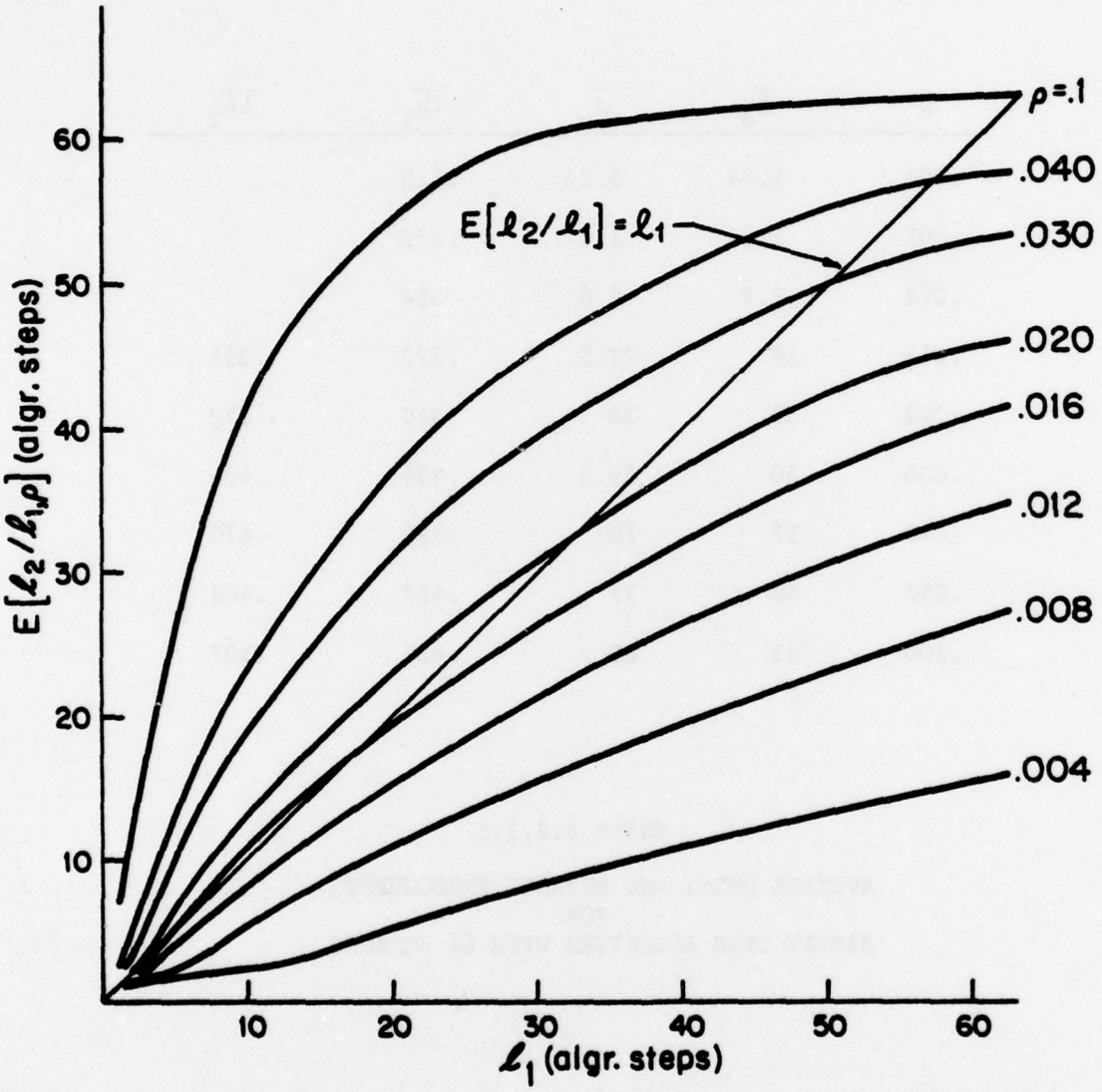


Figure 4.2.1.3 $E\{\ell_2|\ell_1,\rho\}$ versus ℓ_1 for the Binary Tree Algorithm with 64 Sources

ρ	$\overline{\ell}_u$	$\overline{\delta}_u$	$\overline{th}_1$	$\overline{th}_2$
.004	1.04	1.24	.125	
.001	1.2	1.6	.232	
.012	1.9	2.6	.314	
.016	19	22.5	.372	.331
.020	32	36	.410	.370
.030	50	59.5	.436	.436
.040	57	70	.436	.470
.050	60	77	.437	.489
.100	63	89	.439	.507

TABLE 4.2.1.1

AVERAGE DELAY AND AVERAGE THROUGHPUT
FOR
BINARY TREE ALGORITHM WITH 64 SOURCES

$$E\{\ell_2|\ell_1,.004\} \leq .224\ell_1 + .804 \qquad (4.2.1.12)$$

$$E\{\ell_2|\ell_1,.008\} \leq .500\ell_1 + .605 \qquad (4.2.1.13)$$

$$E\{\ell_2|\ell_1,.012\} \leq .750\ell_1 + .473 \; . \qquad (4.2.1.14)$$

These f_v's were used in Eq. (4.2.1.9) to obtain $\overline{\ell}_u$ for the indicated ρ's. The results are listed in Table 4.2.1.1. On that same table we also present $\overline{\delta}_u$, $\overline{th}_1$ and $\overline{th}_2$. $\overline{\delta}_u$ is the upper bound to the E{delay} which is determined in the following subsection and $\overline{th}_1$ and $\overline{th}_2$ are lower bounds to the expected throughput which are derived in Section 4.2.2.

iv. Determination of Upper Bound to E{delay}

In Subsection ii, it was shown $E\{\delta|\ell,\rho\}$ is concave and nondecreasing in ℓ. Therefore, from Jenssen's inequality and the concavity of $E\{\delta|\ell,\rho\}$ it follows that

$$E\{\delta|\rho\} \leq E\{\delta|\overline{\ell},\rho\} \qquad (4.2.1.15)$$

and from the nondecreasing property of $E\{\delta|\ell,\rho\}$ we have the following upper bound.

$$E\{\delta|\rho\} \leq E\{\delta|\overline{\ell}_u,\rho\} \equiv \overline{\delta}_u \qquad (4.2.1.16)$$

Note that the right side of the above equation equals Eq. (4.2.1.4) with

ℓ_1 replaced by $\bar{\ell}_u$. The values of $\bar{\ell}_u(\rho)$ that were calculated in the preceding subsection were substituted into Eq. (4.2.1.16) and the results are presented in Table 4.2.1.1 and in Fig. 4.2.1.4.

One more result will be developed before comcluding this section; it is an exact expression for E{delay} at ρ=1. This is a useful quantity since it is the maximum E{delay} over ρ. When ρ=1, a packet arrives at each source in the first step of each epoch with probability one. Therefore, it can be shown that

$$E\{d_1|\rho=1\} = 2^N-2 \qquad (4.2.1.17)$$

and

$$E\{d_2|\rho=1\} = \frac{N + 2^N - 1}{2} \qquad (4.2.1.18)$$

and we have

$$E\{\delta|\rho=1\} = \frac{3(2^N) + N - 5}{2} \qquad (4.2.1.19)$$

This should be compared with the maximum average delay for the optimum dynamic tree which, as we will see, is given by

$$E_{dyn}\{\delta|\rho=1\} = (3(2^{N-1})-1)/2 \qquad (4.2.1.20)$$

This concludes Section 4.2.1. Next we consider the expected throughput.

4.2.2 Average Throughput

This section has two objectives. The first is the determination of a lower bound on the E{throughput} as a function of ρ, and the second is the determination of an E{delay} vs E{throughput} curve. These two results are displayed in Figs. 4.2.2.1 and 4.2.2.2.

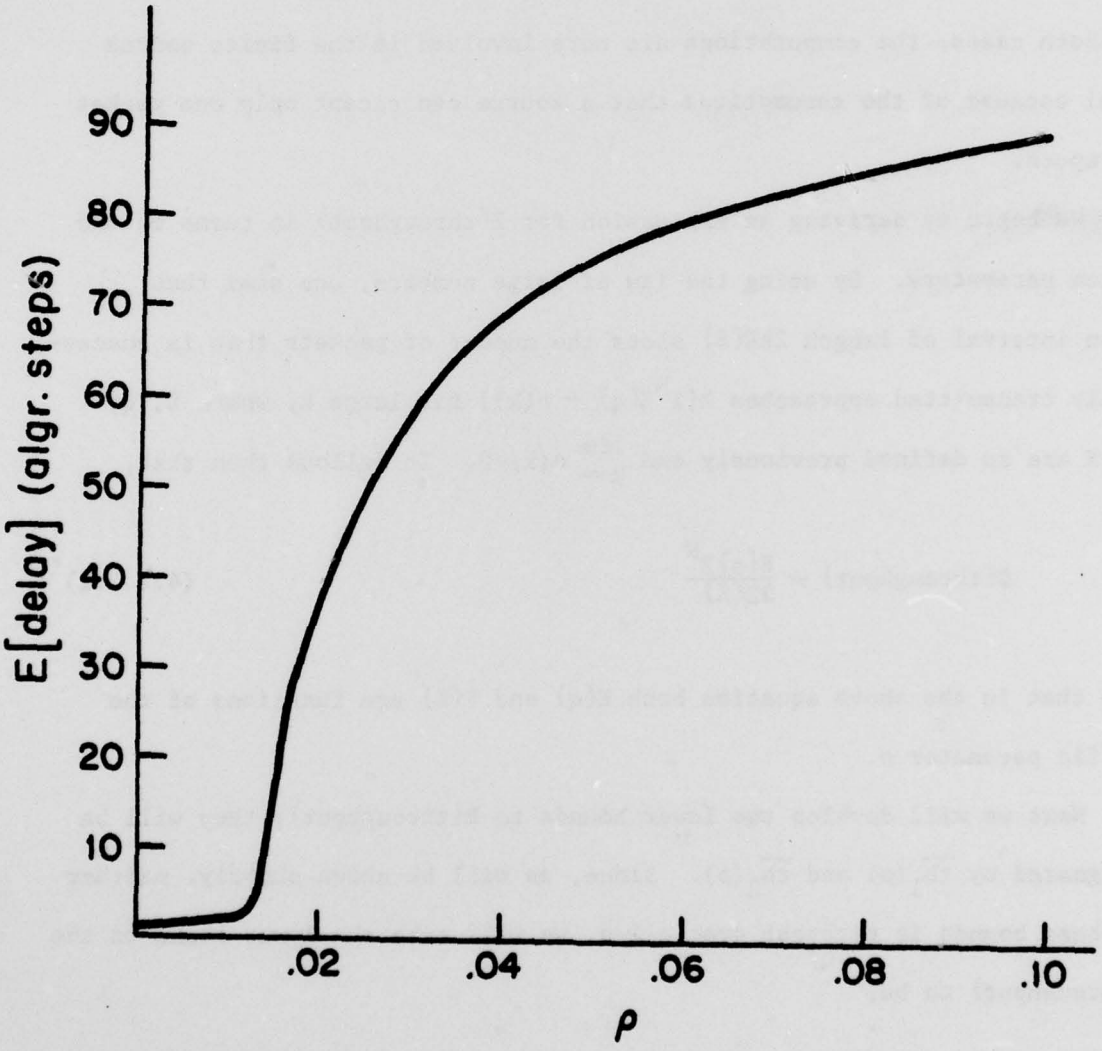


Figure 4.2.1.4 Upper Bound to the E{delay} versus ρ for the Binary Tree Algorithm with 64 Sources

The definition of throughput for the finite source model is the same as it was for the Poisson source model. That is, the average throughput is the fraction of time that the channel contains valid data, i.e., exactly one packet/slot. Even though the definition of E{throughput} is the same for both cases, the computations are more involved in the finite source model because of the assumptions that a source can accept only one packet per epoch.

We begin by deriving an expression for E{throughput} in terms of the system parameters. By using the law of large numbers, one sees that in an interval of length $2kE\{\ell\}$ slots the number of packets that is successfully transmitted approaches $k[2^N E\{q\} + o(k)]$ for large k, where ℓ, q, and N are as defined previously and $\lim_{k\to\infty} o(k)=0$. It follows then that,

$$E\{\text{throughput}\} = \frac{E\{q\}2^N}{2E\{\ell\}} \qquad (4.2.2.1)$$

Note that in the above equation both $E\{q\}$ and $E\{\ell\}$ are functions of the traffic parameter ρ.

Next we will develop two lower bounds to E{throughput}; they will be designated by $\overline{th}_1(\rho)$ and $\overline{th}_2(\rho)$. Since, as will be shown shortly, neither of these bounds is tightest over all ρ, we will take the lower bound to the E{throughput} to be,

$$\overline{th}_\ell = \text{Max}\{\overline{th}_1, \overline{th}_2\} \qquad (4.2.2.2)$$

Now we will derive $\overline{th}_1$. Equation (4.2.2.1) can be rewritten as follows

$$E\{\text{throughput}\} = 2^{N-1} \frac{\sum_{\ell_1} E\{q|\ell_1\}p\{\ell_1\}}{\sum_{\ell_1} E\{\ell_2|\ell_1\}p\{\ell_1\}}$$

$$\geq 2^{N-1} \min_{\ell_1} \frac{E\{q|\ell_1\}}{E\{\ell_2|\ell_1\}} = \overline{th}_1 \qquad (4.2.2.3)$$

Where $E\{q|\ell_1\}$ and $E\{\ell_2|\ell_1\}$ are given by Eqs. (4.1.0.1) and (4.2.1.11).

Next we will derive $\overline{th}_2$. First note that $E\{\ell|q\}$ (see Eq. 4.1.0.1) can be lower bounded as follows:

$$E\{q|\ell\} \geq a\ell - a + \rho \qquad \text{for } \ell=1,2,\ldots,2^N-1 \qquad (4.2.2.4)$$

where

$$a = \frac{1 - (1-\rho)^{2^N-1} - \rho}{2^N-2} \qquad (4.2.2.5)$$

Taking expectations of both sides of Eq. (4.2.2.4) we have

$$E\{q\} \geq a\,E\{\ell\} - a + \rho \qquad (4.2.2.6)$$

Substituting this into Eq. (4.2.2.1) we have

$$E\{th\} \geq [a + \frac{\rho-a}{E\{\ell\}}]2^{N-1} \qquad (4.2.2.7)$$

It can be shown that $\rho - a > 0$, and therefore

$$E\{th\} \geq [a + \frac{\rho-a}{\ell_u}]2^{N-1} \equiv \overline{th}_2 \qquad (4.2.2.8)$$

This is the desired form for $\overline{th}_2$. $\overline{th}_1(\rho)$ and $\overline{th}_2(\rho)$ were calculated for N=6 for several values of ρ. The results are presented in Table 4.2.1.1 and in Fig. 4.2.2.1.

The performance curve ($\overline{\delta}_u$ vs $\overline{th}_\ell$) can be obtained directly from Table 4.2.1.1. This is plotted in Fig. 4.2.2.2.

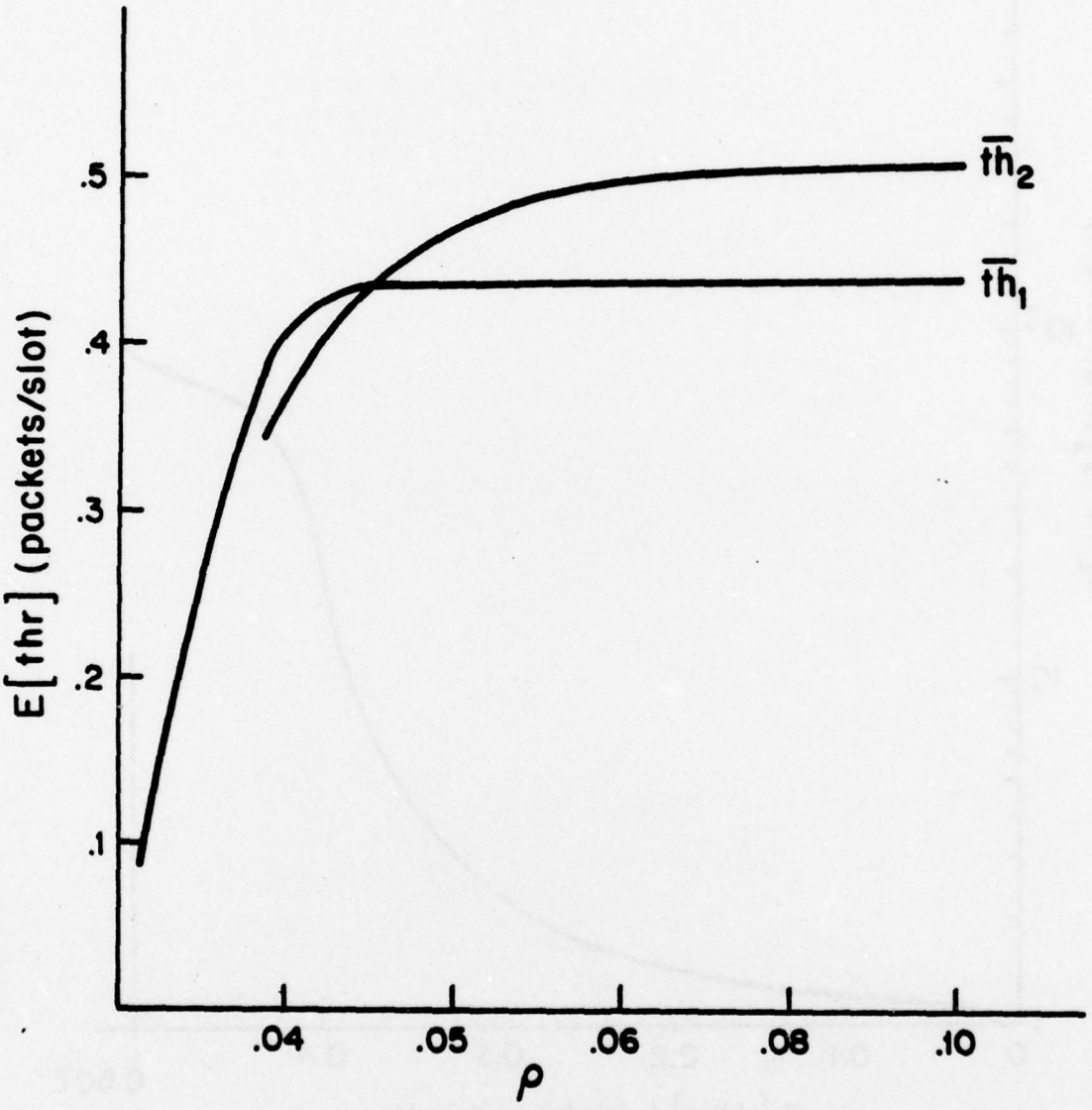


Figure 4.2.2.1 Lower Bounds to the Average Throughput versus ρ for the Binary Tree Algorithm with 64 Sources

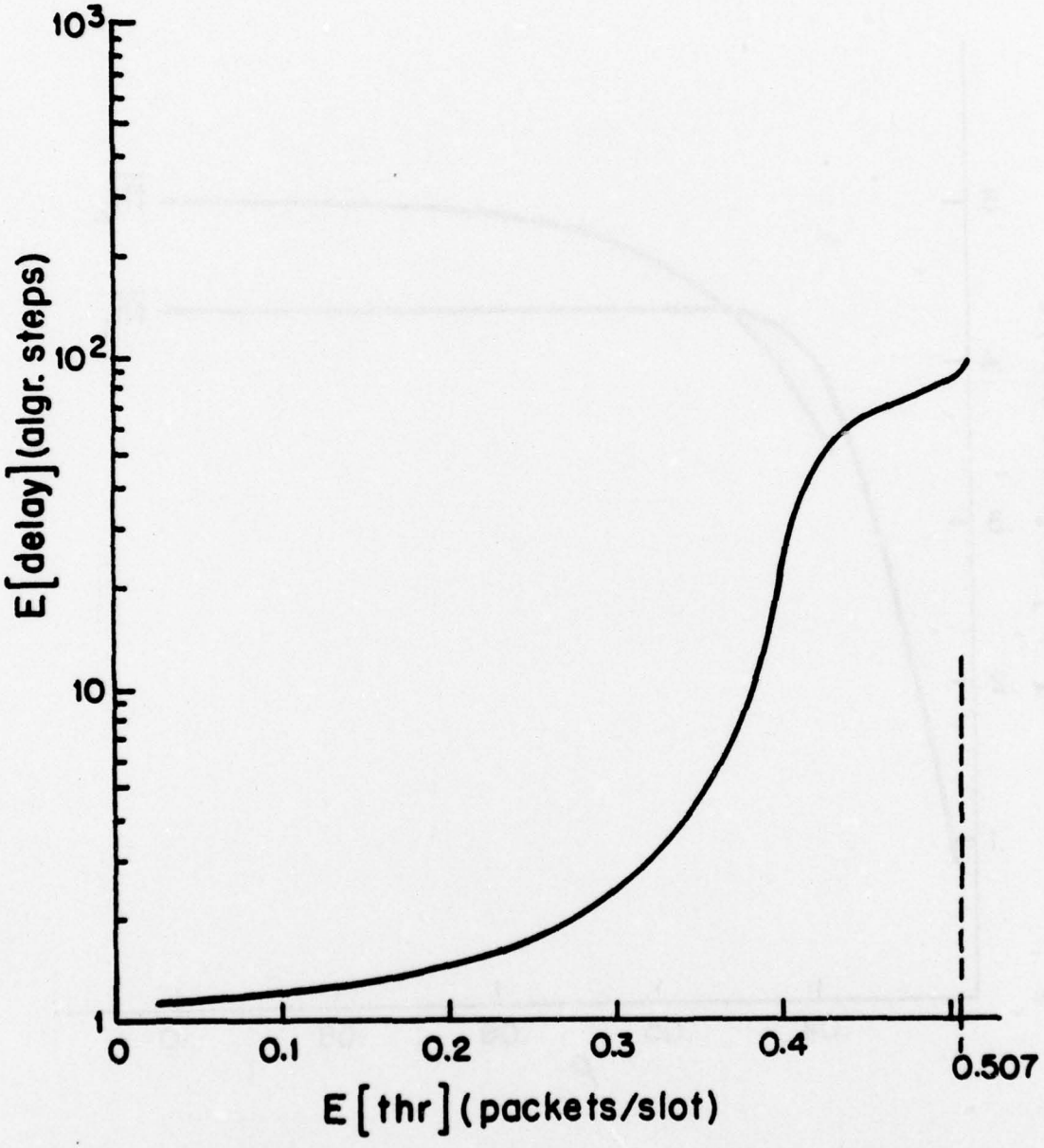


Figure 4.2.2.2 The Upper Bound to E{delay} versus the Lower Bound the E{throughput} for the Binary Tree Algorithm with 64 Sources

Note: An Algr. step equals one round trip delay plus two slots

4.3 Optimum Dynamic Tree Algorithm with Finite Source Model

There are two major objectives to this section; the determination of the optimum finite tree algorithm and the analysis of that algorithm. These two problems are considered in Subsection 4.3.1 and 4.3.2, respectively. In Subsection 4.3.2 we also prove that the optimum dynamic tree algorithm is superior to the TDMA protocol.

4.3.1 Optimum Tree

The criterion of optimality to be used here is the same as that of Chapter 3. That is, the optimum tree is that which minimizes the expected number of slots needed to process the 2^N sources given that the probability that any one of them has a packet to transmit is q. Since the number of sources is finite, the optimization is carried out over a smaller set of trees then it was in the Poisson source model. More specifically, we are going to restrict all nodes to be binary except for the root node which can have a degree that is a power of 2. In other words, if we let $g_0=2^K$, then the problem is to choose K=1,2,...,N so that $2E\{\ell|q,N,K\}$ is minimum.

The reason for restricting the initial degree to be a power of two is easier implementation. Variations in the degree of the root node under this restriction are equivalent to starting the binary tree algorithm at different levels, thus not requiring a different tree each time the degree of the root node is changed. For example, an algorithm whose tree has 2^N leaves and root node degree equal to 2^K is equivalent to a binary tree algorithm that starts with the nodes of depth K.

Now we will determine K^*, the optimum K, as a function of q. More specifically we will obtain an equation whose solution is $\hat{q}(K,N)$, where $\hat{q}(K,N)$ is defined by

$$K^* = K \text{ for } \hat{q}(K,N) < q \leq \hat{q}(K+1,N) \qquad (4.3.1.1)$$

As we will shortly see, $\hat{q}(K,N) = \hat{q}(N-K)$. It can be shown by a procedure analogous to that of Section 3.3.1 that for a tree with 2^N leaves and a root node degree 2^K, that

$$E\{\ell_2|q,N,K\} = 2^{K-1}\, E\{\ell_2|q,N-K+1,1\} \qquad (4.3.1.2)$$

Substituting Eq. (4.2.1.11) into this expression results in,

$$E\{\ell_2|q,N,K\} = 2^{K-1}\{1 + 2D(q,N-K)\} \qquad (4.3.1.3)$$

$\hat{q}(N,K)$ is determined by setting

$$E\{\ell_2|q,N,K-1\} = E\{\ell_2|q,N,K\} \qquad (4.3.1.4)$$

and then solving for q. Substituting Eqs. (4.2.1.3) and (4.3.1.3) into Eq. (4.3.1.4) and then rearranging we have the following expression which defines $\hat{q}$.

$$\psi(\hat{q},\ N-K+1) = 1/2$$

or

$$[1 + \hat{q}(2^{N-K+1}-1)](1-\hat{q})^{(2^{N-K+1}-1)} = 1/2 \qquad (4.3.1.5)$$

Equation (4.3.1.5) was solved for N-K=0,1,2,3,4,5. The results are given in Table 4.3.1.1 below:

N-K	$\hat{q}(N-K)$
0	.707
1	.38
2	.20
3	.10
4	.05
5	.025

TABLE 4.3.1.1

OPTIMUM ROOT NODE DEGREE

From the above table the following expression for $\hat{q}(N-K)$ is evident

$$\hat{q}(N-K) = \begin{cases} .707 & \text{for } N-K=0 \\ .38 & N-K=1 \\ .8/2^{N-K} & N-K \geq 2 \end{cases} \qquad (4.3.1.6)$$

The optimum K for N=6, which follows from Eqs. (4.3.1.1) and (4.3.1.6) was substituted into Eq. (4.3.1.2) and the result is plotted in Fig. 4.2.1.2. As can be seen from that figure, $E\{\ell|q\}$ is the same for both binary and optimum tree for small q. As q increases, however, the optimum protocol is definitely superior. What this suggests is that the optimum algorithm should be used when the traffic is heavy.

When K=N the tree algorithm is equivalent to the TDMA protocol. Since $\hat{q}(N-K=0) = 1/\sqrt{2} = .707$ we see that TDMA is optimum if $q \geq 1/\sqrt{2}$. What is

surprising is that this result does not depend on N. The relationship between the tree algorithm and TDMA is considered in more detail in Theorem 4.3.2.1 in the following section.

We conclude this section with the presentation of the optimal dynamic strategy. This is as follows:

1. Observe ℓ_1, the length of the previous epoch.
2. Substitute ℓ_1 into Eq. (4.1.0.1) to obtain q.
3. Use the q, determined in Step-2, to obtain K^* from Eq. (4.3.1.6).
4. Use the tree, determined in Step-3, to resolve any conflicts that may exist.

4.3.2 Analysis of the Optimum Dynamic Tree Algorithm

In this section we will analyze the optimum dynamic algorithm when it is used in conjunction with the finite source model. First, we will develop an upper bound to the E{delay} in terms of ρ, secondly, we will develop a corresponding lower bound to the E{throughput}, and finally, combine these two bounds to obtain a $\overline{\delta}_u$ vs $\overline{th}_\ell$ performance curve. These three results are illustrated in Figs. 4.3.2.3, 4.3.2.4 and 4.3.2.5. In this section, we will also prove (in Theorem 4.3.2.1) that the E{delay} of the optimum dynamic tree algorithm is smaller than or equal to that of the TDMA protocol. This interesting result should be evident from the work of the preceding section.

i. Upper Bound to E{delay}

As in Section 4.2.1 the delay is decomposed into d_1 and d_2. Therefore, we can write

$$E\{\delta|\ell,\rho,N,K\} = E\{d_1|\ell,\rho,N,K\} + E\{d_2|\ell,\rho,N,K\} \qquad (4.3.2.1)$$

$E\{d_1|\ell,\rho,N,K\}$ in the preceding equation is the same as that of the static algorithm. That is,

$$E\{d_1|\ell,\rho,N,K\} = \ell - 1 - \frac{1-\rho-(1-\rho)^{\ell}(\ell\rho+1-\rho)}{\rho(1-(1-\ell)^{\ell})} \qquad (4.3.2.2)$$

$E\{d_2|\ell,\rho,N,K\}$ can be shown to be

$$E\{d_2|\ell,\rho,N,K\} = \frac{2^{(K-1)}-1}{2}\ E\{\ell_2|q(\ell,\rho),N-K+1,1\} + E\{d_2|q(\ell,\rho),\ N-K+1,1\} \qquad (4.3.2.3)$$

In the preceding equation, q, $E\{d_2|q,m,1\}$, and $E\{\ell_2|q,m,1\}$ are given by Eqs. (4.1.0.1), (4.2.1.8), and (4.2.1.11).

The conditional delay given by Eq. (4.3.2.1) was computed for N=6 and $K=K^*$ (from Table 4.3.1.1). The results are shown in Fig. 4.3.2.1. As can be seen from that figure, $E\{\delta|\ell,\rho,N,K^*\}$ is increasing in ℓ but it is not concave. The nonconcavity of this function is especially evident for $\rho=.4$ around $\ell=10$. Next, we proceed by obtaining $f_{v1}(\ell)$, the tightest concave lower bound to $E\{\delta|\ell,\rho,N,K^*\}$ and then upper bounding $E\{\delta\}$ by

$$E\{\delta\} \leq f_{v1}(\bar{\ell}_u). \qquad (4.3.2.4)$$

The function f_{v1} is determined graphically from Fig. 4.3.2.1 and from a more detailed computer printout.

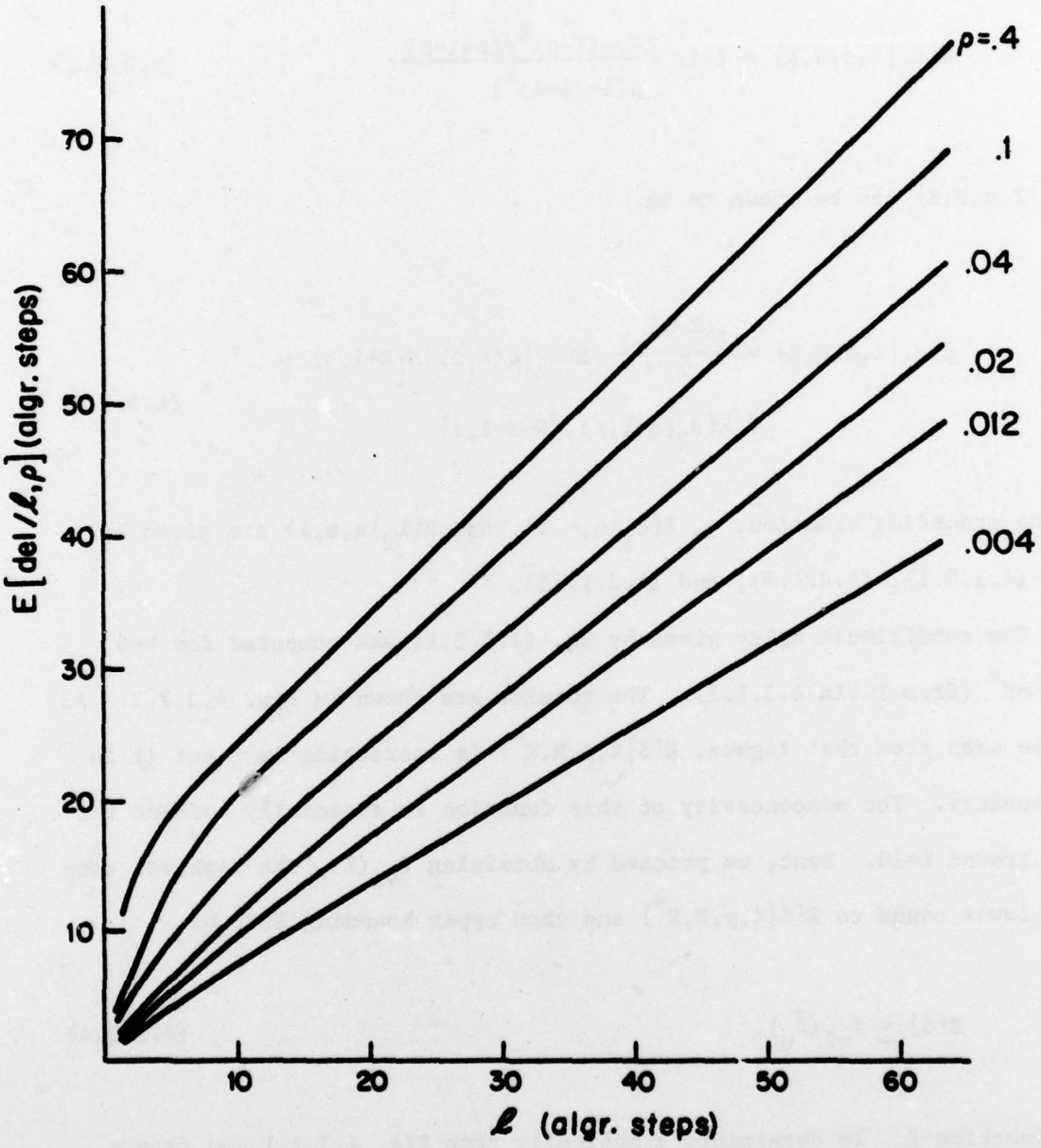


Figure 4.3.2.1 E{delay|ℓ,ρ} versus ℓ for the Optimum Dynamic Tree Algorithm with 64 Sources

The next step is to calculate $\overline{\ell}_u(\rho)$. It can be seen that

$$E\{\ell_2|\ell_1,\rho,N,K^*\} = 2^{K^*-1}\, E\{\ell_2|\ell_1,\rho,N-K^*+1,1\} \qquad (4.3.2.5)$$

where the conditional mean on the right of Eq. (4.3.2.5) is given by Eq. (4.2.1.8). Equation (4.3.2.5) is plotted in Fig. (4.3.2.2) for N=6. By a procedure similar to that of Section 4.2.1, $\overline{\ell}_u(\rho)$ is calculated. The results are listed in Table 4.3.2.1. Finally $\overline{\ell}_u$ is substituted into Eq. (4.3.2.4) and the upper bound to E{δ}, thus derived, is tabulated in Table 4.3.2.1 and plotted in Fig. 4.3.2.3.

ii. Lower Bound to E{throughput}

Here as in Section 4.2.2, we will determine $\overline{th}_1$ and $\overline{th}_2$. These two quantities are defined by Eqs. (4.2.2.3) and (4.2.2.9). They have been calculated using the values of $\overline{\ell}_u$ given in Table 4.3.2.1, and the results are presented on that same table and in Fig. 4.3.2.4. Finally, by using the lower bound to E{throughput} the $\max\{\overline{th}_1, \overline{th}_2\}$ we obtain from Table 4.3.2.1 the $\overline{\delta}_u$ vs $\overline{th}_\ell$ performance curve shown in Fig. 4.3.2.5.

iii. On the Superiority of the Optimum Dynamic Tree Over the TDMA Protocol

The following theorem is based on the observation that $\frac{\partial}{\partial q} E\{\delta|q\} \geq 0$. That this is so follows by showing that

$$\frac{\partial}{\partial \ell} E\{\delta|\ell\}\, \frac{\partial \ell}{\partial q} \geq 0.$$

Theorem 4.3.2.1: Let q be the probability that a source has a packet to transmit and let δ be the delay that a packet undergoes when it is processed by the optimum tree algorithm. Furthermore, assume that,

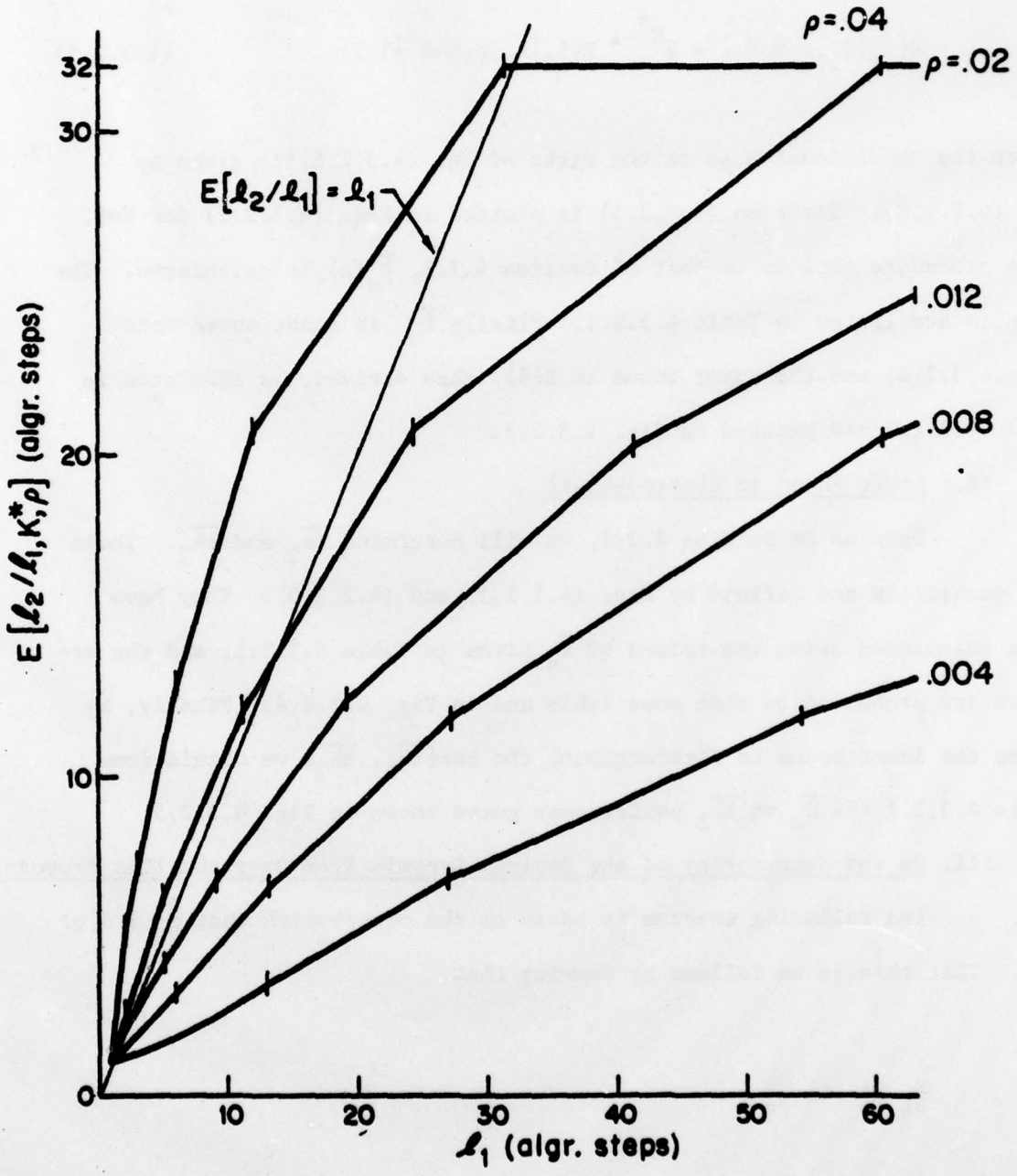


Figure 4.3.2.2 $E\{\ell_2|\ell_1,\rho\}$ versus ℓ_1 for the Optimum Dynamic Tree Algorithm with 64 Sources

ρ	$\overline{\ell}_u$	$\overline{\delta}_u$	$\overline{th}_1$	$\overline{th}_2$
.004	1,04	1.24	.125	
.008	1.20	1.60	.232	
.012	1.72	2.40	.314	
.016	6.8	7.7	.372	.329
.02	14	14.9	.410	.381
.03	19	30.5	.454	.443
.04	32	35.4	.462	.481
.05	32	36.2		.505
.1	32	39.6		.549
.2	32	43.7		.600
.4	32	46.2		.700
.6	32	47.0		.812

TABLE 4.3.2.1

AVERAGE DELAY AND AVERAGE THROUGHPUT FOR OPTIMUM DYNAMIC TREE ALGORITHM WITH 64 SOURCES

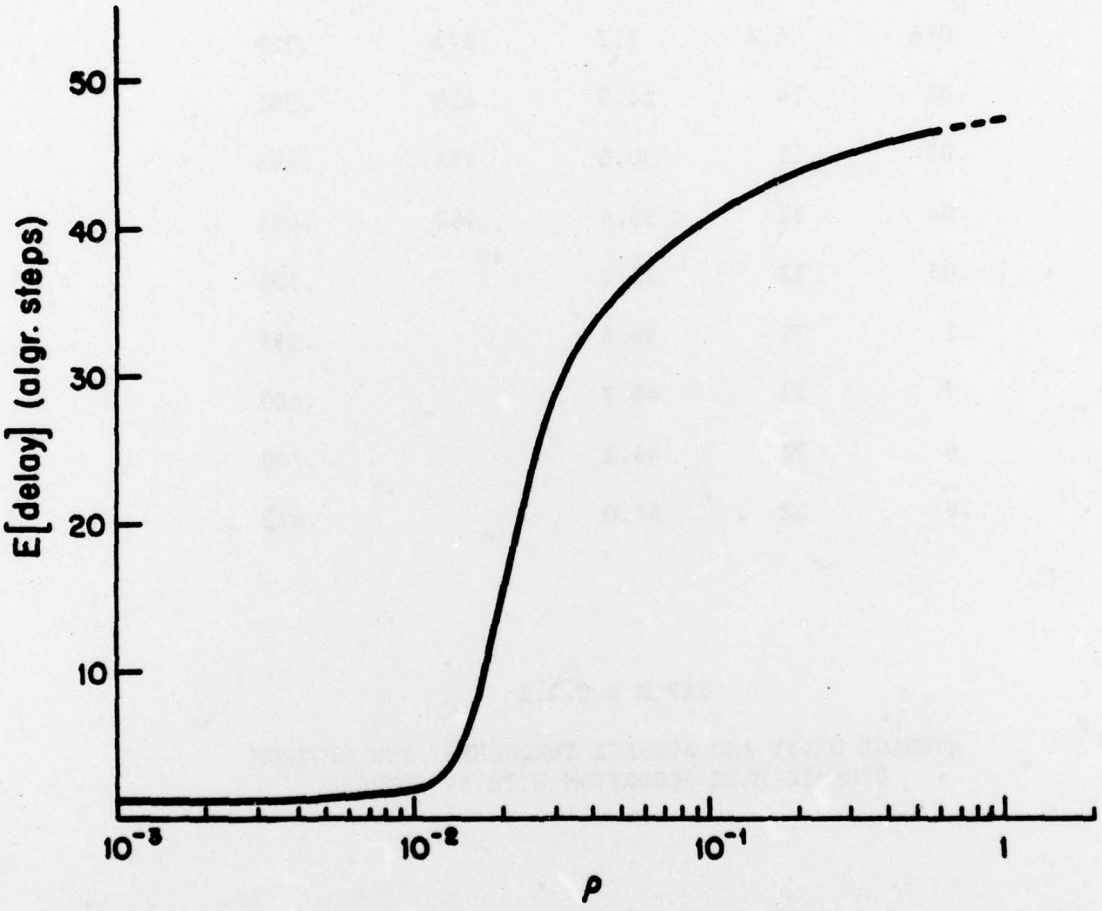


Figure 4.3.2.3 Upper Bound to the E{delay} versus ρ for the Optimum Dynamic Tree Algorithm with 64 Sources

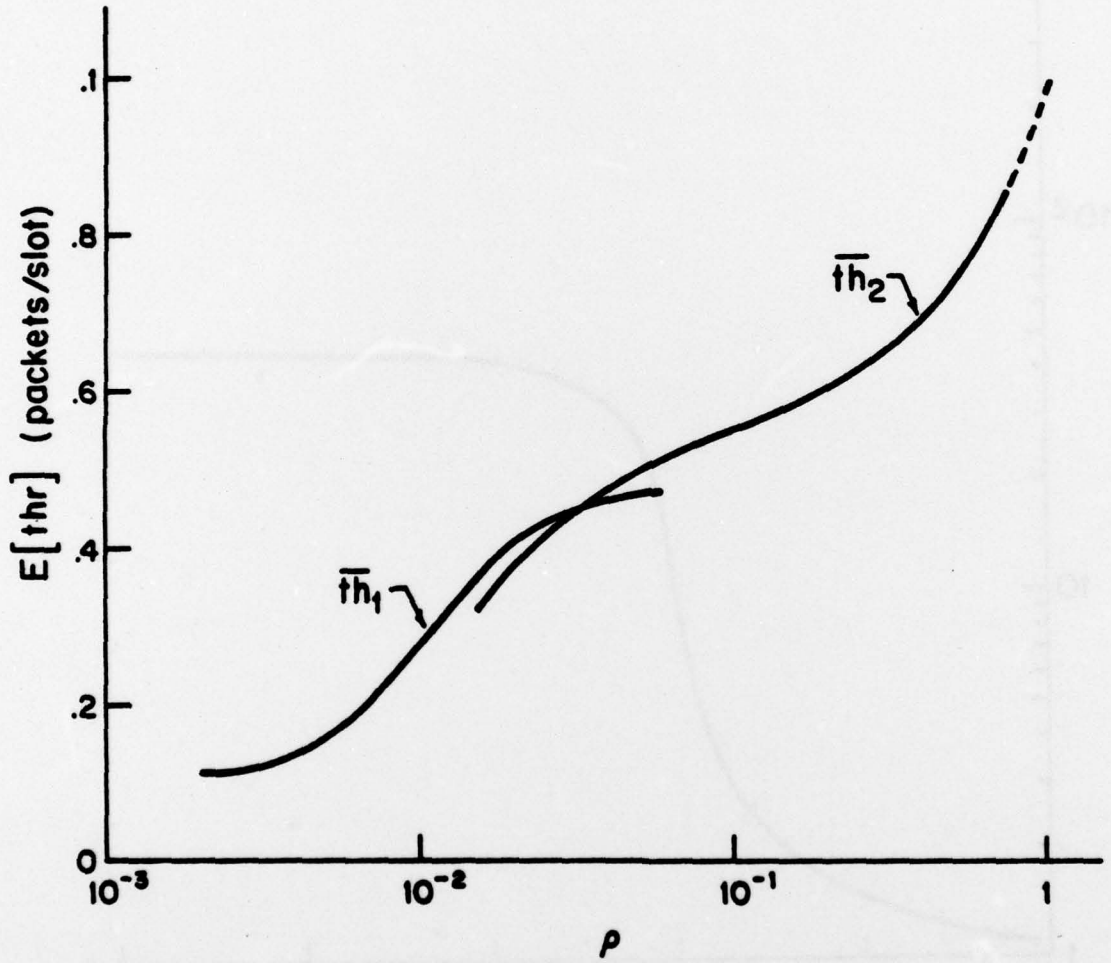


Figure 4.3.2.4 Lower Bounds to E{throughput} versus ρ for the Optimum Dynamic Tree Algorithm with 64 Sources

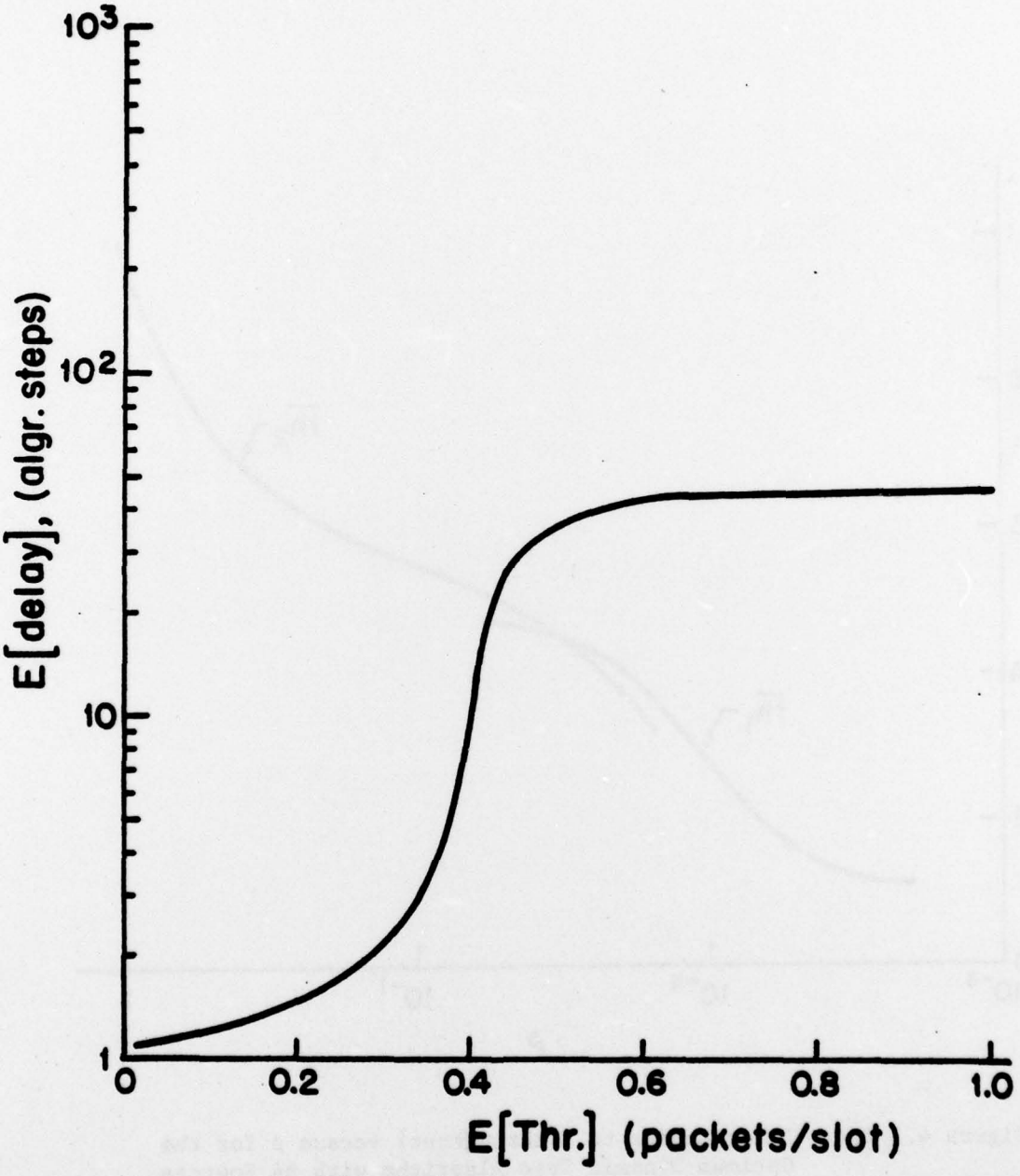


Figure 4.3.2.5 The Upper Bound to E{delay} versus the Lower Bound to E{throughput} for the Optimum Dynamic Tree Algorithm with 64 Sources

Note: An Algr. step equals one round trip delay plus two slots

$$\frac{\partial}{\partial \ell} E\{\delta|\ell\} \frac{\partial \ell}{\partial q} \geq 0.$$

Then the E{delay} of the optimum dynamic tree protocol is less than or equal to the E{delay} of the TDMA protocol.

Proof:

$$E\{\delta\} = \int_0^1 E\{\delta|q\}\, dF(q)$$

integrating by parts we have,

$$E\{\delta\} = E\{\delta|q\}\, F(q)\Big|_0^{1^-} - \int_0^1 \left(\frac{\partial}{\partial q} E\{\delta|q\}\right) F(q)dq.$$

But $\frac{\partial}{\partial q} E\{\delta|q\} \geq 0$, $F(q) \geq 0$, $F(0) = 0$, and $F(1) = 1$; therefore

$$E\{\delta\} \leq E\{\delta|q=1\}$$

But from Eq. (4.3.1.6) it follows that at q=1 the optimum tree algorithm is the TDMA protocol.

QED

APPENDIX A4

PROPERTIES OF THE BINARY TREE/FINITE SOURCE SYSTEM

Let there be 2^N sources in a multiaccess system where each source may be active with probability q, and assume that the multi-access protocol is the static binary tree algorithm. Also let ℓ and d have the same definitions as they did in Chapter 2. Then in this appendix we will derive expressions for $E\{\ell|q\}$ and $E\{d|q\}$. Since the work of this appendix parallels that of Appendix A2, we will not be as detailed here.

A4.1 Derivation of $E\{\ell|q\}$

The number of nodes ℓ visited by the algorithm may be written as follows:

$$\ell = 1 + \sum_{i=1}^{N-1} \sum_{j=0}^{2^i-1} x_{ij} \qquad \text{(A4.1.1)}$$

where

$$x_{ij} = \begin{cases} 1 \text{ if node } n_{ij} \text{ is visited} \\ 0 \text{ otherwise} \end{cases} \qquad \text{(A4.1.2)}$$

Since a node n_{ij} is visited if there are at least two active sources in T_{ij} we have that

$$P(x_{ij}=1/q) = \psi(q,N-i) \qquad \text{(A4.1.3)}$$

where

$$\psi(q,m) \equiv 1 - (1-q)^{2^m} - 2^m q(1-q)^{2^m-1} \qquad \text{(A4.1.4)}$$

and it follows that

$$E\{\ell|q\} = 1 + \sum_{i=1}^{N-1} 2^i \psi(q,N-i) \qquad \text{(A4.1.5)}$$

or

$$E\{\ell|q\} = 1 + 2D(q,N-1) \qquad \text{(A4.1.6)}$$

where

$$D(q,m) = \sum_{i=0}^{m-1} 2^i \psi(q,m-i) \qquad \text{(A4.1.7)}$$

Equations (A4.1.5) through (A4.1.7) are the desired results.

A4.2 Derivation of $E\{d|q\}$

$E\{d|q\}$ is the number of nodes that are visited before a randomly selected packet from the contending set is successfully transmitted. First we will calculate $E\{d_s|q\}$, where d_s is the number of nodes visited before source-s is successfully transmitted. $E\{d_s|q\}$ may be decomposed as

$$E\{d_s|q\} = X_s + Y_s \qquad \text{(A4.2.1)}$$

Where X_s and Y_s have the same definitions as in Appendix A2.6. That is, X_s is the average number of nodes lying on s, and Y_s is the average number of nodes above s that were visited before the successful transmission of s. As in Appendix A2.6 we may write X_s and Y_s as

$$X_s = 1 + \sum_{i=1}^{N-1} \phi(q,N-i) \qquad \text{(A4.2.2)}$$

and

$$Y_s = \sum_{i=1}^{N-1} S_{i-1}\, D(q,N-i)\, \phi(q,N-i) \qquad \text{(A4.2.3)}$$

where

$$\phi(q,m) = 1 - (1-q)^{(2^m-1)}$$

$$= \Pr\{\text{at least 2 active sources in a branch of depth } m, \text{ given that a particular source in that branch is active}\}.$$

To obtain the final result, first substitute Eqs. (A4.2.2) and (A4.2.3) into Eq. (A4.2.1) and then add $\frac{1}{2} E\{d_s|q\}$ and $\frac{1}{2} E\{d_{\bar{s}}|q\}$, as shown in Eq. (A4.2.5). Note that $\bar{s}$ is the ones' complement of s.

$$\frac{1}{2} E\{d_s|q\} + \frac{1}{2} \{d_{\bar{s}}|q\} = 1 + \sum_{i=1}^{N-1} [1 + \frac{1}{2} D(q,N-i)]\, \phi(q,N-i) \qquad \text{(A4.2.5)}$$

$$= E\{d|q\} \qquad \text{(A4.2.6)}$$

The last step follows because $E\{d_s|q\} + E\{d_{\bar{s}}|q\}$ is independent of s.

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BIOGRAPHICAL NOTE

John Ippocratis Capetanakis was born in Greece on December 5, 1944. He attended public schools in New Britain, Connecticut and graduated from New Britain High School in June, 1964.

Mr. Capetanakis received his B.S. and M.S. degrees in electrical engineering from M.I.T. in June, 1970. From September, 1968, to June, 1972, he was supported by a teaching assistantship. In June, 1972, he joined the M.I.T. Lincoln Laboratory staff where he was employed until September, 1975. From September, 1975, to the present time he has been a full-time graduate student at M.I.T. supported by the Lincoln Laboratory staff associate program.

Mr. Capetanakis is a member of Tau Beta Pi, Eta Kappa Nu, and Sigma Xi.